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OPERATIONS AND QUANTUM DOUBLES IN COMPLEX ORIENTED COHOMOLOGY THEORY

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Abstract

We survey recent developments introducing quantum algebraic methods into the study of cohomology operations in complex oriented cohomology theory. In particular, we discuss geometrical and homotopy theoretical aspects of the quantum double of the Landweber-Novikov algebra, as represented by a subalgebra of operations in double complex cobordism. We work in the context of Boardman's eightfold way, which offers an important framework for clarifying the relationship between quantum doubles and the standard machinery of Hopf algebroids of homology cooperations. These considerations give rise to novel structures in double cohomology theory, and we explore the twist operation and extensions of the quantum antipode by way of example.

1. Introduction

In his pioneering work [**8**], Drinfeld introduced the quantum double construction $\mathcal{D}(H)$ for a Hopf algebra H, as an aid to the solution of the Yang-Baxter equations. His work aroused interest in many areas of mathematics, and applications to algebraic topology were begun by Novikov, who proved in [**17**] that whenever H acts appropriately on a ring R, then the smash product $R \# H$ (in the sense of [**24**]) may be represented as a ring of operators on R. Novikov therefore referred to $R\#H$ as the *operator double*, and observed that his construction applied to the adjoint action of H on its dual H^* whenever H is cocommutative, thereby exhibiting $\mathcal{D}(H)$ as the operator double $H^* \# H$. Authors in other fields have recorded similar phenomena, and the subject of doubling constructions is currently very active.

Novikov was motivated by the algebra of cohomology operations in complex cobordism theory, which he constructed as an operator double by choosing H to be the Landweber-Novikov algebra S^*, and R the complex cobordism ring Ω_*^U. This viewpoint was suggested by the description of S^* as an algebra of differential operators on a certain algebraic group, due to Buchstaber and Shokurov [**4**]. Since the Landweber-Novikov algebra is cocommutative, its quantum double is also an operator double, and Buchstaber used this property in [**5**] to prove the remarkable fact that $\mathcal{D}(S^*)$ may be faithfully represented in A_{DU}^*, the ring of operations in *double* complex cobordism theory. In this sense, the algebraic and geometric doubling procedures coincide. The structures associated with $\mathcal{D}(S^*)$ are very different from those traditionally studied by topologists, and one of our aims is to explore their implications for other complex oriented cohomology theories.

By way of historical comment, we recall that complex cobordism originally gained prominence during the late 1960s in the context of stable homotopy theory, but was superseded in the 1970s by Brown-Peterson cohomology because of the computational advantages gained by working with a single prime at a time. Ravenel's book [**20**] gives an exhaustive account

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of these events. Work such as [**13**] has recently led to a resurgence of interest; this has been fuelled by mathematical physics, which was also the driving force behind Drinfeld's original study of quantum groups.

So far as we are aware, double cobordism theories first appeared in [**21**], and in the associated work [**22**] where the double SU-cobordism ring was computed. Since neither source describes the foundations, we begin with a brief but rigorous treatment of the geometric and homotopy theoretic details, establishing double complex cobordism as the natural setting for the dual and the quantum double of the Landweber-Novikov algebra. We couch a major part of our exposition in terms of Boardman's *eightfold way* [**2**], which we believe is still the best available framework for keeping track of all the actions and coactions that we need. We locate the subalgebra $\mathcal{D}(S^*)$ of A^*_{DU} in this context, and illustrate the novelty of the resulting structures by studying endomorphisms which extend its antipode. Much of the algebra may be interpreted more geometrically in terms of double complex structures on manifolds of flags, as described in [**6**].

We use the following notation and conventions without further comment.

We often confuse a complex vector bundle ρ with its classifying map into the appropriate Grassmannian, and write $\xi(m)$ for the universal m-plane bundle over $BU(m)$. We let $\mathbb{C}^m$ denote the trivial m-plane bundle over any base; if ρ has dimension m and lies over a finite CW complex, we write $\rho^\perp$ for the complementary $(p-m)$-plane bundle in some suitably high dimensional $\mathbb{C}^p$.

Our Hopf algebras are intrinsically geometrical and naturally graded by dimension, as are ground rings such as Ω^U_*. Sometimes our algebras are not of finite type, and must therefore be topologized when forming duals and tensor products; this has little practical effect, but is explained with admirable care by Boardman [**3**], for example. Duals are invariably taken in the graded sense and we adapt our notation accordingly. Thus we write A^*_U for the algebra of complex cobordism operations, and A^U_* for its continuous dual $\mathrm{Hom}_{\Omega^U_*}(A^*_U, \Omega^U_*)$, forcing us to write S^* for the graded Landweber-Novikov algebra, and S_* for its dual $\mathrm{Hom}_{\mathbb{Z}}(S^*, \mathbb{Z})$; neither of these notations is entirely standard.

We follow Sweedler's convention of writing coproducts as $\delta(x) = \sum x_1 \otimes x_2$ in any coalgbera.

We refer readers to [**1**], [**23**], and [**25**] for comprehensive coverage of basic information in algebraic topology. So far as Hopf algebras and their actions on rings are concerned, we recommend the books [**12**], [**14**], and [**16**] for background material and a detailed survey of the state of the art.

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2. Double complex cobordism

In this section we outline the theory of double complex cobordism, considering both the manifold and homotopy theoretic viewpoints. We follow the lead of [**5**] by writing unreduced bordism functors as $\Omega_*(\ )$ when emphasizing their geometric origins; if these are of secondary importance, we revert to the notation $T_*(\ )$, where T is the appropriate Thom spectrum.

The theory is based on manifolds M whose stable normal bundle is endowed with a fixed splitting $\nu \cong \nu_\ell \oplus \nu_r$ into a *left* and *right* component. We may invoke standard procedures [**23**] to construct the associated bordism and cobordism functors, and to describe them homotopy theoretically in terms of an appropriate Thom spectrum. Nevertheless we explain some of the details.

Given positive integers m and n, we write $U(m,n)$ for the product of unitary groups $U(m) \times U(n)$, so that the classifying space $BU(m,n)$ is canonically identified with $BU(m) \times BU(n)$.

Thus $BU(m,n)$ carries the complex $(m+n)$-plane bundle $\xi(m,n)$, defined as $\xi(m)\times\xi(n)$ and classified by the Whitney sum map $BU(m,n)\to BU(m+n)$. The standard inclusions of $U(m)$ in $O(2m)$ and of $U(m,n)$ in $U(m+1,n+1)$ induce a doubly indexed version of a (B,f) structure, but care is required to ensure that the maps are sufficiently compatible over m and n. There are product maps

$$BU(m,n)\times BU(p,q)\longrightarrow BU(m+p,n+q) \tag{2.1}$$

induced by Whitney sum, whose compatibility is more subtle, but serves to confirm that the corresponding (B,f) cobordism theory is multiplicative; it is *double complex cobordism* theory, referred to in [**21**] as $U\times U$ theory.

We therefore define a *double U-structure on M* to consist of an equivalence class $(M;\nu_\ell,\nu_r)$ of lifts of ν to $BU(m,n)$, for some values of m and n which are suitably large. The lifts provide the isomorphism $\nu\cong\nu_\ell\oplus\nu_r$, where ν_ℓ and ν_r are classified by the left and right projections onto the respective factors $BU(m)$ and $BU(n)$. If we wish to record a particular choice of m and n, we refer to the resulting *$U(m,n)$-structure*. Given such a structure on M, it is convenient to write $\chi(M)$ for the $U(n,m)$-structure $(M;\nu_r,\nu_\ell)$, induced by the switch map $BU(m,n)\to BU(n,m)$; we emphasise that M and $\chi(M)$ are generally inequivalent. If M has a $U(m,n)$-structure and N has a $U(p,q)$-structure, then the product $U(m+p,n+q)$ structure is defined by $(M\times N;\nu_\ell^M\times\nu_\ell^N,\nu_r^M\times\nu_r^N)$.

A typical example is given by complex projective space CP^n, which admits the double U-structure $(-k\xi,(k-n-1)\xi)$ for each integer k.

The compatibility required of the maps (2.1) is best expressed in the language of May's coordinate-free functors (as described, for example, in [**9**]), which relies on an initial choice of infinite dimensional inner product space Z_∞, known as a *universe*. We assume here that Z_∞ is complex. This language was originally developed to prove that the multiplicative structure of complex cobordism is highly homotopy coherent [**15**]; its usage establishes that the same is true for double complex cobordism, so long as we consistently embed our double U-manifolds in finite dimensional subspaces $V\oplus W$ of the universe $Z_\infty\oplus Z_\infty$. We define the classifying space $B(V,W)$ by appropriately topologizing the set of all subspaces of $V\oplus W$ which are similarly split. If V and W are spanned respectively by m and n element subsets of some predetermined orthonormal basis for $Z_\infty\oplus Z_\infty$, we refer to them as *coordinatized*, and write the classifying space as $BU(m,n)$ to conform with our earlier notation. We then interpret (2.1) as a coordinatized version of Whitney sum, on the understanding that the subspaces of dimension m and p are orthogonal in Z_∞, as are those of dimension n and q. The Grassmannian geometry of the universe guarantees the required compatibility.

In our work below we may safely confine such considerations to occasional remarks, although they are especially pertinent when we define the corresponding Thom spectrum and its multiplicative properties.

The double complex cobordism ring Ω_*^{DU} consists of cobordism classes of double U-manifolds, with the product induced as above. The double complex bordism functor $\Omega_*^{DU}(\ )$ is an unreduced homology theory, defined on spaces X by bordism classes of singular double U-manifolds in X; it admits a canonical involution (also denoted by χ), induced by switching the factors of the normal bundle. Thus $\Omega_*^{DU}(X)$ is always a module over Ω_*^{DU}, which is the coefficient ring of the theory, and the product structure ensures that $\Omega_*^{DU}(X)$ is both a left and a right Ω_*^U-module.

Double complex *co*bordism $\Omega^*_{DU}(\ )$ is the dual cohomology functor, which we define geometrically as in [**19**]. For any double U-manifold X, a cobordism class in $\Omega^*_{DU}(X)$ is represented by an equivalence class of compositions

$$M\xrightarrow{i}E_\ell\oplus E_r\xrightarrow{\pi}X$$

where π is the projection of a complex vector bundle split into left and right components, and i is an embedding of a double U-manifold M whose normal bundle is split compatibly.

If we ignore the given splitting of each normal bundle and simultaneously identify $Z_\infty \oplus Z_\infty$ isometrically with Z_∞, we obtain a forgetful homomorphism $\pi\colon \Omega_*^{DU}(X) \to \Omega_*^U(X)$ for any space X. Conversely, if we interpret a given $U(m)$-structure as either a $U(m,0)$-structure or a $U(0,m)$-structure, we obtain left and right inclusions ι_ℓ and $\iota_r\colon \Omega_*^U(X) \to \Omega_*^{DU}(X)$, which are interchanged by χ. All three transformations have cobordism counterparts, which are multiplicative. Since $\pi \circ \iota_\ell$ and $\pi \circ \iota_r$ give the identity, we deduce that π is an epimorphism and that ι_ℓ and ι_r are monomorphisms, in both bordism and cobordism.

We now turn to the homotopy theoretic viewpoint. In order to allow spectra which consist of a doubly indexed direct systems, as well as to ensure the existence of products which are highly homotopy coherent, it is most elegant to return to the coordinate-free setting. We define the Thom space $M(V,W)$ by the standard construction on $B(V,W)$, and again allow the Grassmannian geometry of the universe to provide the necessary compatibility for both the structure and the product maps. As in (2.1) we give explicit formulae only for coordinatized subspaces.

We write $MU(m,n)$ for the Thom complex of $\xi(m,n)$, which is canonically identified with $MU(m) \wedge MU(n)$. Then the coordinatized structure maps take the form

$$S^{2(p+q)} \wedge MU(m,n) \longrightarrow MU(m+p,n+q), \tag{2.2}$$

given by Thom complexifying the classifying maps of $(\mathbb{C}^p \times \mathbb{C}^q) \oplus \xi(m,n)$. We take this direct system as our definition of the *DU spectrum*, noting that the Thom complexifications of the maps (2.1) provide a product map μ_{DU}, which is highly coherent, and equipped with a unit by (2.2) in the case $m = n = 0$. It is a left and right module spectrum over MU by virtue of the systems of maps

$$\begin{aligned} MU(p) \wedge MU(m,n) \to MU(m+p,n) \quad &\text{and} \\ MU(m,n) \wedge MU(q) \to MU(m,n+q)&, \end{aligned}$$

which are also highly coherent by appeal to the coordinate-free setting.

This setting enables us to define smash products of spectra [**9**], and therefore to write DU as $MU \wedge MU$. The involution χ is then induced by interchanging factors, and we may represent the bimodule structure by maps $MU \wedge DU \to DU$ and $DU \wedge MU \to DU$, induced by applying the MU product μ_U on the left and right copies of $MU \wedge MU$ respectively.

As usual, we define the reduced bordism and cobordism functors by

$$\begin{aligned} DU_k(X) &= \varinjlim{}_{m,n} \pi_{2(n+m)+k}(MU(m,n) \wedge X) \\ \text{and} \quad DU^k(X) &= \varinjlim{}_{m,n} \{S^{2(n+m)-k} \wedge X, MU(m,n)\} \end{aligned} \tag{2.3}$$

respectively, for all integers $k \geqslant 0$, where the brackets $\{\ \}$ denote based homotopy classes of maps. The graded group $DU^*(X)$ becomes a commutative graded ring by virtue of the product structure on DU, and the coefficient ring DU_* (or DU^{-*}) is the homotopy ring $\pi_*(DU)$. The *unreduced* bordism and cobordism functors arise as $DU_*(X_+)$ and $DU^*(X_+)$, by appending a disjoint basepoint. In this context, we write 1 for the element which corresponds to the appropriate generator of DU_0. Given a second homology theory $E_*(\ )$, we define the homology groups $E_k(DU)$ and the cohomology groups $E^k(DU)$ by $\varinjlim_{m,n} E_{2(n+m)+k}(MU(m,n))$ and $\varprojlim_{m,n} E^{2(n+m)+k}(MU(m,n))$ respectively, for all integers $k \geqslant 0$.

The Whitney sum map $BU(m,n) \to BU(m+n)$ induces a forgetful map $\pi\colon DU \to MU$, which coincides with the product μ_U; the inclusions of $BU(m)$ and $BU(n)$ in $BU(m+n)$ induce inclusions ι_ℓ and $\iota_r\colon MU \to DU$ of the left and right factors respectively. All three are maps of ring spectra which extend to the coordinate-free setting, and define multiplicative transformations between the appropriate functors. Clearly ι_ℓ and ι_r are interchanged by χ, and yield the identity map after composition with π. Given an element θ of $MU^*(X)$ or $MU_*(X)$, we shall often write $\iota_\ell(\theta)$ and $\iota_r(\theta)$ as θ_ℓ and θ_r respectively.

Following the descriptions above, we may define the *canonical isomorphism* $i_U\colon MU_*(MU) \to DU_*$ by identifying both sides with $\pi_*(MU \wedge MU)$. We shall explain below how $MU_*(MU)$

is also the Hopf algebroid of cooperations in MU-homology theory; suffice it to say here that its associated homological algebra has been extensively studied in connection with the Adams-Novikov spectral sequence and the stable homotopy groups of spheres. For detailed calculations, however, it has proven more efficient to concentrate on a single prime p at a time, and work with the p-local summand $BP_*(BP)$ given by Brown-Peterson homology [**20**].

There is a natural isomorphism between the manifold and the homotopy theoretic versions of any bordism functor, stemming from the Pontryagin-Thom construction. Given a map $f\colon M^k \to X$, where M^k embedded in $S^{k+2(m+n)}$ with a $U(m,n)$-structure, the construction is accomplished by collapsing the complement of a normal neighbourhood to ∞ and composing with the Thom complexification of $\nu \times f$; the result is a map from $S^{k+2(m+n)}$ to $MU(m,n) \wedge X_+$, and so defines the isomorphism $\Omega_k^{DU}(X_+) \to \pi_k(DU \wedge X_+)$. Verifying the the necessary algebraic properties requires considerable effort, and depends upon Thom's transversality theorems. The isomorphism maps the geometric versions of the transformations χ, π, ι_ℓ, and ι_r to their homotopy theoretic counterparts, and may be naturally extended to the coordinate-free setting.

3. Orientation classes

In this section we characterize DU as the universal example of a spectrum equipped with two complex orientations, and consider the consequences for the double complex bordism and cobordism groups of some well-known spaces in complex geometry. We establish our notation by recalling certain basic definitions and results, which may be found, for example, in [**1**].

We assume throughout that E is a commutative ring spectrum, with E_0 isomorphic to $\mathbb{Z}$. Then E is *complex oriented* if the cohomology group $E^2(CP^\infty)$ contains an *orientation class* x^E whose restriction to $E^2(CP^1)$ is a generator when the latter group is identified with E_0. Under these circumstances, we may deduce that $E^*(CP^\infty)$ consists of formal power series over E_* in the variable x^E, whose powers define dual basis elements β_k^E in $E_{2k}(CP^\infty)$. If we continue to write β_k^E for their image under the inclusion of $BU(1)$ in $BU(m)$ (for any value of m, including ∞), then $E_*(BU(m))$ is the free E_*-module generated by commutative monomials of length at most m in the elements β_k^E. For $1 \leqslant k \leqslant m$, the duals of the powers of β_1^E define the *Chern classes* c_k^E in $E^{2k}(BU(m))$, which generate $E^*(BU(m))$ as a power series algebra over E_*; clearly c_1^E agrees with x^E over CP^∞. When we pass to the direct limit over m, we obtain

$$E_*(BU_+) \cong E_*[\beta_k^E : k > 0] \qquad \text{and} \qquad E^*(BU_+) \cong E_*[[c_k^E : k > 0]], \tag{3.1}$$

where the Pontryagin product in homology is induced by Whitney sum. We write monomial basis elements $\prod_i (\beta_i^E)^{\omega_i}$ as $(\beta^E)^\omega$ for any sequence ω of nonnegative, eventually zero integers, and their duals as c_ω^E. Thus c_ω^E is usually not a monomial, and $c_{(k)}^E$ and c_k^E coincide.

For $m \geqslant 1$ the Thom complex $MU(m)$ of $\xi(m)$ is the cofiber of the inclusion of $BU(m-1)$ in $BU(m)$, allowing $E_*(MU(m))$ and $E^*(MU(m))$ to be computed from the corresponding short exact sequences of E_*-modules. We may best express the consequences in terms of Thom isomorphisms, for which we first identify the pullback of c_m^E in $E^{2m}(MU(m))$ as the *Thom class* $t^E(m)$ of $\xi(m)$, observing that its restriction over the base point is a generator of $E^{2m}(S^{2m})$ when the latter is identified with E_0. Indeed, this property reduces to the defining property for x^E when m is 1, thereby identifying x^E as the Thom class of $\xi(1)$. The Thom isomorphisms

$$\begin{aligned} \phi_*\colon E_{k+2m}(MU(m)) \to E_k(BU(m)_+) \\ \text{and} \quad \phi^*\colon E^k(BU(m)_+) \to E^{k+2m}(MU(m)), \end{aligned} \tag{3.2}$$

for $\xi(m)$ are determined by the relative cap and cup products with $t^E(m)$, and define elements b_k^E in $E_{2(k+m)}(MU(m))$ as $\phi_*^{-1}(\beta_k)$, and elements s_k^E in $E^{2(k+m)}(MU(m))$ as $\phi^*(c_k^E)$; each of these families extends to a set of generators over E_* in the appropriate sense.

When m is 1 the projection $CP^\infty \to MU(1)$ onto the cofiber is a homotopy equivalence, and the induced homomorphism identifies β_k^E with b_{k-1}^E in E-homology. The Thom isomorphisms satisfy $\phi_*(b_k^E) = \beta_k^E$ and $\phi^*((x^E)^{k-1}) = (x^E)^k$ respectively, for all $k \geqslant 1$.

We stabilize (3.2) by allowing m to become infinite, so that (3.1) yields

$$E_*(MU) \cong E_*[b_k^E : k > 0] \qquad \text{and} \qquad E^*(MU) \cong E_*[[s_k^E : k > 0]], \tag{3.3}$$

where b_k^E lies in $E_{2k}(MU)$ and s_k^E in $E^{2k}(MU)$, for all $k \geqslant 0$. The multiplicative structure in homology is induced by μ_U, but exists in cohomology only as an algebraic consequence of ϕ^*, and not as a cup product. We continue to write monomial basis elements in the b_k^E as $(b^E)^\omega$, and their duals as s_ω^E, for any sequence ω. Again, $s_{(k)}^E$ and s_k^E coincide. We write t^E in $E^0(MU)$ for the stable Thom class, which corresponds to the element 1 under (3.3), and is represented by a multiplicative map of ring spectra.

We have therefore described a procedure for constructing t^E from our initial choice of x^E; in fact this provides a bijection between complex orientation classes in E and multiplicative maps $MU \to E$.

Any complex m-plane bundle ρ over a space X has a Thom class $t^E(\rho)$ in $E^{2m}(M(\rho))$, obtained by pulling back the universal example $t^E(m)$ along the classifying map. We use this Thom class as in (3.2) to define isomorphisms

$$\phi_*\colon E_{k+2m}(M(\rho)) \to E_k(X_+) \qquad \text{and} \qquad \phi^*\colon E^k(X_+) \to E^{k+2m}(M(\rho)).$$

If ρ is virtual then $M(\rho)$ is stable, and the Thom isomorphisms assume a similar format to the universal examples (3.1) and (3.3) if we arrange for the bottom cell to have dimension zero.

We remark that MU is complex oriented by letting the homotopy equivalence $CP^\infty \to MU(1)$ represent x^{MU}; the corresponding Thom class t^U is represented by the identity map on MU. In fact MU is the universal example, since any Thom class t^E induces the ring map $MU^*(CP^\infty) \to E^*(CP^\infty)$ uniquely specified by $x^{MU} \mapsto x^E$, and t_*^E on coefficients. In view of these properties, we shall dispense with sub- and superscripts U wherever possible in the universal case. So far as the geometric description of cobordism is concerned, a Thom class $t(\rho)$ in $\Omega_U^{2m}(M(\rho))$ is represented by the inclusion of the zero section $M \subset M(\rho)$, whenever ρ lies over a U-manifold M.

We combine the Thom isomorphism $MU_*(MU) \cong MU_*(BU_+)$ with the canonical isomorphism i to obtain an isomorphism $h\colon DU_* \cong MU_*(BU_+)$ of left MU_*-modules; it has an important geometrical interpretation.

Proposition 3.4. *Suppose that an element of Ω_*^{DU} is represented by a manifold M^k with double U-structure $\nu_\ell \oplus \nu_r$; then its image under h is represented by the singular U-manifold $\nu_r\colon M^k \to BU$.*

Proof. By definition, the image we seek is represented by the composition

$$\begin{aligned} S^{k+2(m+n)} &\to M(\nu) \to MU(m) \wedge MU(n) \\ &\xrightarrow{1\wedge\delta} MU(m) \wedge MU(n) \wedge BU(n)_+ \xrightarrow{\mu\wedge 1} MU(m+n) \wedge BU(n)_+, \end{aligned}$$

where the first map is obtained by applying the Pontryagin-Thom construction to an appropriate embedding $M^k \subset S^{k+2(m+n)}$, and the second classifies the double U-structure on M^k. We may identify the final three maps as the Thom complexification of the composition

$$\begin{aligned} M^k \xrightarrow{\nu_\ell \oplus \nu_r} BU(m) \times BU(n) \xrightarrow{1\times\delta} BU(m) &\times BU(n) \times BU(n) \\ &\xrightarrow{\oplus\times 1} BU(m+n) \times BU(n), \end{aligned}$$

which simplifies to $\nu \oplus \nu_r$, as sought. $\square$

Corollary 3.5. *Suppose that an element of $\Omega_*^U(BU_+)$ is represented by a singular U-manifold $f\colon M^k \to BU(q)$ for suitably large q; then its inverse image under h is represented by the double U-structure $(\nu \oplus f^\perp) \oplus f$ on M.*

Our proof of Proposition 3.4 shows that h is multiplicative, so long as we invest $MU_*(BU_+)$ with the Pontryagin product which arises from the Whitney sum map on BU. Moreover, h conjugates the involution χ so as to act on $MU_*(BU_+)$, where it interchanges the map f of Corollary 3.5 with $\nu \oplus f^\perp$.

The multiplicative maps ι_ℓ and $\iota_r\colon MU \to DU$ both define complex orientations for the spectrum DU, with corresponding Thom classes t_ℓ and t_r and orientation classes x_ℓ and x_r in $D^2(CP^\infty)$. We obtain

$$DU^*(CP^\infty_+) \cong DU_*[[x_\ell]] \cong DU_*[[x_r]],$$

and there are mutually inverse formal power series

$$x_r = \sum_{k\geqslant 0} g_k x_\ell^{k+1} \qquad \text{and} \qquad x_\ell = \sum_{k\geqslant 0} \bar{g}_k x_r^{k+1}, \tag{3.6}$$

written $g(x_\ell)$ and $\bar{g}(x_r)$ respectively. The elements g_k and $\bar{g}_k$ lie in DU_{2k} for all k, and are interchanged by the involution χ; in particular, $g_0 = \bar{g}_0 = 1$. They generate an important subalgebra G_* of DU_*, whose structure we now adress.

Proposition 3.7. *Under the isomorphism h, we have that*

$$h(g_k) = \beta_k$$

in $MU_{2k}(BU_+)$, for all $k \geqslant 0$.

Proof. We may express g_k as the Kronecker product $\langle x_r, \beta_{k+1,\ell}\rangle$ in DU_{2k}, which is represented by the composition

$$S^{2(p+k+1)} \xrightarrow{\beta_{k+1}} MU(p) \wedge CP^\infty \xrightarrow{1\wedge x} MU(p) \wedge MU(1),$$

for suitably large p. This stabilizes to b_k in $MU_{2k}(MU)$, and hence to β_k in $MU_{2k}(BU_+)$, as required. □

Corollary 3.8. *The subalgebra G_* is polynomial over $\mathbb{Z}$.*

Proof. This result follows from the multiplicativity of h and the independence of monomials in the β_k over $\mathbb{Z}$. □

We refer to any spectrum D with two complex orientations which restrict to the same element of D_0 as *doubly complex oriented*, writing

$$x_r^D = g^D(x_\ell^D) \qquad \text{and} \qquad x_\ell^D = \bar{g}^D(x_r^D) \tag{3.9}$$

for the two orientation classes in $D^2(CP^\infty)$. These power series define elements g_k^D and $\bar{g}_k^D$ in D_*, which generate a subalgebra G_*^D. Our main examples are *doubles* of complex oriented spectra E, given by $E \wedge E$ and denoted by $D(E)$; in such cases, we may extend the left and right notation to *all* groups $D(E)_*(X)$ and $D(E)^*(X)$.

Following the example of E, we construct *left* and *right* sets of D_*-generators for $D_*(CP^\infty)$, $D_*(BU(m))$, $D_*(MU(m))$, and for their cohomological counterparts. Thus there are left and right Chern classes $c_{k,\ell}^D$ and $c_{k,r}^D$ in $D^{2k}(BU(m))$ for $k \leqslant m$, and left and right Thom classes $t_\ell^D(m)$ and $t_r^D(m)$ in $D^{2m}(MU(m))$; the latter give rise to left and right Thom isomorphisms associated to an arbitrary complex bundle ρ. The elements g_k^D again provide the link, as the following example shows.

Lemma 3.10. *The left and right Thom classes are related by*

$$t_r^D = t_\ell^D + \sum_\omega (g^D)^\omega s_{\omega,\ell}^D$$

in $D^0(MU)$.

Proof. We dualize (3.9) and pass to $D_*(BU)$, then dualize back again to $D^*(BU)$ and apply the Thom isomorphism. $\square$

When DU is equipped with the orientation classes x_ℓ and x_r it becomes the universal example of a doubly complex oriented spectrum, since the exterior product $t_\ell^D t_r^D$ is represented by a multiplicative spectrum map $t^D\colon DU \to D$ whose induced transformation $DU^2(CP^\infty) \to D^2(CP^\infty)$ maps x_ℓ and x_r to x_ℓ^D and x_r^D respectively. It therefore often suffices to consider the case DU (as we might in Lemma 3.10, for example). We shall continue to omit the superscript DU whenever possible in the universal case. We note from the definitions that the homomorphism of coefficient rings $DU_* \to D_*$ induced by t^D satisfies

$$g_k \mapsto g_k^D \qquad \text{and} \qquad \bar{g}_k \mapsto \bar{g}_k^D \tag{3.11}$$

for all $k \geqslant 0$.

Whenever a complex vector bundle has a prescribed splitting $\rho \cong \rho_\ell \oplus \rho_r$, then $t^D(\rho)$ acts as a canonical Thom class $t_\ell^D(\rho_\ell)t_r^D(\rho_r)$, and so defines a Thom isomorphism which respects the splitting. In the universal case, $t(\rho)$ is represented geometrically by the inclusion of the zero section $M \subseteq M(\rho_\ell \oplus \rho_r)$ whenever ρ lies over a double U-manifold M.

As an example, it is instructive to consider the case when D is MU, doubly oriented by setting $x_\ell^D = x_r^D = x$. The associated Thom class is the forgetful transformation $\pi\colon DU \to MU$, since $\pi(x_\ell) = \pi(x_r) = x$; we therefore deduce from (3.11) that both $\pi(g_k)$ and $\pi(\bar{g}_k)$ are zero, for all $k > 0$.

We also consider the D_*-modules $D_*(BU(m,n))$ and $D_*(MU(m,n))$, together with their cohomological counterparts, which may all be described by applying the Künneth formula. For example $D^*(BU(m,n))$ is a power series algebra, generated by any one of the four possible sets of Chern classes

$$\begin{aligned} &\{c_{j,\ell}^D \otimes 1, 1 \otimes c_{k,r}^D\}, \quad \{c_{j,\ell}^D \otimes 1, 1 \otimes c_{k,\ell}^D\}, \\ &\qquad\qquad\qquad\quad \{c_{j,r}^D \otimes 1, 1 \otimes c_{k,\ell}^D\}, \quad \text{or} \quad \{c_{j,r}^D \otimes 1, 1 \otimes c_{k,r}^D\}, \end{aligned} \tag{3.12}$$

where $1 \leqslant j \leqslant m$ and $1 \leqslant k \leqslant n$. The first of these are the most natural, and we shall choose them whenever possible. The stable versions, in which we take limits over one or both of m and n, are obtained by the obvious relaxation on the range of j and k. We write $(\beta_\ell^D)^\psi \otimes (\beta_r^D)^\omega$ for the dual basis monomials in $D_*(BU(m,n))$, and $(b_\ell^D)^\psi \otimes (b_r^D)^\omega$ for their images in $D_*(MU(m,n))$ under the Thom isomorphism induced by $t^D(m,n)$; we then let $s_{\psi,\ell}^D \otimes s_{\omega,r}^D$ denote the corresponding basis elements in $D^*(MU(m,n))$. As before, it often suffices to consider the universal example DU.

We may translate much of the above into the language of formal group laws [**10**], observing that DU_* is universal amongst rings equipped with two formal group laws linked by a strict isomorphism whose coefficients are the elements g_k.

4. The eightfold way

In this section we consider the operations and cooperations associated with our spectrum E, specializing to MU as required. We study algebraic and homotopy theoretic aspects in the framework of Boardman's eightfold way [**2**], relating the actions and coactions involving $E_*(E)$ to the theory of the double spectrum $D(E)$, with a view to placing the quantum double in the traditional framework. All comments concerning the singly oriented E apply equally well to D unless otherwise stated.

For any integer n, the cohomology group $E^n(E)$ consists of homotopy classes of spectrum maps $s\colon E \to S^n \wedge E$, and therefore encodes E-theory cohomology operations of degree n. Thus $E^*(E)$ is a noncommutative, graded E_*-algebra with respect to composition of maps, and realizes the algebra A_E^* of stable E-cohomology operations. It is important to observe that $E^*(E)$ is actually a bimodule over the coefficients E_*, which act naturally on the left (as used implicitly above), but also on the right. The same remarks apply to $E_*(E)$, on which the product map μ_E induces a commutative E_*-algebra structure; the two module structures are then defined respectively by the left and right inclusions η_ℓ and η_r of the coefficients in $E_*(E) \cong \pi_*(E \wedge E)$. We refer to $E_*(E)$ as the algebra A_*^E of stable E-homology *cooperations*, for reasons described below.

In fact $E_*(X)$ is free and of finite type for all spaces and spectra X that we consider. This reduces the topologizing of $E^*(X)$ to the accommodation of formal power series in certain computations, and ensures that μ_E induces a cocommutative coproduct $\delta^E\colon E^*(E) \to E^*(E)\widehat{\otimes}_{E_*} E^*(E)$; the tensor product must be completed whenever $E^*(E)$ fails to be of finite type [**3**].

We consider the E_*-algebra map $t_*^E\colon E_*(MU) \to E_*(E)$ induced by the Thom class t^E, and define monomials $(p^E)^\omega$ as $t_*^E(b^E)^\omega$. When E is singly oriented we restrict attention to cases where $E_*(E)$ is a free E_*-module and t_*^E is an epimorphism, so that $E_*(E)$ is a quotient of the polynomial algebra $E_*[p_k^E : k \geqslant 0]$. It follows that $E^*(E)$ is given as a subalgebra of the dual by $\mathrm{Hom}_{E_*}(E_*(E), E_*)$, and that we may interpret a generic operation s as such a homomorphism. The composition product dualizes to a noncocommutative coproduct $\delta_E\colon E_*(E) \to E_*(E) \otimes_{E_*} E_*(E)$ (where $\otimes_{E_*}$ is taken over the right action on the left factor), with counit given by projection onto the coefficients. Together with the left and right units η_ℓ and η_r, and the antipode χ_E induced by interchanging the factors in $E_*(E)$, this coproduct turns $E_*(E)$ into a cogroupoid object in the category of E_*-algebras. Such an object generalizes the notion of Hopf algebra, and is known as a *Hopf algebroid*; for a detailed discussion, see [**20**].

Given our assumptions on E, the monomials $(p^E)^\omega$ generate $E_*(E)$ over E_* although they may be dependent; nevertheless, they appear naturally in several formulae below. To describe a *basis* over E_*, we consider each example on its merits and choose an appropriate set of homogeneous polynomials e^α, noting that the new set of indices α may well be smaller. Each $(p^E)^\omega$ may then be expressed as a linear combination of the e^α, whose coefficients generally lie in E_*, rather than $\mathbb{Z}$. Maintaining our earlier conventions, we let e_α denote the topological basis for $E^*(E)$ dual to the e^α.

For any space or spectrum X, we first consider the *standard action*

$$E^*(E) \otimes_{E_*} E^*(X) \longrightarrow E^*(X), \tag{4.1}$$

(where $\otimes_{E_*}$ is taken over the right action on $E^*(E)$). It is the map of left E_*-modules which arises from considering the elements of $E^*(E)$ as selfmaps of E, and we write it functionally; when X is E it reduces to the composition product in $E^*(E)$, and when X is a point (or the sphere spectrum), to the action of $E^*(E)$ on the coefficient ring E_*. The Cartan formula asserts that the product map in $E^*(X)$ is a homomorphism of left $E^*(E)$-modules with respect to the standard action, and was restated by Milnor in the form

$$s(yz) = \sum s_1(y)s_2(z) \qquad \text{where} \qquad \delta^E(s) = \sum s_1 \otimes s_2 \tag{4.2}$$

for all y and z in $E^*(X)$. Following Novikov [**17**], we refer to any such module with property (4.2) as a *Milnor module.*

Given our freeness assumptions we may dualize and conjugate (4.1) to obtain seven alternative structures, whose unification is the aim of Boardman's eightfold way. We consider four of these (together with a fifth and sixth which are different), selecting from [**2**] without comment and ignoring issues of sign because our spaces and spectra have no cells in odd dimensions.

The second structure is the E_*-dual of (4.1), the *Adams coaction*

$$\psi\colon E_*(X) \longrightarrow E_*(E) \otimes_{E_*} E_*(X) \tag{4.3}$$

(where $\otimes_{E_*}$ is taken over the right action on $E_*(E)$), which reduces to the coproduct δ_E when

X is E. For each operation s and each y in $E^*(X)$, the duality is specified by

$$\langle s(y), a\rangle = \sum_\alpha \langle s, e^\alpha \langle y, a^\alpha\rangle\rangle, \tag{4.4}$$

where a lies in $E_*(X)$ and a^α is defined by $\psi(a) = \sum_\alpha e^\alpha \otimes a^\alpha$.

If we assume that X is a spectrum (or stable complex), we may interpret $\pi_*(X \wedge E)$ as $X_*(E)$, and consider the isomorphism $c\colon E_*(X) \cong X_*(E)$ of *conjugation*. Our third structure is the *right coaction*

$$\psi^c\colon X_*(E) \longrightarrow X_*(E) \otimes_{E_*} E_*(E) \tag{4.5}$$

of right E_*-modules (where $\otimes_{E_*}$ is taken over the right action of the scalars on $X_*(E)$); it is evaluated by $\psi(ca) = \sum_\alpha ca^\alpha \otimes \chi_E(e^\alpha)$, conjugating (4.3). When X is E, then c reduces to χ_E and ψ becomes δ_E, as before.

Fourthly, (4.1) dualizes partially over E_* to give the *Milnor coaction*

$$\rho\colon E^*(X) \longrightarrow E^*(X) \widehat{\otimes}_{E_*} E_*(E) \tag{4.6}$$

of E_*-modules. As Milnor famously observed, the Cartan formulae ensure that ρ is an algebra map, making $E^*(X)$ a *Hopf comodule* over $E_*(E)$. For each operation s and each x in $E^*(X)$, the partial duality satisfies

$$s(y) = \sum_\alpha \langle s, e^\alpha\rangle y_\alpha, \tag{4.7}$$

where y_α is defined by $\rho(y) = \sum_\alpha y_\alpha \otimes e^\alpha$; thus $y_\alpha = e_\alpha(y)$. In view of the completion required of the tensor product in (4.6), we describe ρ more accurately as a *formal coaction*.

A fifth possibility is provided by the *left action*

$$E^*(E) \otimes_{E_*} E_*(X) \longrightarrow E_*(X)$$

(where $\otimes_{E_*}$ is taken over the right action of the scalars on $X_*(E)$), which is defined by analogy with (4.1) in terms of spectrum maps. It is evaluated by partially dualizing the Adams coaction, giving

$$s_\ell(a) = \sum_\alpha \langle s, \chi_E(e^\alpha)\rangle a^\alpha, \tag{4.8}$$

with notation as above. Given y and a as before, the left action satisfies

$$\langle y, s_\ell(a)\rangle = \langle s, c(y_* a)\rangle, \tag{4.9}$$

where $y_*\colon E_*(X) \to E_*(E)$ is the induced homomorphism.

For our sixth and seventh structures we again assume that X is stable, so the selfmaps of E induce a left action

$$E^*(E) \otimes_{E_*} X_*(E) \longrightarrow X_*(E) \tag{4.10}$$

(where $\otimes_{E_*}$ is taken over the right action of the scalars on both factors), and a right action

$$X^*(E) \otimes_{E_*} E^*(E) \longrightarrow X^*(E) \tag{4.11}$$

(where $\otimes_{E_*}$ is taken over the right action on $X^*(E)$). Neither of these is discussed explicitly by Boardman, although (4.10) appears regularly in the literature, and coincides with (4.1) when X is the sphere spectrum; it is evaluated by partially dualizing (4.5), to obtain

$$s_r(d) = \sum_\alpha d^\alpha \langle s, e^\alpha\rangle, \tag{4.12}$$

where d lies in $X_*(E)$ with $\psi(d) = \sum_\alpha d^\alpha \otimes e^\alpha$. The actions (4.10) and (4.11) are related by

$$\langle w, s_r(d)\rangle = \langle (w)s, d\rangle, \tag{4.13}$$

where w lies in $X^*(E)$; this should be compared with (4.9), and justifies the interpretation of (4.10) as a right action (on the left!).

When X is E, (4.9) may be rewritten as

$$\langle y, s_\ell(a)\rangle = \langle s, \chi_E(y_r(a))\rangle, \tag{4.14}$$

whilst (4.13) reduces to the right action of $E^*(E)$ on its dual $E_*(E)$. Given any d in $E_*(E)$, we may evaluate the coproduct $\delta_E(d)$ as

$$\sum_\alpha e_{\alpha,r}(d) \otimes e^\alpha = \sum_\alpha \chi_E(e^\alpha) \otimes e_{\alpha,\ell}(d). \tag{4.15}$$

We now translate aspects of the eightfold way into the language of doubles, using the E-theory canonical isomorphism $i_E \colon E_*(E) \to D(E)_*$. The definition of the $(p^E)^\omega$ ensures that the homomorphism $x_*^E \colon E_{*+2}(CP^\infty) \to E_*(E)$ maps β_{k+1}^E to p_k^E. Dualization yields

$$s(x^E) = \sum_{k \geqslant 0} \langle s, p_k^E\rangle (x^E)^{k+1},$$

which combines with (4.7) to confirm that ρ acts by

$$\rho(x^E) = \sum_{k \geqslant 0} (x^E)^{k+1} \otimes p_k^E \tag{4.16}$$

in $E^*(CP^\infty)\widehat{\otimes}_{E_*} E_*(E)$.

Lemma 4.17. *Given any of our spaces or spectra X, the Milnor coaction* (4.6) *extends to $D(E)^*(X)$ by means of the commutative diagram*

$$\begin{tikzcd}
E^*(X) \arrow[r, "\rho"] \arrow[d, "\iota_r"'] & E^*(X)\widehat{\otimes}_{E_*} E_*(E) \arrow[d, "\iota_\ell \otimes i_E"] \\
D(E)^*(X) \arrow[r, "\cong"] & D(E)^*(X)\widehat{\otimes}_{D(E)_*} D(E)_*
\end{tikzcd}.$$

Proof. Commutativity follows from the fact that the Milnor coaction factors through the map $E^*(X) \to \mathrm{Hom}_{E_*}(E^*(X), E^*(E))$, defined by taking induced homorphisms. ☐

Corollary 4.18. *The canonical isomorphism identifies p_n^E with $g_n^{D(E)}$ and $\chi_E(p_n^E)$ with $\bar{g}_n^{D(E)}$ in $D(E)_*$, for all $n \geqslant 0$.*

Proof. Comparing Lemma 4.17 with (4.16), these equations follow immediately from (3.9). ☐

The formulation of ρ in Lemma 4.17 is almost explicit in [**11**], and has obvious mutants for each of the actions and coactions which involves $E_*(E)$.

Formula (4.16) is used by Boardman to *define* the elements p_n^E, and extends to a representation of the entire algebra A_E^* on the bottom cell of the infinite smash product spectrum $\wedge_\infty CP^\infty$. It also leads to a description of the structure maps for A_*^E in terms of the p_n^E.

Proposition 4.19. *The coproduct and antipode of the Hopf algebroid $E_*(E)$ are given by*

$$\delta_E(p_n^E) = \sum_{k \geqslant 0} (p^E)_{n-k}^{k+1} \otimes p_k^E \qquad \textit{and} \qquad \chi_E(p_n^E) = (p^E)_n^{-(n+1)}$$

respectively.

Proof. Since ρ is a coaction, we have that $\rho \otimes 1(\rho(x^E)) = 1 \otimes \delta^E(\rho(x^E))$ as maps $E^*(CP^\infty) \to E^*(CP^\infty)\widehat{\otimes}_{E_*} E_*(E) \otimes_{E_*} E_*(E)$, and the formula for δ^E ensues. Since χ_E applies to (3.9) by interchanging the factors of $D(E)$, we may use Corollary 4.18 to give $i_E(\chi_E(p_n^E)) = \bar{g}_n^E$, and the result follows by Lagrange inversion. ☐

These formulae are of limited use in cases where the $(p^E)^\omega$ are not independent, since they contain a great deal of inbuilt redundancy. Readers will recognize the spectrum BP as a pertinent example.

There is no such problem with the universal example MU, for which the e^ω are simply the original basis monomials b^ω in A^U_*. The dual operations are written s_ω, as described in §3, and are known as the Landweber-Novikov operations. We consider the integral spans S^* and S_* of the s_ω and b^ω respectively, so that $A^*_U \cong \Omega^U_* \otimes S^*$ and $A^U_* \cong \Omega^U_* \otimes S_*$ as Ω^U_*-modules, where S_* is the polynomial algebra $\mathbb{Z}[b_k : k \geqslant 0]$. The Ω^U_*-duality between A^*_U and A^U_* therefore restricts to an integral duality between S^* and S_*, for which no topological considerations are necessary because S^* has finite type.

The coproduct and antipode of A^U_* are given as $\delta(b_n) = \sum_{k\geqslant 0}(b)^{k+1}_{n-k} \otimes b_k$ and $\chi(b_n) = (b)_n^{-(n+1)}$ by Corollary 4.19, and therefore restrict to S_*. Since the left unit and the counit also make sense over $\mathbb{Z}$, we deduce that S_* is a Hopf subalgebra of the Hopf algebroid. Duality ensures that S^* is also a Hopf algebra, with respect to composition of operations and the Cartan formula

$$\delta(s_\omega) = \sum_{\omega_1+\omega_2=\omega} s_{\omega_1} \otimes s_{\omega_2}, \tag{4.20}$$

which is dual to the product of monomials. Of course S^* is the Landweber-Novikov algebra. Alternatively, and following the original constructions, we may use the action of S^* on $\Omega^*_U(\wedge_\infty CP^\infty)$ to prove directly that S^* is a Hopf algebra. Many of our actions and coactions restrict to S^* and S_* and will be important below. We emphasize that A^*_U has no Ω^U_*-linear antipode, and that the antipode in S^* is induced from the antipode in S_* by $\mathbb{Z}$-duality.

Having identified A^*_U additively as $\Omega^U_* \otimes S^*$ and noted that both factors are subalgebras, the remaining multiplicative structure is determined by the *commutation rule* for products sx, where s and x lie in S^* and Ω^U_* respectively. Recalling (4.1), (4.2), and (4.12) we obtain

$$sx = \sum s_{1,r}(x)s_2, \tag{4.21}$$

and write the resulting algebra as $\Omega^U_* \# S^*$, where the right action of S^* on Ω^U_* is understood. This is an important case of the smash product [**24**], and an analogous algebra may be constructed from any Milnor module over a Hopf algebra; it is Novikov's *operator double* [**17**].

Choosing E and X to be MU in (4.13) and (4.14) provides the left and right action of A^*_U on its dual. If s and y lie in A^*_U and u in A^U_*, we have

$$\langle y, s_\ell(u)\rangle = \langle s, \chi(y_r(u))\rangle \qquad \text{and} \qquad \langle y, s_r(u)\rangle = \langle ys, u\rangle. \tag{4.22}$$

Alternatively, by appealing to (4.8) and (4.12) we may write

$$s_\ell u = \sum \langle s, \chi(u_1)\rangle u_2 \qquad \text{and} \qquad s_r u = \sum \langle s, u_2\rangle u_1. \tag{4.23}$$

By restriction we obtain identical formulae for the left and right actions of S^* on A^U_* and on S_*. In the latter case, $\mathbb{Z}$-duality allows us to rewrite the left action as

$$\langle y, s_\ell(u)\rangle = \langle \chi(s)y, u\rangle,$$

thereby (at last) according it equivalent status to the right action.

The *adjoint actions* of S^* on A^U_* and S_* are similarly defined by

$$\begin{aligned} \langle y, \mathrm{ad}(s)(u)\rangle = \sum \langle \chi(s_1)ys_2, u\rangle \qquad &\text{and} \\ \mathrm{ad}(s)(u) = \sum \langle \chi(s_1), u_1\rangle\langle s_2, u_3\rangle u_2&, \end{aligned} \tag{4.24}$$

which give rise to the adjoint Milnor module structure on A^U_* and S_*. By way of example, we combine Corollary 4.19 with (4.15) to produce

$$\begin{aligned} s_{\epsilon(k),\ell}(b_n) =&(k-n-1)b_{n-k}, \qquad s_{\epsilon(k),r}(b_n) = (b)^{k+1}_{n-k} \\ &\text{and} \qquad \mathrm{ad}(s_{\epsilon(k)})(b_n) = (k-n-1)b_{n-k} + (b)^{k+1}_{n-k} \end{aligned} \tag{4.25}$$

for all $0 \leqslant k \leqslant n$.

An alternative interpretation of S_* lies at the heart of our next section, and follows directly from Proposition 4.18 in the case $E = MU$.

Proposition 4.26. *The subalgebra G_* of Ω^{DU}_* is identified with the dual of the Landweber-Novikov algebra S_* in A^U_* under the canonical isomorphism.*

In more general cases we insist that the integral span of the e^α forms a subring S^E_* of A^E_*, which is invariant under χ_E and maps to $\mathbb{Z}$ under the counit. Therefore A^E_* is isomorphic to $E_* \otimes S^E_*$ as left E_*-algebras. The integral span of the e_α is the dual subcolagebra S^*_E of A^*_E, which admits the involution dual to χ_E, and features in the dual isomorphism between A^*_E and $E_* \otimes S^*_E$ as left E_*-coalgebras. We may interpret this decomposition as a weak form of smash product $E_* \# S^*_E$, since S^*_E acts on the coefficient ring E_* according to (4.1) and satisfies the Milnor condition (4.2); the only products not so defined are those internal to S^E_*. A typical example to bear in mind is BP, where S^{BP}_* is the polynomial algebra over $\mathbb{Z}_{(p)}$ on the $2(p^j-1)$-dimensional Adams generators t_j, and S^*_{BP} is the dual coalgebra spanned over $\mathbb{Z}_{(p)}$ by the Quillen operations r_α [**18**]. The subalgebra G^{BP}_* of $D(BP)_*$ is certainly *not* identified with S^{BP}_* under the canonical isomorphism i_{BP}.

5. Double cohomology operations

In this section we consider the algebra of cohomology operations $A^*_{D(E)}$, studying various subalgebras isomorphic to A^*_E and locating an additive subgroup which becomes the quantum double $\mathcal{D}(S^*)$ in the universal example DU. We conclude by investigating extensions of the partial endomorphism on A^*_{DU} provided by the antipode in $\mathcal{D}(S^*)$. Throughout the section we understand that D denotes an arbitrary double spectrum $D(E)$, in which E satisfies the conditions imposed in Section 4.

Since the Künneth formula identifies $E_*(D)$ with $E_*(E) \otimes_{E_*} E_*(E)$, the elements $(p^E)^\alpha$ in $E_*(E)$ give rise to elements $(p^E_\ell)^\alpha$ and $(p^E_r)^\alpha$ in $D_*(E)$, and therefore to four possible choices of generators for $D_*(D)$. These arise from the four sets of generators for $D_*(MU)$ given in (3.12), by applying the map induced in D-homology by the Thom class $t^D_\ell t^D_r$. We deduce that A^D_* is a quotient of the polynomial algebra $D_*[p^D_{j,\ell} \otimes 1, 1 \otimes p^D_{k,r} : j,k \geqslant 0]$, and that the algebra A^*_D of D-theory operations is the appropriate dual $D^*(D)$. These act and coact according to the eightfold way.

Since D is $E \wedge E$, an element s of A^*_E yields operations $s \wedge 1$ and $1 \wedge s$ in A^*_D, which commute by construction; in terms of (3.12) these correspond to $s_\ell \otimes 1$ and $1 \otimes s_r$ respectively, and are therefore consistent with our preferred choice of D_*-basis elements $e_{\alpha,\ell} \otimes 1$ and $1 \otimes e_{\alpha,r}$. We denote the subalgebras consisting of all $s_\ell \otimes 1$ and $1 \otimes s_r$ by $A^*_{E,\ell} \otimes 1$ and $1 \otimes A^*_{E,r}$ respectively, and refer to them as the *left* and *right copies* of A^*_E. They act on the coefficient groups D_* in accordance with (4.1), and feature in the D_*-coalgebra decomposition of A^*_D as $D_* \otimes S^*_{E,\ell} \otimes S^*_{E,r}$; since both actions are Milnor, we may interpret this decomposiiton as a weak form of smash product $D_* \# (S^*_{E,\ell} \otimes S^*_{E,r})$, by analogy with our previous description of A^*_E. These actions have an important alternative description in terms of (4.13) and (4.14), which follows immediately from the definitions.

Proposition 5.1. *The canonical isomorphism identifies the left and right actions of A^*_E on A^E_* with the actions of $A^*_{E,\ell}$ and $A^*_{E,r}$ on D_*, respectively.*

By restriction we obtain Milnor actions of the subcoalgebras $S^*_{E,\ell}$ and $S^*_{E,r}$ on D_*, but the image of S^E_* under i_E is not generally preserved. The coproduct $\delta_E\colon S^*_E \to S^*_{E,\ell}\otimes S^*_{E,r}$ describes a *diagonal copy* $S^*_{E,d}$ of S^*_E in A^*_D, which is a subcoalgebra by virtue of cocommutativity.

The D_*-duality between A^*_D and A^D_* confirms that the latter is isomorphic to $D_*\otimes S^{E,\ell}_*\otimes S^{E,r}_*$ as D_*-algebras, and restricts to the integral duality between $S^*_{E,\ell}\otimes S^*_{E,r}$ and $S^{E,\ell}_*\otimes S^{E,r}_*$.

In the universal case, $S^*_\ell\otimes S^*_r$ is actually a Hopf subalgebra of A^{DU}_*, whilst the dual Hopf algebra $S_{*,\ell}\otimes S_{*,r}$ is a sub-Hopf algebra of the Hopf algebroid A^{DU}_*, and is isomorphic to $\mathbb{Z}[b_{j,\ell}\otimes b_{k,r} : j,k\geqslant 0]$. In consequence, A^*_{DU} is the genuine operator double $\Omega^{DU}_*\,\#\,(S^*_\ell\otimes S^*_r)$, and the actions of S^*_ℓ and S^*_r on Ω^{DU}_* are already familiar, as shown by the following extension of Proposition 5.1.

Proposition 5.2. *The subalgebra G_* of Ω^{DU}_* is closed with respect to the actions of S^*_ℓ and S^*_r; in particular, the action of the diagonal subalgebra S^*_d is identified with the adjoint action of S^* on S_* under the canonical isomorphism.*

Proof. Combining Proposition 4.26 with the fact that S^* is multiplicatively closed, we identify the left and right actions of S^* on S_* (as in (4.22)) with the restrictions to G_* of the respective actions of S^*_ℓ and S^*_r on Ω^{DU}_*. The result then follows from the dual fact that S_* is comultiplicatively closed. □

Corollary 5.3. *The quantum double $\mathcal{D}(S^*)$ is isomorphic to a subalgebra of A^*_{DU} [**5**]: as such, the universal R matrix is given by $\sum_\omega(s_{\omega_1,\ell}\otimes s_{\omega_2,r})\otimes g^\omega$ in $A^*_{DU}\otimes A^*_{DU}$; the square of the antipode acts as conjugation by the element $\sum_\omega \bar{b}^\omega\,\#\,(s_{\omega_1,\ell}\otimes s_{\omega_2,r})$; the ring $\Omega^*_{DU}(X)$ is a crossed S^*-bimodule for any space X; and solutions V_X to the Yang-Baxter equations are given by endomorphisms of the form $V_X(x\otimes y)=\sum_\omega b^\omega y\otimes(s_{\omega_1,\ell}\otimes s_{\omega_2,r})(x)$ of $\Omega^*_{DU}(X)\otimes\Omega^*_{DU}(X)$*

.

Proof. Since S^* is cocommutative we may construct $\mathcal{D}(S^*)$ as the operator double $S_*\,\#\,S^*$, with respect to the adjoint action [**16**], [**17**]. By Proposition 5.2, the latter is isomorphic to the subalgebra $G_*\,\#\,S^*_d$ of $\Omega^{DU}_*\,\#\,(S^*_\ell\otimes S^*_r)$, which we have already identified as A^*_{DU}. The remaining facts then follow directly from the definitions; for terminology, we refer to [**12**]. □

Both Hopf algebras $S^*_\ell\otimes S^*_r$ and $\mathcal{D}(S^*)$ are subalgebras of A^*_{DU}, and also of the operator double $G^*\,\#\,(S^*_\ell\otimes S^*_r)$; this is not itself a Hopf algebra, however, because the action of $S^*_\ell\otimes S^*_r$ on G_* is not a coalgebra map [**12**]. Nevertheless, it has a considerable amount of algebraic structure which extends that of $\mathcal{D}(S^*)$, and gives rise to left Ω^U_*-linear structure maps on A^*_{DU}. We briefly consider this situation with respect to antipodes.

Suppose we are given Hopf algebras H and T with bijective antipodes χ_H and χ_T respectively, and a Milnor action of T on H. Then the algebra $H\,\#\,T$ has an additive basis of elements $h\,\#\,t$, and we may define a unique linear automorphism θ on $H\,\#\,T$ by

$$\theta(h\,\#\,t)=\chi_T(t)\cdot\chi_H(h); \tag{5.4}$$

if T is cocommutative and the Milnor coaction is also a coalgebra map, then $H\,\#\,T$ is a Hopf algebra, and θ is its antipode [**12**]. This analysis applies to $\mathcal{D}(S^*)$ by taking $H=S_*$ and $T=S^*$ with respect to the adjoint action, and shows that the antipode is given by

$$\chi_{\mathcal{D}(S^*)}(b\,\#\,s)=\sum\langle\chi(s_3),b_3\rangle\langle s_2,b_1\rangle\chi(b_2)\,\#\,\chi(s_1). \tag{5.5}$$

Theorem 5.6. *We may extend $\chi_{\mathcal{D}(S^*)}$ to a left Ω^U_*-automorphism of A^*_{DU}; in particular, there is an extension θ satisfying*

$$\theta(g\,\#\,s)=\theta(s)\cdot\theta(g)\quad\textit{and}\quad\theta(s\cdot g)=\sum\theta(s_2)(\theta(s_1)(g))\,\#\,\theta(s_3)$$

and an extension θ' satisfying

$$\theta'(g\,\#\,s)=\sum\theta'(\theta'(s_1)(g))\,\#\,\theta'(s_2)\quad\textit{and}\quad\theta'(s\cdot g)=\theta'(g)\,\#\,\theta'(s),$$

where $g \# s$ is a generator of $G^* \# (S_\ell^* \otimes S_r^*)$.

Proof. We first translate (5.5) into A_{DU}^* by Corollary 5.3, and let θ be given by (5.4) in the case $H = G_*$ and $T = S_\ell^* \otimes S_r^*$; the formula for $\theta(s \cdot g)$ then follows by direct calculation. To obtain θ', we assume that it is antimultiplicative on products $s \cdot g$ and deduce the formula for $\theta'(g \# s)$ by direct calculation. Since the terms $g \# s$ generate A_{DU}^* as a left Ω_*^U-module, these observations suffice. □

The point about θ and θ' is that neither is antimultiplicative on *all* products.

We conclude with two interrelated remarks concerning the detailed structure of A_{DU}^*; first we express the alternative choices (3.12) for Ω_*^{DU}-bases in terms of the preferred basis, and then we consider the description of the involutory operation χ defined by switching the factors of $MU \wedge MU$.

Proposition 5.7. *In A_{DU}^*, we may express $s_{\omega,r} \otimes 1$ and $1 \otimes s_{\omega,\ell}$ as*

$$\sum_\psi g^\psi s_{\psi,\ell} \circ s_{\omega,\ell} \otimes 1 \qquad \text{and} \qquad \sum_\psi \bar{g}^\psi \otimes s_{\psi,r} \circ s_{\omega,r}$$

respectively, in terms of the preferred Ω_*^{DU}-basis.

Proof. Applying Lemma 3.10 in $DU^0(MU)$, we obtain

$$s_\omega^*(t_r) = s_\omega^*\big(\sum_\psi g^\psi s_{\psi,\ell}\big) t_r$$

in $D^0(MU)$, and the first equation follows from the Künneth formula. For the second equation, we apply χ. □

Proposition 5.8. *The involution χ is given by*

$$\sum_{\psi,\omega} g^\psi \bar{g}^\omega s_{\psi,\ell} \otimes s_{\omega,r}$$

as an element of A_{DU}^0.

Proof. The Thom class t in A_{DU}^0 corresponds to $t_\ell \otimes t_r$ in $DU^0(MU) \otimes DU^0(MU)$ under the Künneth formula. Since χ switches t_ℓ and t_r and is determined by its action on t, it suffices to write $t_r \otimes t_\ell$ in terms of $t_\ell \otimes t_r$. By Lemma 3.10 we obtain $\big(\sum_\psi g^\psi s_{\psi,\ell}\big) t_\ell \otimes \big(\sum_\omega \bar{g}^\omega s_{\omega,r}\big) t_r$, as required. □

In the context of (3.6), χ gives rise to the elements g_n which extend Ω_*^U to Ω_*^{DU}; it also represents the permutation group $\mathfrak{S}_2$ in A_{DU}^*.

Propositions 5.2, 5.7, and 5.8 and Theorem 5.6 have direct analogues for those spectra E and $D(E)$ for which $i_E(S_*^E)$ coincides with the subalgebra G_*^E of $D(E)_*$. More generally, the best we can do is to identify the sub-left E_*-module $S_*^E \otimes S_{E,d}^*$ as some sort of weak double for S_E^*, and consider the associated antipode as an antiautomorphism. In these cases the relationship between t_ℓ^D and t_r^D is governed by that between the bases $e_{\alpha,\ell}$ and $e_{\alpha,r}$.

6. Further developments

The quantum algebraic viewpoint highlights novel aspects of A_E^* and A_*^E, and we conclude with a few examples which we shall develop elsewhere [**7**]. We restrict attention to universal cases, and refer to [**16**] for terminology.

We may interpret the dual of the Landweber-Novikov algebra in many ways. For example, the projection $t_*^H \colon \Omega_*^U(MU) \to H_*(MU)$ restricts to an isomorphism on S_*, with $t_*^H(b^\omega) = (b^H)^\omega$. This isomorphism is often implicit in the literature, and invests $H_*(MU)$

with a coproduct and antipode which are purely algebraic; its dual identifies $H^*(MU)$ with the Landweber-Novikov algebra itself. The Hurewicz homomorphism $\Omega_*^U \to H_*(MU)$ therefore realizes A_U^* as a subalgebra of the *Heisenberg double* $S_* \# S^*$ (often written $\mathcal{H}(S^*)$) where S^* acts on its dual by the right action (4.23).

The appearance of $\mathcal{D}(S^*)$ as an algebra of double cobordism operations is actually part of a decomposition theorem, which describes A_{DU}^* as an appropriate smash product of A_U^* and $\mathcal{D}(S^*)$. More generally, for any $N \geqslant 2$ we may introduce *N-fold cobordism theory* $\Omega_*^{NU}(\ )$, represented by the N-fold smash product spectrum $MU^{\wedge N}$; its algebra of cohomology operations A_{NU}^* has many different subalgebras isomorphic to $\mathcal{D}(S^*)$, which contribute to a decomposition theorem expressing A_{NU}^* as an *iterated smash product* of A_U^* and $N-1$ copies of $\mathcal{D}(S^*)$. When N is sufficiently large, A_{NU}^* contains a remarkable array of subalgebras and Hopf subalgebras, such as the dual of $\mathcal{D}(S^*)$ and the quantum and Heisenberg doubles of tensor powers of S^*.

The Hopf algebroids A_*^{NU} decompose dually, as smash *co*products.

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