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THE $\bar{\partial}$ OPERATOR IN HOLOMORPHIC K-THEORY

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Abstract

The holomorphic (or semi-topological) K-theory of a smooth projective variety sits between the algebraic K-theory of the variety and the topological K-theory of the underlying topological space [**6**], [**8**]. We describe how to define a family of $\bar{\partial}$ operators on holomorphic K-theory in a manner analogous to Atiyah's construction of a family of elliptic operators in topological K-theory [**2**]. In the process, we prove a result akin to Bott periodicity for holomorphic mapping spaces. These results first appeared in the author's Stanford University Ph.D. thesis under the direction of Ralph Cohen [**17**].

1. Introduction

The holomorphic (or semi-topological) K-theory of a smooth projective variety interposes between the algebraic K-theory of the variety and the topological K-theory of the underlying topological space. Algebraic K-theory is often difficult to compute. Topological K-theory is much better understood, in part because topological K-theory can be defined in terms of the space of continuous maps from a topological space into the classifying space of the unitary group [**1**]. Cohen and Lima-Filho defined holomorphic K-theory by considering an analogous construction using holomorphic, rather than continuous, maps [**6**]. This paper describes how to construct a family of $\bar{\partial}$ operators in this holomorphic setting. Our approach is modeled on Atiyah's well-known proof of the Bott periodicity theorem using elliptic operators.

For a smooth projective variety X, let $Hol(X; Gr_n(\mathbb{C}^N))$ denote the space of holomorphic maps between the underlying complex manifold X and the Grassmannian of n-planes in $\mathbb{C}^N$. In [**6**], Cohen and Lima-Filho define the holomorphic K-theory space of X to be the Quillen–Segal completion of the union of these mapping spaces. For a topological monoid M, the Quillen–Segal group completion of M, denoted M^+, is defined to be ΩBM, the loop space of the classifying space of M. This group completion is necessary so that the holomorphic mapping space is an infinite loop space.

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Definition 1.1. [**6**]

$$K_{hol}(X) := \left(\bigcup_{n,N} Hol(X; Gr_n(\mathbb{C}^N)) \right)^+ \equiv Hol(X; \mathbb{Z} \times BU)^+$$

As in topological K-theory, we will be interested in the homotopy types of these mapping spaces.

Definition 1.2.

$$K_{hol}^{-q}(X) := \pi_q K_{hol}(X)$$

The inclusion of the holomorphic mapping space into the continuous mapping space induces a natural map $\iota : K_{hol}^{-q}(X) \to K_{top}^{-q}(X)$.

We note that Cohen and Lima-Filho's theory coincides with what Friedlander and Walker termed "semi-topological K-theory." In [**6**], Cohen and Lima-Filho proved that $K_{hol}^0(X)$ is isomorphic to the Grothendieck group completion of the monoid of algebraic bundles over X modulo algebraic equivalence, which is precisely how Friedlander and Walker defined $K_0^{semi}(X)$ in [**8**]. Friedlander and Walker approached semi-topological K-theory from an algebraic K-theoretic perspective. They proved that the holomorphic, or semi-topological, K-theory of a variety sits between its algebraic K-theory and the topological K-theory of the underlying topological space, a result which motivated their terminology.

One of the most important tools for calculating topological K-theory was the Bott periodicity theorem, which establishes that $K_{top}^{-2}(X) \cong K_{top}^0(X)$, or, stated in terms of mapping spaces,

$$\pi_0 \left[Map(X; \Omega^2(BU)) \right] \cong \pi_0 \left[Map(X; \mathbb{Z} \times BU) \right].$$

In [**2**], Atiyah used the index of elliptic operators to give a proof of this theorem. In particular, he defined a homomorphism from $K_{top}^0(X \times \mathbb{CP}^1)$ to $K_{top}^0(X)$ using the index of a family of $\bar{\partial}$ operators. This paper extends Atiyah's construction to a families of $\bar{\partial}$ operators on holomorphic mapping spaces.

Using a delicate refinement of Atiyah's argument involving indices of elliptic operators to the setting of holomorphic maps to Grassmannians, we obtain the following:

Theorem 1.3. *Let X be any smooth projective variety. The index of a family of $\bar{\partial}$ operators defines a homotopy equivalence:*

$$\bar{\partial} : Hol(X; Hol^{\bullet}(\mathbb{CP}^1; BU))^+ \xrightarrow{\simeq} K_{hol}(X).$$

This homotopy equivalence has also been established using techniques of algebraic K-theory: it arises as a special case of Friedlander and Walker's projective bundle theorem [**8**] which states that there is a natural isomorphism

$$K_*^{semi}(X)^{\times n} \cong K_*^{semi}(\mathbb{P}(E)).$$

Here, X is a projective variety and E is a rank n vector bundle on X. In the case of a trivial bundle, the Friedlander–Walker theorem becomes

$$K_{hol}(X) \simeq K_{hol}(X \times \mathbb{CP}^1; X).$$

This paper interprets Friedlander–Walker's projective bundle theorem in terms of Bott periodicity. Furthermore, the use of the $\bar{\partial}$ operator as a tool in the present paper illustrates a connection between holomorphic K-theory and elliptic operators that may be useful in other contexts.

Note that Theorem 1.3 does *not* imply that $K_{hol}^{-q}(X) \cong K_{hol}^{-q+2}(X)$, and in fact, this stronger result fails in general. By studying the appropriate analogue of the Chern character, Cohen and Lima-Filho showed that if a variety X has non-zero Hodge cohomology groups $H^{p,q}(X, \mathbb{C})$ for some $p \neq q$, then $K_{hol}^*(X)$ is not Bott periodic [**5**], [**6**]. Note that the holomorphic K-theory must be distinct from the topological K-theory for any such X.

The outline of this paper is as follows: in Section 2, we define a family of $\bar{\partial}$ operators indexed by spaces of continuous maps. We restrict this map to holomorphic mapping spaces in Section 3. In Section 4 we construct complex coordinates for $Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^N))$, and use these to show the $\bar{\partial}$ map is holomorphic in the appropriate sense in Section 5. Section 6 contains the proof of Theorem 1.3.

2. Families of $\bar{\partial}$ operators coupled to mapping spaces

Let $Map_k^\bullet(\mathbb{CP}^1; BU)$ denote the space of base-point preserving continuous maps of degree k from $\mathbb{CP}^1$ to BU. Note that this is a connected component of $\Omega^2(BU)$. In this section, we explain how to associate an elliptic operator $\bar{\partial}_f$ to each map f in $Map_k^\bullet(\mathbb{CP}^1; BU)$. We describe how this family of elliptic operators describes a well-defined homotopy class of maps from $\Omega^2(BU)$ to $\mathbb{Z} \times BU$. Moreover, we show that this construction yields an inverse to the homotopy equivalence $\mathbb{Z} \times BU \simeq \Omega^2(BU)$ originally proven by Bott [**3**].

We begin by reviewing the definition of the $\bar{\partial}$ operator for any complex manifold M. Recall that the complexified cotangent space to M at each point $z \in M$, $T_{\mathbb{C},z}^*(M)$, can be decomposed into the sum of the holomorphic cotangent space, which we denote $T_z^{1,0}(M)$, and the anti-holomorphic cotangent space, which we denote $T_z^{0,1}(M)$:

$$T_{\mathbb{C},z}^*(M) = T_z^{1,0}(M) \oplus T_z^{0,1}(M).$$

Let $A^n(M)$ denote the space of complex-valued n-forms on M, and let $A^{p,q}(M)$ denote the forms of type (p, q); that is,

$$A^{p,q}(M) = \left\{\phi \in A^n(M) : \phi(z) \in \wedge^p T_z^{1,0}(M) \otimes \wedge^q T_z^{0,1}(M) \quad \text{for all } z \in M\right\}.$$

Finally, let $\pi^{p,q} : A^{p+q} \to A^{p,q}$ denote the projection from $p + q$ forms to forms of type (p, q). The exterior derivative d defines a map from n-forms to $n + 1$ forms.

Restricted to forms of type (p, q), we see that

$$d : A^{p,q}(M) \to A^{p+1,q}(M) \oplus A^{p,q+1}(M).$$

Thus, d splits as $\partial + \bar{\partial}$, where

$$\pi^{p+1,q} d \equiv \partial : A^{p,q}(M) \to A^{p+1,q}$$

and

$$\pi^{p,q+1} d \equiv \bar{\partial} : A^{p,q}(M) \to A^{p,q+1}.$$

Now let E be a holomorphic vector bundle over $\mathbb{CP}^1$, and let $A^{p,q}(E)$ denote the space of E-valued (p, q) forms. If $\sigma_1, \ldots, \sigma_n$ is a local holomorphic frame over an open set $U \subset \mathbb{CP}^1$ and $\sigma = \sum g_i \otimes \sigma_i$, where $g_i \in A^{p,q}(U)$, then

$$\bar{\partial}_E : A^{p,q}(E) \to A^{p,q+1}(E)$$

is defined by

$$\bar{\partial}(\sigma) = \sum \frac{\partial g_i}{\partial \bar{z}} \, d\bar{z} \otimes \sigma_i.$$

This definition of $\bar{\partial}(\sigma)$ does not depend on the choice of frame; see, for example, [**9**]. In the case where $p = q = 0$, $A^{0,0}(E)$ is the space of smooth sections of E, and we have:

$$\bar{\partial}_E : \Gamma_\infty(E) \to \Gamma_\infty(E \otimes T^{0,1}(\mathbb{CP}^1))$$

where $T^{0,1}(\mathbb{CP}^1)$ denotes the anti-holomorphic cotangent bundle of $\mathbb{CP}^1$.

If E is not holomorphic, then there is no canonically defined operator from E-valued (p, q) forms to E-valued $(p, q + 1)$ forms; however, the choice of a connection on E determines an exterior derivative d from E-valued sections to E-valued 1-forms. One may then define $\bar{\partial}_E$ to be $\pi^{0,1} \circ d$.

Recall that for a Fredholm operator T, the index is defined to be the difference between the dimension of the kernel and the dimension of the cokernel:

$$\operatorname{index} T = \dim \ker T - \dim \operatorname{cok} T.$$

The index of $\bar{\partial}_E$ is defined to be the index of any of its Fredholm extensions

$$\Gamma_{L^2_s}(E) \to \Gamma_{L^2_{s-1}}(E \otimes T^{0,1}(\mathbb{CP}^1))$$

where $\Gamma_{L^2_s(E)}$ denotes the completion of the space of smooth sections of E in the Sobolev s-norm. These extensions have a finite-dimensional kernel and cokernel consisting of smooth sections. Furthermore, the kernel and cokernel are independent of the choice of s. For the sake of concreteness, we take $s = 1$. See [**14**] or [**11**] for a general reference on the index of elliptic operators and families of elliptic operators.

Now consider the space $Map^\bullet_k(\mathbb{CP}^1; BU(n))$ of based continuous maps of degree k from $\mathbb{CP}^1$ to $BU(n)$. For each map f, associate the operator $\bar{\partial}_{f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)}$. The bundle $f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)$ is the dual of the pullback of the universal bundle over $BU(n)$, tensored with $\mathcal{O}(-1)$, the universal bundle over $\mathbb{CP}^1$. Although $f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)$ is not generally a holomorphic bundle if f is not holomorphic, it does come equipped with a connection, since we can pull back the canonical connection on γ_n. In this way, we obtain a family of elliptic operators parametrized by f.

Using the Riemann–Roch theorem, we calculate the index of $\bar{\partial}_{f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)}$. Note that while the dimension of the kernel and cokernel of $\bar{\partial}_{f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)}$ may vary, the index is constant for each f in $Map_k^\bullet(\mathbb{CP}^1; BU(n))$ since the space $Map_k^\bullet(\mathbb{CP}^1; BU(n))$ is connected:

$$\begin{aligned} \text{index}\, \bar{\partial}_{f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)} &= c_1(f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)) + \dim\, (f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)) \\ &= (k-n)+n \\ &= k. \end{aligned}$$

Since the index does not depend on n, we see the index is constant on each connected component $Map_k^\bullet(\mathbb{CP}^1; BU)$ of $\Omega^2(BU)$. This defines a map

$$\text{index} : \Omega^2(BU) = Map^\bullet(\mathbb{CP}^1; BU) \to \mathbb{Z}.$$

In fact a more careful analysis gives us, up to homotopy, a map

$$Map^\bullet(\mathbb{CP}^1; BU) \xrightarrow{\bar{\partial}} \mathbb{Z} \times BU.$$

Note that the evaluation map

$$F : Map^\bullet(\mathbb{CP}^1; BU) \times \mathbb{CP}^1 \to \mathbb{Z} \times BU$$

given by

$$F(f,p) = (\text{index}(f), f(p))$$

defines an element of $K^0_{top}(Map^\bullet(\mathbb{CP}^1; BU) \times \mathbb{CP}^1)$. In [**2**], Atiyah uses the index of a family of $\bar{\partial}$ operators to define a homomorphism from $K^0_{top}(Y \times \mathbb{CP}^1)$ to $K^0_{top}(Y)$. Applying Atiyah's homomorphism here, we get

$$\text{Index}\, \bar{\partial}_{F^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)} \in K^0_{top}(Map^\bullet(\mathbb{CP}^1; BU)) = [Map^\bullet(\mathbb{CP}^1; BU); \mathbb{Z} \times BU].$$

Thus we obtain a well-defined homotopy class of maps from $Map^\bullet(\mathbb{CP}^1; BU)$ to $\mathbb{Z} \times BU$ as desired.

Given any space X, $\bar{\partial}$ extends to give

$$\bar{\partial} : Map(X; Map^\bullet(\mathbb{CP}^1; BU)) \to Map(X; \mathbb{Z} \times BU).$$

Again, this is only defined up to homotopy. We shall see it induces an isomorphism on homotopy groups.

Lemma 2.1. *The homotopy class of maps*

$$\bar{\partial} : Map(X; Map^\bullet(\mathbb{CP}^1; BU)) \to Map(X; \mathbb{Z} \times BU)$$

as defined above induces an isomorphism on homotopy groups, and is therefore a homotopy equivalence.

Proof. Note that

$$\pi_q\left(Map(X; Map^\bullet(\mathbb{CP}^1; BU))\right) = K_{top}^{-q-2}(X)$$

and

$$\pi_q\left(Map(X; \mathbb{Z}\times BU)\right) = K_{top}^{-q}(X).$$

Therefore, we need to show that $\bar\partial$ induces an isomorphism $K_{top}^{-2}(X) \cong K_{top}^{0}(X)$. We appeal to the following theorem of Atiyah, which describes sufficient conditions for such a map to be an isomorphism.

Theorem 2.2. ([**2**]) *Suppose there exists an $\alpha : K_{top}^{-q-2}(X) \to K_{top}^{-q}(X)$ defined for all compact X such that α satisfies the following axioms:*

1. *α is functorial in X;*
2. *α is a $K_{top}^0(X)$-module homomorphism;*
3. *$\alpha(b) = 1$, where $b \equiv [1] - [\mathcal{O}(-1)] \in \widetilde{K}_{top}^0(\mathbb{CP}^1) = K_{top}^{-2}(pt)$.*

Then α is inverse to the Bott map β.

In [**2**], Atiyah verifies that for any elliptic operator d, the map defined by $Q \mapsto \text{Index } d_Q$ from $K_{top}^0(X\times\mathbb{CP}^1)$ to $K_{top}^0(X)$ is functorial and is a $K_{top}^0(X)$ module homomorphism, and so it follows immediately that $\bar\partial_\sharp : K_{top}^{q-2}(X) \longrightarrow K_{top}^{q}(X)$ satisfies the first two axioms; however, $\bar\partial_\sharp(b) = \text{ index } \bar\partial_{b^\vee\otimes\mathcal{O}(-1)} = -1$, by the Riemann–Roch formula, so we actually have that $-\bar\partial_\sharp$ satisfies all of Atiyah's formal properties. Thus, $-\bar\partial$ induces an isomorphism in homotopy groups in all dimensions, and therefore is a homotopy equivalence. Certainly, $\bar\partial$ is a homotopy equivalence as well. □

In terms of mapping spaces, the above result implies that $-\bar\partial$ is homotopy inverse to the Bott map $\beta : \mathbb{Z}\times BU \to \Omega^2 BU = Map^\bullet(\mathbb{CP}^1; BU)$.

3. Restricting $\bar\partial$ to holomorphic mapping spaces

We now consider only those maps which are holomorphic. Let V be an infinite-dimensional complex vector space, and let $Gr_n(V)$ be the Grassmannian of n-dimensional subspaces of V. We take $Gr_n(V)$ as our model for $BU(n)$. The topology of $Gr_n(V)$ is determined by

$$Gr_n(V) = \varinjlim_{F}\ Gr_n(F)$$

where the limit is taken over all finite-dimensional subspaces F of V (cf. [**7**]), and the complex structure of $Gr_n(V)$ can be described using Plücker coordinates, analogous to the usual description for finite Grassmannians (cf. [**9**], [**15**]). Let $Hol_k^\bullet(\mathbb{CP}^1; BU(n))$ denote the space of holomorphic, base-point preserving maps of degree k from $\mathbb{CP}^1$ to $BU(n)$, and let $Hol_k^\bullet(\mathbb{CP}^1; BU)$ be the limit of those mapping spaces over the rank n:

$$Hol_k^\bullet(\mathbb{CP}^1; BU) = \lim_{n\to\infty} Hol_k^\bullet(\mathbb{CP}^1; BU(n)).$$

The main goal of this section is to show that the construction described in Section 2 yields a map from $Hol_k^\bullet(\mathbb{CP}^1; BU)$ to $BU(k)$ when restricted to holomorphic mapping spaces. Furthermore, we prove this map realizes the homotopy equivalence $Hol_k^\bullet(\mathbb{CP}^1; BU) \simeq BU(k)$ originally proven in [**7**].

Theorem 3.1. *When restricted to holomorphic maps, $\bar\partial : Map_k^\bullet(\mathbb{CP}^1; BU) \to BU$ defines a map $\bar\partial : Hol_k^\bullet(\mathbb{CP}^1; BU) \to BU(k)$ that makes the following diagram homotopy commute:*

$$\begin{array}{ccc} Hol_k^\bullet(\mathbb{CP}^1; BU) & \xrightarrow{\bar\partial} & BU(k) \\ \downarrow & & \downarrow \\ Map_k^\bullet(\mathbb{CP}^1; BU) & \xrightarrow{\bar\partial} & BU \end{array}$$

Proof. We first show that if f is holomorphic, then Index $\bar\partial_{f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)}$ is a k-dimensional vector space.

Recall that any holomorphic vector bundle over $\mathbb{CP}^1$ can be decomposed into a sum of line bundles; that is, if E is an n-dimensional holomorphic vector bundle over $\mathbb{CP}^1$, then

$$E = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n),$$

where $\mathcal{O}(a_i)$ is a line bundle with first Chern class a_i. A holomorphic bundle E over $\mathbb{CP}^1$ is said to be negative if in the above decomposition, $a_i \leqslant 0$ for all i. We will say E is strictly negative if $a_i < 0$ for all i. Positive and strictly positive are analogously defined. Finally, if $E = f^*\gamma_n$ for some holomorphic map f, then we will say E is representable.

Clearly any representable bundle can be holomorphically embedded in a trivial bundle by pulling back the embedding $\gamma_n \hookrightarrow Gr_n(\mathbb{C}^N) \times \mathbb{C}^N$. Furthermore, embeddable holomorphic bundles are negative (as can be seen directly from the above decomposition), and therefore $f^*\gamma_n \otimes \mathcal{O}(-1)$ is strictly negative. If $j < 0$, $\mathcal{O}(j)$ has no holomorphic sections. This implies that if f is holomorphic, then $f^*\gamma_n \otimes \mathcal{O}(-1)$ has no holomorphic sections and so $H^0(\mathbb{CP}^1, \mathcal{O}(f^*\gamma_n \otimes \mathcal{O}(-1))) = 0$. It then follows from Kodaira–Serre duality that $H^1(\mathbb{CP}^1, \mathcal{O}(f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1))) = 0$, and therefore cok $\bar\partial_{f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)} = 0$.

Since the cokernel is identically zero for all $f \in Hol_k^\bullet(\mathbb{CP}^1; BU)$, we have that Index $\bar\partial$ = Ker $\bar\partial$ is a k-dimensional bundle over $Hol_k^\bullet(\mathbb{CP}^1; BU)$. Bundles of dimension k are classified by maps to $BU(k)$, and so we obtain a well-defined homotopy class of maps from $Hol_k^\bullet(\mathbb{CP}^1; BU)$ to $BU(k)$ as claimed. □

More concretely, we can think of ker $\bar\partial_{f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)}$ as a k-dimensional subspace of a fixed infinite-dimensional space. For each $f \in Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^N))$, we know that $f^*\gamma_n$ embeds holomorphically into an N-dimensional trivial bundle:

$$f^*\gamma_n \hookrightarrow \mathbb{CP}^1 \times \mathbb{C}^N.$$

This yields a holomorphic surjection of the dual bundles:

$$\mathbb{CP}^1 \times (\mathbb{C}^N)^\vee \to f^*\gamma_n^\vee.$$

Now tensor both bundles with $\mathcal{O}(-1)$:

$$(C^N)^\vee \otimes \mathcal{O}(-1) \to f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1).$$

This induces a surjection of the corresponding spaces of sections

$$\Gamma_{L_1^2}((C^N)^\vee \otimes \mathcal{O}(-1)) \to \Gamma_{L_1^2}(f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)).$$

For any complex vector space V, the Grassmannian of co-dimension k vector spaces of V, $Gr^k(V)$, may be thought of as a quotient of the space of linear surjections from V onto $\mathbb{C}^k$:

$$Gr^k(V) = \operatorname{Lin}\operatorname{Surj}(V; \mathbb{C}^k)/GL(k).$$

It follows that a surjection $W \to V$ induces a contravariant map between the Grassmannians:

$$Gr^k(V) \to Gr^k(W).$$

Thus, we have a map

$$\varphi : Gr^k\left(\Gamma_{L_1^2}(f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1))\right) \to Gr^k\left(\Gamma_{L_1^2}((C^N)^\vee \otimes \mathcal{O}(-1))\right).$$

Finally, $Gr_k(V)$ and $Gr^k(V)$ may be identified by sending a k-dimensional subspace $S \in V$ to $S^\perp$ (similarly for $Gr_k(W)$ and $Gr^k(W)$). Composing this with φ yields a map

$$Gr_k\left(\Gamma_{L_1^2}(f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1))\right) \hookrightarrow Gr_k\left(\Gamma_{L_1^2}((\mathbb{C}^N)^\vee \otimes \mathcal{O}(-1))\right).$$

Let $\bar{\partial}(f)$ be the image of $\ker(\bar{\partial}_{f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)})$ under this map. Then we have

$$\bar{\partial} : Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^N)) \to Gr_k\left(\Gamma_{L_1^2}((\mathbb{C}^N)^\vee \otimes \mathcal{O}(-1))\right).$$

Next, we take the limit as n and N approach infinity. We define

$$Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^N)) \to Hol_k^\bullet(\mathbb{CP}^1; Gr_{n+1}(\mathbb{C}^{N+1}))$$

using the inclusion $Gr_n(\mathbb{C}^N) \to Gr_{n+1}(\mathbb{C}^{N+1})$ given by $V \mapsto V \oplus \mathbb{C}$ and we define

$$Gr_k\left(\Gamma_{L_1^2}((\mathbb{C}^N)^\vee \otimes \mathcal{O}(-1))\right) \to Gr_k\left(\Gamma_{L_1^2}((\mathbb{C}^{N+1})^\vee \otimes \mathcal{O}(-1))\right)$$

by $V \mapsto V \oplus 0$. We have

$$\lim_{n,N\to\infty} Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^N)) = Hol_k^\bullet(\mathbb{CP}^1; BU)$$

and

$$\lim_{N\to\infty} Gr_k\left(\Gamma_{L_1^2}((\mathbb{C}^N)^\vee \otimes \mathcal{O}(-1))\right) = BU(k).$$

The diagram

$$\begin{array}{ccc} Hol^\bullet_k(\mathbb{CP}^1; Gr_n(\mathbb{C}^N)) & \longrightarrow & Hol^\bullet_k(\mathbb{CP}^1; Gr_{n+1}(\mathbb{C}^{N+1})) \\ \bar\partial \downarrow & & \bar\partial \downarrow \\ Gr_k\left(\Gamma_{L^2_1}((\mathbb{C}^N)^\vee \otimes \mathcal{O}(-1))\right) & \longrightarrow & Gr_k\left(\Gamma_{L^2_1}((\mathbb{C}^{N+1})^\vee \otimes \mathcal{O}(-1))\right) \end{array}$$

commutes, and so the $\bar\partial$ map extends to the limit. Thus, we have explicitly defined

$$\bar\partial : Hol^\bullet_k(\mathbb{CP}^1; BU) \to BU(k)$$

for each k. Furthermore, this construction of $\bar\partial$ extends to the Quillen–Segal completion:

Proposition 3.2. *There exists a well-defined extension of $\bar\partial$ to the Quillen–Segal group completion:*

$$\bar\partial : Hol^\bullet(\mathbb{CP}^1; BU)^+ \to \left(\coprod BU(k)\right)^+ = \mathbb{Z} \times BU.$$

Proof. The spaces

$$\bigcup_{k,n,N} Hol^\bullet_k(\mathbb{CP}^1; Gr_n(\mathbb{C}^N))$$

and

$$\bigcup_{k,N} Gr_k\left(\Gamma_{L^2_1}((\mathbb{C}^N)^\vee \otimes \mathcal{O}(-1))\right)$$

are monoids: If $f \in Hol^\bullet_k(\mathbb{CP}^1; Gr_n(\mathbb{C}^N))$ and $g \in Hol^\bullet_l(\mathbb{CP}^1; Gr_m(\mathbb{C}^M))$, then

$$f + g \in Hol^\bullet_{k+l}(\mathbb{CP}^1; Gr_{n+m}(\mathbb{C}^{N+M}))$$

and if $V \in Gr_k\left(\Gamma_{L^2_1}((\mathbb{C}^N)^\vee \otimes \mathcal{O}(-1))\right)$ and $W \in Gr_l\left(\Gamma_{L^2_1}((\mathbb{C}^M)^\vee \otimes \mathcal{O}(-1))\right)$ then

$$V \oplus W \in Gr_{k+l}\left(\Gamma_{L^2_1}((\mathbb{C}^{N+M})^\vee \otimes \mathcal{O}(-1))\right).$$

Furthermore,

$$\left(\bigcup_{k,n,N} Hol^\bullet_k(\mathbb{CP}^1; Gr_n(\mathbb{C}^N))\right)^+ = Hol^\bullet(\mathbb{CP}^1; BU)^+$$

and

$$\left(\bigcup_{k,N} Gr_k\left(\Gamma_{L^2_1}((\mathbb{C}^N)^\vee \otimes \mathcal{O}(-1))\right)\right)^+ = \mathbb{Z} \times BU.$$

Recall that for any monoid $\mathcal{M}$, the Quillen–Segal completion is $\Omega B\mathcal{M}$, the loop space of the classifying space of $\mathcal{M}$. Any map between two monoids $\mathcal{M}_1$ and $\mathcal{M}_2$ which preserves the monoid structure extends to a map of the classifying spaces (using the simplicial bar construction), and subsequently extends to the loop spaces of the classifying spaces. Therefore, to prove Proposition 3.2, it suffices to show that $\bar\partial$ preserves the monoid structure.

If $f \in Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^N))$ and $g \in Hol_l^\bullet(\mathbb{CP}^1; Gr_m(\mathbb{C}^M))$, then we see that

$$\Gamma_{hol}((f\oplus g)^*\gamma_{n+m}^\vee \otimes \mathcal{O}(-1)) = \Gamma_{hol}(f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)) \oplus \Gamma_{hol}(g^*\gamma_m^\vee \otimes \mathcal{O}(-1)).$$

This implies that the following diagram commutes:

$$\begin{array}{ccc}
Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^N)) \times Hol_l^\bullet(\mathbb{CP}^1; Gr_m(\mathbb{C}^M)) & \xrightarrow{\oplus} & Hol_{k+l}^\bullet(\mathbb{CP}^1; Gr_{n+m}(\mathbb{C}^{N+M})) \\
\downarrow{\scriptstyle \bar\partial\times\bar\partial} & & \downarrow{\scriptstyle \bar\partial} \\
Gr_k\left(\Gamma_{L_1^2}((\mathbb{C}^N)^\vee \otimes \mathcal{O}(-1))\right) \times Gr_l\left(\Gamma_{L_1^2}((\mathbb{C}^M)^\vee \otimes \mathcal{O}(-1))\right) & \xrightarrow{\oplus} & Gr_{k+l}\left(\Gamma_{L_1^2}((\mathbb{C}^{N+M})^\vee \otimes \mathcal{O}(-1))\right)
\end{array}$$

Thus, $\bar\partial$ preserves the monoid structure and therefore extends to the Quillen–Segal completion as required. □

In fact, $\bar\partial$ is a homotopy equivalence:

Proposition 3.3. *Let*

$$\bar\partial : Hol_k^\bullet(\mathbb{CP}^1; BU) \to BU(k)$$

be as defined above. Then $\bar\partial$ is a homotopy equivalence.

Note this proposition immediately implies that $Hol^\bullet(\mathbb{CP}^1; BU)^+ \simeq K_{hol}(pt)$, a result which originally follows from work of Kirwan, who proved that the map $\mathrm{Hol}^\bullet(\mathbb{CP}^1; BU)^+ \to Map^\bullet(\mathbb{CP}^1; \mathbb{Z}\times BU)$ induced by the inclusion of the space of holomorphic maps into the space of continuous maps is a homotopy equivalence [**10**].

Proof. The result that $Hol_k^\bullet(\mathbb{CP}^1; BU)$ is homotopy equivalent to $BU(k)$ was first proved in [**7**]; what is new here is that the homotopy equivalence can be realized by the $\bar\partial$ operator. In [**7**], Cohen, Lupercio, and Segal constructed a homotopy equivalence

$$\gamma : Hol_k^\bullet(\mathbb{CP}^1; BU) \to BU(k)$$

using the Mitchell–Segal filtration $F_{k,n}$ of $\Omega SU(n)$; see [**13**]. Their result immediately implies that γ realizes the homotopy equivalence $\mathrm{Hol}^\bullet(\mathbb{CP}^1; BU)^+ \simeq K_{hol}(pt)$.

Furthermore, their map γ fits into the following commutative diagram:

$$\begin{array}{ccc}
\coprod_k Hol^{\bullet}_k(\mathbb{CP}^1; BU) & \xrightarrow{\gamma} & \coprod_k BU(k) \\
{\scriptstyle +}\downarrow & & \downarrow{\scriptstyle +} \\
Hol^{\bullet}(\mathbb{CP}^1; BU)^+ & \xrightarrow{\gamma} & \mathbb{Z} \times BU \\
{\scriptstyle \simeq}\downarrow & & \downarrow{\scriptstyle =} \\
Map^{\bullet}(\mathbb{CP}^1; \mathbb{Z} \times BU) & \underset{\beta}{\overset{\simeq}{\rightleftarrows}} & \mathbb{Z} \times BU
\end{array}$$

By Theorem 3.1 and the proof of Lemma 2.1, this diagram also commutes if we replace γ by $-\bar{\partial}$. (Recall that we must use $-\bar{\partial}$ rather than $\bar{\partial}$ since the index of $\bar{\partial}_b$ is -1 rather than 1.) It follows that the compositions

$$Hol^{\bullet}_k(\mathbb{CP}^1; BU) \xrightarrow{\gamma} BU(k) \hookrightarrow BU$$

and

$$Hol^{\bullet}_k(\mathbb{CP}^1; BU) \xrightarrow{-\bar{\partial}} BU(k) \hookrightarrow BU$$

are homotopic. We also note that after the Quillen–Segal completion, $-\bar{\partial}$ and γ are homotopic; that is,

$$\begin{array}{ccc}
Hol^{\bullet}(\mathbb{CP}^1; BU)^+ & \xrightarrow{\phi} & \mathbb{Z} \times BU \\
& \searrow^{\bar{\partial}} & \downarrow{\scriptstyle -1} \\
& & \mathbb{Z} \times BU
\end{array}$$

homotopy commutes. We conjecture that the maps $\bar{\partial}$ and γ are also closely related before group completion, but we shall not need that here. It suffices to observe that

$$\bar{\partial} : Hol^{\bullet}_k(\mathbb{CP}^1; BU) \to BU(k)$$

must induce a map which is injective in homology. Since $Hol^{\bullet}_k(\mathbb{CP}^1; BU)$ and $BU(k)$ are known to be homotopy equivalent, this map must be isomorphism in homology, and so $\bar{\partial}$ is a homotopy equivalence. □

4. Complex coordinates for $Hol^{\bullet}_k(\mathbb{CP}^1; Gr_n(\mathbb{C}^{n+m}))$

The based holomorphic mapping space $Hol^{\bullet}_k(\mathbb{CP}^1; Gr_n(\mathbb{C}^{n+m}))$ is a connected, complex manifold of complex dimension $(n+m)k$ [**4**]. In this section, we explicitly describe how to obtain complex coordinates on open sets of this manifold. We use this result in Section 5 to prove that

$$\bar{\partial} : Hol^{\bullet}_k(\mathbb{CP}^1; Gr_n(\mathbb{C}^{n+m})) \to Gr_k\left(\Gamma_{L^2_1}((\mathbb{C}^{n+m})^{\vee} \otimes \mathcal{O}(-1))\right)$$

is holomorphic.

Mann and Milgram [**12**] proved that given any $f \in Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^{n+m}))$, f may be expressed as a matrix of polynomials in the following unique normal form:

$$[P, Q] = \begin{bmatrix} p_{11}(z) & p_{12}(z) & \dots & p_{1n}(z) & q_{11}(z) & \dots & q_{1m}(z) \\ 0 & p_{22}(z) & \dots & p_{2n}(z) & q_{21}(z) & \dots & q_{2n}(z) \\ \vdots & \ddots & \dots & \vdots & \vdots & \ddots & \vdots \\ 0 & & \dots & p_{nn}(z) & q_{n1}(z) & \dots & q_{nm}(z) \end{bmatrix}$$

where P is an upper triangular $n \times n$ matrix of polynomials and Q is an $n \times m$ matrix of polynomials such that the polynomials p_{ii} are monic. Let k_i denote the degree of p_{ii}. Then $\sum_{i=1}^n k_i = k$, deg $p_{ij} < k_j$ if $i < j$, and deg $q_{nj} < k_n$.

For each $f \in Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^{n+m}))$, we have that

$$f(\infty) = \mathbb{C}^n \oplus \mathbf{0} \in Gr_n(\mathbb{C}^{n+m}).$$

This base point condition imposes additional constraints on the q_{ij}. In terms of the above normal form, this implies that the elements of the matrix $P^{-1}Q$ must tend to 0 as z tends to infinity. In particular, this condition forces the polynomials q_{nj} to have degree less than k_n since

$$\frac{q_{nj}}{p_{nn}} \to 0.$$

More generally, the condition that each component of $P^{-1}Q$ tends to zero determines each polynomial q_{ij} in terms of the matrix of polynomials P and the polynomials q_{kj} for $k > i$, up to an arbitrary polynomial s_{ij} of degree less than k_i.

From the above description, we note that the jth column of P has jk_j degrees of freedom, and each column of Q has k degrees of freedom (the ith row of Q has mk_i degrees of freedom). Given a partition $K = (k_1, k_2, \dots, k_n)$ such that $\sum_{i=1}^n k_i = k$, Mann and Milgram showed that the collection of $f \in Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^{n+m}))$ such that the normal form of f satisfies deg $p_{ii} = k_i$ is a complex submanifold of $Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^{n+m}))$ of dimension $d_K = km + \sum_{j=1}^n jk_j$.

We are interested, however, not in the complex coordinates of this submanifold, but rather in an open set of dimension $k(n+m)$. Let $f \in Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^{n+m}))$, and suppose the normal form of f satisfies $\deg p_{ii} = k_i$ for some fixed partition $K = (k_1, k_2, \dots, k_n)$. In order to define an open set $\mathcal{U}_K$ containing f, we consider matrices of polynomials P and Q, where P is $n \times n$ and Q is $n \times m$, with the following conditions:

1. The diagonal terms p_{ii} of P are monic with $\deg p_{ii} = k_i$.
2. All off-diagonal terms in a given column of P must have degree smaller than that of the diagonal term; that is, $\deg p_{ij} < k_j$ for all $i \neq j$.
3. The elements q_{ij} of Q must be compatible with the basepoint condition as described above; that is, the entries of the matrix of rational functions $P^{-1}Q$ must tend to 0 as z tends to infinity.
4. The rows of the $n \times (n+m)$ matrix $[P, Q]$ must be linearly independent for all z.

This description differs from the normal form in that the matrix P is no longer required to be upper triangular, so we obtain in this way an open set rather than one of the complex submanifolds considered by Mann and Milgram. While each element of the open set as defined above certainly has an equivalent description in the Mann–Milgram normal form, we will find this description more convenient. Note that the determinant of P is a monic polynomial of degree k. The jth column of P has nk_j degrees of freedom given by the coefficients of the p_{ij}, and, as in the case of the normal form, each element q_{ij} has k_i degrees of freedom since we shall see that q_{ij} is completely determined by the base point condition up to a polynomial s_{ij} of degree less than k_i.

Theorem 4.1. *Let $K = (k_1, k_2, \ldots, k_n)$ be a fixed partition of $k = \sum_{i=1}^{n} k_i$ and let $[P, Q]$ be an element of $\mathcal{U}_K$. There exist polynomials s_{il} with $deg\, s_{il} < k_i$ for all $1 \leqslant i \leqslant n$ and $1 \leqslant l \leqslant m$ such that the coefficients of the polynomials in the matrix Q are polynomial in the coefficients of the polynomials $\{p_{ij}\}$ and $\{s_{il}\}$. The map*

$$\mathcal{U}_K \longrightarrow V \subset \mathbb{C}^{k(n+m)}$$

defined by sending an element of $\mathcal{U}_K$ to the coefficients of the polynomials $\{p_{ij}\}$ and $\{s_{il}\}$ is a biholomorphic map onto an open set V in $\mathbb{C}^{k(n+m)}$.

Before proving Theorem 4.1, we consider the following example.

Example 4.2. Suppose that $f \in Hol_2^\bullet(\mathbb{CP}^1; Gr_2(\mathbb{C}^3))$, and that the normal form of f corresponds to the partition $K = (0, 2)$, so that $k_1 = 0, k_2 = 2, k = k_1 + k_2 = 2, n = 2, m = 1$. Then we have that $Hol_2^\bullet(\mathbb{CP}^1; Gr_2(\mathbb{C}^3))$ is six-dimensional, and $\mathbb{C}^6 \supset V \longrightarrow \mathcal{U}_K$ by:

$$(a, b, c, d, e, f) \mapsto \left[\begin{array}{cc|c} 1 & az + b & ae \\ 0 & z^2 + cz + d & ez + f \end{array}\right]$$

Here, V is the open set of $\mathbb{C}^6$ such that $z^2 + cz + d$ and $ez + f$ have no common roots; that is, $cef \neq f^2 + de^2$. To understand how we obtained q_{11} and q_{21} as above, note that once the matrix P is determined, we have

$$\begin{bmatrix} 1 & az + b \\ 0 & z^2 + cz + d \end{bmatrix}^{-1} \begin{bmatrix} q_{11}(z) \\ q_{21}(z) \end{bmatrix} = \begin{bmatrix} q_{11}(z) - \dfrac{(az + b)q_{21}}{z^2 + cz + d} \\ \dfrac{q_{21}}{z^2 + cz + d} \end{bmatrix}$$

Since each element of the resulting matrix must go to zero as z goes to infinity, we see that $q_{21} = s_{21}$ can be any polynomial of degree less than 2, but then q_{11} is completely determined by the polynomials p_{ij} and q_{21}.

Proof of Theorem 4.1. It suffices to consider the case when $m = 1$, so that the matrix Q consists of a single column vector. Define

$$e_{ij} = (-1)^{i+j} \det\, P_{ji}$$

where P_{ji} denotes the $(n - 1) \times (n - 1)$ matrix obtained by removing the jth row and ith column from P. Then $e_{ij}/\det P$ denotes the ijth component of P^{-1}. Note that we have the following constraints on the degree of e_{ij}:

$$\begin{aligned} \deg e_{ij} &\leqslant k - k_i - 1 \quad \text{if } i \neq j \\ \deg e_{ii} &= k - k_i \end{aligned}$$

Furthermore, the polynomials e_{ii} are monic.

Using the above notation, the condition that the elements of $P^{-1}Q$ tend to zero as z tends to infinity is equivalent to the following restriction for all i:

$$\deg\left(e_{i1}q_1 + e_{i2}q_2 + \cdots + e_{in}q_n\right) < k.$$

Isolating q_i in the ith such inequality, we obtain

$$q_i = \text{polynomial part of } \left\{\frac{-\sum_{i\neq l} e_{il}q_l}{e_{ii}}\right\} + s'_i$$

where s'_i is an arbitrary polynomial of degree less than k_i. This implies that

$$\deg q_i \leqslant \max\nolimits_{l\neq i}(\deg e_{il} + \deg q_l - \deg e_{ii}, k_i - 1) \leqslant \max\left(\deg q_l - 1,\ \ k_i - 1\right).$$

Let $\mathcal{S}$ denote the set of i for which $k_i \equiv k_{\max} \geqslant k_j$ for all $1 \leqslant j \leqslant n$. Then it follows immediately that $\deg q_i \leqslant k_{\max} - 1$ for all i, with equality possible only if $i \in \mathcal{S}$. Consequently, for any $i \in \mathcal{S}$, $q_i = s_i$ can be an arbitrary polynomial of degree less than $k_i = k_{\max}$.

In the case when $i \notin \mathcal{S}$, then we may have that $k_i - 1 < \deg q_i < k_{\max} - 1$. We want to show that the coefficients of the terms of degree greater than $k_i - 1$ are completely determined by those that are already known. Consider the polynomial $e_{i1}q_1 + e_{i2}q_2 + \cdots + e_{in}q_n$. The highest possible degree of any of the terms is $k - k_i + k_{\max} - 2$, and the only summands which possibly contain terms of this degree are $e_{ii}q_i$ and $e_{ij}q_j$ for $j \in \mathcal{S}$. Therefore, we must have that the coefficient for $z^{k_{\max}-2}$ in q_i is equal to the negative of the sum of the coefficients for $z^{k-k_i+k_{\max}-2}$ in $e_{ij}q_j$, since e_{ii} is monic. Note that this coefficient will certainly be polynomial in the coefficients for the p_{ij} and s_j, for $j \in \mathcal{S}$.

Similarly, the coefficient for $z^{k_{\max}-3}$ in q_i is determined using the terms of order $k - k_i + k_{\max} - 3$ in the polynomial $e_{i1}q_1 + e_{i2}q_2 + \cdots + e_{in}q_n$. Coefficients of the terms of order $k - k_i + k_{\max} - 3$ are obtained from the coefficients of $z^{k_{\max}-2}$ in q_j for $j \notin \mathcal{S}$, which have already been determined, from the coefficient of $z^{k_{\max}-1}$ in $q_j = s_j$ for $j \in \mathcal{S}$, and from the coefficients of the e_{ij}. Again, since e_{ii} is monic, the coefficient for $z^{k_{\max}-3}$ will be a polynomial combination of the previously determined coefficients.

In general, for $i \neq n$, the coefficients of z^l in q_i for $l \geqslant k_i$ will be a polynomial combination of the coefficients of the e_{ij} and the coefficients of z^r for $r > l$ in q_j, where $1 \leqslant j \leqslant n$. For $l < k_i$, the coefficients are arbitrary; that is, they agree with the coefficients of the arbitrary polynomial s_i. Therefore, by induction, the coefficients of z^l in q_i are polynomial in the coefficients of the p_{ij} and s_j, which completes the proof of the theorem. $\square$

5. The $\bar{\partial}$ map is holomorphic

We have seen that the map

$$\bar{\partial} : Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^{n+m})) \to Gr_k\left(\Gamma_{L_1^2}((\mathbb{C}^{n+m})^\vee \otimes \mathcal{O}(-1))\right)$$

yields a homotopy equivalence $Hol_k^\bullet(\mathbb{CP}^1; BU)^+ \simeq BU(k)$. In this section, we prove the following theorem:

Theorem 5.1. *The map*

$$\bar{\partial} : Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^{n+m})) \to Gr_k\left(\Gamma_{L_1^2}((\mathbb{C}^{n+m})^\vee \otimes \mathcal{O}(-1))\right)$$

is holomorphic.

As we have seen, the holomorphic mapping space $Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^{n+m}))$ is a connected, complex manifold of complex dimension $(n+m)k$. The Grassmannian $Gr_k\left(\Gamma_{L_1^2}((\mathbb{C}^{n+m})^\vee \otimes \mathcal{O}(-1))\right)$ is a complex Hilbert manifold; the complex structure of $Gr_k\left(\Gamma_{L_1^2}((\mathbb{C}^{n+m})^\vee \otimes \mathcal{O}(-1))\right)$ is obtained by thinking of it as a colimit of finite Grassmannians, for which the complex structure is well-understood.

Let the space $\mathcal{G}$ be defined by:

$$\mathcal{G} = \left\{(f, V) : f \in Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^{n+m})) \text{ and } V \in Gr_k(\Gamma_{L_1^2}(f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)))\right\}.$$

We will show that $\mathcal{G} \to Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^{n+m}))$ is a holomorphic fibration with fiber over $f = Gr_k(\Gamma_{L_1^2}(f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)))$. Then the map $\bar{\partial}$ is the composite

$$Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^{n+m})) \longrightarrow \mathcal{G} \longrightarrow Gr_k\left(\Gamma_{L_1^2}((\mathbb{C}^{n+m})^\vee \otimes \mathcal{O}(-1))\right)$$

where the first map is given by

$$f \mapsto \Gamma_{hol}(f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)) = \ker \bar{\partial}_{f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)}$$

and the second map is the map

$$Gr_k\left(\Gamma_{L_1^2}(f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1))\right) \hookrightarrow Gr_k\left(\Gamma_{L_1^2}((\mathbb{C}^N)^\vee \otimes \mathcal{O}(-1))\right)$$

defined in Section 3. We want to show that each of these maps is holomorphic, and hence the composition is holomorphic.

Lemma 5.2. *The map*

$$\mathcal{G} \to Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^{n+m}))$$

which sends (g, V) to g is a holomorphic fibration.

Proof of Lemma 5.2. First, we construct an open cover of $\mathbb{CP}^1$ so that the bundle $f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)$ is trivial over each open set and the transition functions are as simple as possible. We do this by pulling back an open cover of $Gr_n(\mathbb{C}^{n+m})$. Recall that the finite Grassmannian $Gr_n(\mathbb{C}^{n+m})$ of n-dimensional subspaces of $\mathbb{C}^{n+m}$ is a quotient of the Stiefel manifold of linear monomorphisms from $\mathbb{C}^n$ into $\mathbb{C}^{n+m}$. The

Stiefel manifold can be thought of as the collection of $(n+m) \times n$ matrices of rank n, and the column space of such a matrix represents the corresponding element of $Gr_n(\mathbb{C}^{n+m})$. Given an $(n+m) \times n$ matrix A and a multi-index $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ with $1 \leqslant a_1 < a_2 < \ldots < a_n \leqslant n+m$, one may consider the $n \times n$ matrix $A_{\mathbf{a}}$, where the ith row of $A_{\mathbf{a}}$ is the a_ith row of A. Let $U_{\mathbf{a}}$ be the subset of $Gr_n(\mathbb{C}^{n+m})$ such that if A is any matrix with image $V \in U_{\mathbf{a}}$, then the determinant of $A_{\mathbf{a}}$ is non-zero. In other words, if $\{e_1, e_2, \ldots, e_{n+m}\}$ is the standard basis for $\mathbb{C}^{n+m}$, then $U_{\mathbf{a}}$ consists of those subspaces of $\mathbb{C}^{n+m}$ such that the projection onto the span of $\{e_{a_1}, \ldots, e_{a_n}\}$ is an isomorphism:

$$U_{\mathbf{a}} = \{V \in Gr_n(\mathbb{C}^{n+m}) |\ pr_{\mathbf{a}} : V \xrightarrow{\cong} \mathrm{span}\{e_{a_1}, \ldots, e_{a_n}\}\}.$$

Given a particular $f \in Hol_k^{\bullet}(\mathbb{CP}^1; Gr_n(\mathbb{C}^{n+m}))$, the sets $f^{-1}(U_{\mathbf{a}})$ cover $\mathbb{CP}^1$. Furthermore, the bundle $f^*\gamma_n$ is trivial when restricted to such an open set.

From here on, we will identify f with the $(n+m) \times n$ matrix $[f]$ which is the transpose of its matrix description $[P, Q]$ in the Mann–Milgram normal form. Let $K = (k_1, \cdots, k_n)$, where $k_i = \deg p_{ii}$. As we have seen in Section 4, we can describe an open set $\mathcal{U}_K \subset Hol_k^{\bullet}(\mathbb{CP}^1; Gr_n(\mathbb{C}^{n+m}))$ in terms of matrices of polynomials with certain relations among the coefficients and restrictions on the degrees. We identify $g \in \mathcal{U}_K$ with the matrix $[g]$ which is the transpose of the matrix description given in Section 4. Since we will think of $[f]$ and $[g]$ as linear monomorphisms from $\mathbb{C}^n$ to $\mathbb{C}^{n+m}$, it is more convenient to use the transpose of the previously described normal form. Consider the $n \times n$ matrices $[f(z)_{\mathbf{a}}]$. We can describe a local trivialization $\phi_{\mathbf{a}}$ of $f^*\gamma_n$ as follows:

$$\phi_{\mathbf{a}} : f^*\gamma_n \mid_{f^{-1}(U_{\mathbf{a}})} \xrightarrow{\cong} f^{-1}(U_{\mathbf{a}}) \times \mathbb{C}^n$$

$$(z, v) \mapsto (z, [f(z)_{\mathbf{a}}][f(z)]^{-1}v).$$

Here, $v \in \mathbb{C}^{n+m}$ is an element of the subspace $f(z) \in Gr_n(\mathbb{C}^{n+m})$, or equivalently of the column space of the matrix $[f(z)]$. Although the transformation $[f(z)]$ is not invertible, it is injective, therefore $[f(z)]^{-1}v$ is well-defined. Essentially, the above trivialization takes an element v of the column space of $[f(z)]$ and maps it to the row space of $[f(z)]$ (which is precisely $\mathbb{C}^n$, since the matrix $[f(z)]$ is rank n), so we may think of this trivialization as giving an explicit isomorphism from the column space of $[f(z)]$ to the row space, with inverse

$$\phi_{\mathbf{a}}^{-1} : f^{-1}(U_{\mathbf{a}}) \times \mathbb{C}^n \xrightarrow{\cong} f^*\gamma_n \mid_{f^{-1}(U_{\mathbf{a}})}$$

$$(z, w) \mapsto (z, [f(z)][f(z)_{\mathbf{a}}]^{-1}w).$$

Note that both $\phi_{\mathbf{a}}$ and its inverse are holomorphic.

Consider the decomposition of $\mathbb{CP}^1$ as the union of two disks, $D_0 = \mathbb{CP}^1 - \infty = \mathbb{C}$ and $D_\infty = \mathbb{CP}^1 - 0 = \{\mathbb{C} - 0\} \cup \infty$. Then we have holomorphic trivializations of the bundle $\mathcal{O}(-1)$ over each disk. Recall that

$$\mathcal{O}(-1) = \{(z, (w_1, w_2)) \in \mathbb{CP}^1 \times \mathbb{C}^2 |\ w_1/w_2 = z \in \mathbb{CP}^1 = \mathbb{C} \cup \infty\}$$

and so the trivialization $\mathcal{O}(-1)|_{D_0} \cong D_0 \times \mathbb{C}$ is given by $(z, (w_1, w_2)) \mapsto (z, w_1)$, and the trivialization $\mathcal{O}(-1)|_{D_\infty} \cong \mathbb{CP}^1 \times \mathbb{C}$ is given by $(z, (w_1, w_2)) \mapsto (z, w_2)$.

Thus, we may cover $\mathbb{CP}^1$ with open sets $f^{-1}(U_{\mathbf{a}}) \cap D_i$ so that the bundle $f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)$ is trivial when restricted to one of these open sets. Actually, we wish to replace each of these open sets with a smaller open set such that the collection still covers $\mathbb{CP}^1$. The determinant of $[f(z)_{\mathbf{a}}]$ is a polynomial in z of degree at most k, and therefore it has at most k distinct zeros. This implies that the set $f^{-1}(U_{\mathbf{a}}) = \{z : \det[f(z)_{\mathbf{a}}] \neq 0\}$ is either empty (if $\det[f(z)_{\mathbf{a}}] \equiv 0$) or consists of $\mathbb{CP}^1$ with finitely many punctures. It follows that there exists some $\epsilon > 0$ so that the open sets

$$V_{f,\mathbf{a}} = f^{-1}(U_{\mathbf{a}}) - \{\text{closed } \epsilon \text{ ball around each puncture}\}$$

will cover $\mathbb{CP}^1$.

Proposition 5.3. *Let $N = \frac{(n+m)!}{n!m!}$, and let the open sets $V_{f,\mathbf{a}}$ and D_i be defined as above. Then an element of $\Gamma_{L_1^2}(f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1))$ can be expressed in the following way as a 2N-tuple of functions and relations:*

$$\Gamma_{L_1^2}(f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)) = \{(\sigma_1, \dots, \sigma_N, \quad \tau_1, \dots, \tau_N)|$$

$$\sigma_j : V_{f,\mathbf{a}_j} \cap D_0 \longrightarrow \mathbb{C}^{n\vee}$$

$$\tau_j : V_{f,\mathbf{a}_j} \cap D_\infty \longrightarrow \mathbb{C}^{n\vee}$$

$$\sigma_i(z) \circ [f(z)_{\mathbf{a}_i}] = \sigma_j(z) \circ [f(z)_{\mathbf{a}_j}], \quad z \in V_{f,\mathbf{a}_i} \cap V_{f,\mathbf{a}_j} \cap D_0$$

$$\tau_i(z) \circ [f(z)_{\mathbf{a}_i}] = \tau_j(z) \circ [f(z)_{\mathbf{a}_j}], \quad z \in V_{f,\mathbf{a}_i} \cap V_{f,\mathbf{a}_j} \cap D_\infty$$

$$\sigma_i(z) = 1/z\tau_i(z), \quad z \in V_{f,\mathbf{a}_i} \cap D_0 \cap D_\infty\}$$

The proposition follows immediately from the above description of the local trivializations of the bundle $f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)$.

Define

$$\mathcal{U}_f = \mathcal{U}_K \cap \{g \in Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^{n+m})) : V_{f,\mathbf{a}} \subset g^{-1}(U_{\mathbf{a}}) \ \forall \ \mathbf{a}\}.$$

Note that for every $g \in \mathcal{U}_f$, the space of sections of the bundle associated to g can be described as above, in terms of 2N-tuples of functions from $V_{f,\mathbf{a}_j} \cap D_i$ to $\mathbb{C}$, which leads to the following lemma:

Lemma 5.4. *The space of L^2 sections $\Gamma_{L_1^2}(f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1))$ is isomorphic to the corresponding space of L^2 sections associated to g, $\Gamma_{L_1^2}(g^*\gamma_n{}^\vee \otimes \mathcal{O}(-1))$, with isomorphism ψ_g given by $(\sigma_1, \dots, \sigma_N, \tau_1, \dots, \tau_N) \mapsto (\sigma_1', \dots, \sigma_N', \tau_1', \dots, \tau_N')$ where*

$$\sigma_i'(z) = \sigma_i(z) \circ [f(z)_{\mathbf{a}_i}][g(z)_{\mathbf{a}_i}]^{-1}$$

and

$$\tau_i'(z) = \tau_i(z) \circ [f(z)_{\mathbf{a}_i}][g(z)_{\mathbf{a}_i}]^{-1}.$$

Furthermore, ψ_g restricts to the space of holomorphic sections to give an isomorphism

$$\psi_g : \Gamma_{hol}(f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)) \to \Gamma_{hol}(g^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)).$$

Using the map ψ_g, we see that the spaces $\mathcal{U}_f \times Gr_k(\Gamma_{L^2_1}(f^*{\gamma_n}^\vee \otimes \mathcal{O}(-1)))$ and $\mathcal{G}|_{\mathcal{U}_f}$ are in bijective correspondence. The complex structure on $\mathcal{G}$ is induced locally by the complex structure of the product spaces $\mathcal{U}_f \times Gr_k(\Gamma_{L^2_1}(f^*{\gamma_n}^\vee \otimes \mathcal{O}(-1)))$, so that $\mathcal{G}$ a holomorphic fibration. This completes the proof of Lemma 5.2. □

In order to prove Theorem 5.1, it remains to show the map

$$\mathcal{G} \to Gr_k\left(\Gamma_{L^2_1}((\mathbb{C}^{n+m})^\vee \otimes \mathcal{O}(-1))\right)$$

induced by the surjection

$$\pi_g : (C^{n+m})^\vee \otimes \mathcal{O}(-1) \to g^*{\gamma_n}^\vee \otimes \mathcal{O}(-1)$$

is holomorphic.

Again, we may work locally, so our goal is to show that

$$\mathcal{U}_f \times Gr_k(\Gamma_{L^2_1}(f^*{\gamma_n}^\vee \otimes \mathcal{O}(-1))) \to Gr_k\left(\Gamma_{L^2_1}((\mathbb{C}^{n+m})^\vee \otimes \mathcal{O}(-1))\right)$$

is holomorphic, or equivalently, that the adjoint map

$$\mathcal{U}_f \to Hol\left(Gr_k(\Gamma_{L^2_1}(f^*{\gamma_n}^\vee \otimes \mathcal{O}(-1))); Gr_k\left(\Gamma_{L^2_1}((\mathbb{C}^{n+m})^\vee \otimes \mathcal{O}(-1))\right)\right)$$

is holomorphic.

We shall need the following lemma:

Lemma 5.5. *The map*

$$\mathcal{U}_f \to \operatorname{Lin\,Surj}\left(\Gamma_{L^2_1}((C^{n+m})^\vee \otimes \mathcal{O}(-1)); \Gamma_{L^2_1}(f^*{\gamma_n}^\vee \otimes \mathcal{O}(-1))\right)$$

induced by the composition

$$g \mapsto \left\{\Gamma_{L^2_1}((C^{n+m})^\vee \otimes \mathcal{O}(-1)) \xrightarrow{\pi_g} \Gamma_{L^2_1}(g^*{\gamma_n}^\vee \otimes \mathcal{O}(-1)) \xrightarrow{\cong} \Gamma_{L^2_1}(f^*{\gamma_n}^\vee \otimes \mathcal{O}(-1))\right\}$$

is holomorphic.

A linear surjection between vector spaces immediately yields a holomorphic map between Grassmannians of fixed codimension:

$$\begin{array}{ccc} \mathcal{U}_f & \longrightarrow & \operatorname{Lin\,Surj}\left(\Gamma_{L^2_1}((C^{n+m})^\vee \otimes \mathcal{O}(-1)); \Gamma_{L^2_1}(f^*{\gamma_n}^\vee \otimes \mathcal{O}(-1))\right) \\ & & \downarrow \\ & & Hol\left(Gr^k(\Gamma_{L^2_1}(f^*{\gamma_n}^\vee \otimes \mathcal{O}(-1))); Gr^k\left(\Gamma_{L^2_1}((C^{n+m})^\vee \otimes \mathcal{O}(-1))\right)\right) \end{array}$$

Therefore, Lemma 5.5 yields a holomorphic map

$$\mathcal{U}_f \times Gr^k(\Gamma_{L^2_1}(f^*{\gamma_n}^\vee \otimes \mathcal{O}(-1))) \to Gr^k(\Gamma_{L^2_1}((C^N)^\vee \otimes \mathcal{O}(-1))).$$

For any Hilbert space $\mathcal{H}$, there is an anti-holomorphic map from $Gr_k(\mathcal{H})$ to $Gr^k(\mathcal{H})$. Letting $\mathcal{H} = \Gamma_{L^2_1}(f^*{\gamma_n}^\vee \otimes \mathcal{O}(-1))$ produces an anti-holomorphic map

$$\mathcal{U}_f \times Gr_k(\Gamma_{L^2_1}(f^*{\gamma_n}^\vee \otimes \mathcal{O}(-1))) \to Gr^k(\Gamma_{L^2_1}((C^{n+m})^\vee \otimes \mathcal{O}(-1))).$$

Letting $\mathcal{H} = \Gamma_{L^2_1}((C^{n+m})^\vee \otimes \mathcal{O}(-1))$ yields an anti-holomorphic map

$$Gr^k(\Gamma_{L^2_1}((C^{n+m})^\vee \otimes \mathcal{O}(-1))) \to Gr_k(\Gamma_{L^2_1}((C^{n+m})^\vee \otimes \mathcal{O}(-1))).$$

Since the composition of two anti-holomorphic maps is holomorphic, the result is a holomorphic map

$$\mathcal{U}_f \times Gr_k(\Gamma_{L^2_1}(f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1))) \to Gr_k(\Gamma_{L^2_1}((C^{n+m})^\vee \otimes \mathcal{O}(-1))),$$

as required.

Proof of Lemma 5.5. We will treat elements of $\Gamma_{L^2_1}(f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1))$ as local sections $(\sigma_1, \cdots, \sigma_n, \tau_1, \cdots, \tau_n)$ with relations as described in Proposition 5.3. Let V be one of the open sets $V_{f,\mathbf{a}_j} \cap D_0$, and let

$$s_j : V \to \mathbb{C}^{N^\vee}$$

be the local trivialization of some section $s \in \Gamma_{L^2_1}((C^{n+m})^\vee \otimes \mathcal{O}(-1))$ restricted to V. (The case for $V = V_{f,\mathbf{a}_j} \cap D_\infty$ is analogous.)

We map s_j to σ'_j, a local section of $g^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)$, as follows:

$$s \mapsto \{z \mapsto \sigma'_j(z) = s_j(z) \circ [g(z)][g(z)_{\mathbf{a_j}}]^{-1}.$$

By Lemma 5.4, we know that $\Gamma_{L^2_1}((f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1))|_V) \cong \Gamma_{L^2_1}((g^*\gamma_n{}^\vee \otimes \mathcal{O}(-1))|_V)$ with isomorphism

$$\sigma(z) \mapsto \sigma'(z) = \sigma(z) \circ [f(z)_{\mathbf{a}}][g(z)_{\mathbf{a}}]^{-1}\},$$

therefore the composite

$$\Gamma_{L^2_1}((C^{n+m})^\vee \otimes \mathcal{O}(-1)) \to \Gamma_{L^2_1}(f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1))$$

is given locally by

$$s \mapsto \{z \mapsto \sigma_j(z) = s(z) \circ [g(z)][f(z)_{\mathbf{a_j}}]^{-1}\}.$$

This is clearly holomorphic as g varies, and furthermore yields a well-defined section of $f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1)$ as can be immediately verified from the description of $\Gamma_{L^2_1}(f^*\gamma_n{}^\vee \otimes \mathcal{O}(-1))$ given in Proposition 5.3. □

This completes the proof of Theorem 5.1, and so we have that

$$\bar{\partial} : Hol^\bullet_k(\mathbb{CP}^1; Gr_n(\mathbb{C}^{n+m})) \to Gr_k\left(\Gamma_{L^2_1}((\mathbb{C}^{n+m})^\vee \otimes \mathcal{O}(-1))\right)$$

is holomorphic, as required. We can immediately extend $\bar{\partial}$ for all smooth projective varieties X to

$$\bar{\partial} : Hol\left(X; Hol^\bullet_k(\mathbb{CP}^1; Gr_n(\mathbb{C}^{n+m}))\right) \to Hol\left(X; Gr_k\left(\Gamma_{L^2_1}((\mathbb{C}^{n+m})^\vee \otimes \mathcal{O}(-1))\right)\right)$$

for all k,m, and n.

Proposition 5.6. *There exists a well-defined extension of $\bar{\partial}$ to the Quillen–Segal group completion:*

$$\bar{\partial} : Hol(X; Hol^\bullet(\mathbb{CP}^1; BU))^+ \to Hol(X; \mathbb{Z} \times BU)^+.$$

Proof. The spaces

$$\bigcup_{k,n,m} Hol\left(X; Hol^{\bullet}_k(\mathbb{CP}^1; Gr_n(\mathbb{C}^{n+m}))\right)$$

and

$$\bigcup_{k,n+m} Hol\left(X; Gr_k\left(\Gamma_{L^2_1}((\mathbb{C}^{n+m})^{\vee}\otimes\mathcal{O}(-1))\right)\right)$$

are monoids, and so this follows immediately from the proof of Proposition 3.2. □

6. Proof of Theorem 1.3

In this section, we prove that

$$\bar\partial : Hol(X; Hol^{\bullet}_k(\mathbb{CP}^1; BU))^+ \to Hol(X; \mathbb{Z}\times BU)^+$$

is a homotopy equivalence, proving Theorem 1.3. We do this by first showing that $-\bar\partial\circ\beta$ is homotopic to the identity and then proving that β (and therefore $\bar\partial$) induces an isomorphism in homology, where β is the Bott map

$$\beta : Hol(X; \mathbb{Z}\times BU)^+ \to Hol(X; Hol^{\bullet}_k(\mathbb{CP}^1; BU))^+.$$

We may view the Bott class b as an element of $Hol^{\bullet}(\mathbb{CP}^1; BU)^+$ by thinking of b as the difference of two holomorphic maps, f_0 and f_1, $f_i : \mathbb{CP}^1 \to BU(1)$, such that $f_0^*\gamma_1 = [1]$ and $f_1^*\gamma_1 = \mathcal{O}(-1)$. Then the Bott map is defined as before (up to homotopy) by tensoring with b.

$$\begin{array}{ccc} K_{hol}(X) & \xrightarrow{\beta} & Hol(X; Hol^{\bullet}(\mathbb{CP}^1; BU))^+ \\ \downarrow & & \downarrow \\ K_{top}(X) & \xrightarrow{\beta} & Map(X; \Omega^2(BU)) \end{array}$$

The vertical maps are induced by the obvious inclusions.

Proposition 6.1. *The composition*

$$-\bar\partial\circ\beta : Hol\left(X; Hol^{\bullet}(\mathbb{CP}^1; BU)\right)^+ \to Hol\left(X; Hol^{\bullet}(\mathbb{CP}^1; BU)\right)^+$$

is homotopic to the identity.

Proof. The argument at the end of Section 3 proves that $-\bar\partial\circ\beta\simeq 1$ when X is a point. For general X, we note that $\bar\partial$ satisfies the following module homomorphism-like property.

Lemma 6.2. *The diagram*

$$\begin{array}{ccc} K_{hol}(X)\times Hol^{\bullet}(\mathbb{CP}^1; BU)^+ & \xrightarrow{\otimes} & Hol(X; Hol^{\bullet}(\mathbb{CP}^1; BU))^+ \\ {\scriptstyle 1\times\bar\partial}\downarrow & & {\scriptstyle\bar\partial}\downarrow \\ K_{hol}(X)\times K_{hol}(pt) & \xrightarrow{\otimes} & K_{hol}(X) \end{array}$$

commutes up to homotopy.

Proof of Lemma 6.2. Recall that $K_{hol}(X) \equiv Hol(X; \mathbb{Z} \times BU)^+$. Since the tensor product and $\bar{\partial}$ both extend to the group completion, it suffices to consider the analogous diagram for maps to finite Grassmannians. On the level of finite Grassmannians, we have defined

$$\bar{\partial} : Hol\left(X; Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^N))\right) \to Hol\left(X; Gr_k\left(\Gamma_{L_1^2}((\mathbb{C}^N)^\vee \otimes \mathcal{O}(-1))\right)\right).$$

Since there exist natural biholomorphic equivalences

$$Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^N)) \cong Hol_k^\bullet(\mathbb{CP}^1; Gr^{N-n}(\mathbb{C}^N))$$

and

$$Gr_k\left(\Gamma_{L_1^2}((\mathbb{C}^N)^\vee \otimes \mathcal{O}(-1))\right) \cong Gr^k\left(\Gamma^*_{L_1^2}((C^N)^\vee \otimes \mathcal{O}(-1))\right)$$

we can equivalently think of $\bar{\partial}$ as

$$\bar{\partial} : Hol\left(X; Hol_k^\bullet(\mathbb{CP}^1; Gr^{N-n}(\mathbb{C}^N))\right) \to Hol\left(X; Gr^k\left(\Gamma^*_{L_1^2}((C^N)^\vee \otimes \mathcal{O}(-1))\right)\right).$$

It follows from the definition of $\bar{\partial}$ that the following diagram commutes:

$$\begin{array}{ccc}
Gr^m(\mathbb{C}^M) \times Hol_k^\bullet(\mathbb{CP}^1; Gr^{N-n}(\mathbb{C}^N)) & \xrightarrow{\otimes} & Hol_{mk}^\bullet(\mathbb{CP}^1; Gr^{m(N-n)}(\mathbb{C}^M \otimes \mathbb{C}^N)) \\
\Big\downarrow {\scriptstyle 1\times\bar{\partial}} & & \Big\downarrow {\scriptstyle \bar{\partial}} \\
Gr^m(\mathbb{C}^M) \times Gr^k\left(\Gamma^*_{L_1^2}((\mathbb{C}^N)^\vee \otimes \mathcal{O}(-1))\right) & \xrightarrow{\otimes} & Gr^{mk}\left(\Gamma^*_{L_1^2}((\mathbb{C}^M \otimes \mathbb{C}^N)^\vee \otimes \mathcal{O}(-1))\right)
\end{array}$$

Since all maps involved are holomorphic, we can apply $Hol(X; -)$ to the above diagram. Taking the Quillen–Segal completion of all spaces leads to the desired result. □

Recall that β is defined by tensoring with the Bott class $b \in Hol^\bullet(\mathbb{CP}^1; BU)^+$. The above lemma implies that

$$-\bar{\partial} \circ \beta : K_{hol}(X) \longrightarrow K_{hol}(X)$$

is homotopic to tensoring with $-\bar{\partial}(b) \in K_{hol}(pt)$:

$$K_{hol}(X) \xrightarrow{\otimes -\bar{\partial}(b)} K_{hol}(X).$$

Since $-\bar{\partial} \circ \beta \simeq 1$ on $K_{hol}(pt)$, we see that tensoring with $-\bar{\partial}(b)$ is homotopic to the identity, and therefore $-\bar{\partial} \circ \beta \simeq 1$ on $K_{hol}(X)$ as claimed. □

Next, we prove that β induces an isomorphism in homology. We approach this indirectly, by defining a map η with well-defined homotopy type

$$\eta : Hol(X; Hol^{\bullet}(\mathbb{CP}^1; BU))^+ \to K_{hol}(X)$$

such that

$$\beta \circ \eta : Hol(X; Hol^{\bullet}(\mathbb{CP}^1; BU))^+ \to Hol(X; Hol^{\bullet}(\mathbb{CP}^1; BU))^+$$

is homotopic to the identity. Since $-\bar{\partial} \circ \beta$ is also homotopic to the identity, β is a homotopy equivalence. Consequently, $\bar{\partial}$ must also be a homotopy equivalence, completing the proof of Theorem 1.3.

Proposition 6.3. *There exists a homotopy class*

$$[\eta] \in \pi_0 Map\left(Hol(X; Hol^{\bullet}(\mathbb{CP}^1; BU))^+; K_{hol}(X)\right)$$

such that if η is any representative of $[\eta]$, then

$$(\beta \circ \eta)_* : H_*\left(Hol(X; Hol^{\bullet}(\mathbb{CP}^1; BU))^+\right) \to H_*\left(Hol(X; Hol^{\bullet}(\mathbb{CP}^1; BU))^+\right)$$

is an isomorphism.

We define $[\eta]$ using the projective bundle theorem for algebraic K-theory:

Theorem 6.4. ([**16**]) *Let Y be a smooth quasi-projective scheme, let E be a vector bundle of rank r over Y, and let $\mathbb{P}(E)$ be the associated projective bundle. If $z \in K^0_{alg}(\mathbb{P}(E))$ is the class of the canonical line bundle $\mathcal{O}(-1)$, then we have an isomorphism*

$$(K^0_{alg}(Y))^r \xrightarrow{\cong} K^0_{alg}(\mathbb{P}(E))$$

given by

$$[x_i]_{0 \leqslant i < r} \mapsto \sum_{i=0}^{r-1} z^i f^* x_i$$

where $f : \mathbb{P}(E) \to Y$ is the structural map.

In the case when E is the two-dimensional trivial bundle, this theorem states that $K^0_{alg}(Y \times \mathbb{CP}^1) \cong K^0_{alg}(Y)^2$. In turn, this yields an isomorphism

$$\beta : K^0_{alg}(Y) \to K^0_{alg}(Y \times \mathbb{CP}^1, Y \times \infty)$$

for any smooth quasi-projective scheme Y, where β is precisely the Bott map.

Let $Y = Hol^{\bullet}_k(\mathbb{CP}^1; Gr_n(\mathbb{C}^N))$. Then the evaluation map

$$Y \times \mathbb{CP}^1 \to BU$$

defines a canonical element $[e]$ in $K^0_{alg}(Y \times \mathbb{CP}^1, Y \times \infty)$.

Define $[\eta]$ by $[\eta] = \beta^{-1}[e]$, and consider the images of $[e]$ and $[\eta]$ in holomorphic K-theory. By definition, we have that

$$[\eta] \in \pi_0 Hol\,(Y; \mathbb{Z} \times BU)^+ = K^0_{hol}(Y).$$

We know that the set of path components of the Quillen–Segal group completion of a monoid is the Grothendieck group completion of the monoid of path components [**6**].

Therefore, $[\eta]$ can be represented as the difference of two classes

$$[\eta] = [\eta_1] - [\eta_2]$$

where each $\eta_i \in Hol\left(Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^N)); \mathbb{Z} \times BU\right)$. Composing with β yields maps

$$\beta \circ \eta_i : Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^N)) \to Hol^\bullet(\mathbb{CP}^1; BU)^+.$$

Finally, since $Hol^\bullet(\mathbb{CP}^1; BU)^+$ is an infinite loop space (see [**6**]), subtraction of elements is well-defined, and so $\beta \circ \eta_1 - \beta \circ \eta_2 = \beta \circ \eta$ represents a well-defined homotopy class:

$$\beta \circ \eta \in \pi_0 Hol\left(Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^N)); Hol^\bullet(\mathbb{CP}^1; BU)\right)^+.$$

Since $[\beta \circ \eta]$ is adjoint to the class $\beta[\eta] = [e]$, and $[e]$ is clearly adjoint to the element $[\iota]$ representing the inclusion of $Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^N))$ into $Hol^\bullet(\mathbb{CP}^1; BU)$:

$$[\beta \circ \eta] = [\iota] \in \pi_0 Hol\left(Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^N)); Hol^\bullet(\mathbb{CP}^1; BU)\right)^+.$$

Up to homotopy, this induces a map

$$\beta \circ \eta \simeq \iota : Hol(X; Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^N))) \to Hol(X; Hol^\bullet(\mathbb{CP}^1; BU))^+$$

for all smooth projective varieties X.

Since the diagram

$$\begin{array}{ccc} Hol(X; Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^N))) & \xrightarrow{\beta\circ\eta} & Hol(X; Hol^\bullet(\mathbb{CP}^1; BU))^+ \\ \downarrow & \nearrow_{\beta\circ\eta} & \\ Hol(X; Hol_k^\bullet(\mathbb{CP}^1; Gr_n(\mathbb{C}^{N+1}))) & & \end{array}$$

commutes, we obtain

$$\beta \circ \eta \simeq \iota : \sqcup_{n \geqslant 0} Hol(X; Hol^\bullet(\mathbb{CP}^1; BU(n))) \to Hol(X; Hol^\bullet(\mathbb{CP}^1; BU))^+$$

which extends to the group completion

$$\left(\sqcup_n Hol(X; Hol^\bullet(\mathbb{CP}^1; BU(n)))\right)^+ \xrightarrow{\simeq} Hol(X; Hol^\bullet(\mathbb{CP}^1; BU))^+.$$

Thus, $\beta \circ \eta$ is homotopic to the identity as required, and β, and hence $\bar{\partial}$, are homotopy equivalences, completing the proof of Theorem 1.3.

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