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METHODS OF CALCULATING COHOMOLOGICAL AND HOCHSCHILD–MITCHELL DIMENSIONS OF FINITE PARTIALLY ORDERED SETS

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Abstract

Mitchell characterized all finite partially ordered sets with incidence ring of Hochschild dimension 0, 1, and 2. Cheng characterized all finite partially ordered sets of cohomological dimension one. There are no conjectures in other dimensions. This article contains the algorithms for calculating the dimensions of finite partially ordered sets by elementary operations over rows and columns of matrices with integer entries.

1. Cohomological dimension

Denote by $\mathbf{N}$ the set of nonnegative integers, $\mathbf{Z}$ the additive group of integers, Ab the category of abelian groups and homomorphisms. For any subset $M \subseteq \mathbf{N}$ the sup will be considered in $\{-1\} \cup \mathbf{N} \cup \{\infty\}$. Thus we let $\sup \emptyset = -1$.

Let $(X, \leqslant)$ be a partially ordered set (shortly *poset*). We consider X as a small category with objects $ObX = X$, in which for every $x, y \in X$ the set $X(x,y)$ of morphisms $x \to y$ consists of one element if $x \leqslant y$ in X, and $X(x,y) = \emptyset$, otherwise. If $x \leqslant y$ and $x \neq y$, then we write $x < y$. If neither $x \leqslant y$ nor $y \leqslant x$, then we say that x and y are *incompatible*. If every pair of distinct elements in X are incompatible, then we say that X is *discrete*. If for every $x, y \in X$ there exists a sequence $x_0 \leqslant x_1 \geqslant x_2 \leqslant \cdots \geqslant x_{2n}$ of elements in X such that $x_0 = x$ and $x_{2n} = y$, then X is *connected*. Every subset $Y \subseteq X$ will be considered as the poset in which $y_1 \leqslant y_2$ for $y_1, y_2 \in Y$ if and only if $y_1 \leqslant y_2$ in X. The maximal connected subsets of X are called the *connected components*.

Denote by Ab^X the category of functors $X \to Ab$. For every $F \in Ab^X$ and $n \in \mathbf{N}$ we have the abelian groups

$$C^n(X,F) = \prod_{x_0 < \cdots < x_n} F(x_n).$$

Considering elements of $C^n(X,F)$ as functions with $\varphi(x_0 < \cdots < x_n) \in F(x_n)$, we define homomorphisms $d^n : C^n(X,F) \to C^{n+1}(X,F)$ by the formulas

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$$(d^n\varphi)(x_0 < \cdots < x_{n+1}) = \sum_{i=0}^{n}(-1)^i\varphi(x_0 < \cdots < \hat{x}_i < \cdots < x_{n+1}) + (-1)^{n+1}F(x_n < x_{n+1})(\varphi(x_0 < \cdots < x_n)).$$

Here $(x_0 < \cdots < \hat{x}_i < \cdots < x_{n+1}) = (x_0 < \cdots < x_{i-1} < x_{i+1} \cdots < x_{n+1})$, for $0 \leqslant i \leqslant n+1$.

It is well known that $d^{n+1} \circ d^n = 0$ for all $n \in \mathbf{N}$.

Let $C^n(X,F) = 0$ for $n < 0$. We obtain the complex $C^*(X,F) = \{C^n(X,F), d^n\}_{n\in\mathbf{Z}}$. Let $\lim_X : Ab^X \to Ab$ be the limit functor, $\lim_X^n : Ab^X \to Ab$ its n-th right derived functors, $n \geqslant 0$. Then $\lim_X^n F$ are natural isomorphic to groups $H^n(C^*(X,F)) = Ker d^n / Im d^{n-1}$.

Definition 1.1. *Let X be a poset. The cohomological dimension* cd X *is the sup of $n \in \mathbf{N}$ for which* $\lim_X^n \neq 0$.

If $X = \emptyset$ then the set of such n is empty, hence cd $\emptyset = -1$.

Cheng [2] characterized all finite posets of cohomological dimension one.

For each abelian group A we denote by $\Delta_X A : X \to Ab$ the functor with $\Delta_X A(x) = A$ for all $x \in X$, and $\Delta_X A(x \leqslant y) = 1_A$ for all $x \leqslant y$ in X. We denote $H^n(X,A) = \lim_X^n \Delta_X A$. Let pt be a poset consisting of one element. For every poset X there is precisely one map $S : X \to pt$. This map induces the homomorphisms $H^n(pt,A) \to H^n(X,A)$. Let $\tilde{H}^n(X,A)$ be cokernels of these homomorphisms. It is not hard to see that $\tilde{H}^n(X,A) \cong H^n(X,A)$ for all $n > 0$, and $\tilde{H}^0(X,A)$ is the cokernel of the homomorphism $A \to \lim_X \Delta_X A$ which carries each $a \in A$ into the thread $\{a\}_{x\in X}$.

Let $\tilde{H}^{-1}(\emptyset, A) = A$, and $\tilde{H}^n(\emptyset, A) = 0$ for all $n \neq -1$. When $n < -1$ we let $\tilde{H}^n(X,A) = 0$ for all posets X.

For any $z \in X$ we denote $W(z,X) = \{x \in X : x < z\}$, $X/z = \{x \in X : x \leqslant z\}$. For every $x, y \in X$ the *closed interval* is the subset $[x,y] = \{z \in X : x \leqslant z \leqslant y\}$ of X. A subset $W \subseteq X$ is *convex* if for any $x, y \in W$ the closed interval $[x,y]$ is included in W.

Let A be an abelian group, W a convex subset of X. We denote by $A[W] : X \to Ab$ the functor such that $A[W]|_W = \Delta_W A$ and $A[W](z) = 0$ for all $z \notin W$. If $W = \{z\}$ consists of one element z we denote $A[\{z\}]$ by $A[z] : X \to Ab$.

Lemma 1.2. *Let A be an Abelian group, X a poset, $z \in X$ an element, $n \in \mathbf{N}$ a number. Then there is an isomorphism*

$$\lim_X^n A[z] \cong \tilde{H}^{n-1}(W(z,X), A), \quad n \geqslant 0. \tag{1}$$

PROOF. For $n \geqslant 2$ the assertion is proved in [8, Lemma 3.1]. We will prove it for $n = 0$ and $n = 1$. If z is a minimal element in X then $\lim_X A[z] = A$ and $\lim_X^n A[z] = 0$ for all $n > 0$. If z is not minimal, then $\lim_X A[z] = 0$. Therefore, the assertion is true for $n = 0$. For $V = X/z$ and $W = W(z,X)$ there is an exact sequence

$$0 \to A[z] \to \Delta_V A \to A[W] \to 0.$$

The long exact sequence associated with $\lim_V^k$ gives an exact sequence

$$0 \to \lim_V A[z] \to \lim_V \Delta_V A \to \lim_V A[W] \to \lim_V^1 A[z] \to 0,$$

where $\lim_V \Delta_V A = A$. It follows that $\lim^1_V A[z]$ is a cokernel of the homomorphism $A \to \lim_V A[W]$. Hence $\lim^1_V A[z] \cong \tilde{H}^0(W, A)$. It was remarked in the proof of [**8**, Lemma 3.1] that $\lim^n_V A[z] \cong \lim^n_X A[z]$. Therefore $\lim^1_X A[z] \cong \tilde{H}^0(W, A)$, and (1) is true for all $n \geqslant 0$. Q.E.D.

Definition 1.3. *The cohomological height c.h.X of a poset X is the sup of $n \in \mathbf{N}$, for which $\tilde{H}^n(X, \mathbf{Z}) \neq 0$.*

If $X = \emptyset$, or X is a nonempty totally ordered set then $c.h.X = -1$.

Let $C_n X$ be the free abelian group generated by the set of all sequences $x_0 < \cdots < x_n$ in X. For each $n \geqslant 0$ we define the homomorphism $C_{n+1}X \xrightarrow{\partial_n} C_n X$ on elements of basis as $\partial_n(x_0 < \cdots < x_{n+1}) = \sum_{k=0}^{n+1}(-1)^k(x_0 < \cdots < \hat{x}_k < \cdots < x_{n+1})$. It is well known that homology groups of C_*X are isomorphic to the integer homology groups of the *nerve* of X [**4**]. We call it the *integer homology groups* of X and denote by $H_n X$. The complex $Ab(C_*X, \mathbf{Z})$ is isomorphic to $C^*(X, \Delta_X \mathbf{Z})$. Hence we have the exact sequence of Universal Coefficient Theorem [**10**, Ch.III, Theorem 4.1]

$$0 \longrightarrow Ext(H_{n-1}X, A) \longrightarrow H^n(X, A) \longrightarrow Ab(H_n X, A) \longrightarrow 0 \tag{2}$$

Proposition 1.4. *Let X be a nonempty finite poset. Then*

$$\text{cd } X = 1 + \sup\{c.h.W(z, X) : z \in X\}. \tag{3}$$

PROOF. Let $n = \text{cd } X$. The poset X is nonempty, hence $n \geqslant 0$. There is $F \in Ab^X$ such that $\lim^n_X F \neq 0$. It follows by [**8**, Corollary 2.2] that there are $A \in Ab$ and $z \in X$ satisfying $\lim^n_X A[z] \neq 0$, and hence $\tilde{H}^{n-1}(W(z, X), A) \neq 0$. For $k > n$ the groups $\lim^k_X A[z]$ are trivial, therefore $\tilde{H}^{k-1}(W(z, X), A) = 0$. Let $n > 1$. For $W = W(z, X)$ the following sequence is exact by (2)

$$0 \longrightarrow Ext(H_{n-2}W, A) \longrightarrow H^{n-1}(W, A) \longrightarrow Ab(H_{n-1}W, A) \longrightarrow 0$$

For every Abelian group B we have $H^n(W, B) \cong \lim^{n+1}_X B[z] = 0$. Therefore, either $H_{n-1}W$ is a nontrivial free group, or $H_{n-2}W$ has torsion. Each one of these conditions implies $H^{n-1}(W, \mathbf{Z}) \neq 0$. If $n = 1$ then $\tilde{H}^0(W, A) \neq 0$, hence W is nonempty and has more than one components, and $\tilde{H}^0(W, \mathbf{Z}) \neq 0$ in this case. Therefore, if $n = \text{cd } X \geqslant 1$ then $\tilde{H}^{n-1}(W(z, X), \mathbf{Z})$ for some $z \in X$, and consequently $n \leqslant 1 + c.h.W(z, X)$. There are isomorphisms $\lim^k_X \mathbf{Z}[x] \cong \tilde{H}^{k-1}(W(x, X), \mathbf{Z})$, hence $\tilde{H}^{k-1}(W(x, X), \mathbf{Z})$ for all $x \in X$ and $k > n$. Thus, if $\text{cd } X \geqslant 1$ then the inequality $\text{cd } X \geqslant 1 + c.h.W(x, X)$ holds for all $x \in X$.

We have proved $\text{cd } X = 1 + \sup\{c.h.W(z, X) : z \in X\}$ when $\text{cd } X \geqslant 1$. If $\text{cd } X = 0$ then by Laudal [**9**] each connected component of X contains the smallest element. In this case $\tilde{H}^n(W(z, X), \mathbf{Z}) = 0$ for all $n \geqslant 0$, we obtain $\text{cd } W(z, X) = -1$ for all $z \in X$. Therefore, if $\text{cd } X = 0$ then $\sup\{c.h.W(z, X) : z \in X\} = -1$. Thus (3) is true for all nonempty posets X. Q.E.D.

2. Hochschild–Mitchell dimension

Let X be a poset. Let X' be denoted the set of all closed intervals in X ordered by the inclusion:

$$[x_1, y_1] \leqslant [x_2, y_2] \Leftrightarrow [x_1, y_1] \subseteq [x_2, y_2].$$

If it is convenient we consider closed intervals as pairs $a \leqslant b$ ordered by $(x_1 \leqslant y_1) \leqslant (x_2 \leqslant y_2) \Leftrightarrow x_2 \leqslant x_1 \leqslant y_1 \leqslant y_2$. Baues and Wirshing [**1**] call X' the *factorization category*.

Using [**6**] we take the following definition of the Hochschild–Mitchell dimension for posets:

Definition 2.1. *Let X be a poset. The Hochschild–Mitchell dimension* $\dim X$ *of X is the cohomological dimension* $\operatorname{cd} X'$ *of the poset of closed intervals in X.*

It has been proved in [**6**] that Mitchell's definition of the Hochschild–Mitchell dimension is equivalent to the one above for partially ordered sets.

Mitchell [**11**] characterized the finite posets of Hochschild–Mitchell dimensions 0, 1 and 2; $\dim X = 0$ if and only if X is discrete, $\dim X \leqslant 1$ if and only if X is isomorphic to a free category. Related questions was studied in term of the incidence algebra in [**3**], [**5**], [**13**].

Let X and Y be posets. The *ordinal sum* $X + Y$ is the set $X \coprod Y$ ordered in such a way that $x \leqslant y$ for every $x \in X$ and $y \in Y$, and orders on X and Y are preserved. We consider the product $X^{op} \times Y$ as the subset of $(X+Y)'$ consisting of pairs $x < y$ with $x \in X$ and $y \in Y$. A nondecreasing map $f : X \to Y$ is called *strong coinitial* if for all $y \in Y$ the integer homology groups of the subset $\{x \in X : f(x) \leqslant y\}$ are isomorphic to homology groups of the point.

Lemma 2.2. *Let X and Y are posets, $T : (X+Y)' \to Ab$ a functor. If $T|_{X'} = 0$ and $T|_{Y'} = 0$ then $\lim_{(X+Y)'} T = 0$, $\lim^1_{(X+Y)'} T \cong \lim T|_{X^{op}\times Y}$, $\cdots$, $\lim^n_{(X+Y)'} T \cong \lim^{n-1} T|_{X^{op}\times Y}$, $\cdots$, $\forall n \geqslant 1$.*

Proof. It is proved in [**7**] that inclusions $X' \subseteq X' \cup (X^{op} \times Y)$ and $Y' \subseteq Y' \cup (X^{op} \times Y)$ are strong coinitial. Applying [**7**, Lemma 5.1] to the cover of $(X+Y)'$ by $U = X' \cup (X^{op} \times Y)$ and $V = Y' \cup (X^{op} \times Y)$ we obtain the exact sequence

$$0 \to \lim_{(X+Y)'} T \to \lim T|_{X'} \oplus \lim T|_{Y'} \to \lim T|_{X^{op}\times Y} \cdots$$

$$\to \lim^n_{(X+Y)'} T \to \lim^n T|_{X'} \oplus \lim^n T|_{Y'} \to \lim^n T|_{X^{op}\times Y} \cdots$$

which with the conditions $T|_{X'} = 0$ and $T|_{Y'} = 0$ gives the desired isomorphisms. Q.E.D.

The subset $]x, y[= \{z \in X : x < z < y\}$ is called the *open interval*. We denote $[x, y[= \{z \in X : x \leqslant z < y\}$ and $]x, y] = \{z \in X : x < z \leqslant y\}$.

Lemma 2.3. *Let X be a poset, A an abelian group. Then for every $a < b$ in X there exist isomorphisms*

$$\lim^n_{X'} A[a < b] \cong \tilde{H}^{n-2}(]a, b[, A)$$

for all $n \geqslant 0$. If $a = b$ then $\lim_{X'} A[a \leqslant b] = A$ and $\lim^n_{X'} A[a \leqslant b] = 0$ for $n > 0$.

Remark 2.4. *In particular, if $]a, b[= \emptyset$ then $\lim^1 A[a < b] \cong \tilde{H}^{-1}(\emptyset, A) = A$.*

Proof. For $n \geqslant 3$ this was proved in [**8**, Lemma 3.2]. It follows from [**8**, Lemma 3.1] that $\lim^n_{X'} A[a \leqslant b] \cong \lim^n_{[a,b]'} A[a \leqslant b]$. Let $a < b$. Application of Lemma 2.2 to $[a, b[= \{x \in X : a \leqslant x < b\}$ and $\{b\}$ provides isomorphisms

$$\lim^n_{[a,b]'} A[a < b] \cong \lim^{n-1}_{[a,b[^{op}\times\{b\}} A[a < b] \cong \lim^{n-1}_{[a,b[^{op}} A[a] \cong \tilde{H}^{n-2}(]a, b[, A),$$

for $a \neq b$. If $a = b$ then $\lim_{[a,b]'} A[a \leqslant b] = 0$ for $n > 0$. The assertion is evident for $n = 0$, since $\lim_{X'} A[a < b] = 0$. Q.E.D.

Theorem 2.5. *Let X be a poset which is not discrete. Then*

$$\dim X = 2 + \sup\{c.h.(]a,b[) : a < b\}.$$

PROOF. The poset X is not discrete, hence $\dim X \geqslant 1$ [**11**].By Lemma 2.3 there are isomorphisms $\lim^n_{X'} A[a < b] \cong \tilde{H}^{n-2}(]a,b[, A)$ for all $n \in \mathbf{N}$, $a < b$ in X, and $A \in Ab$. The application of [**8**, Corollary 2.2] to X' gives that the inequality $n \geqslant \dim X$ is equivalent to the existence of $A \in Ab$ and $a < b$ for which $\tilde{H}^{n-2}(]a,b[, A) \neq 0$. Hence, if $n = \dim X$ then there exists $a < b$ and A, for which $\tilde{H}^{n-2}(]a,b[, A) \neq 0$. Remark that if $k > n$ then the groups $\tilde{H}^{k-2}(]a,b[, B)$ are trivial for all Abelian groups B. If $n = \dim X = 1$ then $\tilde{H}^i(]a,b[, A) = 0$ for $i \geqslant 0$ and so $c.h.(]a,b[) = -1$. If $n = \dim X = 2$ then we obtain $\tilde{H}^0(]a,b[, A) \neq 0$, it follows that $]a,b[$ is nonconnected and consequently $\tilde{H}^0(]a,b[, \mathbf{Z}) \neq 0$. If $n = \dim X > 2$ then $\tilde{H}^{n-1}(]a,b[, B) = 0$ for all Abelian groups B, and it follows from the exact sequence of Universal Coefficient Theorem

$$0 \to Ext(H_{n-3}(]a,b[), A) \longrightarrow H^{n-2}((]a,b[), A) \longrightarrow Ab(H_{n-2}(]a,b[), A) \to 0$$

that either $\tilde{H}_{n-2}(]a,b[)$ is non trivial free, or $\tilde{H}_{n-3}(]a,b[)$ has a torsion. Every of the conditions implies that $\tilde{H}^{n-2}((]a,b[), \mathbf{Z}) \neq 0$. Therefore $\dim X \leqslant 2 + \sup\{c.h.(]a,b[) : a < b\}$. On the other hand if $\tilde{H}^{n-2}((]a,b[), \mathbf{Z}) \neq 0$, for $n \geqslant 2$, then cd $X' \geqslant n$. Hence $\dim X \geqslant 2 + \sup\{c.h.(]a,b[) : a < b\}$. Thus the equality holds.

3. Method of calculating the cohomological height

Let $\mathcal{P}$ be the set of all prime numbers $p > 1$, $\mathbf{Z}_p$ the additive group of integers *mod p*. We introduce a *p-cohomological height* for the description of calculating the cohomological height.

Definition 3.1. *For $p \in \mathcal{P}$ the p-cohomological height of a poset X is the sup of $n \in \mathbf{N}$ for which $\tilde{H}^n(X, \mathbf{Z}_p) \neq 0$.*

Remark 3.2. *The equality $c.h._pX = -1$ holds for every $p \in \mathcal{P}$ if and only if the groups $H^i(X, \mathbf{Z}_p)$ are zero for all $i > 0$ and X is connected or empty. If $H^i(X, \mathbf{Z}_p) = 0$ for all $i > 0$, then the rank of the vector space $H^0(X, \mathbf{Z}_p)$ over the field $\mathbf{Z}_p$ is equal to $\sum_{i\geqslant 0}(-1)^i|X^{(i)}|$ where $X^{(k)}$ is the set of all sequences $x_0 < x_1 < \cdots < x_k$, and $|X^{(k)}|$ the number of these sequences. Hence in this case $c.h._pX = -1$ if and only if $\sum_{i\geqslant 0}(-1)^i|X^{(i)}| \leqslant 1$, and the equality $\sum_{i\geqslant 0}(-1)^i|X^{(i)}| = 0$ holds if and only if $X = \emptyset$.*

Lemma 3.3. *Let X be an arbitrary finite poset. Then*

$$c.h.X = \sup\{c.h._pX : p \in \mathcal{P}\}.$$

PROOF. If $X = \emptyset$ then the equality is clear. Let $X \neq \emptyset$ and $c.h.X = -1$. Then $H^i(X, \mathbf{Z}) = 0$ for all $i > 0$ and $H^0(X, \mathbf{Z}) = \mathbf{Z}$. Using the exact sequences

$$0 \longrightarrow Ext(H_{n-1}X, \mathbf{Z}) \longrightarrow H^n(X, \mathbf{Z}) \longrightarrow Ab(H_nX, \mathbf{Z}) \longrightarrow 0 \qquad (4)$$

we establish that $H_iX = 0$ for all $i > 0$ and $H_0X = \mathbf{Z}$. Applying Universal Coefficient Theorem (2) we obtain $c.h._pX = -1$. If $c.h.X = 0$, then $H^i(X, \mathbf{Z}) = 0$ for all $i > 0$, it follows from Universal Coefficient Theorem that $H_iX = 0$ for all $i > 0$, and consequently $H^i(X, \mathbf{Z}_p) = 0$ for all $i > 0$. In this case we obtain $c.h._pX = 0$. Considering (4) in the case of $c.h.X > 0$ we obtain that either H_nX is free, or $H_{n-1}X$ includes $\mathbf{Z}_p$ for some $p \in \mathcal{P}$. It follows that $H^n(X, \mathbf{Z}_p) \neq 0$ for some $p \in \mathcal{P}$. Hence $\sup\{c.h._pX : p \in \mathcal{P}\} \geqslant c.h.X$. On the other hand, the following sequences are exact for all $k \geqslant n$

$$0 \longrightarrow Ext(H_kX, \mathbf{Z}_p) \longrightarrow H^{k+1}(X, \mathbf{Z}_p) \longrightarrow Ab(H_{k+1}X, \mathbf{Z}_p) \longrightarrow 0,$$

therefore $H^{k+1}(X, \mathbf{Z}_p) = 0$. It leads to $c.h._pX \leqslant c.h.X$ for all $p \in \mathcal{P}$. Thus, the equality is true. Q.E.D.

For arbitrary matrix with integer entries $A = (a_{ij})$ the *p-rank* is the rank of the matrix $A \ mod \ p = (a_{ij} modp)$ over the field $\mathbf{Z}_p$. The *absolute rank* is the number $\operatorname{rk} A = inf\{rk_pA : p \in \mathcal{P}\}$. It is well known (see [**14**, §I.6(D)], for example) that for every matrix A there are invertible matrices S and T, with the determinants ± 1, for which the matrix $\tilde{A} = S \circ A \circ T$ has entries $\tilde{a}_{ij} = 0$ for $i \neq j$, where $\tilde{a}_{ii} > 0$ divide $\tilde{a}_{i+1 i+1}$ for all $1 \leqslant i \leqslant n$, where n is a largest number for which $\tilde{a}_{nn} \neq 0$.

Remark that $\operatorname{rk} A$ is the number of ones in $\tilde{A}$.

It is true for all $n \in \mathbf{N}$ that $\operatorname{rk} d^{n-1} \leqslant \sum_{i \geqslant n} (-1)^{i-n} |X^{(i)}|$. If X is a nonempty poset then there exists $n \in \mathbf{N}$ for which this inequality is strong, since otherwise the equalities $\operatorname{rk} H^0(X, \mathbf{Z}) = \operatorname{rk} d^{-1} = 0$ lead to $X = \emptyset$.

Lemma 3.4. *Let X be a nonempty finite poset. Denote*

$$K_n = \sum_{i \geqslant n} (-1)^{i-n} |X^{(i)}|.$$

Let n be the largest number for which the strong inequality $\operatorname{rk} d^{n-1} < K_n$ holds. If $n > 0$ then $c.h.X = n$. If $n = 0$ then

$$c.h.X = \begin{cases} -1, & \text{if } K_0 = 1, \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. Let N be largest for which the set of sequences $x_0 < x_1 < \cdots < x_N$ is nonempty. Consider complex of abelian groups $C^n = C^n(X, \Delta_X \mathbf{Z}_p)$ and homomorphisms

$$0 \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{N-1}} C^N \xrightarrow{d^N} 0$$

We have $Kerd^N = C^N$. Hence $rk_pKerd^N = |X^{(N)}|$ for all $p \in \mathcal{P}$. If there is $p \in \mathcal{P}$ such that $rk_pd^{N-1} < |X^{(N)}|$ then the equality $H^N(X, \mathbf{Z}_p) = C^N / Im \ d^{N-1}$ implies that $c.h._pX = N$ and consequently $c.h.X = N$. Otherwise for every $p \in \mathcal{P}$ the exact sequence

$$0 \longrightarrow Kerd^{N-1} \longrightarrow C^{N-1} \longrightarrow Imd^{N-1} \longrightarrow 0$$

gives the equality $rk_pKerd^{N-1} = |X^{(N-1)}| - |X^{(N)}|$. If $n > c.h.X$, then $H^n(X, \mathbf{Z}_p) = 0$ for all $p \in \mathcal{P}$ and hence the following sequences are exact

$$0 \longrightarrow Kerd^n \longrightarrow C^n \xrightarrow{d^n} \cdots \xrightarrow{d^{N-1}} C^N \longrightarrow 0,$$

consequently $rk_p Kerd^n = |X^{(n)}| - \cdots + (-1)^{N-n}|X^{(N)}|$. For $n = c.h.X$ there exists $p \in \mathcal{P}$, such that $rk_p Kerd^n > rk_p Imd^n$. Therefore, for $n = c.h.X$ there is the inequality

$$rk_p Imd^{n-1} < |X^{(n)}| - \cdots + (-1)^{N-n}|X(N)|.$$

Thus the lemma is proved for $n > 0$. Let $n = 0$. Then $H^i(X, \mathbf{Z}) = 0$ for all $i > 0$, consequently $\operatorname{rk} H^0(X, \mathbf{Z}) = K_0$. The poset X is not empty, hence $K_0 > 0$. If $K_0 = 1$, then $\tilde{H}^0(X, \mathbf{Z}) = 0$ and $c.h.X = -1$. If $K_0 > 1$ then $\tilde{H}^0(X, \mathbf{Z}) \neq 0$ and $c.h.X = 0$. Q. E. D.

4. Algorithms

Calculating the absolute rank $\operatorname{rk} A$ of integer matrix. We transform the matrix A to the diagonal form $\tilde{A}$ by the well-known algorithm (see **[14]**, for example). Then the absolute rank $\operatorname{rk} A$ is equal to the number of ones in $\tilde{A}$. Hence the process is finished if we see an element on the diagonal of $\tilde{A}$ which is not equal to 1.

An elementary column (row) operation is a composition of the following transformations:

(1) interchange any two columns (rows),

(2) multiply any column (row) by -1,

(3) add of one column (row) to another.

If S and T are square matrices with $detS = \pm 1$, $detT = \pm 1$, for which is defined $S \circ A \circ T$, then the matrix $\tilde{A} = S \circ A \circ T$ may be obtained from A by elementary column and row operations.

The transformation to the diagonal form consists of the following actions:

(a) If a_{11} divides all elements of the first column and of the first row, then we subtract from the i-th row the first row multiplied by a_{i1}/a_{11} for each $i \neq 1$. Then we obtain a matrix with $a_{i1} = 0$ for all $i \neq 1$. Analogously we make $a_{1j} = 0$ for all $j \neq 1$.

(b) Let $||A||$ be smallest of the $|a_{ij}|$ among $a_{ij} \neq 0$. We interchange two columns and two rows in such a way that we obtain $|a_{11}| = ||A||$. If $a_{11} < 0$ then we multiply the first row by -1. We consider the case when (a) does not hold. Let i be a number for which a_{11} does not divide a_{i1}. If $a_{i1} < 0$ then we multiply the i-th row by -1. There are $q > 0$ and $0 < r < a_{11}$ such that $a_{i1} = a_{11}q + r$. We subtract the first row multiplied by q from the i-th row and have obtained a matrix A' with $||A'|| < ||A||$. We do analogous actions over columns if a_{11} does not divide a_{1j} for some j.

(c) Thus we can suppose that $a_{i1} = 0$ for $i > 1$ and $a_{1j} = 0$ for $j > 1$.

If a_{ij} is not divided by a_{11} for some (i, j) then we add the j-th column to the first column and subtract from i-th row the first row multiplied by q where $a_{i1} = a_{11}q + r$, $0 < r < a_{11}$. We obtain A' with $||A'|| < ||A||$ and then apply (b).

We obtain that all entries are divided by a_{11}. Now, if $a_{11} = 1$, then we apply the process to the submatrix without the first column and row, and if $a_{11} \neq 1$, then we stop.

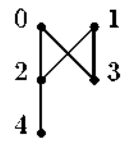
Calculating the cohomological height. Let X be a finite poset. If $X = \emptyset$ then $c.h.X = -1$. We will work with $X \neq \emptyset$. Let N be the largest number for which the set of sequences $x_0 < x_1 < \cdots < x_N$ is not empty.

Step 0. Let $n = N$, $K_n = |X^{(N)}|$.

Step 1. If $\operatorname{rk} d^{n-1} = K_n$ and $n \geqslant 0$, then we let $K_{n-1} = |X^{(n-1)}| - K_n$, $n = n-1$ and go to step 1.

Step 2. If $n > 0$, then we let $c.h.X = n$. If $n = 0$, then $c.h.X = -1$ in the case of $K_0 = 1$, and $c.h.X = 0$ in the case of $K_0 > 1$.

Example 4.1. *Let X be the poset depicted in the following Hasse diagram (directed downward)*



The set $X^{(k)}$ consists of all sequences $x_0 < x_1 < \cdots < x_k$, for every $k \geqslant 0$. So $X^{(0)} = \{0,1,2,3,4\}$; $X^{(1)} = \{0<2, 0<3, 0<4, 1<2, 1<3, 1<4, 2<4\}$; $X^{(2)} = \{0<2<4, 1<2<4\}$; $X^{(3)} = \emptyset$. The homomorphism $d^k : C^{(k)} \to C^{(k+1)}$ is given by $|C^{(k)}| \times |C^{(k+1)}|$ matrix of integers, which is transpose of the matrix defined by the action

$$\partial_k(x_0 < \cdots < x_{k+1}) = \sum_{i=0}^{k+1} (-1)^i (x_0 < \cdots < \hat{x}_i < \cdots < x_{i+1}).$$

The entries of d^0 and d^1 are given in the tables

	0<2	0<3	0<4	1<2	1<3	1<4	2<4
0	-1	-1	-1	0	0	0	0
1	0	0	0	-1	-1	-1	0
2	1	0	0	1	0	0	-1
3	0	1	0	0	1	0	0
4	0	0	1	0	0	1	1

	0<2<4	1<2<4
0<2	1	0
0<3	0	0
0<4	-1	0
1<2	0	1
1<3	0	0
1<4	0	-1
2<4	1	1

We remark that d^0 and d^1 will be reduced to the following diagonal matrices

$$\tilde{d^1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \tilde{d^0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 0. $n = 2$, $K_2 = |X^{(2)}| = 2$.

Step 1. We reduce d^1 by elementary row and column operators to a diagonal matrix $\tilde{d^1}$ and obtain $\operatorname{rk} d^1 = 2$. We have $\operatorname{rk} d^1 = K_2$, let $n = 2 - 1$, $K_1 = |X^{(1)}| - K_2 = 5$, and go to step 1.

Step 1. We reduce d^0 to a diagonal matrix $\tilde{d}^0$ and obtain $\operatorname{rk} d^0 = 4$. *We have* $\operatorname{rk} d^0 = 4 < K_1$, *and take the next step.*

Step 2. $n = 1 > 0$. *Therefore,* $c.h.X = 1$.

Calculating the cohomological dimension. For every $x \in X$ we compute $c.h.W(x, X)$ where $W(x, X) = \{y \in X : x < y\}$. If $X = \emptyset$ then we let cd $X = -1$, otherwise cd X is equal to $1 + m$, where m is the largest among $c.h.W(x, X)$.

Calculating the Hochschild–Mitchell dimension. $X = \emptyset$ if and only if $\dim X = -1$. If X is discrete then $\dim X = 0$. Otherwise we compute $c.h.]x, y[$ for all pairs $x < y$. If m is the largest of $c.h.(]x, y[)$ then we let $\dim X = m + 2$.

Calculating the cohomological dimension of a finite topological space. Let $X_{top} = (X, \tau_X)$ be a topological space. The *cohomological dimension* cd X_{top} *of a topological space* is the sup of $n \in \mathbf{N}$, for which $H^n(X_{top}, -) : Sh(X_{top}) \to Ab$ are not zeros. Here $Sh(X_{top})$ is denoted the category of sheaves over X_{top}, and $H^n(X_{top}, \mathcal{F})$ cohomology groups of X_{top} with coefficients in the sheaf.

Let $O(x)$ be the smallest open subset containing $x \in X$. We let $x \leqslant y \Leftrightarrow O(x) \subseteq O(y)$. Then we obtain the preordered set X. It is known that cd $X_{top} = \operatorname{cd}(X, \leqslant)$. Cohomological dimensions of equivalent sets are equal, hence $\operatorname{cd}(X, \leqslant)$ is equal to the cohomological dimension of the corresponding poset.

Calculating the Hochschild dimension of incidence ring. Let X be a small category. The *incidence ring* $\mathbf{Z}(X)$ consists of sums $\sum_{\alpha \in MorX} n_\alpha \alpha$ where $n_\alpha \in \mathbf{Z}$ are integers almost all of which are zeros. If X is a category with finite set of objects, then by **[11]** the dimension $\dim X$ is equal to Hochschild dimension of the ring $\mathbf{Z}(X)$. Therefore, we can calculate the Hochschild dimension of the incidence ring of a finite poset as the Hochschild–Mitchell dimension.

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