Homology, Homotopy and Applications, vol.4(2), 2002, pp.29–62

SECONDARY COHOMOLOGY AND THE STEENROD SQUARE

HANS–JOACHIM BAUES

(*communicated by Larry Lambe*)

Abstract

We introduce and study various properties of the secondary cohomology of a space. Certain Steenrod squares are shown to be related to the action of the symmetric groups on secondary cohomology.

To Jan–Erik Roos on his sixty–fifth birthday

For a field k we choose the Eilenberg–MacLane space $Z^n = K(k,n)$ by the realization of the simplicial k–vector space generated by the non–basepoint singular simplices of the n–sphere $S^n = S^1 \wedge \ldots \wedge S^1$. The permutation of the smash product factors S^1 yields an action of the symmetric group σ_n on S^n and hence on Z^n. Moreover the quotient map $S^n \times S^m \to S^{n+m}$ induces a cup product map $\mu : Z^n \times Z^m \to Z^{m+n}$ with $n, m \geqslant 1$; see the Appendix below.

It is well known that the (reduced) cohomology $\widetilde{H}^n(X,k)$ of a path-connected pointed space X is the same as the set $[X, Z^n]$ of homotopy classes $\{x\}$ of pointed maps $x : X \to Z^n$. Moreover the cup product of the cohomology algebra $H^* = H^*(X,k) = \widetilde{H}^* \oplus k$ is induced by the map μ, that is $\{x\} \cup \{y\} = \{\mu(x,y)\}$. The cohomology algebra is *graded commutative* in the sense that

$$\{x\} \cup \{y\} = (-1)^{nm}\{y\} \cup \{x\}$$

In this paper we replace the homotopy set $[X, Z^n]$ by the groupoid $[\![X, Z^n]\!]$. The objects of this groupoid are the pointed maps $x : X \to Z^n$ and the morphisms $x \Rightarrow y$ in $[\![X, Z^n]\!]$ are the homotopy classes of homotopies $x \simeq y$ termed tracks. The set of path components of $[\![X, Z^n]\!]$ is

$$\pi_0[\![X, Z^n]\!] = [X, Z^n] = \widetilde{H}^n$$

and the group of tracks $0 \Rightarrow 0$ of the trivial map $0 : X \to * \to Z^n$ in $[\![X, Z^n]\!]$ is

$$\pi_1[\![X, Z^n]\!] = [X, \Omega Z^n] = \widetilde{H}^{n-1}$$

We associate with $[\![X, Z^n]\!]$ the exact sequence $\mathcal{H}^n(X)$:

$$0 \to \widetilde{H}^{n-1} \to \mathcal{H}^n(X)_1 \xrightarrow{\partial} \mathcal{H}^n(X)_0 \to \widetilde{H}^n \to 0$$

Here $\mathcal{H}^n(X)_0$ is the set of all pointed maps $X \to Z^n$ and $\mathcal{H}^n(X)_1$ is the set of pairs (x, H) with $H : x \Rightarrow 0$ and $\partial(x,H) = x$.

The Eilenberg–MacLane spaces Z^n have the following *basic properties*:

Received November 14, 2000, revised February 14, 2002; published on July 12, 2002.
2000 Mathematics Subject Classification: 55N99.
Key words and phrases: secondary module, secondary cohomology, Hochschild cohomology.

(a) Z^n is a k–vector space object in the category $\mathbf{Top}^*$ of pointed spaces.

(b) The symmetric group σ_n acts on Z^n via linear automorphisms inducing the sign of a permutation on $H_n(Z^n)$.

(c) The cup product map $\mu : Z^n \times Z^m \to Z^{n+m}$ is k–bilinear and equivariant with respect to the inclusion $\sigma_n \times \sigma_m \subset \sigma_{n+m}$. Moreover μ is associative in the obvious sense and the following diagram commutes.

$$\begin{tikzcd} Z^n \times Z^m \arrow[r, "\mu"] \arrow[d, "T"'] & Z^{m+n} \arrow[d, "\tau_{n,m}"] \\ Z^m \times Z^n \arrow[r, "\mu"] & Z^{m+n} \end{tikzcd}$$

The map T is the interchange map $T(x,y) = (y,x)$ and $\tau_{n,m} \in \sigma_{n+m}$ is the element interchanging the first n–block with the second m–block.

Properties (b) and (c) imply that $\{Z^n, n \geqslant 0\}$ is a "symmetric spectrum" in the sense of Hovey–Shipley–Smith [**HSS**] 1.2.5. We use the poperties (a), (b) and (c) of Z^n to show that the graded object

$$\mathcal{H}^*(X) = \{\mathcal{H}^n(X), n \geqslant 1\}$$

has the structure of a "secondary algebra" which we call the *secondary cohomology* of the space X. Using secondary algebras we introduce the third *cohomology* SH^3 of a graded commutative algebra and we show that the secondary cohomology $\mathcal{H}^*(X)$ represents an element

$$\langle\!\langle \mathcal{H}^*(X) \rangle\!\rangle \ \in \mathsf{SH}^3(H^*, \widetilde{H}^*[1])$$

which is an invariant of the homotopy type of X. There is a natural transformation from the symmetric cohomology SH^3 to the Hochschild cohomology HH^3 which carries the class $\langle\!\langle \mathcal{H}^*(X) \rangle\!\rangle$ to the class

$$\langle\!\langle C^*(X) \rangle\!\rangle \ \in \mathsf{HH}^*(H^*, \widetilde{H}^*[1])$$

defined by the algebra of cochains $C^*(X)$ of the space X. It is known that the class $\langle\!\langle C^*(X) \rangle\!\rangle$ determines all triple Massey products in the cohomology $H^*(X,k)$, see for example Berrick Davydov [**BD**] or Baues–Minian [**BM**]. The new class $\langle\!\langle \mathcal{H}^*(X) \rangle\!\rangle$ in addition determines for $k = \mathbb{F}_2$ the Steenrod operations

$$\mathrm{Sq}^{n-1} : H^n \to H^{2n-1}, \ n \geqslant 1.$$

The Hochschild cohomology HH^* is defined for algebras and graded algebras in general while the symmetric cohomology SH^3 is only defined for commutative graded algebras.

1. Secondary modules

Motivated by properties of Eilenberg–MacLane spaces in topology we introduce the algebraic concept of a secondary module. Later we will consider functors from the category of spaces to the categories of secondary modules and secondary algebras respectively.

Let k be a field and let R be a k–*algebra* with *unit* i and *augmentation* ε

$$k \xrightarrow{i} R \xrightarrow{\varepsilon} k. \tag{1.1}$$

Here i and ε are algebra maps with $\varepsilon i = 1$. For example let G be a group together with a homomorphism $\varepsilon : G \to k^*$ where k^* is the group of units in the field k. Then ε induces an augmentation

$$\varepsilon : k[G] \to k \tag{1}$$

where $k[G]$ is the *group algebra* of G. Here $k[G]$ is a vector space with basis G and ε carries the basis element $g \in G$ to $\varepsilon(g)$. In particular we have for the *symmetric group* σ_n (which is the group of bijections of the set $\{1, \dots, n\}$) the sign–homomorphism

$$\mathsf{sign} : \sigma_n \to \{1, -1\} \to k^* \tag{2}$$

which induces the *sign–augmentation*

$$\varepsilon = \varepsilon_{\mathsf{sign}} : k[\sigma_n] \to k \tag{3}$$

These examples play a special role in applications to topology below.

For k–vector spaces A, B we use the *tensor product*

$$A \otimes B = A \otimes_k B \tag{1.2}$$

A homomorphism $f : A \to B$ is termed a k–*linear* map. If A and B are R–modules then the map f is R–*linear* if in addition $f(r \cdot x) = r \cdot f(x)$ for $r \in R, x \in A$. If R and K are k–algebras then also $R \otimes K$ is a k–algebra with augmentation

$$\varepsilon : R \otimes K \xrightarrow{\varepsilon \otimes \varepsilon} k \otimes k = k$$

The multiplication in $R \otimes K$ is defined as usual by $(\alpha \otimes \beta) \cdot (\alpha' \otimes \beta') = (\alpha\alpha') \otimes (\beta\beta')$. Moreover if X is an R–module and Y is a K–module then $X \otimes Y$ is an $R \otimes K$–module by $(\alpha \otimes \beta) \cdot (x \otimes y) = (\alpha x) \otimes (\beta y)$. The following definition of a secondary module is motivated by the examples in section 3. Therefore the definition may be considered as a result of calculation derived from these examples, see (2.6) and (2.10). Since, however, secondary modules play a central role in this paper we define them right away as follows.

Definition 1.3. Let R be a k–algebra as in (1.1). A *secondary module* $X = X_R$ *over* R consists of a diagram

$$R \otimes X_0 \xrightarrow{\Gamma} X_1 \xrightarrow{\partial} X_0$$

where X_0 and X_1 are R–modules and ∂ is R–linear and Γ is k–linear such that for $r, r' \in R, a \in X_1, x \in X_0$ the following equations hold.

$$\partial\Gamma(r \otimes x) = (r - \varepsilon(r))x \tag{1}$$

$$\Gamma(r \otimes \partial a) = (r - \varepsilon(r))a \tag{2}$$

$$\Gamma((r \cdot r') \otimes x) = r\Gamma(r' \otimes x) + \varepsilon(r')\Gamma(r \otimes x) \tag{3}$$

$$\Gamma((r \cdot r') \otimes x) = \Gamma(r \otimes r'x) + \varepsilon(r)\Gamma(r' \otimes x) \tag{4}$$

Now let X_R and Y_K be secondary modules over R and over K respectively. A *map* between secondary modules

$$f = f_h : X_R \to Y_K \tag{5}$$

consists of an augmented algebra map $h : R \to K$ and a commutative diagram

$$\begin{tikzcd} R \otimes X_0 \arrow[r, "\Gamma"] \arrow[d, "h\otimes f_0"'] & X_1 \arrow[r, "\partial"] \arrow[d, "f_1"] & X_0 \arrow[d, "f_0"] \\ K \otimes Y_0 \arrow[r, "\Gamma"] & Y_1 \arrow[r, "\partial"] & Y_0 \end{tikzcd} \tag{6}$$

where f_1 and f_0 are k–linear and h–*equivariant*, i. e. $f_i(r \cdot b) = h(r) \cdot f_i(b)$ for $r \in R, b \in X_i$ and $i = 0, 1$. Let **secmod** be the category of secondary modules and let **secmod**(R) be the subcategory of secondary modules over R and R–*equivariant* maps $f = f_h$ for which h is the identity of R.

One readily checks that **secmod**(R) is an *additive* category (in fact an *abelian* category) with the direct sum $X_R \oplus Y_R$ given by

$$R \otimes (X_0 \oplus Y_0) \xrightarrow{\Gamma\oplus\Gamma} X_1 \oplus Y_1 \xrightarrow{\partial\oplus\partial} X_0 \oplus Y_0 \tag{1.4}$$

Moreover for a map $f : X_R \to Y_R$ in **secmod**(R) the secondary modules kernel(f) and cokernel(f) are defined in **secmod**(R) by using kernel(f_i) and cokernel(f_i) for $i = 0, 1$ in the obvious way.

Remark 1.5. Let $R = k[G]$ be a group algebra augmented by $\varepsilon : G \to k^*$ as in (1.1). Then a secondary module X_R over R can be identified with a diagram

$$G \times X_0 \xrightarrow{\Gamma} X_1 \xrightarrow{\partial} X_0$$

where X_1, X_0 are k–vector spaces with an action of G via k–linear automorphisms and where ∂ is k–linear and G–equivariant. Moreover $G \times X_0$ is the product set and Γ is a function between sets which is k–linear in X_0 (i. e. for $g \in G$ the function $X_0 \to X_1, x \mapsto \Gamma(g, x)$ is k–linear). Moreover for $g, g' \in G$ the following equations hold.

$$\partial\Gamma(g, x) = (g - \varepsilon(g))x \tag{1}$$
$$\Gamma(g, \partial a) = (g - \varepsilon(g))a \tag{2}$$
$$\Gamma(gg', x) = g\Gamma(g', x) + \varepsilon(g')\Gamma(g, x) \tag{3}$$
$$\Gamma(gg', x) = \Gamma(g, g'x) + \varepsilon(g)\Gamma(g', x) \tag{4}$$

Let $I(R) = \text{kernel}(\varepsilon : R \to k)$ be the *augmentation ideal* considered as an R–bimodule. For an R–module M the tensor product $I(R) \otimes_R M$ over R is defined and this tensor product is an R–module by $r \cdot (\overline{r} \otimes m) = (r\overline{r}) \otimes m$ for $r \in R, \overline{r} \in I(R)$ and $m \in M$. We have the equation $(\overline{r} \cdot r) \otimes m = \overline{r} \otimes (r \cdot m)$ in $I(R) \otimes_R M$.

Lemma 1.6. *A secondary R–module X can be equivalently described by a commu-*

tative diagram of R–linear maps:

$$\begin{tikzcd} I(R)\otimes_R X_1 \arrow[r, "1\otimes\partial"] \arrow[d, "\mu"'] & I(R)\otimes_R X_0 \arrow[d, "\mu"] \arrow[dl, "\widetilde{\Gamma}"'] \\ X_1 \arrow[r, "\partial"] & X_0 \end{tikzcd}$$

Here μ *is given by* $\mu(\overline{r}\otimes m)=\overline{r}\cdot m$ *for* $m\in X_1$ *or* $m\in X_0$.

This characterization of a secondary R–module is more appropriate than definition (1.3) which is motivated by topological examples below. In [**B**] we consider modules over crossed algebras generalising secondary modules in (1.6).

Proof of (1.6). Given (1.3) we observe that $\Gamma(1\otimes x)=0$ for $x\in X_0$ by (3). Hence Γ in (1.3) is determined by the restriction

$$\Gamma' : I(R)\otimes X_0 \subset R\otimes X_0 \xrightarrow{\Gamma} X_1$$

Now (4) shows that Γ' induces a map

$$\widetilde{\Gamma} : I(R)\otimes_R X_0 \to X_1$$

which is R–linear by (3). By (1) and (2) we see that the diagram in (1.6) commutes. Conversely given such a diagram we define Γ in (1.3) by

$$\Gamma(r\otimes x)=\widetilde{\Gamma}((r-\varepsilon r)\otimes x)$$

Now it is easy to show that equations (1),...,(4) are satisfied. □

We use (1.6) for the following construction of free secondary R–modules.

Definition 1.7. Let $d : V \to X_0$ be an R–linear map. Then the *free secondary R–module* X with basis (V,d) is obtained by the following push out in the category of R–modules and R–linear maps:

$$\begin{tikzcd} I(R)\otimes_R V \arrow[r, "1\otimes d"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\text{push}"] & I(R)\otimes_R X_0 \arrow[d, "\widetilde{\Gamma}"] \arrow[ddr, bend left, "\mu"] & \\ V \arrow[r, "i"] \arrow[drr, bend right, "d"'] & X_1 \arrow[dr, "\partial"] & \\ & & X_0 \end{tikzcd}$$

We also write $X_1 = X_1(d)$ and $X = X(d)$. Since $\mu(1\otimes d) = d\mu$ the R–linear map ∂ is well defined. Moreover we show that X is a well defined secondary R–module:

Proof. By (1.6) we have to show that $\widetilde{\Gamma}(1\otimes\partial)=\mu$ on $I(R)\otimes_R X_1$. This holds if $\widetilde{\Gamma}(1\otimes\partial)(1\otimes i)=\mu(1\otimes i)$ on $I(R)\otimes_R V$ and $\widetilde{\Gamma}(1\otimes\partial)(1\otimes\widetilde{\Gamma})=\mu(1\otimes\widetilde{\Gamma})$ on $I(R)\otimes_R (I(R)\otimes_R X_0)$. Now the first equation holds since

$$\widetilde{\Gamma}(1\otimes\partial)(1\otimes i)=\widetilde{\Gamma}(1\otimes(\partial i))=\widetilde{\Gamma}(1\otimes d)=i\mu=\mu(1\otimes i)$$

Here $i\mu = \mu(1 \otimes i)$ holds since i is R–linear. For the second equation we get

$$\widetilde{\Gamma}(1 \otimes \partial)(1 \otimes \widetilde{\Gamma}) = \widetilde{\Gamma}(1 \otimes (\partial\widetilde{\Gamma})) = \widetilde{\Gamma}(1 \otimes \mu) = \mu(1 \otimes \widetilde{\Gamma})$$

Here the last equation holds since for $r, \overline{r} \in I(R), x \in X_0$ we have

$$\begin{aligned} \widetilde{\Gamma}(1 \otimes \mu)(r \otimes \overline{r} \otimes x) &= \widetilde{\Gamma}(r \otimes (\overline{r} \cdot x)) \\ &= \widetilde{\Gamma}((r \cdot \overline{r}) \otimes x) \\ &= r\widetilde{\Gamma}(\overline{r} \otimes x) \\ &= \mu(1 \otimes \widetilde{\Gamma})(r \otimes \overline{r} \otimes x) \end{aligned}$$

Here we use the fact the $\widetilde{\Gamma}$ is R–linear. □

One readily checks that the free secondary module $X(d)$ has the following *universal property*: Let X be an object in **secmod**(R) and let

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr, "d"'] & & X_1 \arrow[dl, "\partial"] \\ & X_0 & \end{tikzcd} \tag{1.8}$$

be a commutative diagram of R–linear maps. Then there is a unique map $\overline{f} : X(d) \to X$ in **secmod**(R) of the form

$$\begin{tikzcd} I(R) \otimes_R X_0 \arrow[r, "\widetilde{\Gamma}"] \arrow[d, equal] & X_1(d) \arrow[r, "\partial"] \arrow[d, "\overline{f}_1"] & X_0 \arrow[d, equal, "\overline{f}_0=\text{identity}"] \\ I(R) \otimes_R X_0 \arrow[r, "\widetilde{\Gamma}"] & X_1 \arrow[r, "\partial"] & X_0 \end{tikzcd}$$

such that $\overline{f}_1 i = f$ for $i : V \to X_1(d)$ defined in (1.7).

2. Examples of secondary modules in topology

We describe examples of secondary modules which arise in topology. Let $\mathbf{Top}^*$ be the category of pointed topological spaces with base point. This is a *groupoid enriched category* in the following sense. For objects X, Y in $\mathbf{Top}^*$ the morphism object $[\![X; Y]\!]$ is the groupoid given as follows. Objects in $[\![X, Y]\!]$ are the pointed maps $X \to Y$ and for pointed maps $f, g : X \to Y$ the morphisms $H : f \Rightarrow g$ in $[\![X, Y]\!]$ are the *tracks* from f to g, that is H is a homotopy class of homotopies $f \simeq g$. The composite of tracks

$$h \overset{G}{\Longleftarrow} g \overset{H}{\Longleftarrow} f$$

is denoted by $G\Box H$ where $G\Box H$ is defined by adding homotopies in the usual way. The inverse of the track H is denoted by $H^{op} : g \Rightarrow f$ with $H^{op}\Box H = \widehat{0}_f$ where $\widehat{0}_f$ denotes the identity track of f.

If $Y \times Z$ is a product in $\mathbf{Top}^*$ then

$$[\![X, Y \times Z]\!] = [\![X, Y]\!] \times [\![X, Z]\!] \tag{2.1}$$

is a product of groupoids. This shows that for an algebraic object Y in $\mathbf{Top}^*$ the groupoid $[\![X, Y]\!]$ is a corresponding algebraic object in the category $\mathbf{Grd}$ of (small) groupoids. For example if Y is an abelian group object in $\mathbf{Top}^*$ (i. e. an abelian topological group) then $[\![X, Y]\!]$ is a abelian group object in the category $\mathbf{Grd}$. A map between abelian group objects which is a homomorphism of the group structure is termed a linear map.

Let $\mathbf{C}$ be a category. Then the category of pairs in $\mathbf{C}$ denoted by $\mathsf{pair}(\mathbf{C})$ is defined. Objects are morphisms $f : A \to B$ in $\mathbf{C}$ and morphisms $(\alpha, \beta) : f \to g$ in $\mathsf{pair}(\mathbf{C})$ are commutative diagrams in $\mathbf{C}$

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{\beta} & B' \end{array}$$

Let $\mathbf{Ab}$ be the category of abelian groups. The following result is well known.

Proposition 2.2. *The category of abelian group objects in* $\mathbf{Grd}$ *and linear maps is equivalent to the category* $\mathsf{pair}(\mathbf{Ab})$.

In order to fix notation we give a proof of this result. Given an abelian group object G in $\mathbf{Grd}$ we obtain the object

$$\partial : G_1^0 \to G_0$$

in $\mathsf{pair}(\mathbf{Ab})$ as follows. Here G_0 is the set of objects of G which is an abelian group since G is an abelian group object in $\mathbf{Grd}$. Let $0 \in G_0$ be the neutral object in the abelian group G_0. Then G_1^0 is the set of all morphisms $f : a \Rightarrow 0$ in G with $a \in G_0$ and $\partial f = a$. The abelian group structure of G_1^0 is defined by

$$(f : a \Rightarrow 0) + (g : b \Rightarrow 0) = (f + g : a + b \to 0 + 0 = 0)$$

where the right hand side is defined since G is an abelian group object in $\mathbf{Grd}$.

Conversely given an object $\partial : A_1 \to A_0$ in $\mathsf{pair}(\mathbf{Ab})$ we define the abelian group object $G(\partial)$ in $\mathbf{Grd}$ as follows. The set of objects of $G(\partial)$ is the set A_0. The set of morphisms of $G(\partial)$ is the product set $A_1 \times A_0$ where $(a_1, x) \in A_1 \times A_0$ is a morphism $(a_1, x) : \partial a_1 + x \to x$ in $G(\partial)$ also denoted by $(a_1, x) = a_1 + x$. The identity of x is $(0, x) : x = \partial 0 + x \to x$. Composition of

$$1x \xleftarrow{(a_1,x)} \partial a_1 + x \xleftarrow{(b_1,\partial a_1 + x)} \partial b_1 + \partial a_1 + x$$

is $(b_1 + a_1, x)$ for $a_1, b_1 \in A_1$ and $x \in A_0$. Now it is readily seen that this way one gets an equivalence of categories.

There are well known generalizations of (2.2). In particular the category of unital groups in $\mathbf{Grd}$ is equivalent to the category of crossed modules in the sense of J.H.C. Whitehead, see for example Porter [**P**].

Now given an abelian group object Y in $\mathbf{Top}^*$ the abelian group object $G = [\![X,Y]\!]$ in $\mathbf{Grd}$ is given via (2.2) by a homomorphism

$$G_1^0 = [\![X,Y]\!]_1^0 \xrightarrow{\partial} G_0 = [\![X,Y]\!]_0 \tag{2.3}$$

where G_0 is the set of all pointed maps $f : X \to Y$ and where G_1^0 is the set of all tracks $H : f \Rightarrow 0$ with $f \in G_0$ and $\partial H = f$. The group structure of Y induces the group structure on G_0 and G_1^0 in the obvious way.

Definition 2.4. Let R be a k–algebra with augmentation $\varepsilon : R \to k$ as in (1.1). A *topological track module* Y *over* R is a R–module object Y in $\mathbf{Top}^*$ (i. e. a topological R–module) for which each map $r : Y \to Y$ given by $r \in R$ admits a *unique* track $\Gamma_r : r \Rightarrow \varepsilon r$ where $\varepsilon r : Y \to Y$ is defined by the k–vector space structure of Y.

Example 2.5. Let $Z^n = K(k,n)$ be an Eilenberg–MacLane space of the underlying abelian group of the field k with the properties in the introduction, see Appendix A. This shows that for $R = k[\sigma_n]$ the space Z^n is a topological R–module, in fact, a topological track module over R since for $r \in R$ there is a unique track $r \Rightarrow \varepsilon r$ (with $r, \varepsilon r : Z^n \to Z^n$). Here ε is the sign–augmentation as in (1.1)(3).

Proposition 2.6. *Let* Y *be a topological track module over* R*. Then for each* X *in* $\mathbf{Top}^*$ *one obtains canonically a secondary module over* R

$$R \otimes G_0 \xrightarrow{\Gamma} G_1^0 \xrightarrow{\partial} G_0$$

where ∂ *is given by the groupoid* $G = [\![X,Y]\!]$ *as in* (2.3) *and where* Γ *is defined by the composite*

$$\Gamma(r \otimes f) = \Gamma_{r-\varepsilon r} f$$

Here the track $\Gamma_{r-\varepsilon r} : r - \varepsilon r \Rightarrow 0$ *is given for* $r - \varepsilon r \in R$ *by* (2.4).

Using (2.6) we obtain for each track module Y over R a functor

$$\mathcal{H}(-;Y) : \mathbf{Top}^* \to \mathbf{secmod}(R) \tag{2.7}$$

which carries X to the track module $\mathcal{H}(X;Y) = (G_1^0, G_0, \partial, \Gamma)$ given by $[\![X,Y]\!]$ in (2.6). Of course we have

$$\pi_0\mathcal{H}(X,Y) = \mathrm{cokernel}(\partial) = [X,Y] \tag{1}$$

$$\pi_1\mathcal{H}(X,Y) = \mathrm{kernel}(\partial) = [X,\Omega Y] \tag{2}$$

Here $[X,Y]$ denotes the set of homotopy classes of pointed maps $X \to Y$ and ΩY is the loop space of Y. The elements $H \in [X,\Omega Y]$ are identified with the tracks $H : 0 \Rightarrow 0$ where $0 : X \to * \to Y$ is the *zero map*. By (1) and (2) we see that the functor (2.7) carries homotopy equivalences in $\mathbf{Top}^*$ to weak equivalences between secondary modules as defined in the next section.

We are mainly interested in the secondary module

$$\mathcal{H}^n(X) = \mathcal{H}(X, Z^n) \tag{2.8}$$

over $R_n = k[\sigma_n]$ given by (2.5) with $\pi_0\mathcal{H}^n(X) = \widetilde{H}^n(X,k)$ and $\pi_1\mathcal{H}^n(X) = \widetilde{H}^{n-1}(X,k)$. This corresponds to the boundary map ∂ in the introduction.

Proof of (2.6). By definition of ∂ we have $\partial(\Gamma_{r-\varepsilon r} f) = (r - \varepsilon r)f$ so that (1.3)(1) is satisfied. Now let $H : f \Rightarrow 0$ be an element in G_1 with $\partial H = f$. Then we get

$$\begin{tikzcd} \bullet & \bullet \arrow[l, "0"'] & \bullet \arrow[l, "0"'] \\ \bullet \arrow[u, phantom, ""] & \bullet \arrow[l, "r-\varepsilon r"] \arrow[ul, phantom, "\Uparrow \Gamma_{r-\varepsilon r}"] & \bullet \arrow[l, "f"] \arrow[ul, phantom, "\Uparrow H"] \end{tikzcd}$$

so that for $\Gamma = \Gamma_{r-\varepsilon r}$

$$\begin{aligned} \Gamma * H &= 0H\square\Gamma f = \Gamma 0\square(r - \varepsilon r)H \\ &= \widehat{0}_0\square\Gamma f = \widehat{0}_0\square(r - \varepsilon r)H \\ &= \Gamma f = (r - \varepsilon r)H \end{aligned}$$

and this implies (1.3)(2). Next we have by uniqueness of tracks in $[\![Y, Y]\!]$ the equations

$$\begin{aligned} \Gamma_{rr'-\varepsilon(rr')} &= r\Gamma_{r-\varepsilon r'} + \varepsilon(r')\Gamma_{r-\varepsilon r} \\ &= \Gamma_{r-\varepsilon r}r' + \varepsilon(r)\Gamma_{r'-\varepsilon r'} \end{aligned}$$

and these equations imply (1.3)(3),(4). $\square$

Remark 2.9. Let Y be given as in (2.6) and let $\widehat{R} \subset [\![Y, Y]\!]$ be the full subgroupoid with objects given by maps $r : Y \to Y$ for $r \in R$. Then we obtain the action

$$\mu_R : \widehat{R} \times [\![X, Y]\!] \subset [\![Y, Y]\!] \times [\![X, Y]\!] \xrightarrow{\circ} [\![X, Y]\!]$$

where the second arrow is composition in the groupoid enriched category $\mathbf{Top}^*$. The action μ_R determines Γ in the secondary module $\mathcal{H}(X, Y)$ given by (2.6) and conversely $\mathcal{H}(X, Y)$ determines uniquely the action μ_R. In this sense a secondary module is a $\widehat{R}$–module in the category **Grd** of groupoids. Here $\widehat{R}$ is the groupoid with objects R, path components $\varepsilon^{-1}(x)$ with $x \in k$ and all automorphism groups in $\widehat{R}$ are trivial. The algebra structure of R yields a corresponding structure of $\widehat{R}$.

As pointed out by the referee this remark corresponds to the followoing result generalizing (2.2).

Proposition 2.10. *Let $\widehat{R}$ be the internal k-algebra in the category of groupoids* **Grd** *given by R similarly as in (2.9). Then the category of $\widehat{R}$-internal modules in* **Grd** *is equivalent to the category* **secmod**(R) *of secondary modules over R.*

The proof of (2.10) uses similar arguments to the proof of (2.6). We leave details to the reader. A generalization of (2.10) is proved in [**B**].

3. Weak equivalences

We can consider a secondary module X as a chain complex of k–vector spaces concentrated in degree 0 and 1. The homology of this chain complex is denoted by

$$\begin{aligned} \pi_0(X) &= \text{cokernel}(\partial : X_1 \to X_0) \\ \pi_1(X) &= \text{kernel}(\partial : X_1 \to X_0) \end{aligned} \tag{3.1}$$

A map $F = F_h : X_R \to Y_K$ between secondary modules is a *weak equivalence* if $h : R \to K$ is an isomorphism and f induces isomorphisms

$$f_* : \pi_0(X_R) \cong \pi_0(Y_K)$$
$$f_* : \pi_1(X_R) \cong \pi_1(Y_K)$$

We point out that $\pi_0(X)$ and $\pi_1(X)$ are also R–modules for which, however, by (1.3)(1),(2) the R–module structure is induced by the augmentation ε, that is $r \cdot x = \varepsilon(r) \cdot x$ for $r \in R, x \in \pi_0(X), \pi_1(X)$. Hence $\pi_0(X)$ and $\pi_1(X)$ are just k–vector spaces with an action of R via ε. Such R–modules are termed *ε–modules.* If X_R is a secondary module with $\Gamma = 0$ then X_0 and X_1 are also ε–modules. Hence in this case X_R is given by a chain complex $\partial : X_1 \to X_0$ of k–vector spaces. We say that X_R is of *trivial type* if $\Gamma = 0$ and $\partial = 0$ so that in this case $X_0 = \pi_0$ and $X_1 = \pi_1$.

Two secondary modules X_R, Y_R are *weakly equivalent* if there exists a chain of R–equivariant weak equivalences

$$X_R \xleftarrow{\sim} X_1 \xrightarrow{\sim} X_2 \xleftarrow{\sim} \dots X_n \xrightarrow{\sim} Y_R.$$

Proposition 3.2. *Each secondary module is weakly equivalent to a secondary module of trivial type.*

We prove this in (3) below. Hence the only invariants of the weak equivalence class of a secondary module X are $\pi_0 X$ and $\pi_1 X$.

Remark 3.3. For the secondary module $\mathcal{H}^n(X)$ over $R_n = k[\sigma_n]$ in (2.7) we know by (3.2) that the weak equivalence type of $\mathcal{H}^n(X)$ is trivial. This can also be seen by the following topological argument. By Baues [**B**] there exists a sequence

$$Z^n \xleftarrow{f} Y_1 \xrightarrow{g} Y_2 \xleftarrow{h} Y_3$$

of topological R_n–modules and R_n–linear maps f, g, h with the following properties. The action of R_n an Y_3 satisfies $r \cdot y = \varepsilon(r) \cdot y$ for $r \in R_n$ and $y \in Y_3$ where ε is the sign augmentation of R_n. Moreover f, g and h are homotopy equivalences on $\mathbf{Top}^*$. Hence we obtain weak equivalences of secondary modules over R_n

$$\mathcal{H}^n(X) = \mathcal{H}(X, Z^n) \xleftarrow{\sim} \mathcal{H}(X; Y_1) \xrightarrow{\sim} \mathcal{H}(X, Y_2) \xleftarrow{\sim} \mathcal{H}(X, Y_3)$$

where $\mathcal{H}(X, Y_3)$ is easily seen to be weakly equivalent to a secondary module of trivial type.

For the proof of (3.2) in (3) below we need the following pull back construction for secondary modules. Let X_R be a secondary module and let Y_0 be an R–module and let $f : Y_0 \to X_0$ be a R–linear map. Then we obtain the following commutative diagram in which the subdiagram 'pull' is a pull back in the category of vector spaces.

$$\begin{tikzcd}
R \otimes Y_0 \arrow[r, "\overline{\Gamma}"'] \arrow[d, "R\otimes f"] \arrow[rr, bend left, "\gamma"] & f^*X_1 \arrow[r, "\overline{\partial}"'] \arrow[d, "\overline{f}"] \arrow[dr, phantom, "\text{pull}"] & Y_0 \arrow[d, "f"] \\
R \otimes X_0 \arrow[r, "\Gamma"] & X_1 \arrow[r, "\partial"] & X_0
\end{tikzcd}$$

Here $\overline{\Gamma}$ is defined by $\overline{\partial}\,\overline{\Gamma} = \gamma$ with $\gamma(r \otimes y) = (r - \varepsilon r) \cdot y$ and $\overline{f}\,\overline{\Gamma} = \Gamma(R \otimes f)$. Then f^*X_1 is an R–module and $\overline{\partial}$ is R–linear. Moreover we get the following fact.

Lemma 3.4. *The top row $Y_R = (\overline{\partial}, \overline{\Gamma}) = f^*X_R$ of the diagram is a secondary module over R and $(\overline{f}, f) : Y_R \to X_R$ is a map in* **secmod**(R) *which is a weak equivalence if $(\partial, f) : X_1 \oplus Y_0 \to X_0$ is surjective.*

The map $f^*X_R \to X_R$ has the following property. Let $i : K \to R$ be an augmented map between k–algebras and let $g : Z_K \to X_R$ be an i–equivariant map between secondary modules for which a commutative diagram

$$\begin{tikzcd}
 & Y_0 \arrow[d, "f"] \\
Z_0 \arrow[ur, "h_0"] \arrow[r, "g_0"'] & X_0
\end{tikzcd} \tag{1}$$

is given. Here h_0 and g_0 are i–equivariant. Then there exists a unique i–equivariant map $h : Z_K \to f^*X_R$ for which the diagram

$$\begin{tikzcd}
 & f^*X_R \arrow[d, "(\overline{f}, f)"] \\
Z_K \arrow[ur, "h"] \arrow[r, "g"'] & X_R
\end{tikzcd} \tag{2}$$

commutes in **secmod**.

Proof of (3.4). The elements of f^*X_1 are pairs (x_1, y) with $\partial x_1 = fy$. We define $r(x_1, y) = (rx_1, ry)$ so that $\overline{\partial}$ and $\overline{f}$ are R–linear with $\overline{\partial}(x_1, y) = y$ and $\overline{f}(x_1, y) = x_1$. Moreover

$$\overline{\Gamma}(r \otimes y) = (\Gamma(r \otimes fy), (r - \varepsilon r)y)$$

Hence (1.3)(1) holds for Y_R. Moreover

$$\begin{aligned}\overline{\Gamma}(r \otimes \overline{\partial}(x_1, y)) &= \overline{\Gamma}(r \otimes y)\\ &= (\Gamma(r \otimes fy), (r - \varepsilon r)y)\\ &= (\Gamma(r \otimes \partial x_1), (r - \varepsilon r)y)\\ &= ((r - \varepsilon r)x_1, (r - \varepsilon r)y)\\ &= (r - \varepsilon r)(x_1, y)\end{aligned}$$

This shows (1.3)(2) for Y_R. Next we consider (1.3)(3) for Y_R and we get

$$\begin{aligned}\overline{\Gamma}(r \cdot r' \otimes y) &= (\Gamma(r \cdot r' \otimes fy), (r \cdot r' - \varepsilon(rr'))y)\\ r\overline{\Gamma}(r' \otimes y) + \varepsilon(r')\overline{\Gamma}(r \otimes y) &= r(\Gamma(r' \otimes fy), (r' - \varepsilon r')y) + \varepsilon(r')(\Gamma(r \otimes fy), (r - \varepsilon r)y)\\ &= (\Gamma(rr' \otimes fy), r(r' - \varepsilon r')y + \varepsilon(r')(r - \varepsilon r)y)\\ &= (\Gamma(rr' \otimes fg), (rr' - \varepsilon(rr'))y).\end{aligned}$$

Similarly one checks (1.3)(4) for Y_R. Hence $(f, \overline{f}) : Y_R \to X_R$ is a well defined map between secondary modules. If (∂, f) is surjective then the pull back is also a push out and therefore $(\overline{f}, f)$ is a weak equivalence. □

Now given a secondary module X_R we can choose a k–linear section $s : \pi_0 \to X_0$ of the quotient map $q : X_0 \to X_0/\operatorname{im}(\partial) = \pi_0$. Hence the R–linear map

$$f : R \otimes \pi_0 \to X_0 \tag{3.5}$$

with $f(r \otimes x) = r \cdot sx$ is defined with $qf(r \otimes x) = \varepsilon(r) \cdot x$. Here $R \otimes \pi_0$ is a *free* R–module with the action of R given by $r \cdot (r' \otimes x) = (r \cdot r') \otimes x$ and qf coincides with the R–linear map

$$qf = \varepsilon \otimes 1 : R \otimes \pi_0 \to k \otimes \pi_0 = \pi_0 \tag{1}$$

For $I(R) = \ker(\varepsilon : R \to k)$ we have $\ker(\varepsilon \otimes 1) = I(R) \otimes \pi_0$. Using f in (3.5) we get as in (3.4) an R–linear map between secondary modules

$$(\overline{f}, f) : Y_R = f^* X_R \to X_R \tag{2}$$

which is a weak equivalence since $(\partial, f) : X_1 \oplus Y_0 = X_1 \oplus R \otimes \pi_0 \to X_0$ is surjective. Here Y_R is a secondary module which is special in the following sense. We say that a secondary module X_R is *special* if for $\pi_0 = \pi_0 X$ one has

$$\begin{cases} X_0 = R \otimes \pi_0 \text{ and} \\ \operatorname{im}(\partial : X_1 \to X_0) = I(R) \otimes \pi_0. \end{cases}$$

Proposition 3.6. *A special secondary R–module X_R admits an R–linear section $t : I(R) \otimes \pi_0 \to X_1$ of $\partial : X_1 \to I(R) \otimes \pi_0$.*

Proof. We define t by the map

$$\Gamma : R \otimes X_0 = R \otimes R \otimes \pi_0 \to X_1,$$

namely for $r' \in I(R)$ and $x \in \pi_0$ let

$$t(r' \otimes x) = \Gamma(r' \otimes 1 \otimes x).$$

Then we have $\partial t(r' \otimes x) = \partial\Gamma(r' \otimes 1 \otimes x) = (r' - \varepsilon r')(1 \otimes x) = r' \otimes x$ since $\varepsilon(r') = 0$. Moreover t is R–linear since for $r \in R$

$$\begin{aligned} t(r \cdot r' \otimes x) &= \Gamma(r \cdot r' \otimes 1 \otimes x) \\ &= r\Gamma(r' \otimes 1 \otimes x) + \varepsilon(r')\Gamma(r \otimes 1 \otimes x) \\ &= r\Gamma(r' \otimes 1 \otimes x) \\ &= r \cdot t(r' \otimes x). \end{aligned}$$

□

Remark 3.7. A converse of (3.6) is also true. Let π_0 and π_1 be k–vector spaces and let

$$\partial : X_1 \to I(R) \otimes \pi_0$$

be a surjective R–linear map for which $\pi_1 = \ker(\partial)$ is an ε–module and let t be an R–linear section of ∂. Then a special secondary R–module X_R is defined in terms of ∂ and t as follows. Let $X_0 = R \otimes \pi_0$ and let

$$\Gamma : R \otimes X_0 = R \otimes R \otimes \pi_0 \to X_1$$

be given by $(r, r' \in R, x \in \pi_0)$

$$\Gamma(r \otimes r' \otimes x) = (r - \varepsilon r)t((r' - \varepsilon r') \otimes x) + \varepsilon(r')t((r - \varepsilon r) \otimes x)$$

Then one can check that Γ satisfies all the axioms in (1.3) so that X_R is a well defined special secondary module. Hence by (3.6) special secondary modules are up to isomorphism determined by π_0 and π_1 with $X_0 = R \otimes \pi_0$ and $X_1 = \pi_1 \oplus I(R) \otimes \pi_0$.

Proof of (3.2). Let X_R be a secondary module. Then we obtain by (3.5) the special secondary module $Y_R = f^* X_R$ and the weak equivalence $Y_R \xrightarrow{\sim} X_R$. Moreover by (3.6) and (3.7) we have $Y_0 = R \otimes \pi_0$ and $Y_1 = \pi_1 \oplus I(R) \otimes \pi_0$ and

$$\partial : Y_1 = \pi_1 \oplus I(R) \otimes \pi_0 \to I(R) \otimes \pi_0 \subset R \otimes \pi_0 = Y_0$$

is given by the projection and the inclusion. Now we obtain a weak equivalence g with

$$\begin{tikzcd} Y_1 \arrow[r, "\partial"] \arrow[d, "g_1"'] & Y_0 \arrow[d, "g_0"] \\ \pi_1 \arrow[r, "0"] & \pi_0 \end{tikzcd}$$

where $g_0 = \varepsilon \otimes 1$ and g_1 is the projection. By the definition of Γ in (3.7) in terms of the section $t : I(R) \otimes \pi_0 \subset Y_1$ we see that $g_1\Gamma = 0$ so that g is a well defined map between secondary modules where $0 : \pi_1 \to \pi_0$ is the secondary module of trivial type given by π_0 and π_1. □

4. Tensor products of secondary modules

Here we introduce the tensor product of secondary modules which is needed for the definition of secondary algebras in the next section. For secondary modules X_R

and Y_K the tensor product of the underlying chain complexes is given by the chain complex of k–vector spaces

$$\begin{aligned} X_1 \otimes Y_1 \xrightarrow{d_2} X_1 \otimes Y_0 \oplus X_0 \otimes Y_1 &\xrightarrow{d_1} X_0 \otimes Y_0 \\ d_2(a \otimes b) &= (\partial a) \otimes b - a \otimes (\partial b) \\ d_1(a \otimes y) &= (\partial a) \otimes y \\ d_1(x \otimes b) &= x \otimes (\partial b) \end{aligned}$$

with $x \in X_0, y \in Y_0, a \in X_1, b \in Y_1$. Hence d_1 induces the boundary map

$$\partial_\otimes : (X_1 \otimes Y_0 \oplus X_0 \otimes Y_1)/\operatorname{im}(d_2) \to X_0 \otimes Y_0 \tag{4.1}$$

Since k is a field we get by the Künneth formula

$$\begin{aligned} \pi_0 \partial_\otimes &= \operatorname{cok}(\partial_\otimes) = \pi_0(X) \otimes \pi_0(Y) \\ \pi_1 \partial_\otimes &= \ker(\partial_\otimes) = \pi_1(X) \otimes \pi_0(Y) \oplus \pi_0(X) \otimes \pi_1(Y) \end{aligned}$$

One readily checks that $\partial_\otimes$ is an $R \otimes K$–equivariant k–linear map.

Definition 4.2. We define the *tensor product* $X_R \otimes Y_K = (X \otimes Y)_{R \otimes K}$ of secondary modules X_R and Y_K by the diagram

$$R \otimes K \otimes X_0 \otimes Y_0 \xrightarrow{\Gamma_\otimes} (X_1 \otimes Y_0 \oplus X_0 \otimes Y_1)/\operatorname{im} d_2 \xrightarrow{\partial_\otimes} X_0 \otimes Y_0$$

Here $\partial_\otimes$ is defined as in (4.1) and $\Gamma_\otimes$ is defined by the following formula

$$\begin{aligned} \Gamma_\otimes(\alpha \otimes \beta \otimes x \otimes y) &= \Gamma(\alpha \otimes x) \otimes (\beta y) + (\varepsilon(\alpha)x) \otimes \Gamma(\beta \otimes y) \\ &= (\alpha x) \otimes \Gamma(\beta \otimes y) + \Gamma(\alpha \otimes x) \otimes (\varepsilon(\beta) y) \end{aligned}$$

Here the second equation is a consequence of the first equation since $(\partial a) \otimes b = a \otimes (\partial b)$ by (4.1).

Lemma 4.3. *The tensor product $X_R \otimes Y_K$ of secondary modules X_R and Y_K is a well defined secondary module over $R \otimes K$.*

Proof. The map $\partial_\otimes$ is $R \otimes K$–linear and $\Gamma_\otimes$ is a well defined k–linear map. Hence we have to check the equations (1)…(4) in (1.3): We first check (1).

$$\begin{aligned} \partial_\otimes \Gamma_\otimes(\alpha \otimes \beta \otimes x \otimes y) &= \partial\Gamma(\alpha \otimes x) \otimes \beta y + \varepsilon(\alpha) x \otimes \partial\Gamma(\beta \otimes y) \\ &= (\alpha - \varepsilon(\alpha)) x \otimes \beta y + \varepsilon(\alpha) x \otimes (\beta - \varepsilon(\beta)) y \\ &= \alpha x \otimes \beta y - \varepsilon(\alpha)\varepsilon(\beta) x \otimes y \\ &= (\alpha \otimes \beta - \varepsilon(\alpha \otimes \beta))(x \otimes y) \end{aligned}$$

Next we check (2) for $\Gamma_\otimes$.

$$\begin{aligned} \Gamma_\otimes(\alpha \otimes \beta \otimes \partial_\otimes(a \otimes y)) &= \Gamma_\otimes(\alpha \otimes \beta \otimes \partial a \otimes y) \\ &= \Gamma(\alpha \otimes \partial a) \otimes \beta y + \varepsilon(\alpha) \partial a \otimes \Gamma(\beta \otimes y) \\ &= (\alpha - \varepsilon(\alpha)) a \otimes \beta y + \varepsilon(\alpha) a \otimes \partial\Gamma(\beta \otimes y) \quad \text{, see (4.1),} \\ &= (\alpha - \varepsilon(\alpha)) a \otimes \beta y + \varepsilon(\alpha) a \otimes (\beta - \varepsilon(\beta)) y \\ &= \alpha a \otimes \beta y - \varepsilon(\alpha)\varepsilon(\beta) a \otimes y \\ &= (\alpha \otimes \beta - \varepsilon(\alpha \otimes \beta))(a \otimes y) \end{aligned}$$

$$\begin{aligned}\Gamma_\otimes(\alpha\otimes\beta\otimes\partial_\otimes(x\otimes b)) &= \Gamma_\otimes(\alpha\otimes\beta\otimes x\otimes\partial b)\\ &= \Gamma(\alpha\otimes x)\otimes\beta\partial b+\varepsilon(\alpha)x\otimes\Gamma(\beta\otimes\partial b)\\ &= \Gamma(\alpha\otimes x)\otimes\partial(\beta b)+\varepsilon(\alpha)x\otimes(\beta-\varepsilon(\beta))b\\ &= \partial\Gamma(\alpha\otimes x)\otimes\beta b+\varepsilon(\alpha)x\otimes(\beta-\varepsilon(\beta))b \quad \text{, see (4.1),}\\ &= (\alpha-\varepsilon(\alpha))x\otimes\beta b+\varepsilon(\alpha)x\otimes(\beta-\varepsilon(\beta))b\\ &= \alpha x\otimes\beta b-\varepsilon(\alpha)\varepsilon(\beta)x\otimes b\\ &= (\alpha\otimes\beta-\varepsilon(\alpha\otimes\beta))(x\otimes b)\end{aligned}$$

Now we check (3) for $\Gamma_\otimes$.

$$\begin{aligned}\Gamma_\otimes((\alpha\otimes\beta)(\alpha'\otimes\beta')\otimes x\otimes y) &= \Gamma_\otimes(\alpha\alpha'\otimes\beta\beta'\otimes x\otimes y)\\ &= \Gamma(\alpha\alpha'\otimes x)\otimes\beta\beta' y+\varepsilon(\alpha\alpha')x\otimes\Gamma(\beta\beta'\otimes y)=\text{(i)}\end{aligned}$$

Now we get by (3) that (i)=(ii) coincides with

$$\text{(ii)}=(\alpha\Gamma(\alpha'\otimes x)+\varepsilon(\alpha')\Gamma(\alpha\otimes x))\otimes\beta\beta' y+\varepsilon(\alpha\alpha')x\otimes(\beta\Gamma(\beta'\otimes y)+\varepsilon(\beta')\Gamma(\beta\otimes y))$$

On the other hand we have

$$\begin{aligned}\text{(iii)} &= (\alpha\otimes\beta)\Gamma_\otimes(\alpha'\otimes\beta'\otimes x\otimes y)+\varepsilon(\alpha'\otimes\beta')\Gamma_\otimes(\alpha\otimes\beta\otimes x\otimes y)\\ &= \alpha\Gamma(\alpha'\otimes x)\otimes\beta\beta' y+\varepsilon(\alpha')\alpha x\otimes\beta\Gamma(\beta'\otimes y)\\ &\quad +\varepsilon(\alpha'\otimes\beta')(\Gamma(\alpha\otimes x)\otimes\beta y+\varepsilon(\alpha)x\otimes\Gamma(\beta\otimes y)\end{aligned}$$

We have to check (ii)=(iii). But this is equivalent to

$$\varepsilon(\alpha')\Gamma(\alpha\otimes x)\otimes\beta\beta' y+\varepsilon(\alpha\alpha')x\otimes\beta\Gamma(\beta'\otimes y)=\varepsilon(\alpha')\alpha x\otimes\beta\Gamma(\beta'\otimes y)+\varepsilon(\alpha'\otimes\beta')\Gamma(\alpha\otimes x)\otimes\beta y$$

This equation is equivalent to

$$\begin{tikzcd}
\Gamma(\alpha\otimes x)\otimes\beta(\beta'-\varepsilon\beta')y \arrow[r, equal] \arrow[d, equal] & (\alpha-\varepsilon\alpha')x\otimes\beta\Gamma(\beta'\otimes y) \arrow[d, equal] \\
\Gamma(\alpha\otimes x)\otimes\beta\partial\Gamma(\beta'\otimes y) & \partial\Gamma(\alpha\otimes x)\otimes\beta\Gamma(\beta'\otimes y)
\end{tikzcd}$$

By (4.1) we know that

$$\begin{aligned}\Gamma(\alpha\otimes x)\otimes\beta\partial\Gamma(\beta'\otimes y) &= \Gamma(\alpha\otimes x)\otimes\partial\beta\Gamma(\beta'\otimes y)\\ &= \partial\Gamma(\alpha\otimes x)\otimes\beta\Gamma(\beta'\otimes y)\end{aligned}$$

This completes the proof that (ii)=(iii) and hence (i)=(iii) and hence (3) holds for $X\otimes Y$. Finally we have to check (4). For this we apply (4) to (i) above and we get (i)=(iv) where

$$\text{(iv)}=(\Gamma(\alpha\otimes\alpha' x)+\varepsilon(\alpha)\Gamma(\alpha'\otimes x))\otimes\beta\beta' y+\varepsilon(\alpha\alpha')x\otimes(\Gamma(\beta\otimes\beta' y)+\varepsilon(\beta)\Gamma(\beta'\otimes y))$$

On the other hand we have

$$\begin{aligned}\text{(v)} &= \Gamma_\otimes(\alpha\otimes\beta\otimes\alpha' x\otimes\beta' y)+\varepsilon(\alpha\otimes\beta)\Gamma_\otimes(\alpha'\otimes\beta'\otimes x\otimes y)\\ &= \Gamma(\alpha\otimes\alpha' x)\otimes\beta\beta' y+\varepsilon(\alpha)\alpha')x\otimes\Gamma(\beta\otimes\beta' y)\\ &\quad +\varepsilon(\alpha\otimes\beta)(\Gamma(\alpha'\otimes x)\otimes\beta' y+\varepsilon(\alpha')x\otimes\Gamma(\beta'\otimes y))\end{aligned}$$

We have to check (iv)=(v). This is the case if and only if the following equation holds.

$$\begin{aligned}&\varepsilon(\alpha)\Gamma(\alpha'\otimes x)\otimes\beta\beta' y+\varepsilon(\alpha\alpha')x\otimes\Gamma(\beta\otimes\beta' y)\\&\qquad=\varepsilon(\alpha)\alpha' x\otimes\Gamma(\beta\otimes\beta' y)+\varepsilon(\alpha\otimes\beta)\Gamma(\alpha'\otimes x)\otimes\beta' y\end{aligned}$$

This equation holds if and only if the following equation is true

$$\begin{tikzcd}
\Gamma(\alpha'\otimes x)\otimes(\beta-\varepsilon\beta)\beta' y \arrow[r, equal] \arrow[d, equal] & (\alpha'-\varepsilon(\alpha'))x\otimes\Gamma(\beta\otimes\beta' y) \arrow[d, equal] \\
\Gamma(\alpha'\otimes x)\otimes\partial\Gamma(\beta\otimes\beta' y) & \partial\Gamma(\alpha'\otimes x)\otimes\Gamma(\beta\otimes\beta' y)
\end{tikzcd}$$

Now again (4.1) shows that this equation is true. Hence we have shown (i)=(v) and this corresponds to equation (4) for $X\otimes Y$. Now the proof of the lemma is complete. □

Lemma 4.4. *The tensor product of secondary modules is associative and bilinear, that is:*

$$\begin{aligned}(X_R\otimes Y_K)\otimes Z_L&=X_R\otimes(Y_K\otimes Z_L)\\(X_R\oplus Y_R)\otimes Z_L&=X_R\otimes Z_L\oplus Y_R\otimes Z_L\\Z_L\otimes(X_R\oplus Y_R)&=Z_L\otimes X_R\oplus Z_L\otimes Y_R\end{aligned}$$

We point out that the chain complex $k=(0\to k)$ is a unit for the tensor product, that is

$$X_R\otimes k=X_R=k\otimes X_R \tag{4.5}$$

Here we use the obvious identification $V\otimes k=V=k\otimes V$ for a k–vector space V.

We shall use the tensor product of secondary modules mainly for the next result.

Proposition 4.6. *The cup product map $\mu: Z^n\times Z^m\to Z^{n+m}$ induces an $i_{n,m}$–equivariant map between secondary modules*

$$\mu_*:\mathcal{H}^n(X)\otimes\mathcal{H}^m(X)\to\mathcal{H}^{n+m}(X)$$

where $i_{n,m}: k[\sigma_n]\otimes k[\sigma_m]\to k[\sigma_{n+m}]$ is induced by the inclusion $\sigma_n\times\sigma_m\subset\sigma_{n+m}$, $n\geqslant 1$.

Proof. The map μ_* carries $f\otimes g\in\mathcal{H}^n_0\otimes\mathcal{H}^m_0$ with $f: X\to Z^n, g: X\to Z^m$ to the composite $\mu(f,g): X\to Z^n\times Z^m\to Z^{n+m}$. Since μ is k–bilinear and $i_{n,m}$–equivariant $\mu_*:\mathcal{H}^n_0\otimes\mathcal{H}^m_0\to\mathcal{H}^{n+m}_0$ is well defined. Moreover μ_* is induced on $(\mathcal{H}^n\otimes\mathcal{H}^m)_1$ by the map

$$\bar\mu:\mathcal{H}^n_0\otimes\mathcal{H}^m_1\oplus\mathcal{H}^n_1\otimes\mathcal{H}^m_0\to\mathcal{H}^{n+m}_1 \tag{1}$$

which carries $f\otimes G$ with $G: g\Rightarrow 0\in\mathcal{H}^m_1$ to $\mu(f,G):\mu(f,g)\Rightarrow 0$ and carries $H\otimes g$ with $H: f\Rightarrow 0\in\mathcal{H}^n_1$ to $\mu(H,g):\mu(f,g)\Rightarrow 0$. Here we use the fact that the bilinearity of μ implies that $\mu(f,0)=0=\mu(0,g)$. If $H: f\Rightarrow 0$ and $G: g\Rightarrow 0$ are given then in fact

$$\bar\mu(f\otimes G)=\bar\mu(H\otimes g) \tag{2}$$

so that μ_* is well defined and $i_{n,m}$–equivariant. In fact, we have for the track $(H,G) : (f,g) \Rightarrow (0,0)$ with $(f,g) : X \to Z^n \times Z^m$ given by the homotopy (H_t, G_t) the formula

$$\begin{aligned} \mu(H,G) &= \mu((0,G)\square(H,g)) \\ &= \mu((H,0)\square(f,G)) \end{aligned} \tag{4.6}$$

where $\square$ denotes addition of tracks. Hence we get

$$\begin{aligned} \mu(H,g) &= 0\square\mu(H,g) = \mu((0,G))\square\mu(H,g) \\ &= \mu((0,G)\square(H,g)) = \mu(H,G) \\ &= \mu((H,0)\square(f,G)) \\ &= \mu(H,0)\square\mu(f,G) \\ &= 0\square\mu(f,G) = \mu(f,G) \end{aligned}$$

and this proves (2). Finally we have to show that μ_* is compatible with the Γ–operator. For this let $r \in R_n$ and $s \in R_m$ and let

$$\begin{aligned} \Gamma_{r-\varepsilon r} &: r - \varepsilon r \Rightarrow 0 : Z^n \to Z^n \\ \Gamma_{s-\varepsilon s} &: s - \varepsilon s \Rightarrow 0 : Z^m \to Z^m \\ \Gamma_{r\odot s-\varepsilon(r)\cdot\varepsilon(s)} &: r \odot s - \varepsilon(r \odot s) \Rightarrow 0 : Z^{n+m} \to Z^{n+m} \end{aligned} \tag{4.6}$$

where $r \odot s = i_{n,m}(r,s) \in R_{n+m}$. We observe that in R_{n+m} we have the following equations

$$\begin{aligned} &(r - \varepsilon r) \odot s + \varepsilon(r)(1_n \odot (s - \varepsilon s)) = \\ &\qquad (r \odot s) - \varepsilon(r)(1_n \odot s) + \varepsilon(r)(1_n \odot s) - \varepsilon(r)\varepsilon(s)(1_n \odot 1_m) \\ &\qquad = r \odot s - \varepsilon(r \odot s) \in R_{n+m} \end{aligned} \tag{4.6}$$

Let $Z^n \wedge Z^m = Z^n \times Z^m / Z^n \times \{0\} \cup \{0\} \times Z^m$ be the smash product. Then the cup product map μ induces a map $\widetilde{\mu} : Z^n \wedge Z^m \to Z^{n+m}$ and we get the composites $a, b, c : Z^n \wedge Z^m \to Z^{n+m}$ by

$$\begin{aligned} a &= \widetilde{\mu}(r - \varepsilon r) \wedge s, \\ b &= \varepsilon(r)\widetilde{\mu}(1_n \wedge (s - \varepsilon s)), \\ c &= (r \odot s - \varepsilon(r \odot s))\widetilde{\mu}. \end{aligned} \tag{4.6}$$

Then (5) shows that $a + b = c$. Hence

$$\begin{aligned} A &= \widetilde{\mu}(\Gamma_{r-\varepsilon r} \wedge s) + \varepsilon(r)\widetilde{\mu}(1_n \wedge \Gamma_{s-\varepsilon s}) \text{ and} \\ B &= \Gamma_{r\odot s-\varepsilon(r\odot s)}\widetilde{\mu} \end{aligned} \tag{4.6}$$

are both tracks $c \Rightarrow 0$. Now obstruction theory shows that these tracks A and B coincide since the set of homotopy classes $[\Sigma Z^n \wedge Z^m, Z^{n+m}]$ is trivial. The equation $A = B$ implies that μ satisfies

$$\mu_*\Gamma(r \otimes s \otimes f \otimes g) = \Gamma(r \odot s \otimes \mu(f,g)) \tag{8}$$

by definition on Γ in (2.6) and by formula (2.4), which exactly corresponds to $A = B$.
$\square$

5. Secondary cohomology

Using the tensor product of secondary modules we introduce the notion of a secondary algebra. We define a functor which associates with each space X a secondary cohomology algebra $\mathcal{H}^*(X)$.

We consider a sequence R_* of augmented k–algebras $R_n, n \geqslant 0$, together with augmented algebra maps

$$i_{n,m} = \odot : R_n \otimes R_m \to R_{n+m} \tag{5.1}$$

carrying $\alpha \otimes \beta$ to $\alpha \odot \beta$ such that for $\gamma \in R_k$ we have

$$(\alpha \odot \beta) \odot \gamma = \alpha \odot (\beta \odot \gamma)$$

in R_{n+m+k}. Since $\odot$ is an algebra map we have

$$(\alpha \cdot \alpha') \odot (\beta \cdot \beta') = (\alpha \odot \beta) \cdot (\alpha' \odot \beta')$$

where $\alpha \cdot \alpha'$ denotes the product in R_n. Let $1_n \in R_n$ be the unit element of R_n with $1_n \odot 1_m = 1_{n+m}$. For $n = 0$ we have $R_0 = k$ and $1_0 \in R_0$ satisfies $1_0 \odot \alpha = \alpha \odot 1_0 = \alpha$. We call $R_* = (R_*, \odot)$ a *coefficient algebra*.

Of course we have the *trivial* coefficient algebra k with $R_n = k$ for $n \geqslant 0$. On the other hand we shall use the *symmetric coefficient algebra* $k[\sigma_*]$ given by the augmented group algebras $R_n = k[\sigma_n]$ where σ_n is the symmetric group and R_n has the sign augmentation (1.1)(3). Moreover $\odot = i_{m,n}$ is induced by the inclusion of groups $\sigma_n \times \sigma_m \subset \sigma_{n+m}$.

Definition 5.2. An *algebra* V *over a coefficient algebra* R_* is a sequence of R_n–modules $V^n, n \geqslant 0$, together with k–linear maps

$$V^n \otimes V^m \to V^{n+m}$$

carrying $x \otimes y$ to $x \cdot y$. For $z \in V^k$ we have in V^{n+m+k}

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

and for $\alpha \in R_n, \beta \in R_m$ we have

$$(\alpha x) \cdot (\beta y) = (\alpha \odot \beta)(x \cdot y).$$

We do not assume that the algebra V has a unit. Let V and W be such algebras over R_*. Then a *map* $f : V \to W$ over R_* is given by an R_n–linear map $f = f^n : V^n \to W^n, n \geqslant 0$, with $f(x \cdot y) = f(x) \cdot f(y)$. This defines the category of algebras over R_*.

If $R_* = k$ is the trivial coefficient algebras then V in (5.2) is just a *graded algebra* over k. A graded algebra V is *commutative* if for $x \in V^n, y \in V^m$ we have

$$y \cdot x = (-1)^{nm} x \cdot y \tag{5.3}$$

For example the reduced cohomology $\widetilde{H}^*(X, k)$ of a pointed space X with coefficients in k is a commutative graded algebra. We generalize this notion of commutative algebras as follows.

Definition 5.4. Let R_* be a coefficient algebra and assume elements

$$\tau_{m,n} \in R_{n+m} \ (n, m \geqslant 0)$$

are given with the following properties ($m, n, k \geqslant 0$).

$$\begin{aligned}
\tau_{m,n}\tau_{n,m} &= 1_{n+m} \\
\tau_{m,0} = \tau_{0,m} &= 1_m \\
\tau_{n,m}(\alpha \odot \beta) &= (\beta \odot \alpha)\tau_{n,m} \text{ for } \alpha \in R_n, \beta \in R_m \\
\tau_{m+n,k} &= (\tau_{m,k} \odot 1_n)(1_m \odot \tau_{n,k}) \\
\varepsilon(\tau_{m,n}) &= (-1)^{m,n} \in k
\end{aligned}$$

Then we say that an algebra V over R_* is τ*–commutative* if for $x \in V^n, y \in V^m$

$$y \cdot x = \tau_{n,m}(x \cdot y)$$

in V^{n+m}.

For example we have the interchange elements $\tau_{n,m} \in k[\sigma_{n+m}]$ with $\tau_{n,m}(1) = m+1$ in the symmetric coefficient algebra for which a τ–commutative algebra is the same as a "commutative twisted algebra" in the sense of Stover [**St**]. On the other hand we can define the interchange elements $\tau_{n,m} = (-1)^{nm} \in k$ in the trivial coefficient algebra so that in this case a τ–commutative algebra is the same as a commutative graded algebra in (5.3). We now are ready to introduce the notion of a secondary algebra.

Definition 5.5. Let R_* be a coefficient algebra. A *secondary algebra* $\mathcal{H}^*$ *over* R_* consists of a sequence of secondary modules $\mathcal{H}^n$ over R_n, $n \geqslant 1$, together with $i_{n,m}$–equivariant maps

$$\mu = \mu_{n,m} : \mathcal{H}^n \otimes \mathcal{H}^m \to \mathcal{H}^{n+m} \tag{1}$$

for $n, m \geqslant 1$ which are associative in the sense that the diagram

$$\begin{tikzcd}
\mathcal{H}^n \otimes \mathcal{H}^m \otimes \mathcal{H}^r \arrow[r, "1\otimes\mu"] \arrow[d, "\mu\otimes 1"'] & \mathcal{H}^n \otimes \mathcal{H}^{m+r} \arrow[d, "\mu"] \\
\mathcal{H}^{n+m} \otimes \mathcal{H}^r \arrow[r, "\mu"] & \mathcal{H}^{n+m+r}
\end{tikzcd} \tag{2}$$

commutes. Here we use the tensor product of secondary modules. If elements $\tau_{n,m} \in R_{n+m}$ are given as in (5.4) we say that the secondary algebra $\mathcal{H}^*$ is τ*–commutative* if the diagram

$$\begin{tikzcd}
\mathcal{H}^n \otimes \mathcal{H}^m \arrow[r, "\mu"] \arrow[rr, bend right, "T"'] & \mathcal{H}^{n+m} & \mathcal{H}^m \otimes \mathcal{H}^n \arrow[l, "\mu"']
\end{tikzcd} \tag{3}$$

commutes. Here T carries $x \otimes y$ to $\tau_{n,m}(y \otimes x)$ for $(x \in \mathcal{H}_0^n, y \in \mathcal{H}_0^m)$ or $(x \in \mathcal{H}_0^n, y \in \mathcal{H}_1^m)$ or $(x \in \mathcal{H}_1^n, y \in \mathcal{H}_0^m)$. We study τ–commutative secondary algebras in more detail in section §7 below.

We say that the secondary algebra $\mathcal{H}^*$ is *w–closed* if one has k–linear isomorphisms

$$w = w^n : \pi_1 \mathcal{H}^n \cong \pi_0 \mathcal{H}^{n-1} \tag{4}$$

for $n \geqslant 1$ which satisfy

$$\begin{aligned} w^{n+m}(y_1 \cdot z) &= w^n(y_1) \cdot z \\ w^{n+m}(y \cdot z_1) &= (-1)^n y \cdot w^m(z_1) \end{aligned} \tag{5.5}$$

for $y \in \pi_0\mathcal{H}^n, z \in \pi_0\mathcal{H}^m, y_1 \in \pi_1\mathcal{H}^n, z_1 \in \pi_1\mathcal{H}^m$. In (5) the multiplication is defined by the maps (1). A *map* $f : \mathcal{H}^* \to \mathcal{G}^*$ between secondary algebras is given by a sequence $f^n : \mathcal{H}^n \to \mathcal{G}^n$ of R_n–equivariant maps between secondary modules such that f^n is compatible with μ in (1) and w in (4). Such a map f is a *weak equivalence* if f^n is a weak equivalence in **secmod** for $n \geqslant 0$.

Let **secalg** be the category of secondary algebras over the symmetric coefficient algebra $k[\sigma_*]$ which are τ–commutative and w–closed. For an object $\mathcal{H}^*$ in **secalg** we obtain a commutative graded algebra H^* by

$$H^n = \pi_0\mathcal{H}^n \text{ for } n \geqslant 0 \tag{5}$$

with the multiplication $H^n \otimes H^m \to H^{n+m}$ induced by μ in (5.5)(1). We see that H^* is commutative since $\mathcal{H}^*$ is τ–commutative. Assume $sign(\tau_{n,m}) = 1$ in k then $\mu : \mathcal{H}^n \otimes \mathcal{H}^n \to \mathcal{H}^{2n}$ for $n \geqslant 1$ yields the *squaring operation*

$$\mathrm{Sq}^{n-1} = w^{2n-1}\,\mathrm{Sq} : H^n \to H^{2n-1} \tag{5.7}$$

with $2\,\mathrm{Sq}^{n-1} = 0$ as follows. For this we use the assumption that $\mathcal{H}^*$ is w–closed. The k-linear map

$$Sq : \pi_0\mathcal{H}^n \longrightarrow \pi_1\mathcal{H}^{2n}$$

carries the element $\{y\}$ represented by $y \in \mathcal{H}^n_0$ to the element

$$Sq\{y\} = \Gamma(\tau_{n,n}, y \cdot y)$$

wich satisfies $\partial\Gamma(\tau_{n,n}, y \cdot y) = 0$. One can check that Sq is well defined, see also [**B**]. The next result describes the *secondary cohomology algebra* $\mathcal{H}^*(X)$ of a path-connected pointed space X. Let $\mathbf{Top}^*_0$ be the category of path-connected pointed spaces and pointed maps.

Theorem 5.8. *There is a contravariant functor*

$$\mathcal{H}^* : \mathbf{Top}^*_0 \to \mathbf{secalg}$$

which carries a space X to a secondary algebra $\mathcal{H}^*(X)$ which is τ–commutative and w–closed. See (4.6). Moreover the algebra $\widetilde{H}^* = \pi_0\mathcal{H}^*(X)$ is (5.6) *coincides with the reduced cohomology algebra $\widetilde{H}^*(X;k)$ and for $k = \mathbb{F}_2$ the squaring operation* Sq^{n-1} *in* (5.7) *coincides with the corresponding Steenrod operation.*

Proof. This is a consequence of (3.7) and property (c) of the cup product maps in the introduction. The result on Steenrod squares is a consequence of a result of Kristensen, lemma 2.5 in [**K**]. Compare [**B**]. □

6. Crossed modules and Hochschild cohomology

we show that Hochschild cohomology can be deduced from the concept of secondary algebra in (5.5). More precisely, a secondary algebra over $R_* = k$ corresponds to the notion of a "crossed module" which is used to define "crossed extensions". Moreover weak equivalence classes of crossed extensions are in fact the elements in the Hochschild cohomology. In a similar way we shall deduce from the concept of a τ–commutative secondary algebra in (5.5) the notion of symmetric cohomology; see §9 below.

We introduce the concept of a crossed module in the context of algebras and we show that (in the graded case) a crossed module is the same as a secondary algebra over the trivial coefficient algebra $R_* = k$. A crossed module and equivalently a secondary algebra over k represent an element in the third Hochschild cohomology. This leads to the notion of a characteristic class of a differential algebra.

We here consider the graded and the non–graded case at the same time. A graded vector space V is assumed to be non-negatively graded, i. e. $V^i = 0$ for $i < 0$.

We use the following notation. An *algebra* $\widetilde{A}$ is given by a (graded) k–vector space $\widetilde{A}$ and a multiplication map $\widetilde{A} \otimes \widetilde{A} \to \widetilde{A}$ which is associative. On the other hand a *k–algebra* A is an algebra with unit $k \to A$ and augmentation $\varepsilon : A \to k$. Hence a k–algebra A is an algebra under and over k. Then the augmentation ideal

$$\widetilde{A} = \mathrm{kernel}(\varepsilon : A \to k)$$

is an algebra which determines the k–algebra $A = \widetilde{A} \oplus k$ completely. Moreover an $\widetilde{A}$–module is also an A–module and vice versa.

Let A be a (graded) k–algebra. An *A–bimodule* V is a (graded) k–vector space which is a left and a right A–module such that for $a, b \in A, x \in V$ we have $(a \cdot x) \cdot b = a \cdot (x \cdot b)$. For example A can be considered as an A–bimodule via the multiplication in A.

Definition 6.1. A *crossed module* is a map of A–bimodules

$$\partial : V \to A$$

satisfying $\varepsilon\partial = 0$ and $(\partial v) \cdot w = v \cdot (\partial w)$ for $v, w \in V$.

Let $\pi_0(\partial) = \mathrm{cokernel}(\partial)$ and $\pi_1(\partial) = \mathrm{kernel}(\partial)$ in the category of (graded) vector spaces. Then the algebra structure of A induces an algebra structure of $\pi_0(\partial)$ and the A–bimodule structure of V induces a $\pi_0(\partial)$–bimodule structure of $\pi_1(\partial)$. In fact for $\{a\} \in \pi_0(\partial)$ the multiplication $\{a\} \cdot v = a \cdot v$ with $v \in \pi_1(\partial)$ is well defined since $(a + \partial w) \cdot v = a \cdot v + (\partial w) \cdot v = a \cdot v + w \cdot \partial v = a \cdot v$ where $\partial v = 0$. Hence a crossed module yields the exact sequence

$$0 \longrightarrow \pi_1(\partial) \xrightarrow{i} V \xrightarrow{\partial} A \xrightarrow{q} \pi_0(\partial) \longrightarrow 0$$

in which all maps are A–bimodule morphisms. Here the A–bimodule structure of $\pi_0(\partial)$ and $\pi_1(\partial)$ is induced by the algebra map q.

Lemma 6.2. *A secondary algebra $\mathcal{H}^*$ over the trivial coefficient algebra k as defined in* (5.5) *is the same as a crossed module ∂.*

Proof. Given $\mathcal{H}^*$ we obtain

$$\partial : \mathcal{H}^*_1 \to \mathcal{H}^*_0$$

where $\mathcal{H}^*_0$ is an algebra by the multiplication μ in (5.5)(1). Moreover $\mathcal{H}^*_1$ is a $\mathcal{H}^*_0$–bimodule by the multiplication (5.5)(1). Using (4.1) we see that ∂ yields the crossed module

$$\partial = (\partial, 0) : \mathcal{H}^*_1 \to \mathcal{H}^*_0 \oplus k$$

where $\mathcal{H}^*_0 \oplus k$ is the k–algebra given by $\mathcal{H}^*_0$. Conversely it is easy to see that a crossed module (6.1) defines a secondary algebra over k. □

We now use crossed modules (or equivalently secondary algebras over k) for the definition of Hochschild cohomology.

Definition 6.3. Let H be a (graded) k–algebra and let M be an H–bimodule. A *crossed extension* of H by M is an exact sequence in the category of (graded) k–vector spaces

$$\mathcal{E} : 0 \longrightarrow M \xrightarrow{\gamma} V \xrightarrow{\partial} A \xrightarrow{q} H \longrightarrow 0$$

where ∂ is a crossed module. Moreover all maps are A–bimodule maps with the A–bimodule structure induced by the algebra map $q : A \to H$. A *weak equivalence* between two such extensions is a commutative diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d, equal] & V \arrow[r] \arrow[d, "f_1"] & A \arrow[r] \arrow[d, "f_0"] & H \arrow[d, equal] \\ M \arrow[r] & W \arrow[r] & B \arrow[r] & H \end{tikzcd}$$

where f_0 is an algebra map and f_1 is a f_0–biequivariant homomorphism.

induces an isomorphism kernel(q') = kernel(q) and hence we get for an n–fold extension (6.3) the following diagram of G by f^*M. Now f^* in (6.4) carries the weak–equivalence class of the extension $\mathcal{E}$ to the weak–equivalence class of the extension $f^*\mathcal{E}$ in the top row of the diagram. $(n \geqslant 2)$ weak equivalence class of the extension $\mathcal{E}$ to the weak equivalence class of the extension $g_*\mathcal{E}$ in the bottom row of the diagram.

$M, n \geqslant 2$.

Proposition 6.4. *The third Hochschild cohomology $HH^3(H, M)$ of H with coefficients in M coincides naturally with the set of weak equivalence classes of crossed extensions of H by M.*

This result is proved in [**BM**], see also Loday [**L**] or Lue [**Lu**]. The proposition holds in the graded and in the non–graded case.in order to define a crossed resolution

of a k–algebra A. tensor product of V. Given a (graded) k–algebra A and a k–linear map $d : V \to A$ with $\varepsilon d = 0$ we obtain the *free crossed modul* with basis (V, d) as follows. Let is the free crossed module with basis (V, d). Finally we define for a k–algebra H the *free H–bimodule* with basis V by $H \otimes V \otimes H$. choose a commutative diagram ($n \geqslant 2$) this surjection we define the k–vector space structure of $\mathsf{HH}^{n+1}(H, M)$ so that addition of crossed extensions in $\mathsf{HH}^{n+1}(H, M)$ is given by the "Baer sum" of extensions. 0
$h_0(v) = 1 \otimes v \otimes 1$ for $v \in V_0$ and $h_0(a \cdot b) = (qa) \cdot h_0(b) + h_0(b) \cdot q(a)$ for $a, b \in T$. One can check that there is a unique H–bimodule map d_2 for which

We have seen in (6.2) that each secondary algebra $\mathcal{H}^*$ over k yields a canonical crossed extension

$$0 \longrightarrow \pi_1(\mathcal{H}^*) \longrightarrow \mathcal{H}_1^* \xrightarrow{\partial} \mathcal{H}_0^* \oplus k \longrightarrow \pi_0(\mathcal{H}^*) \oplus k \longrightarrow 0$$

Here $H = \pi_0(\mathcal{H}^*) \oplus k$ is a k–algebra and $M = \pi_1(\mathcal{H}^*)$ is an H–bimodule. Hence the crossed extension represents an element

$$\langle\mathcal{H}^*\rangle \in \mathsf{HH}^3(H, M) \tag{6.5}$$

which is termed the *characteristic class* of the secondary algebra $\mathcal{H}^*$. On the other hand a differential algebra C (like the cochain algebra of a space) as well yields a crossed extension representing a characteristic class $\langle C\rangle$ as in the following example.

Example 6.6. Let C be a differential graded k–algebra, that is, $C = \{C^i, i \geqslant 0\}$ with $C^i C^j \subseteq C^{i+j}$ and $d : C \to C$ of degree $+1$ satisfying $d(xy) = (dx)y + (-1)^{|x|} x d(y)$ and $dd = 0$ and $\varepsilon d = 0$. Then d induces the map of graded k–vector spaces

$$V = \operatorname{coker}(\widetilde{d})[1] \xrightarrow{\partial} \ker(d) = A \tag{1}$$

Here we define for a graded vector space W the *shifted* graded vector space $W[1]$ by

$$W^n = (W[1])^{n+1}, w \mapsto s(w), \tag{2}$$

Hence for the cokernel of the differential $\operatorname{coker}(d) = \widetilde{C}/\operatorname{im}(\widetilde{d})$ we obtain the shifted object $V = \operatorname{coker}(\widetilde{d})[1]$. Since d is of degree $+1$ the boundary induces ∂ by $\partial s\{v\} = d(v)$ for $\{v\} \in \operatorname{coker}(d), v \in C$. The algebra C induces an algebra structure of $A = \ker(d)$. Moreover it induces the structure of an A–bimodule on V by setting

$$\begin{aligned} a \cdot (s\{v\}) &= (-1)^{|a|} s\{a \cdot v\} \\ (s\{v\}) \cdot b &= s\{v \cdot b\} \end{aligned} \tag{6.6}$$

One can check that $\partial : V \to A$ is a crossed module in the sense of (6.1), see [**BM**]. This proves that $\partial : V \to A$ is crossed module and therefore we obtain by (6.2) a secondary algebra $\partial : V \to \widetilde{A}$ over k which is, in fact, w–closed (see (5.5)) by defining

$$w : \pi_1(\partial) = \widetilde{H}^*(C)[1] \cong \widetilde{H}^*(C) = \pi_0(\partial) \tag{4}$$

with $ws(x) = x$ for $x \in H^*(C) = \ker(d)/\operatorname{im}(d)$. The equations in (3) for the A–bimodule structure of V correspond exactly to the equations in (5.5)(5).

According to (3) we define for a graded algebra H^* the H^*–*bimodule* $\widetilde{H}^*[1]$ by setting

$$\begin{aligned} a\cdot(sx) &= (-1)^{|a|}s(a\cdot x)\\ (sx)\cdot b &= s(x\cdot b) \end{aligned} \tag{6.6}$$

Then we obtain by (1) and (4) the crossed 2–extension

$$0 \longrightarrow \widetilde{H}^*[1] \longrightarrow V \xrightarrow{\partial} A \longrightarrow H^* \longrightarrow 0$$

which by (6.4) represents an element

$$\langle\!\langle C \rangle\!\rangle \in \mathsf{HH}^*(H^*, \widetilde{H}^*[1]) \tag{6}$$

where $H^* = H^*C$ is the cohomology algebra of the differential algebra C. A cocycle θ representing $\langle\!\langle C \rangle\!\rangle$ is considered in Berrick–Davydov [**BD**].

As a special case we obtain for a pointed space X the augmented algebra of cochains on X denoted by C^*X for which

$$H^*(C^*X) = H^*(X)$$

is the cohomology algebra of X. Hence we get by (6.6)(6) the class

$$\langle\!\langle C^*(X) \rangle\!\rangle \in \mathsf{HH}^*(H^*(X), \widetilde{H}^*(X)[1]) \tag{6.7}$$

which is an invariant of the homotopy type of X in the sense that a pointed map $f : X \to Y$ satisfies

$$(f^*)^*\langle\!\langle C^*(X) \rangle\!\rangle = (f^*[1])_*\langle\!\langle C^*(Y) \rangle\!\rangle$$

in $\mathsf{HH}^3(H^*(Y), \widetilde{H}^*(X)[1])$ where $f^* : H^*(Y) \to H^*(X)$ yields the structure of an $H^*(Y)$–bimodule on $\widetilde{H}^*(X)[1]$.

We now compare the class (6.7) with the secondary cohomology algebra $\mathcal{H}^*(X)$ in (5.8).

Proposition 6.8. *By forgetting structure we obtain from the secondary cohomology* $\mathcal{H}^*(X)$ *a secondary algebra over* k *denoted by* $\mathcal{H}^*(X)_{(k)}$. *Then the classes*

$$\langle\!\langle C^*X \rangle\!\rangle = \langle\!\langle \mathcal{H}^*(X)_{(k)} \rangle\!\rangle \in \mathsf{HH}^3(H^*(X), \widetilde{H}^*(X)[1])$$

given by (6.5) *and* (6.7) *coincide.*

Proof. Using (3.3) and the definition of Y_3 in Baues [**B**] we see that $\langle\!\langle \mathcal{H}(X, Y_3) \rangle\!\rangle = \langle\!\langle C^*X \rangle\!\rangle$ for a simplicial set X. Here we use the universal property of Y_3 which says that a simplicial map $X \to Y_3$ can be identified with a cocycle in C^*X. □

7. τ–crossed modules for commutative graded algebras

directly be obtained by crossed n–fold extensions of H by M. A crossed module which by (6.2) is the same as a secondary algebra over k was the crucial ingredient of a crossed extension. We now simply replace the "secondary algebra over k" in a crossed extension by a "τ–commutative secondary algebra" and we then obtain τ–crossed extensions which represent elements in the symmetric cohomology $\mathsf{SH}^*(H, M)$.

We have seen in §7 that a secondary algebra over the trivial coefficient algebra k is the same as a crossed module. We here show that in a similar way a τ–commutative secondary algebra over R_* is the same as a τ–crossed module. Weak equivalence classes of τ–crossed modules yield an abelian group generalizing the Hochschild cohomology in (6.4).

Let R_* be a coefficient algebra with interchange elements $\tau_{m,n} \in R_{m+n}$, for example let $R_* = k[\sigma_*]$ be the symmetric coefficient algebra. An *R_*–module* V is a sequence of (left) R_n–modules $V^n, n \geqslant 0$. A *map* or an *R_*–linear map* $f : V \to W$ between R_*–modules is given by a sequence of R_n–linear maps $f^n : V^n \to W^n$ for $n \geqslant 0$. The field k (concentrated in degree 0) is an R_*–module. Moreover using the augmentation ε of $R_n, n \geqslant 0$, we see that each graded k–vector space M is an R_*–module which we call an *ε–module.* For $x \in M^m$ we write $|x| = m$ where $|x|$ is the *degree* of x.

Given R_*–modules $V_1, \ldots, V_k$ we define the *R_*–tensor product* $V_1 \overline{\otimes} \ldots \overline{\otimes} V_k$ by

$$(V_1 \overline{\otimes} \ldots \overline{\otimes} V_k)_n = \bigoplus_{n_1+\ldots+n_k=n} R_n \otimes_{R_{n_1}\otimes\ldots\otimes R_{n_k}} V_1^{n_1} \otimes \ldots \otimes V_k^{n_k} \tag{7.1}$$

where we use the algebra map $\odot : R_{n_1} \otimes \ldots \otimes R_{n_k} \to R_n$ given by the structure of the coefficient algebra R_* in (5.1). One readily checks associativity

$$(V_{1,1} \overline{\otimes} \ldots \overline{\otimes} V_{1,k_1}) \overline{\otimes} \ldots \overline{\otimes} (V_{s,1} \overline{\otimes} \ldots \overline{\otimes} V_{s,k_s}) = V_{1,1} \overline{\otimes} \ldots \overline{\otimes} V_{1,k_1} \overline{\otimes} \ldots \overline{\otimes} V_{s,1} \overline{\otimes} \ldots \overline{\otimes} V_{s,k_s} \tag{1}$$

Compare Stover [**St**] 2.9. Moreover the interchange element τ in R_* yields the isomorphism

$$T : V \overline{\otimes} W \cong W \overline{\otimes} V \tag{2}$$

which carries $v \otimes w$ to $\tau_{w,v} w \otimes v$ where

$$\tau_{w,v} = \tau_{m,n} \in R_{m+n}$$

for $w \in W^m, v \in V^n$. Of course we have $k \overline{\otimes} V = V = V \overline{\otimes} k$.

Definition 7.2. An *algebra A over R_** is given by an R_*–linear map $\mu : A \overline{\otimes} A \to A, \mu(a \otimes b) = a \cdot b$, which is associative in the sense that the diagram

$$\begin{tikzcd} A\overline{\otimes}A\overline{\otimes}A \arrow[r, "1\overline{\otimes}\mu"] \arrow[d, "\mu\overline{\otimes}1"'] & A\overline{\otimes}A \arrow[d, "\mu"] \\ A\overline{\otimes}A \arrow[r, "\mu"] & A \end{tikzcd}$$

commutes. Moreover A is *τ–commutative* if

$$\begin{tikzcd} A\overline{\otimes}A \arrow[r, "\mu"] \arrow[d, "T"'] & A \arrow[d, equal] \\ A\overline{\otimes}A \arrow[r, "\mu"] & A \end{tikzcd}$$

commutes. One readily checks that this coincides with the notation in (5.2) and (5.3). We say that A is a *k–algebra over R_** if algebra maps $k \xrightarrow{i} A \xrightarrow{\varepsilon} k$ are given with $\varepsilon i = 1$. Such a k–algebra A over R_* is completely determined by the algebra $\widetilde{A}$ over R_* with $\widetilde{A} = \mathrm{kernel}(\varepsilon : A \to k)$ and $A = k \oplus \widetilde{A}$.

For an R_*–module V let $V_{(k)}$ be the underlying graded k–vectorspace. If A is a k–algebra over R_* then $A_{(k)}$ is a k–algebra (over k) in the sense of §7 above.

Definition 7.3. Given an algebra A over R_* we say that an R_*–module V is an *A–module* if a map $m : A\overline{\otimes}V \to V$ is given such that

$$\begin{tikzcd} A\overline{\otimes}A\overline{\otimes}V \arrow[r, "1\overline{\otimes}\mu"] \arrow[d, "\mu\overline{\otimes}1"'] & A\overline{\otimes}V \arrow[d, "\mu"] \\ A\overline{\otimes}V \arrow[r, "\mu"] & V \end{tikzcd}$$

commutes. Hence for $a \cdot x = \mu(a \otimes x)$ with $a \in A, x \in V$ we have $(\alpha a) \cdot (\beta x) = (\alpha \odot \beta)(a \cdot x)$ and $(a \cdot b) \cdot x = a \cdot (b \cdot x)$. If A is a k–algebra over R_* we also assume that $1 \cdot x = x$ for $1 \in k, x \in V$. Then the A–module V is an $\widetilde{A}$–module and vice versa. In particular the algebra A is also an A–module in the obvious way.

Lemma 7.4. *Let A be a τ–commutative k–algebra over R_* and let V be an A–module. Then $V_{(k)}$ is a $A_{(k)}$–bimodule by defining*

$$a \cdot x \cdot b = a \cdot \tau_{b,x}(b \cdot x)$$

for $a, b \in A, x \in V$.

Proof. We write $1_x = 1_n \in R_n$ for $x \in V^n$. Now we have for $a, b \in A$

$$\begin{aligned} (a \cdot x) \cdot b &= \tau_{b,a\cdot x} b \cdot (a \cdot x) = \tau_{b,a\cdot x}(b \cdot a) \cdot x \\ &= \tau_{b,a\cdot x}(\tau_{a,b} a \cdot b) \cdot x \\ &= \tau_{b,a\cdot x}(\tau_{a,b} \odot 1_x)(a \cdot b \cdot x) \\ a \cdot (x \cdot b) &= a \cdot \tau_{b,x}(b \cdot x) = (1_a \odot \tau_{b,x})(a \cdot b \cdot x) \end{aligned}$$

Here we have $\tau_{b,a\cdot x}(\tau_{a,b} \odot 1_x) = 1_a \odot \tau_{b,x}$ by one of the equations in (6.4). □

If A and V in (7.4) are ε–modules then (7.4) corresponds to the following special case.

Lemma 7.5. *Let H be a commutative graded k–algebra and let M be an H–module. Then M is an H–bimodule by defining*

$$a \cdot x \cdot b = a \cdot (-1)^{|b||x|} b \cdot x$$

for $a, b \in H$ and $x \in M$.

For an R_*–module V we obtain as in (1.6) the R_*–linear map

$$I(R_*) \odot_{R_*} V \xrightarrow{\mu} V$$

Here the left hand side is the R_*–module given in degree n by $I(R_n) \otimes_{R_n} V^n$ ond μ carries $a \otimes x$ to $a \cdot x$.

Definition 7.6. Let A be a τ–commutative k–algebra over R_*. A *τ–crossed module* ∂ is given by a commutative diagram of R_*–linear maps

$$\begin{tikzcd} I(R_*) \odot_{R_*} V \arrow[r, "1\odot\partial"] \arrow[d, "\mu"'] & I(R_*) \odot_{R_*} A \arrow[d, "\mu"] \arrow[dl, "\widetilde{\Gamma}"'] \\ V \arrow[r, "\partial"] & A \end{tikzcd}$$

with the following properties (1) and (2). The R_*–module V is an A–module and ∂ is an A–module morphism, that is $\partial(a \cdot x) = a \cdot (\partial x)$ for $a \in A, x \in V$. Moreover $\varepsilon\partial = 0$ and for $x, y \in V$

$$(\partial x) \cdot y = \tau_{\partial y, x}(\partial y) \cdot x. \tag{7.7}$$

The R_*–linear map $\widetilde{\Gamma}$ satisfies for $\beta \in I(R_*)$ and $a, b \in A$ the equation

$$a \cdot \widetilde{\Gamma}(\beta \otimes b) = \widetilde{\Gamma}(1 \odot \beta \otimes a \cdot b) \tag{2}$$

Equation (2) shows that $\widetilde{\Gamma}$ is a map of left A–modules. Moreover (1) and (2) imply that for $\alpha \in I(R_*)$ the following equation holds.

$$\widetilde{\Gamma}(\alpha \otimes a) \cdot b = \widetilde{\Gamma}(\alpha \odot 1 \otimes a \cdot b) \tag{3}$$

Here the right hand action of b on $x = \widetilde{\Gamma}(\alpha \otimes a) \in V$ is defined as in (7.4) by $x \cdot b = \tau_{b,x} b \cdot x$. Using this notation (1) is equivalent to $(\partial x) \cdot y = x \cdot (\partial y)$, compare (6.1).

Proof of (3). For $x \in \widetilde{\Gamma}(\alpha \otimes a)$ we have $|x| = |a|$ and hence $\tau_{b,x} = \tau_{b,a}$. Therefore we get

$$\begin{aligned} \widetilde{\Gamma}(\alpha \otimes a) \cdot b &= \tau_{b,x} b \cdot \widetilde{\Gamma}(\alpha \otimes a) \\ &= \tau_{b,x} \widetilde{\Gamma}(1 \odot \alpha \otimes b \cdot a) \\ &= \widetilde{\Gamma}(\tau_{b,a}(1 \odot \alpha) \otimes b \cdot a) \\ &= \widetilde{\Gamma}((\alpha \odot 1)\tau_{b,a} \otimes b \cdot a) \\ &= \widetilde{\Gamma}((\alpha \odot 1) \otimes \tau_{b,a} b \cdot a) \\ &= \widetilde{\Gamma}((\alpha \odot 1 \otimes a \cdot b). \end{aligned}$$

□

Lemma 7.8. *A τ–crossed module ∂ as in* (7.6) *yields for the underlying k–vector spaces a crossed module in the sense of* (6.1)

$$\partial : V_{(k)} \to A_{(k)}$$

where we use (7.4). *Moreover $\pi_0(\partial) = H$ is a commutative graded k–algebra and $\pi_1(\partial)$ is an H–module with the H–bimodule structure in* (7.5).

The lemma is based on the crucial property of a τ–crossed module, namely that $\pi_0(\partial) = \text{cokernel}(\partial)$ and $\pi_1(\partial) = \text{kernel}(\partial)$ are only ε–modules though V and A are

R_*–modules. A τ–crossed module yields the exact sequence

$$0 \longrightarrow \pi_1(\partial) \xrightarrow{i} V \xrightarrow{\partial} A \xrightarrow{q} \pi_0(\partial) \longrightarrow 0 \tag{7.9}$$

in which all maps are A–module morphisms. Here the A–module structures of $\pi_0(\partial)$ and $\pi_1(\partial)$ are induced by the algebra map q. Moreover for the underlying k–vector spaces this is by (7.8) a crossed extension as in (6.1), (6.3).

The next result generalizes lemma (6.2) on crossed modules.

Lemma 7.10. *A τ–commutative secondary algebra $\mathcal{H}^*$ over the coefficient algebra R_* as defined in* (5.5) *is the same as a τ–crossed module ∂ in* (7.6).

Proof. Given $\mathcal{H}^*$ we obtain

$$\partial : \mathcal{H}_1^* \to \mathcal{H}_0^* \oplus k = A$$

where A is a τ–commutative k–algebra by the multiplication μ in (5.5)(1). Moreover $\mathcal{H}_1^* = V$ is an A–module and one now readily checks by (4.2) and (1.6) that ∂ satisfies the properties of a τ–crossed module. Conversely given $(\partial, \widetilde{\Gamma})$ as in (7.6) we obtain the secondary module $\mathcal{H}^n$ over R_n by the commutative diagram (see (1.6))

$$\begin{tikzcd}
I(R_n) \otimes_{R_n} V^n \arrow[r, "1\otimes\partial"] \arrow[d, "\mu"'] & I(R_n) \otimes_{R_n} \widetilde{A}^n \arrow[d, "\mu"] \arrow[dl, "\widetilde{\Gamma}"'] \\
V^n \arrow[r, "\partial"] & \widetilde{A}^n
\end{tikzcd}$$

which we deduce from the diagram in (7.6). Moreover we define

$$\mu : \mathcal{H}^n \otimes \mathcal{H}^m \to \mathcal{H}^{n+m}$$

by the multiplication

$$\widetilde{A}^n \otimes \widetilde{A}^m = \mathcal{H}_0^n \otimes \mathcal{H}_0^m \xrightarrow{\mu} \widetilde{A}^{n+m} = \mathcal{H}_0^{n+m}$$

of the algebra $\widetilde{A}$ over R_* and by the map

$$\begin{tikzcd}
(\mathcal{H}^n \otimes \mathcal{H}^m)_1 \arrow[r, "\mu"] \arrow[d, equal] & \mathcal{H}_1^{n+m} \arrow[d, equal] \\
(V^n \otimes \widetilde{A}^m \oplus \widetilde{A}^n \otimes V^m)/\operatorname{im} d_2 \arrow[r, "\mu"] & V^{n+m}
\end{tikzcd}$$

with $\mu(x \otimes a) = x \cdot a$ and $\mu(b \otimes y) = b \cdot y$ where $x \cdot a = \tau_{a,x} a \cdot x$. By (7.6)(1) this map is trivial on $\mathrm{im}(d_2)$. Now it is easy to show that the multiplication μ on $\mathcal{H}^*$ is associative (compare the proof on (7.4)) and τ–commutative. Finally we have to check that μ on $\mathcal{H}^*$ is compatible with the equation in (4.2). This follows from (7.8)(2),(3) since for $\alpha' = \alpha + \alpha_k \in I(R^n) \oplus k = R^n$ and $\beta' = \beta + \beta_k \in I(R^m) \oplus k = R^m$ we have

$$\mu\Gamma_\otimes(\alpha' \otimes \beta' \otimes x \otimes y) = \widetilde{\Gamma}(\alpha \odot \beta \otimes x \cdot y) + \beta_k \widetilde{\Gamma}(\alpha \odot 1 \otimes x \cdot y) + \alpha_k \widetilde{\Gamma}(1 \odot \beta \otimes x \cdot y)$$

Here we have $\alpha \odot \beta = (\alpha \odot 1)(1 \odot \beta)$ and hence in $I(R^{n+m}) \otimes_{R^{n+m}} V^{n+m}$ we have

$$\begin{aligned} \alpha \odot \beta \otimes x \cdot y &= (\alpha \odot 1)(1 \odot \beta) \otimes x \cdot y \\ &= \alpha \odot 1 \otimes (1 \odot \beta) x \cdot y \\ &= \alpha \odot 1 \otimes x \cdot \beta y \end{aligned}$$

Therefore we get by (7.8)(2),(3)

$$\begin{aligned} \mu \Gamma_{\otimes}(\alpha' \otimes \beta' \otimes x \otimes y) &= \widetilde{\Gamma}(\alpha \otimes x) \cdot (\beta y + \beta_k y) + \alpha_k x \cdot \widetilde{\Gamma}(\beta \otimes y) \\ &= \mu(\Gamma(\alpha' \otimes x) \otimes \beta' y + (\varepsilon(\alpha')x) \otimes \Gamma(\beta' \otimes y)) \end{aligned}$$

Hence μ is compatible with the equation in (4.2). $\square$

We now use τ-crossed modules (or by (7.10) equivalently τ–commutative secondary algebras) for the following definition of symmetric cohomology which is a symmetric analogue of Hochschild cohomology in (6.3).

Definition 7.11. Let R_* be a coefficient algebra with interchange elements τ for example let $R_* = k[\sigma_*]$ be the symmetric coefficient algebra. Let H be a commutative graded k–algebra and let M be an H–module. A *τ–crossed extension* $\mathcal{E}$ of H by M is an exact sequence of graded k–vector spaces

$$\mathcal{E} : 0 \longrightarrow M \longrightarrow V \xrightarrow{\partial} A \xrightarrow{q} H \longrightarrow 0$$

Here ∂ is a τ–crossed module as in (7.6). Moreover all maps are A–module morphisms with the A–module structure induced by the algebra map $q : A \to H$. A *weak equivalence* between two such τ–crossed extensions is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & V & \longrightarrow & A & \longrightarrow & H & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & W & \longrightarrow & B & \longrightarrow & H & \longrightarrow & 0 \end{array}$$

Here f_0 is a morphism of k–algebras over R_* and f_1 is a f_0–equivariant homomorphism such that (f_0, f_1) is compatible with $\widetilde{\Gamma}$. Let $\mathsf{SH}^3(H, M)$ be the set of weak equivalence classes of τ–crossed extensions of H by M. Below we show that $\mathsf{SH}^3(H, M)$ is a well defined set with the structure of a k–vector space.

homomorphism

At this moment we do not know a "cohomology theory" for commutative graded algebras H which yields the cohomology $\mathsf{SH}^3(H, M)$ above.

We have the canonical natural homomorphism

$$\mathsf{SH}^3(H, M) \to \mathsf{HH}^3(H, M) \tag{7.12}$$

which carries the weak equivalence class of the τ–crossed extension $\mathcal{E}$ to the weak equivalence class of the underlying crossed extension $\mathcal{E}_{(k)}$ given by (7.8), (7.4) and (7.5). We need the following "free" objects.

Definition 7.13. Let V be a graded k–vector space. Then the free R_*–module $R_* \odot V$ generated by V is given by

$$(R_* \odot V)^n = R_n \otimes V^n.$$

For an R_*–module W we obtain the *tensor algebra over* R_* by

$$\overline{T}(W) = \bigoplus_{n \geqslant 0} W^{\overline{\otimes} n}$$

where $W^{\overline{\otimes} 0} = k$ and $W^{\overline{\otimes} n}$ is the n–fold $\overline{\otimes}$–product $W \overline{\otimes} \dots \overline{\otimes} W$ defined in (7.1). For the tensor algebra $T(V)$ over k we get

$$\overline{T}(R_* \odot V) = R_* \odot T(V)$$

so that $R_* \odot T(V)$ is the free k–algebra over R_* generated by V with the multiplication

$$(\alpha \otimes x) \cdot (\beta \otimes y) = \alpha \odot \beta \otimes x \cdot y$$

for $\alpha, \beta \in R_*$ and $x, y \in T(V)$. Let

$$K_\tau \subset R_* \odot T(V) = A$$

be the R_*–submodule generated by elements $1 \otimes y \cdot x - \tau_{x,y} \otimes x \cdot y$ for $x, y \in T(V)$. Then K_τ generates the ideal $A \cdot K_\tau \cdot A$ and the R_*–quotient module

$$\Lambda = A / A \cdot K_\tau \cdot A$$

is the *free τ–commutative k–algebra over R_**.

Given a τ–commutative k–algebra A over R_* and a k–linear map $d : V \to A$ with $\varepsilon d = 0$ we obtain the following push out diagram in the category of R_*–modules

$$\begin{tikzcd}
I(R_*) \odot V \arrow[r, "1 \odot d"] \arrow[d, "i \odot 1"'] \arrow[dr, phantom, "\text{push}"] & I(R_*) \odot_{R_*} A \arrow[d, "i'"] \arrow[ddr, bend left, "\mu"] & \\
R_* \odot V \arrow[r] \arrow[drr, bend right, "d''"'] & Y \arrow[dr, "d'"] & \\
 & & A
\end{tikzcd}$$

Here d'' is defined by $d''(\alpha \otimes x) = \alpha \cdot d(x)$. The pair (μ, d'') induces the R_*–linear map d' which thus determines the map of A–modules ∂' and ∂ in the following commutative diagram

$$\begin{tikzcd}
 & I(R_*) \odot_{R_*} A \arrow[r, "\mu"] \arrow[d, "i'"] \arrow[dddl, "\tilde{\Gamma}"'] & A \arrow[d, equal] \\
 & Y \arrow[r, "d'"] \arrow[d, "i''"] & A \arrow[d, equal] \\
 & A \overline{\otimes} Y \arrow[r, "\partial'"] \arrow[d] & A \arrow[d, equal] \\
X_1(d) \arrow[r, equal] & A \overline{\otimes} Y / U \arrow[r, "\partial"] & A
\end{tikzcd}$$

with $\widetilde{\widetilde{\Gamma}} = i''i'$ and $\widetilde{\Gamma} = p\widetilde{\widetilde{\Gamma}} = pi''i'$. Here i'' is defined by $i''(y) = 1 \otimes y$ and ∂' is defined by $\partial'(a \otimes y) = a \cdot d'(y)$. Let U be the A–submodule of $A\overline{\otimes}Y$ generated by the elements

$$(\partial' x) \cdot y - \tau_{\partial' y, x}(\partial' y) \cdot x,$$
$$a \cdot \widetilde{\widetilde{\Gamma}}(\beta \otimes b) - \widetilde{\widetilde{\Gamma}}(1 \odot \beta \otimes a \cdot b),$$

with $x, y \in A\overline{\otimes}Y$ and $a, b \in A, \beta \in I(R_*)$. One readily checks that U is in the kernel of ∂' so that ∂' induces the A–module map ∂ on the quotient $X_1(d) = A\overline{\otimes}Y/U$. We claim that $(\partial, \widetilde{\Gamma})$ is a well define τ–crossed module which is the *free τ–crossed module* with basis (V, d).

Finally we define for a commutative graded k–algebra the *free H–module* with basis V by $H \otimes V$.

Definition 7.14. Let H be a commutative graded k–algebra and consider a long exact sequence

$$\mathcal{R} : \ldots \longrightarrow C_2 \longrightarrow C_1 \xrightarrow{\partial} \Lambda \xrightarrow{q} H \xrightarrow{0}$$

Here Λ is a free τ–commutative k–algebra over R_* and $(\partial, \widetilde{\Gamma})$ is a free τ–crossed module and $C_i, i \geqslant 2$, is a free H–module. All maps in the sequence are Λ–module morphisms where C_i and H are Λ–modules via the algebra map q. Then we call $\mathcal{R}$ a *free τ–crossed resolution* of H.

It is easy to see that free τ–crossed resolutions exist. Moreover given a τ–crossed extension $\mathcal{E}$ as in (7.11) we can choose a commutative diagram ($n \geqslant 2$)

$$\begin{tikzcd} & C_3 \arrow[r, "\partial"] \arrow[d] & C_2 \arrow[r] \arrow[d, "f_2"] & C_1 \arrow[r] \arrow[d, "f_1"] & \Lambda \arrow[r] \arrow[d, "f_0"] & H \arrow[r] \arrow[d, equal] & 0 \\ \mathcal{E}: & 0 \arrow[r] & M \arrow[r] & V \arrow[r] & A \arrow[r] & H \arrow[r] & 0 \end{tikzcd}$$

so that one gets a weak equivalence

$$(f_2)_* \mathcal{R}_2 \to \mathcal{E}$$

where $\mathcal{R}_2$ is the τ–crossed extension

$$0 \to C_2/\partial C_3 \to C_1 \to \Lambda \to H \to 0$$

given by $\mathcal{R}$. This implies that $\mathsf{SH}^3(H, M)$ is actually a set. In fact, the function

$$\psi : \mathsf{Hom}_H(C_2/\partial C_3, M) \to \mathsf{SH}^3(H, M) \tag{7.15}$$

which carries $f_2 : C_2/\partial C_3 \to M$ to the weak equivalence class of $(f_2)_* \mathcal{R}_2$ is surjective. Using this surjection it is possible to define the k–vector space structure of $\mathsf{SH}^3(H, M)$.

Appendix: Eilenberg–MacLane spaces

Let k be a commutative ring, for example a field. For the definition of $Z^n = K(k, n)$ in (2.5) we shall need the following categories and functors; compare Goerss–

Jardine [**GJ**]. Let **Set** and **Mod** be the category of sets and k–modules respectively and let Δ**Set** and Δ**Mod** be the corresponding categories of simplicial objects in **Set** and **Mod** respectively. We have functors

$$\mathbf{Top}^* \xrightarrow{\mathsf{Sing}} (\Delta\mathbf{Set})^* \xrightarrow{|\ |} \mathbf{Top}^*$$

given by the *singular set* functor Sing and the *realization* functor $|\ |$. Moreover we have

$$\Delta\mathbf{Set} \xrightarrow{R} \Delta\mathbf{Mod} \xrightarrow{\Phi} (\Delta\mathbf{Set})^*$$

where k carries the simplicial set X to the *free k–module* generated by X and where Φ is the *forgetful* functor which carries the simplicial module A to the underlying simplicial set. Moreover we need the Dold–Kan functors

$$\mathbf{Ch}_+ \xrightarrow{\Gamma} \Delta\mathbf{Mod} \xrightarrow{N} \mathbf{Ch}_+$$

where $\mathbf{Ch}_+$ is the category of chain complexes in **Mod** concentrated in degree $\geqslant 0$. Here N is the *normalization* functor which by the Dold–Kan theorem is an equivalence of categories with inverse Γ. For a pointed space V let

$$K(V) = |\Phi S(V)| \text{ with } S(V) = \frac{k\,\mathsf{Sing}(V)}{k\,\mathsf{Sing}(*)}$$

Hence $K : \mathbf{Top}^* \to \mathbf{Top}^*$ carries a pointed space to a topological k–module. We define the binatural map

$$\bar{\otimes} : K(V) \times K(W) \to K(V \wedge W) \tag{*}$$

as follows. We have

$$\mathsf{Sing}(V \times W) = \mathsf{Sing}(V) \times \mathsf{Sing}(W)$$

and this bijection induces a commutative diagram in Δ**Mod**

$$\begin{tikzcd}
R\,\mathsf{Sing}(V) \otimes R\,\mathsf{Sing}(W) \arrow[r, equal] \arrow[d] & R\,\mathsf{Sing}(V \times W) \arrow[d] \\
S(V) \otimes S(W) \arrow[r, "\Lambda"] & S(V \wedge W)
\end{tikzcd}$$

The vertical arrows are induced by quotient maps. For k–modules A, B let $\otimes : A \times B \to A \otimes_k B$ be the map in **Set** which carries (a, b) to the tensor product $a \otimes b$. Of course this map $\otimes$ is bilinear. Moreover for A, B in Δ**Mod** the map $\otimes$ induces the map $\otimes : \Phi(A \times B) \to \Phi(A \otimes B)$ in **Set** and the realization functor yields

$$|\otimes| : |\Phi A| \times |\Phi B| = |\Phi(A \times B)| \to |\Phi(A \otimes B)|$$

Hence for $A = S(V)$ and $B = S(W)$ we get the composite

$$|\Phi S(V)| \times |\Phi S(W)| \xrightarrow{|\otimes|} |\Phi(S(V) \otimes S(W))| \xrightarrow{|\Phi\Lambda|} |\Phi S(V \wedge W)|$$

and this is the map $\bar{\otimes}$ above. One readily checks that $\bar{\otimes}$ is bilinear with respect to the topological k–module structure of $K(V), K(W)$ and $K(V \wedge W)$ respectively.

Moreover the following diagram commutes

$$\begin{tikzcd} K(V) \times K(W) \arrow[rr, "\bar{\otimes}"] & & K(V \wedge W) \\ {|\mathsf{Sing}V| \times |\mathsf{Sing}W|} \arrow[u, "h \times h"] \arrow[r, equal] & {|\mathsf{Sing}(V \times W)|} \arrow[r] & {|\mathsf{Sing}(V \wedge W)|} \arrow[u, "h"'] \end{tikzcd} \qquad (**)$$

Here the *Hurewicz map* h is the realization of the map in $\mathbf{\Delta Set}$

$$\mathsf{Sing}(V) \to \Phi k\,\mathsf{Sing}(V) \to \Phi S(V)$$

which carries an element x in $\mathsf{Sing}(V)$ to the corresponding generator in $k\,\mathsf{Sing}(V)$.

Let $S^n = S^1 \wedge \ldots \wedge S^1$ be the n–fold smash product of the 1–sphere S^1. Then the symmetric group σ_n acts on S^n by permuting the factors S^1. It is well known that this action of σ_n on S^n induces the sign–action of σ_n on the homology $H_n(S^n) = k$. We define the Eilenberg–MacLane space Z^n by

$$Z^n = K(S^n) = \left| \Phi \frac{k\mathsf{Sing}(S^n)}{k\mathsf{Sing}(*)} \right|$$

Since K is a functor we see that σ_n also acts on $K(S^n)$ via k–linear automorphisms. We define the multiplication map $\mu_{m,n}$ by

$$\mu : Z^m \times Z^n = K(S^m) \times K(S^n) \xrightarrow{\bar{\otimes}} K(S^m \wedge S^n) = Z^{m+n}$$

where $S^m \wedge S^n = S^{m+n}$ and where we use (*). Diagram (**) implies that μ induces the cup product in cohomology. The Eilenberg–MacLane spaces Z^n together with the multiplication map μ satisfy the axioms of Karoubi [**Ka**].

References

[**B**] Baues, H.–J.: The algebra of secondary cohomology operations, chapter 6, Preprint 2001.

[**BD**] Berrick, A.–J. and Davydov, A. A.: Splitting of Gysin extensions. Preprint (1999) group. K–theory 3 (1989) 307–338.

[**BM**] Baues, H.–J. and Minian, E. G.: Crossed extensions of algebras and Hochschild cohomology. To appear in this proceedings.

[**GJ**] Goerss, P. and Jardine, J. F.: Simplicial homotopy theory. Birkhäuser Verlag, Progress in Math. 174, (1999) 510 pages

[**H**] Huebschmann, J.: Crossed n–fold extensions of groups and cohomology. Comment. Math. Helvetici 55 (1980) 302–314

[**HSS**] Hovey, M. and Shipley, B. and Smith, J.: Symmetric spectra. Journal of the AMS 13 (2000) 149–208

[**K**] Kristensen, L.: On secondary cohomology operations, Math. Scand. 12 (1963) 57–82

[**L**] Loday, J.–L.: Cyclic Homology, Springer Verlag (1992)

[**Lu**] Lue, S.–T.: Cohomology of algebras relative to a variety. Math. Z. 121 (1971) 220-232

[**Ka**] Karoubi, M.: Formes differentielle non commutatives et operation de Steenrod. Topology 34 (1995) 699–715

[**P**] Porter, T.: Extensions, crossed modules and internal categories in categories of groups with operations. Proc. Edinburgh Mayh. Soc. (2) 30 (1987) 373-381

[**St**] Stover, C. R.: The equivalence of certain categories of twisted Lie and Hopf algebras over a commutative ring. Journal of pure and applied Algebra 86 (1993) 289–326

[**W**] Whitehead, J. H. C.: Combinatorial homotopy II, Bull. AMS 55 (1949) 213–245

This article may be accessed via WWW at http://www.rmi.acnet.ge/hha/ or by anonymous ftp at

ftp://ftp.rmi.acnet.ge/pub/hha/volumes/2002/n2a2/v4n2a2.(dvi,ps,pdf)

Hans–Joachim Baues `baues@mpim-bonn.mpg.de`

Max–Planck–Institut für Mathematik
Vivatsgasse 7
D–53111 Bonn
Germany