Homology, Homotopy and Applications, vol.5(1), 2003, pp.387–406

Dedicated to Professor Hvedri Inassaridze
on the occasion of his seventieth birthday

EXTENSIONS OF SEMIMODULES AND THE TAKAHASHI FUNCTOR $\mathrm{Ext}_{\Lambda}(C, A)$

ALEX PATCHKORIA

(*communicated by George Janelidze*)

Abstract

Let Λ be a semiring with 1. By a Takahashi extension of a Λ-semimodule X by a Λ-semimodule Y we mean an extension of X by Y in the sense of M. Takahashi [**10**]. Let A be an arbitrary Λ-semimodule and C a Λ-semimodule which is normal in Takahashi's sense, that is, there exist a projective Λ-semimodule P and a surjective Λ-homomorphism $\varepsilon : P \longrightarrow C$ such that ε is a cokernel of the inclusion $\mu : \mathrm{Ker}(\varepsilon) \hookrightarrow P$. In [**11**], following the construction of the usual satellite functors, M. Takahashi defined $\mathrm{Ext}_{\Lambda}(C, A)$ by

$$\mathrm{Ext}_{\Lambda}(C, A) = \mathrm{Coker}(\mathrm{Hom}_{\Lambda}(\mu, A))$$

and used it to characterize Takahashi extensions of normal Λ-semimodules by Λ-modules.

In this paper we relate $\mathrm{Ext}_{\Lambda}(C, A)$ with other known satellite functors of the functor $\mathrm{Hom}_{\Lambda}(-, A)$.

Section 1 is concerned with preliminaries. The purpose of Section 2 is to characterize $\mathrm{Ext}_{\Lambda}(C, A)$ in terms of Janelidze's general $\mathrm{Ext}^n_{\mathscr{C}}$-functors [**5**]. In Section 3 we show that $\mathrm{Ext}_{\Lambda}(C, A)$ with A cancellative can be described directly by Takahashi extensions of C by A. The last section is devoted to $\mathrm{Ext}_{\Lambda}(C, G)$ with G a Λ-module. We relate $\mathrm{Ext}_{\Lambda}(C, G)$ with Inassaridze extensions of C by G [**4**]. This allows to relate $\mathrm{Ext}_{\Lambda}(C, G)$ and $S^1 \mathrm{Hom}_{K(\Lambda)}(-, G)(K(C))$, where $K(\Lambda)$ is the Grothendieck ring of Λ, $K(C)$ the Grothendieck $K(\Lambda)$-module of C, and $S^1 \mathrm{Hom}_{K(\Lambda)}(-, G)$ the usual right satellite functor of the functor $\mathrm{Hom}_{K(\Lambda)}(-, G)$.

Partially supported by INTAS grants 97-31961 and 00-566, and grant No 1.5 of the Georgian Academy of Sciences.
Received July 7, 2003, revised September 26, 2003; published on October 1, 2003.
2000 Mathematics Subject Classification: 18G99, 18E25, 16Y60, 20M50.
Key words and phrases: Semiring, semimodule, projective semimodule, normal semimodule, extension of semimodule, satellite functor.

1. There are several concepts of semirings and semimodules (see for example, [**2,3,9**]). In this paper we use the following ones. A semiring $\Lambda = (\Lambda, +, 0, \cdot, 1)$ is an algebraic structure in which $(\Lambda, +, 0)$ is an abelian monoid, $(\Lambda, \cdot, 1)$ a monoid, and

$$\begin{aligned}
&\lambda \cdot (\lambda' + \lambda'') = \lambda \cdot \lambda' + \lambda \cdot \lambda'',\\
&(\lambda' + \lambda'') \cdot \lambda = \lambda' \cdot \lambda + \lambda'' \cdot \lambda,\\
&\lambda \cdot 0 = 0 \cdot \lambda = 0,
\end{aligned}$$

for all $\lambda, \lambda', \lambda'' \in \Lambda$. An abelian monoid $A = (A, +, 0)$ together with a map $\Lambda \times A \longrightarrow A$, written as $(\lambda, a) \longmapsto \lambda a$, is called a (left) Λ-semimodule if

$$\begin{aligned}
&\lambda(a + a') = \lambda a + \lambda a',\\
&(\lambda + \lambda')a = \lambda a + \lambda' a,\\
&(\lambda \cdot \lambda')a = \lambda(\lambda' a),\\
&1a = a, \quad 0a = 0,
\end{aligned}$$

for all $\lambda, \lambda' \in \Lambda$ and $a, a' \in A$. It immediately follows that $\lambda 0 = 0$ for any $\lambda \in \Lambda$.

Let us also recall:

A map $f : A \longrightarrow B$ between Λ-semimodules A and B is called a Λ-homomorphism if $f(a + a') = f(a) + f(a')$ and $f(\lambda a) = \lambda f(a)$, for all $a, a' \in A$ and $\lambda \in \Lambda$. It is obvious that any Λ-homomorphism carries 0 into 0. The abelian monoid of all Λ-homomorphisms from A to B is denoted by $\mathrm{Hom}_\Lambda(A, B)$. (Example: Let N be the semiring of non-negative integers. An N-semimodule A is simply an abelian monoid, and an N-homomorphism $f : A \longrightarrow B$ is just a homomorphism of abelian monoids.)

A Λ-subsemimodule A of a Λ-semimodule B is a subsemigroup of $(B, +)$ such that $\lambda a \in A$ for all $a \in A$ and $\lambda \in \Lambda$. Clearly $0 \in A$. The quotient Λ-semimodule B/A is defined as the quotient Λ-semimodule of B by the smallest congruence on the Λ-semimodule B some class of which contains A. Denote the congruence class of $b \in B$ by $[b]$. Then $[b_1] = [b_2]$ if and only if $a_1 + b_1 = a_2 + b_2$ for some $a_1, a_2 \in A$. The Λ-homomorphism $p : B \longrightarrow B/A$ that carries $b \in B$ into $[b]$ is called the canonical surjection.

A Λ-semimodule A is cancellative if $a + a' = a + a''$ for $a, a', a'' \in A$ implies $a' = a''$. Obviously, A is a cancellative Λ-semimodule if and only if A is a cancellative $C(\Lambda)$-semimodule, where $C(\Lambda)$ denotes the largest cancellative homomorphic image of Λ under addition. A Λ-semimodule A is called a Λ-module if $A = (A, +, 0)$ is an abelian group. It is clear that A is a Λ-module if and only if A is a $K(\Lambda)$-module, where $K(\Lambda)$ denotes the Grothendieck ring of Λ.

The categories of Λ-semimodules, cancellative Λ-semimodules, Λ-modules, abelian monoids, abelian groups, and sets are denoted by Λ-**SMod**, Λ-**CSMod**, Λ-**Mod**, **Abm**, **Ab**, and **Set**, respectively.

A cokernel of a Λ-homomorphism $f : A \longrightarrow B$ is defined to be a Λ-homomorphism $u : B \longrightarrow C$ such that (i) $uf = 0$, and (ii) for any Λ-homomorphism $g : B \longrightarrow D$ with $gf = 0$ there is a unique Λ-homomorphism $g' : C \longrightarrow D$

with $g = g'u$. One dually defines a kernel of f. Clearly, the canonical projection $p : B \longrightarrow B/f(A)$ is a cokernel of f, and the inclusion $\mathrm{Ker}(f) \hookrightarrow A$, where $\mathrm{Ker}(f) = \{a \in A | f(a) = 0\}$, is a kernel of f.

A sequence $E : A \overset{\lambda}{\rightarrowtail} B \overset{\tau}{\twoheadrightarrow} C$ of Λ-semimodules and Λ-homomorphisms is called a *short exact sequence* if λ is injective, τ is surjective, and $\lambda(A) = \mathrm{Ker}(\tau)$ (cf. [**9**]). The following assertion is plain and well-known.

Proposition 1.1. *If* $E : A \rightarrowtail B \twoheadrightarrow C$ *is a short exact sequence, then* B *is a* Λ*-module if and only if* A *and* C *are both* Λ*-modules.*

A *morphism* from $E : A \overset{\lambda}{\rightarrowtail} B \overset{\tau}{\twoheadrightarrow} C$ to $E' : A' \overset{\lambda'}{\rightarrowtail} B' \overset{\tau'}{\twoheadrightarrow} C'$ is a triple of Λ-homomorphisms (α, β, γ) such that

$$\begin{tikzcd} E: & A \arrow[r, tail, "\lambda"] \arrow[d, "\alpha"'] & B \arrow[r, two heads, "\tau"] \arrow[d, "\beta"'] & C \arrow[d, "\gamma"'] \\ E': & A' \arrow[r, tail, "\lambda'"] & B' \arrow[r, two heads, "\tau'"] & C' \end{tikzcd}$$

is a commutative diagram. For a morphism of the form

$$\begin{tikzcd} E: & A \arrow[r, tail] \arrow[d, equal, "1_A"'] & B \arrow[r, two heads] \arrow[d, "\beta"'] & C \arrow[d, equal, "1_C"'] \\ E': & A \arrow[r, tail] & B' \arrow[r, two heads] & C, \end{tikzcd}$$

we write $E \succ E'$. If in addition β is a Λ-isomorphism, we write $E \equiv E'$ and say that E is *equivalent* to E'.

Next, suppose given a short exact sequence $E : A \overset{\lambda}{\rightarrowtail} B \overset{\tau}{\twoheadrightarrow} C$ and a Λ-homomorphism $\gamma : C' \longrightarrow C$. Then

$$E\gamma : A \overset{\lambda^{\gamma}}{\rightarrowtail} B^{\gamma} \overset{\tau^{\gamma}}{\twoheadrightarrow} C',$$

where $B^{\gamma} = \{(b, c') \in B \oplus C' | \tau(b) = \gamma(c')\}$, $\lambda^{\gamma}(a) = (\lambda a, 0)$, $\tau^{\gamma}(b, c') = c'$, is a short exact sequence of Λ-semimodules. Besides, if one defines a Λ-homomorphism $\xi^{\gamma} : B^{\gamma} \longrightarrow B$ by $\xi^{\gamma}(b, c') = b$, then $(1_A, \xi^{\gamma}, \gamma)$ is a morphism from $E\gamma$ to E. From the construction of $E\gamma$ it follows that

$$E \equiv E\,1_C, \quad (E\gamma)\gamma' \equiv E(\gamma\gamma'), \tag{1.2}$$

$$E \equiv E' \Longrightarrow E\gamma \equiv E'\gamma, \tag{1.3}$$

$$E \succ E' \Longrightarrow E\gamma \succ E'\gamma. \tag{1.4}$$

We will also use sequences of the form $S : A \overset{f}{\longleftarrow} X \rightarrowtail Y \twoheadrightarrow C$, where $f : X \longrightarrow A$ is a Λ-homomorphism and $E : X \rightarrowtail Y \twoheadrightarrow C$ a short exact sequence of Λ-semimodules. It will be convenient to denote S by $f \circ E$, and E by $\overline{S}$.

A surjective Λ-homomorphism $\tau : B \longrightarrow C$ is said to be a *normal* Λ*-epimorphism* if it is a cokernel of the inclusion $\mathrm{Ker}(\tau) \hookrightarrow B$. One can easily see that τ is normal if and only if it is kernel-regular in the sense of [**9**]: if $\tau(b_1) = \tau(b_2)$, then $k_1 + b_1 = k_2 + b_2$ for some k_1, k_2 in $\mathrm{Ker}(\tau)$.

Proposition 1.5. *Any surjective Λ-homomorphism $\tau : B \longrightarrow H$ with H a Λ-module is normal.*

Proof. Suppose $\tau(b_1) = \tau(b_2)$. Take $b \in B$ with $\tau(b) = -\tau(b_1)$. Then $(b_2 + b)$, $(b_1 + b) \in \mathrm{Ker}(\tau)$ and $(b_2 + b) + b_1 = (b_1 + b) + b_2$. □

Note also that for any Λ-subsemimodule A of a Λ-semimodule B, the canonical projection $p : B \longrightarrow B/A$ is normal.

Let A and C be Λ-semimodules. By a *Takahashi* (or *normal*) *extension* of C by A we mean an extension of C by A in the sense of [**10**], that is, a short exact sequence $E : A \rightarrowtail^{\lambda} B \xrightarrow{\tau} \!\!\!\!\rightarrow C$ of Λ-semimodules with τ normal. Clearly, a short exact sequence of Λ-semimodules $A \rightarrowtail B \twoheadrightarrow H$ with H a Λ-module and a sequence of the form $\mathrm{Ker}(p) \hookrightarrow B \xrightarrow{p} B/A$ provide examples of Takahashi extensions. (Note that in general $\mathrm{Ker}(p) \neq A$.) Let $\mathscr{T}$ denote the class of all Takahashi extensions of Λ-semimodules. Then

$$E \in \mathscr{T} \Longrightarrow E\gamma \in \mathscr{T}, \tag{1.6}$$

$$E, E' \in \mathscr{T} \Longrightarrow E \oplus E' \in \mathscr{T}. \tag{1.7}$$

Here $E \oplus E'$ denotes $A \oplus A' \xrightarrow{\lambda\oplus\lambda'} B \oplus B' \xrightarrow{\tau\oplus\tau'} C \oplus C'$, the usual direct sum of E and E'. Two extensions E_1 and E_2 are *equivalent* if $E_1 \equiv E_2$, i.e., if they are equivalent as short exact sequences. Following [**10**], we denote by $E_\Lambda(C, A)$ the set of equivalence classes of Takahashi extensions of C by A. It contains at least the 0, the class of

$$0 : A \xrightarrow{i_A} A \oplus C \xrightarrow{\pi_C} C,$$

where $i_A(a) = (a, 0)$ and $\pi_C(a, c) = c$.

A Λ-semimodule P is *projective* if it satisfies the usual lifting property: Given a surjective Λ-homomorphism $\tau : B \longrightarrow C$ and a Λ-homomorphism $f : P \longrightarrow C$, there is a Λ-homomorphism $g : P \longrightarrow B$ such that $f = \tau g$.

Proposition 1.8. *Let $\tau : B \longrightarrow C$ be a normal Λ-epimorphism and let $f_1, f_2 : P \longrightarrow B$ be Λ-homomorphisms with P projective. If $\tau f_1 = \tau f_2$ then there exist Λ-homomorphisms $g_1, g_2 : P \longrightarrow B$ satisfying $\tau g_1 = 0 = \tau g_2$ and $g_1 + f_1 = g_2 + f_2$. That is, the functor $\mathrm{Hom}_\Lambda(P, -)$ preserves normal epimorphisms.*

This fact, proved in [**1**] (and first mentioned in [**8**]), implies

Proposition 1.9 (cf. [11]). *Suppose given a diagram of Λ-semimodules and Λ-ho-*

momorphisms

$$\begin{tikzcd} A' \arrow[r, "\lambda'"] \arrow[d, shift right, "\varphi_1"'] \arrow[d, shift left, "\varphi_2"] & P \arrow[r, "\tau'"] \arrow[dl, dashed, shift right, "g_1"'] \arrow[dl, dashed, shift left, "g_2"] \arrow[d, shift right, "\psi_1"'] \arrow[d, shift left, "\psi_2"] & C' \arrow[d, "\gamma"] \\ A \arrow[r, tail, "\lambda"] & B \arrow[r, two heads, "\tau"] & C, \end{tikzcd}$$

where the bottom row is a Takahashi extension, P is projective, $\tau'\lambda' = 0$, and $\tau\psi_i = \gamma\tau'$, $\lambda\varphi_i = \psi_i\lambda'$ for $i = 1, 2$. Then there are Λ-homomorphisms $g_1, g_2 : P \longrightarrow A$ such that $g_1\lambda' + \varphi_1 = g_2\lambda' + \varphi_2$.

A Λ-semimodule C is called *normal* if there exist a projective Λ-semimodule P and a normal Λ-epimorphism $\varepsilon : P \longrightarrow C$ [**11**]. In other words, C is normal if there is a Takahashi extension of Λ-semimodules $R \rightarrowtail P \twoheadrightarrow C$ with P projective, called a *projective presentation* of C. It follows from Proposition 1.5 that every Λ-module H is normal, since one has a free Λ-semimodule F and a surjective Λ-homomorphism $F \longrightarrow H$. Any quotient Λ-semimodule P/A of a projective Λ-semimodule P is also normal. Moreover, since the class of normal epimorphisms of Λ-semimodules is closed under composition, any quotient Λ-semimodule B/A of a normal Λ-semimodule B is normal [**11**]. We denote the category of normal Λ-semimodules and their Λ-homomorphisms by Λ-**NSMod**.

In [**11**] M. Takahashi has constructed $\mathrm{Ext}_\Lambda(C, A)$ as follows. Let (C, A) be an object of $(\Lambda\text{-}\mathbf{NSMod})^{\mathrm{op}} \times (\Lambda\text{-}\mathbf{SMod})$. Choose a projective presentation $\mathbb{P} : R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} C$ of C and define $\mathrm{Ext}_\Lambda(C, A)$ to be $\mathrm{Coker}(\mathrm{Hom}_\Lambda(\mu, A) : \mathrm{Hom}_\Lambda(P, A) \longrightarrow \mathrm{Hom}_\Lambda(R, A))$. That is,

$$\mathrm{Ext}_\Lambda(C, A) = \mathrm{Hom}_\Lambda(R, A)/\,\mathrm{Hom}_\Lambda(\mu, A)(\mathrm{Hom}_\Lambda(P, A)).$$

If $\alpha : A \longrightarrow A'$ is a homomorphism of Λ-semimodules, one defines $\mathrm{Ext}_\Lambda(C, \alpha) : \mathrm{Ext}_\Lambda(C, A) \longrightarrow \mathrm{Ext}_\Lambda(C, A')$ by $\mathrm{Ext}_\Lambda(C, \alpha)([\varphi]) = [\alpha\varphi]$. Obviously, $\mathrm{Ext}_\Lambda(C, \alpha)$ is well defined. Next, any homomorphism $\gamma : C' \longrightarrow C$ of normal Λ-semimodules can be lifted to a morphism

$$\begin{tikzcd} \mathbb{P}' : & R' \arrow[r, tail, "\mu'"] \arrow[d, "f"'] & P' \arrow[r, two heads, "\varepsilon'"] \arrow[d, "g"'] & C' \arrow[d, "\gamma"'] \\ \mathbb{P} : & R \arrow[r, tail, "\mu"] & P \arrow[r, two heads, "\varepsilon"] & C, \end{tikzcd}$$

and $\mathrm{Ext}_\Lambda(\gamma, A) : \mathrm{Ext}_\Lambda(C, A) \longrightarrow \mathrm{Ext}_\Lambda(C', A)$ is defined by $\mathrm{Ext}_\Lambda(\gamma, A)([\varphi]) = [\varphi f]$. It follows from Proposition 1.9 that $\mathrm{Ext}_\Lambda(\gamma, A)$ is also well defined. Now one can easily see that $\mathrm{Ext}_\Lambda(C, A)$ is a functor from the category $(\Lambda\text{-}\mathbf{NSMod})^{\mathrm{op}} \times (\Lambda\text{-}\mathbf{SMod})$ to the category **Abm** ($[\varphi] \longmapsto [\alpha\varphi]$, $[\varphi] \longmapsto [\varphi f]$), additive in both its arguments. Further, Proposition 1.9 implies that a different choice of the projective presentations would yield a new functor $\widetilde{\mathrm{Ext}}_{\Lambda}(C,A)$ which is naturally isomorphic to the functor $\mathrm{Ext}_{\Lambda}(C,A)$.

It is evident that if $T:\Lambda\text{-}\mathbf{Mod} \longrightarrow \mathbf{Abm}$ is an additive functor, i.e., $T(f+g) = T(f)+T(g)$ and $T(0)=0$, then $T(G)$ is an abelian group for any Λ-module G. Therefore, $\mathrm{Ext}_{\Lambda}(C,A)$ is an abelian group whenever either A or C is a Λ-module.

Next, the Grothendieck functor K carries any short exact sequence of Λ-semimodules $X \rightarrowtail Y \twoheadrightarrow M$ with M a Λ-module into a short exact sequence $K(X) \rightarrowtail K(Y) \twoheadrightarrow M$ of $K(\Lambda)$-modules. Therefore, if $\mathbb{Q}: V \rightarrowtail Q \twoheadrightarrow H$ is a Λ-projective presentation of a Λ-module H, then $K(\mathbb{Q}): K(V) \rightarrowtail K(Q) \twoheadrightarrow H$ is a $K(\Lambda)$-projective presentation of H. Consequently, for any Λ-semimodule A, one has a natural homomorphism

$$\mathrm{Ext}_{\Lambda}(H,A) \xrightarrow{K(H,A)} S^1\,\mathrm{Hom}_{K(\Lambda)}(-,K(A))(H)\,,\ K(H,A)([\varphi]) = [K(\varphi)], \tag{1.10}$$

where $S^1\,\mathrm{Hom}_{K(\Lambda)}(-,K(A))$ denotes the usual right satellite functor of the functor $\mathrm{Hom}_{K(\Lambda)}(-,K(A))$. A straightforward verification shows that $K(H,A)$ is one-to-one whenever A is cancellative. Furthermore, it immediately follows from the universal property of K that

$$K(H,G): \mathrm{Ext}_{\Lambda}(H,G) \longrightarrow S^1\,\mathrm{Hom}_{K(\Lambda)}(-,G)(H) \tag{1.11}$$

ia an isomorphism for any Λ-modules G and H.

2. In [**5**] G. Janelidze introduced and studied general $\mathrm{Ext}^n_{\mathscr{C}}$-functors, where $\mathscr{C}$ is an arbitrary class of diagrams of the form $X \longrightarrow Y \longrightarrow Z$ in an arbitrary category. Suppose $n=1$ and $\mathscr{C} = \mathscr{T}$, the class of all Takahashi extensions of Λ-semimodules. Then $\mathrm{Ext}^1_{\mathscr{T}}(C,A)$ is a functor from the category $(\Lambda\text{-}\mathbf{SMod})^{\mathrm{op}} \times (\Lambda\text{-}\mathbf{SMod})$ to the category $\mathbf{Abm}$. In this section we prove that the restriction of $\mathrm{Ext}^1_{\mathscr{T}}(C,A)$ to the category $(\Lambda\text{-}\mathbf{NSMod})^{\mathrm{op}} \times (\Lambda\text{-}\mathbf{SMod})$ is naturally isomorphic to the functor $\mathrm{Ext}_{\Lambda}(C,A)$.

The functor $\mathrm{Ext}^1_{\mathscr{T}}(C,A)$, denoted by $\mathrm{Ext}_{\Lambda}\,JT(C,A)$ in this paper, is defined as follows. Let $\mathcal{E}xt_{\Lambda}\,JT(C,A)$ be the category of sequences of the form $S = f \circ \overline{S}$: $A \xleftarrow{f} X \rightarrowtail Y \twoheadrightarrow C$, where f is a Λ-homomorphism and $\overline{S}: X \rightarrowtail Y \twoheadrightarrow C$ a Takahashi extension of Λ-semimodules. Define $\mathrm{Ext}_{\Lambda}\,JT(C,A)$ to be the set of connected components of $\mathcal{E}xt_{\Lambda}\,JT(C,A)$, that is,

$$\mathrm{Ext}_{\Lambda}\,JT(C,A) = \mathcal{E}xt_{\Lambda}\,JT(C,A)/\sim,$$

where $\sim$ is the smallest equivalence relation under which $S: A \longleftarrow X \rightarrowtail Y \twoheadrightarrow C$ is equivalent to $S': A \longleftarrow X' \rightarrowtail Y' \twoheadrightarrow C$ whenever there exists a commutative diagram of the form

$$\begin{tikzcd} S: A & X \arrow[l] \arrow[r, tail] \arrow[d, "\varphi"'] & Y \arrow[r, two heads] \arrow[d, "\psi"'] & C \arrow[d, equal] \\ S': A \arrow[u, equal] & X' \arrow[l] \arrow[r, tail] & Y' \arrow[r, two heads] & C. \end{tikzcd}$$

Further, for any Λ-homomorphisms $\alpha : A \longrightarrow A'$ and $\gamma : C' \longrightarrow C$ and any object $S : A \xleftarrow{f} X \rightarrowtail Y \twoheadrightarrow C$ of $\mathcal{E}xt_{\Lambda} JT(C, A)$, define αS and $S\gamma$ by

$$\alpha S = \alpha f \circ \overline{S} \quad \text{and} \quad S\gamma = f \circ \overline{S}\gamma, \tag{2.1}$$

respectively. From (1.2), (1.6) and the fact that the above commutative diagram induces the morphism $(\varphi, \psi^{\gamma}, 1_{C'}) : \overline{S}\gamma \longrightarrow \overline{S}'\gamma$, $\psi^{\gamma}(y, c') = (\psi(y), c')$, it follows that these operations make $\mathrm{Ext}_{\Lambda} JT(C, A)$ a functor from $(\Lambda\text{-}\mathbf{SMod})^{\mathrm{op}} \times (\Lambda\text{-}\mathbf{SMod})$ to **Set**. Next, $(\, 0 \longrightarrow Z = Z\,) \in \mathscr{T}$ for every Λ-semimodule Z. This together with (1.7) implies that $\mathrm{Ext}_{\Lambda} JT(C,-)$ preserves all finite products [**5**]. Therefore $\mathrm{Ext}_{\Lambda} JT(C, A)$ is actually an abelian monoid-valued functor, additive in both its arguments. Observe that the addition in $\mathrm{Ext}_{\Lambda} JT(C, A)$ can be described by

$$cl(S) + cl(S') = cl(\triangledown_A (S \oplus S') \triangle_C),$$

where $\triangledown_A : A \oplus A \longrightarrow A$ and $\triangle_C : C \longrightarrow C \oplus C$ are the codiagonal and diagonal maps, respectively. The class of $0 = 1_A \circ 0 : A = A \xrightarrow{i_A} A \oplus C \xrightarrow{\pi_C} C$ coincides with the class of $A \longleftarrow 0 \longrightarrow C = C$ and serves as a neutral element, i.e., $cl(S) + cl(1_A \circ 0) = cl(S)$.

For every Takahashi extension $F : K \rightarrowtail L \twoheadrightarrow M$ and every Λ-semimodule A, one has a natural connecting homomorphism of abelian monoids

$$\delta(F, A) : \mathrm{Hom}_{\Lambda}(K, A) \longrightarrow \mathrm{Ext}_{\Lambda} JT(M, A)$$

defined by

$$\delta(F, A)(f : K \longrightarrow A) = cl(A \xleftarrow{f} K \rightarrowtail L \twoheadrightarrow M)\ .$$

If L is a projective Λ-semimodule, than $\delta(F, A)$ is surjective [**6**]. Indeed, in this case any object $S : A \xleftarrow{g} X \rightarrowtail Y \twoheadrightarrow M$ of $\mathcal{E}xt_{\Lambda} JT(M, A)$ admits a commutative diagram

$$\begin{tikzcd} g\varphi \circ F: A & K \arrow[l, "g\varphi"'] \arrow[r, tail] \arrow[d, "\varphi"'] & L \arrow[r, two heads] \arrow[d, "\psi"'] & M \arrow[d, equal] \\ S: A \arrow[u, equal] & X \arrow[l, "g"'] \arrow[r, tail] & Y \arrow[r, two heads] & M, \end{tikzcd}$$

i.e., $\delta(F, A)(g\varphi) = cl(S)$.

Proposition 2.2. *Let C be a normal Λ-semimodule and $\mathbb{P} : R \xrightarrow{\mu} P \xrightarrow{\varepsilon} C$ a* *projective presentation of* C. *Then* $\delta(\mathbb{P}, A) : \mathrm{Hom}_\Lambda(R, A) \longrightarrow \mathrm{Ext}_\Lambda\, JT(C, A)$ *is a cokernel of* $\mu^* = \mathrm{Hom}_\Lambda(\mu, A) : \mathrm{Hom}_\Lambda(P, A) \longrightarrow \mathrm{Hom}_\Lambda(R, A)$ *for every* Λ*-semimodule* A.

Proof. Consider a diagram

$$\begin{tikzcd} & & W & \\ \mathrm{Hom}_\Lambda(P, A) \arrow[r, "\mu^*"] & \mathrm{Hom}_\Lambda(R, A) \arrow[ur, "\omega"] \arrow[rr, "\delta(\mathbb{P}, A)"] & & \mathrm{Ext}_\Lambda\, JT(C, A), \arrow[ul, dashed, "\omega'"'] \end{tikzcd}$$

where ω is a homomorphism of abelian monoids such that $\omega\mu^* = 0$. Take $cl(S : A \xleftarrow{f} X \rightarrowtail Y \twoheadrightarrow C) \in \mathrm{Ext}_\Lambda\, JT(C, A)$. There is a morphism $(\varphi, \psi, 1_C) : \mathbb{P} \longrightarrow \overline{S}$. Define $\omega'(cl(S)) = \omega(f\varphi)$. If $(\varphi', \psi', 1_C) : \mathbb{P} \longrightarrow \overline{S}$ is another morphism it follows from Proposition 1.9 that $\varphi + \beta\mu = \varphi' + \beta'\mu$ for some Λ-homomorphisms $\beta, \beta' : P \longrightarrow X$. Whence

$$\begin{aligned} \omega(f\varphi') &= \omega(f\varphi') + \omega\mu^*(f\beta') = \omega(f\varphi' + f\beta'\mu) \\ &= \omega(f\varphi + f\beta\mu) = \omega(f\varphi) + \omega\mu^*(f\beta) = \omega(f\varphi), \end{aligned}$$

i.e., $\omega(f\varphi') = \omega(f\varphi)$. On the other hand, a commutative diagram

$$\begin{tikzcd} S: A \arrow[d, equal] & X \arrow[l, "f"'] \arrow[r, tail] \arrow[d, "g"] & Y \arrow[r, two heads] \arrow[d, "h"] & C \arrow[d, equal] \\ S': A & X' \arrow[l, "f'"'] \arrow[r, tail] & Y' \arrow[r, two heads] & C, \end{tikzcd}$$

where $S' \in \mathcal{E}xt_\Lambda JT(C, A)$, yields the morphism $(g\varphi, h\psi, 1_C) : \mathbb{P} \longrightarrow \overline{S}'$. Therefore $\omega'(cl(S')) = \omega(f'g\varphi) = \omega(f\varphi)$. Thus ω' is well defined. Clearly $\omega = \omega'\delta(\mathbb{P}, A)$. This completes the proof since $\delta(\mathbb{P}, A)$ is surjective. □

As a consequence, we obtain

Theorem 2.3. *Assume* A *is a* Λ*-semimodule and* C *a normal* Λ*-semimodule. Let* $\mathbb{P} : R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} C$ *be the chosen projective presentation of* C. *Then*

$$\theta(C, A) : \mathrm{Ext}_\Lambda(C, A) \longrightarrow \mathrm{Ext}_\Lambda\, JT(C, A)$$

defined by $\theta(C, A)([\varphi]) = cl(\varphi \circ \mathbb{P}) = cl(A \xleftarrow{\varphi} R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} C)$ *is a natural isomorphism of abelian monoids.*

Proof. Since $\delta(\mathbb{P}, A) : \mathrm{Hom}_\Lambda(R, A) \longrightarrow \mathrm{Ext}_\Lambda\, JT(C, A)$ and the canonical projection $p : \mathrm{Hom}_\Lambda(R, A) \longrightarrow \mathrm{Ext}_\Lambda(C, A)$ are both cokernels of $\mu^* = \mathrm{Hom}_\Lambda(\mu, A)$: $\mathrm{Hom}_\Lambda(P, A) \longrightarrow \mathrm{Hom}_\Lambda(R, A)$, we only note that $\theta(C, A)([\varphi]) = \delta(\mathbb{P}, A)(\varphi)$ and $p(\varphi) = [\varphi]$, and that δ and p are natural in (C, A). □

3. In this section we concentrate on $\mathrm{Ext}_{\Lambda}(C,A)$ with A cancellative. It will be shown that this additional condition enables one to give a direct description of $\mathrm{Ext}_{\Lambda}(C,A)$ by Takahashi extensions of C by A (cf. Theorem 2.3).

Let $E : A \rightarrowtail^{\lambda} B \xrightarrow{\tau} \!\!\!\!\twoheadrightarrow C$ be a short exact sequence of Λ-semimodules and $\alpha : A \longrightarrow A'$ a Λ-homomorphism. Following [**10**], denote by B_α the Λ-semimodule $A' \oplus B$ modulo the following congruence relation: $(a_1', b_1)\rho_\alpha(a_2', b_2)$ if there are $a_1, a_2 \in A$ such that $\lambda(a_1) + b_1 = \lambda(a_2) + b_2$ and $\alpha(a_2) + a_1' = \alpha(a_1) + a_2'$; and define $\lambda_\alpha : A' \longrightarrow B_\alpha$, $\tau_\alpha : B_\alpha \longrightarrow C$ and $\xi_\alpha : B \longrightarrow B_\alpha$ by $\lambda_\alpha(a') = [a', 0]$, $\tau_\alpha([a', b]) = \tau(b)$ and $\xi_\alpha(b) = [0, b]$, respectively.

Proposition 3.1 ([10]). *Suppose given a short exact sequence* $E : A \xrightarrow{\lambda} B \xrightarrow{\tau} C$ *of* Λ*-semimodules and a* Λ*-homomorphism* $\alpha : A \longrightarrow A'$ *with* A' *cancellative. Then*

$$\alpha E : A' \xrightarrow{\lambda_\alpha} B_\alpha \xrightarrow{\tau_\alpha} C$$

is a short exact sequence of Λ*-semimodules and* $(\alpha, \xi_\alpha, 1_C)$ *a morphism from* E *to* αE*. Furthermore, if* $E \in \mathscr{T}$ *then* $\alpha E \in \mathscr{T}$*.*

Also note that

$$E \equiv E' \Longrightarrow \alpha E \equiv \alpha E', \tag{3.2}$$

$$E \succ E' \Longrightarrow \alpha E \succ \alpha E'. \tag{3.3}$$

It is directly verified in [**10**] that

$$(\alpha'\alpha)E \equiv \alpha'(\alpha E), \quad 1_G E \equiv E \quad \text{and} \quad \alpha(E\gamma) \equiv (\alpha E)\gamma, \tag{3.4}$$

where $E : G \rightarrowtail B \twoheadrightarrow C$ is a short exact sequence with G a Λ-module, $\gamma : C' \longrightarrow C$ a homomorphism of Λ-semimodules, and $\alpha : G \longrightarrow G'$ and $\alpha' : G' \longrightarrow G''$ are homomorphisms of Λ-modules. These equivalences together with (1.2), (1.3) and (3.2) show that $E_{\Lambda}(C, G)$ is a functor from $(\Lambda\text{-}\mathbf{SMod})^{\mathrm{op}} \times (\Lambda\text{-}\mathbf{Mod})$ to $\mathbf{Set}$ $(E \longmapsto \alpha E, E \longmapsto E\gamma)$ [**10**].

Definition 3.5. *We say that a short exact sequence* $E : A \xrightarrow{\lambda} B \xrightarrow{\tau} C$ *of* Λ*-semimodules is proper if* $\lambda(a) + b_1 = \lambda(a) + b_2$*,* $a \in A$*,* $b_1, b_2 \in B$ *implies* $b_1 = b_2$*.*

Note that it immediately follows from the definition that if $E : A \rightarrowtail B \twoheadrightarrow C$ is proper, then A is a cancellative Λ-semimodule. $0 : A \xrightarrow{i_A} A \oplus C \xrightarrow{\pi_C} C$ is proper if and only if A is cancellative. Also observe that any short exact sequence $G \rightarrowtail B \twoheadrightarrow C$ with G a Λ-module is proper.

Proposition 3.6. *For every short exact sequence* $E : A \xrightarrow{\lambda} B \xrightarrow{\tau} C$ *of* Λ*-semimodules and every* Λ*-homomorphism* $\alpha : A \longrightarrow A'$ *with* A' *cancellative,* $\alpha E : A' \xrightarrow{\lambda_\alpha} B_\alpha \xrightarrow{\tau_\alpha} C$ *is proper.*

Proof. Assume $\lambda_\alpha(a') + [a'_1, b_1] = \lambda_\alpha(a') + [a'_2, b_2]$, $a', a'_1, a'_2 \in A'$, $b_1, b_2 \in B$, i.e., $[a' + a'_1, b_1] = [a' + a'_2, b_2]$. By definition of ρ_α, there are $a_1, a_2 \in A$ such that $\lambda(a_1) + b_1 = \lambda(a_2) + b_2$ and $\alpha(a_2) + a' + a'_1 = \alpha(a_1) + a' + a'_2$. Whence, since A' is cancellative, $\lambda(a_1) + b_1 = \lambda(a_2) + b_2$ and $\alpha(a_2) + a'_1 = \alpha(a_1) + a'_2$. That is, $[a'_1, b_1] = [a'_2, b_2]$. □

Remark 3.7. For a short exact sequence $E : A \rightarrowtail B \twoheadrightarrow C$ and a Λ-homomorphism $\gamma : C' \longrightarrow C$, $E\gamma$ is proper if and only if $\lambda(a) + b_1 = \lambda(a) + b_2$, $a \in A$, $b_1, b_2 \in \tau^{-1}(\gamma(C'))$ implies $b_1 = b_2$. In particular, it follows that if E is proper, then $E\gamma$ is proper.

Lemma 3.8. *Suppose given a short exact sequence* $E : A \overset{\lambda}{\rightarrowtail} B \overset{\tau}{\twoheadrightarrow} C$ *of* Λ*-semimodules and a* Λ*-homomorphism* $\alpha : A \longrightarrow A'$ *with* A' *cancellative. And assume that* $E' : A' \overset{\lambda'}{\rightarrowtail} B' \overset{\tau'}{\twoheadrightarrow} C'$ *is a proper short exact sequence and that* (α, β, γ) *is a morphism from* E *to* E'*. Then there exists a unique* Λ*-homomorphism* $\beta' : B_\alpha \longrightarrow B'$ *such that* $(1_{A'}, \beta', \gamma)$ *is a morphism from* αE *to* E' *and* $\beta = \beta'\xi_\alpha$*. In particular, if* $\gamma = 1_C$ *then* $\alpha E \rightarrowtail E'$*.*

Proof. Define $\beta' : B_\alpha \longrightarrow B'$ by $\beta'([a', b]) = \lambda'(a') + \beta(b)$. Assume $[a'_1, b_1] = [a'_2, b_2]$, i.e., $\lambda(a_1) + b_1 = \lambda(a_2) + b_2$ and $\alpha(a_2) + a'_1 = \alpha(a_1) + a'_2$ for some $a_1, a_2 \in A$. Then, since $(\alpha, \beta, \gamma) : E \longrightarrow E'$ is a morphism, we have $\lambda'\alpha(a_1) + \beta(b_1) = \lambda'\alpha(a_2) + \beta(b_2)$ and $\lambda'\alpha(a_2) + \lambda'(a'_1) = \lambda'\alpha(a_1) + \lambda'(a'_2)$. These equations give $\lambda'\alpha(a_2) + \lambda'(a'_1) + \beta(b_1) = \lambda'\alpha(a_1) + \lambda'(a'_2) + \beta(b_1) = \lambda'\alpha(a_2) + \lambda'(a'_2) + \beta(b_2)$. Whence $\lambda'(a'_1) + \beta(b_1) = \lambda'(a'_2) + \beta(b_2)$ since E' is proper. Hence β' is well defined. Clearly, β' is a Λ-homomorphism with $\beta'\lambda_\alpha = \lambda'$, $\gamma\tau_\alpha = \tau'\beta'$ and $\beta = \beta'\xi_\alpha$. If $\beta'' : B_\alpha \longrightarrow B'$ is another Λ-homomorphism such that $\beta = \beta''\xi_\alpha$ and $(1_{A'}, \beta'', \gamma)$ is a morphism from αE to E', then $\beta''([a', b]) = \beta''([a', 0] + [0, b]) = \beta''\lambda_\alpha(a') + \beta''\xi_\alpha(b) = \lambda'(a') + \beta(b) = \beta'([a', b])$. □

Let $E : A \overset{\lambda}{\rightarrowtail} B \overset{\tau}{\twoheadrightarrow} C$ be a short exact sequence with A cancellative, and let $\alpha : A \longrightarrow A'$, $\alpha' : A' \longrightarrow A''$ and $\gamma : C' \longrightarrow C$ be Λ-homomorphisms with A' and A'' cancellative. Then Propositions 3.1 and 3.6 and Lemma 3.8 immediately provide the morphisms

$$(1_{A''}, \nu, 1_C) : (\alpha'\alpha)E \longrightarrow \alpha'(\alpha E), \quad (1_A, \xi_{1_A}, 1_C) : E \longrightarrow 1_A E$$
$$\text{and} \quad (1_{A'}, \iota, 1_{C'}) : \alpha(E\gamma) \longrightarrow (\alpha E)\gamma,$$

where $\nu : B_{\alpha'\alpha} \longrightarrow (B_\alpha)_{\alpha'}$ and $\iota : (B^\gamma)_\alpha \longrightarrow (B_\alpha)^\gamma$ are defined by $\nu([a'', b]) = [a'', [0, b]]$ and $\iota\big([a', (b, c')]\big) = ([a', b], c')$, respectively. Thus

$$(\alpha'\alpha)E \rightarrowtail \alpha'(\alpha E), \quad E \rightarrowtail 1_A E \quad \text{and} \quad \alpha(E\gamma) \rightarrowtail (\alpha E)\gamma. \tag{3.9}$$

Remark 3.10. One can easily verify that ν and ι are in fact Λ-isomorphisms. Hence $(\alpha'\alpha)E \equiv \alpha'(\alpha E)$, $E \rightarrowtail 1_A E$ and $\alpha(E\gamma) \equiv (\alpha E)\gamma$ (cf. (3.4)). Furthermore, ξ_{1_A} is a Λ-isomorphism if and only if E is proper. (Indeed, assume that E is proper.

Let $\xi_{1_A}(b_1) = \xi_{1_A}(b_2)$, i.e., $[0,b_1] = [0,b_2]$. Then $\lambda(a_1) + b_1 = \lambda(a_2) + b_2$ and $1_A(a_2)+0 = 1_A(a_1)+0$ for some $a_1, a_2 \in A$. Whence, since E is proper, $b_1 = b_2$. On the other hand, ξ_{1_A} is always surjective ($[a,b] = [0, \lambda(a)+b] = \xi_{1_A}(\lambda(a)+b)$). Hence ξ_{1_A} is a Λ-isomorphism. The converse immediately follows from Proposition 3.6.) Therefore $1_A E \equiv E$ if and only if E is proper. Denote by $E_\Lambda P(C,A)$ the set of $\equiv$-equivalence classes of proper Takahashi extensions of C by A. Then, by Proposition 3.6 and Remark 3.7, $E_\Lambda P(C,A)$ is a functor from $(\Lambda\text{-}\mathbf{SMod})^{\mathrm{op}} \times (\Lambda\text{-}\mathbf{CSMod})$ to $\mathbf{Set}$ which canonically extends the functor $E_\Lambda(-,-) : (\Lambda\text{-}\mathbf{SMod})^{\mathrm{op}} \times (\Lambda\text{-}\mathbf{Mod}) \longrightarrow \mathbf{Set}$.

Now let $\mathcal{E}xt_\Lambda T(C,A)$ be the category of Takahashi extensions of a Λ-semimodule C by a cancellative Λ-semimodule A. Define

$$\mathrm{Ext}_\Lambda T(C,A) = \mathcal{E}xt_\Lambda T(C,A)/\langle \succ \rangle\,,$$

where $\langle \succ \rangle$ is the smallest equivalence relation containing the relation $\succ$. By (1.2), (1.4), (1.6), Proposition 3.1, (3.3) and (3.9), the rules $E \longmapsto E\gamma$ and $E \longmapsto \alpha E$ make $\mathrm{Ext}_\Lambda T(C,A)$ a functor from $(\Lambda\text{-}\mathbf{SMod})^{\mathrm{op}} \times (\Lambda\text{-}\mathbf{CSMod})$ to $\mathbf{Set}$.

Remark 3.11. It follows from Proposition 3.6 and Remark 3.7 that one can similarly introduce the functor

$$\mathrm{Ext}_\Lambda PT(C,A) = \mathcal{E}xt_\Lambda PT(C,A)/\langle \succ \rangle\,,$$

where $\mathcal{E}xt_\Lambda PT(C,A)$ denotes the category of proper Takahashi extensions of C by A. Obviously, the maps

$$\mathrm{Ext}_\Lambda PT(C,A) \underset{\Gamma'(C,A)}{\overset{\Gamma(C,A)}{\rightleftarrows}} \mathrm{Ext}_\Lambda T(C,A)\,, \quad \Gamma(cl(E)) = cl(E), \quad \Gamma'(cl(E)) = cl(1_A E)$$

are natural, and $\Gamma'\Gamma = 1$ and $\Gamma\Gamma' = 1$.

In order to prove the following theorem, note that

$$\alpha \circ E \sim 1_{A'} \circ \alpha E \tag{3.12}$$

for any short exact sequence $E : A \overset{\lambda}{\rightarrowtail} B \overset{\tau}{\twoheadrightarrow} C$ and any Λ-homomorphism $\alpha : A \longrightarrow A'$ with A' cancellative. Indeed, the morphism $(\alpha, \xi_\alpha, 1_C) : E \longrightarrow \alpha E$ gives the commutative diagram

$$\begin{tikzcd}
\alpha \circ E : & A' \arrow[d, equal] & A \arrow[l, "\alpha"'] \arrow[r, tail, "\lambda"] \arrow[d, "\alpha"'] & B \arrow[r, two heads, "\tau"] \arrow[d, "\xi_\alpha"'] & C \arrow[d, equal] \\
1_{A'} \circ \alpha E : & A' \arrow[r, equal] & A' \arrow[r, tail, "\lambda_\alpha"] & B_\alpha \arrow[r, two heads, "\tau_\alpha"] & C.
\end{tikzcd}$$

Theorem 3.13. *Let C be a Λ-semimodule and A a cancellative Λ-semimodule. Then*

$$\chi(C,A) : \mathrm{Ext}_\Lambda T(C,A) \longrightarrow \mathrm{Ext}_\Lambda JT(C,A)$$

defined by

$$\chi(C,A)\big(cl(E\colon A \rightarrowtail B \twoheadrightarrow C)\big) = cl\big(1_A \circ E\colon\ A = A \rightarrowtail B \twoheadrightarrow C\big)$$

is a natural bijection.

Proof. Define

$$\chi'(C,A) : \mathrm{Ext}_\Lambda\, JT(C,A) \longrightarrow \mathrm{Ext}_\Lambda\, T(C,A)$$

by

$$\chi'(C,A)\big(cl(S : A \xleftarrow{f} X \overset{\varkappa}{\rightarrowtail} Y \overset{\sigma}{\twoheadrightarrow} C)\big) = cl(f\overline{S}).$$

This definition is independent of the chosen representative sequence S. Indeed, suppose given a commutative diagram

$$\begin{tikzcd}
S: & A \arrow[d, equal] & X \arrow[l, "f"'] \arrow[r, tail, "\varkappa"] \arrow[d, "\varphi"'] & Y \arrow[r, two heads, "\sigma"] \arrow[d, "\psi"'] & C \arrow[d, equal] \\
S': & A & X' \arrow[l, "g"] \arrow[r, "\varkappa'"] & Y' \arrow[r, two heads, "\sigma'"] & C
\end{tikzcd}$$

with $S' \in \mathcal{E}xt_\Lambda JT(C,A)$. This commutative diagram and the morphism $(g, \xi_g, 1_C)$: $\overline{S}\,' \longrightarrow g\overline{S}\,'$ yield the following commutative diagram

$$\begin{tikzcd}
\overline{S}: & X \arrow[r, tail, "\varkappa"] \arrow[d, "f"'] & Y \arrow[r, two heads, "\sigma"] \arrow[d, "\xi_g\psi"'] & C \arrow[d, equal] \\
g\overline{S}\,': & A \arrow[r, "\varkappa'_g"] & Y'_g \arrow[r, two heads, "\sigma'_g"] & C.
\end{tikzcd}$$

Whence, by Proposition 3.6 and Lemma 3.8, $f\overline{S} \succ g\overline{S}\,'$. Hence $\chi'(C,A)$ is well defined. Further, by (3.12), $\chi\chi'(cl(S)) = \chi(cl(f\overline{S})) = cl(1_A \circ f\overline{S}) = cl(f \circ \overline{S}) = cl(S)$. On the other hand, $\chi'\chi(cl(E)) = \chi'(cl(1_A \circ E)) = cl(1_A E) = cl(E)$ since $E \succ 1_A E$. Thus $\chi(C,A)$ is a bijection. Finally, consider the diagram

$$\begin{tikzcd}
\mathrm{Ext}_\Lambda\, T(C,A) \arrow[r, "\chi(C,A)"] \arrow[d, "\mathrm{Ext}_\Lambda\, T(\gamma,\alpha)"'] & \mathrm{Ext}_\Lambda\, JT(C,A) \arrow[d, "\mathrm{Ext}_\Lambda\, JT(\gamma,\alpha)"] \\
\mathrm{Ext}_\Lambda\, T(C',A') \arrow[r, "\chi(C',A')"] & \mathrm{Ext}_\Lambda\, JT(C',A'),
\end{tikzcd}$$

where $\alpha : A \longrightarrow A'$ is a homomorphism of cancellative Λ-semimodules and γ : $C' \longrightarrow C$ a homomorphism of Λ-semimodules. Using (3.12) and (2.1), we obtain

$$\begin{aligned}
&\chi(C',A')\,\mathrm{Ext}_\Lambda\, T(\gamma,\alpha)(cl(E)) = cl(1_{A'} \circ \alpha E\gamma) = cl(\alpha \circ E\gamma) \\
&= cl\big((\alpha \circ E)\gamma\big) = cl\big(\alpha(1_A \circ E)\gamma\big) = \mathrm{Ext}_\Lambda\, JT(\gamma,\alpha)\ \chi(C,A)(cl(E)),
\end{aligned}$$

i.e., the diagram is commutative. Thus $\chi(C,A)$ is natural. $\square$

Remark 3.14. More general results than Theorem 3.13 are discussed in [**5**,**6**]. However Theorem 3.13 is not merely a consequence of those since the span of Takahashi extensions is not regular in the sense of N. Yoneda [**12**].

It is evident that Theorem 3.13 remains valid for any class $\mathscr{C}$ of short exact sequences $E : A \rightarrowtail B \twoheadrightarrow C$ with A cancellative which satisfies the following conditions: $(0 \longrightarrow Z = Z) \in \mathscr{C}$ for every Λ-semimodule Z; if $E \in \mathscr{C}$ then $\alpha E,\ E\gamma \in \mathscr{C}$.

Theorem 3.13 shows that $\mathrm{Ext}_{\Lambda}\, T(C,A)$ is in fact an abelian monoid-valued functor, additive in each of its arguments; the addition in $\mathrm{Ext}_{\Lambda}\, T(C,A)$ obviously coincides with the Bear addition:

$$cl(E) + cl(E') = cl\big(\ \triangledown_A\ (E \oplus E')\ \triangle_C\ \big).$$

As a corollary of Theorems 2.3 and 3.13 we have

Theorem 3.15. *Assume A is a cancellative Λ-semimodule and C a normal Λ-semimodule. Let $\mathbb{P} : R \overset{\nu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} C$ be the chosen projective presentation of C. Then*

$$w(C,A) : \mathrm{Ext}_{\Lambda}(C,A) \longrightarrow \mathrm{Ext}_{\Lambda}\, T(C,A)$$

defined by $w(C,A)([\varphi]) = cl(\varphi\mathbb{P})$ is a natural isomorphism of abelian monoids.

Proof. $w(C,A) = \chi'(C,A)\theta(C,A)$. □

Let G and H be Λ-modules, i.e., $K(\Lambda)$-modules. It immediately follows from Proposition 1.1 that

$$\mathrm{Ext}_{\Lambda}\, T(H,G) = \mathrm{Ext}^1_{K(\Lambda)}(H,G) = E_{\Lambda}(H,G),$$

where $\mathrm{Ext}^1_{K(\Lambda)}$ is the usual Ext functor. From this and Theorem 3.13 one has

Corollary 3.16. *Let G and H be Λ-modules. The map*

$$\mathrm{Ext}^1_{K(\Lambda)}(H,G) \longrightarrow \mathrm{Ext}_{\Lambda}\, JT(H,G)\ ,$$

$$cl(G \rightarrowtail B \twoheadrightarrow H) \longmapsto cl(G = G \rightarrowtail B \twoheadrightarrow H)$$

is a natural isomorphism of abelian groups.

Note that in [**6**] Janelidze proved this for $\Lambda = N$, the semiring of non-negative integers.

Remark 3.17. Let Λ and Λ' be additively cancellative semirings. In [**8**], for any contravariant additive functor $T : (\Lambda\text{-}\mathbf{CSMod}) \longrightarrow (\Lambda'\text{-}\mathbf{CSMod})$, we constructed and studied right derived functors $\mathbb{R}^n T : (\Lambda\text{-}\mathbf{CSMod}) \longrightarrow (\Lambda'\text{-}\mathbf{CSMod})$, $n = 0, 1, 2, \ldots$. In particular, we described $\mathbb{R}^n \mathrm{Hom}_{\Lambda}(-,A)(C)$ by means of certain n-fold extensions of C by A. According to that description, for $n = 1$, $H \in (\Lambda\text{-}\mathbf{Mod})$ and $A \in (\Lambda\text{-}\mathbf{CSMod})$, one has a natural isomorphism of abelian groups

$$\mathbb{R}^1 \mathrm{Hom}_{\Lambda}(-,A)(H) \cong \mathcal{E}xt^1_{\Lambda}(H,A)/\langle \rightarrowtail \rangle,$$

where $\mathcal{E}xt^1_\Lambda(H,A)$ denotes the category of short exact sequences $A \rightarrowtail B \twoheadrightarrow H$ of Λ-semimodules with B cancellative. (Note that $R^1 \operatorname{Hom}_\Lambda(-,A)(H) = R^1 \operatorname{Hom}_\Lambda(-,U(A))(H) \cong \operatorname{Ext}^1_{K(\Lambda)}(H,U(A))$, where $U(A)$ is the maximal Λ-submodule of A, and $R^1 \operatorname{Hom}_\Lambda(-,A)$ denotes the usual right derived functor of the functor $\operatorname{Hom}_\Lambda(-,A) : (\Lambda\text{-}\mathbf{Mod}) \longrightarrow \mathbf{Ab}$.) On the other hand, if $A \overset{\varkappa}{\rightarrowtail} X \overset{\sigma}{\twoheadrightarrow} H$ is a proper short exact sequence of Λ-semimodules with $H \in (\Lambda\text{-}\mathbf{Mod})$, then X is cancellative. (To see this, suppose $x + x_1 = x + x_2$, $x, x_1, x_2 \in X$. Take $x' \in X$ such that $\sigma(x') = -\sigma(x)$. Then $x' + x = \varkappa(a)$ for some $a \in A$; and we obtain $\varkappa(a) + x_1 = x' + x + x_1 = x' + x + x_2 = \varkappa(a) + x_2$. Whence $x_1 = x_2$ since $A \overset{\varkappa}{\rightarrowtail} X \overset{\sigma}{\twoheadrightarrow} H$ is proper.) Hence there is a natural isomorphism

$$\mathbb{R}^1 \operatorname{Hom}_\Lambda(-,A)(H) \cong \operatorname{Ext}_\Lambda PT(H,A).$$

Consequently, by Remark 3.11 and Theorems 3.13 and 3.15, each of the abelian groups $\operatorname{Ext}_\Lambda JT(H,A)$, $\operatorname{Ext}_\Lambda T(H,A)$ and $\operatorname{Ext}_\Lambda(H,A)$, where H is a Λ-module and A a cancellative Λ-semimodule, is naturally isomorphic to $\mathbb{R}^1 \operatorname{Hom}_\Lambda(-,A)(H)$.

4. Let G be a Λ-module. In this section, continuing the investigation started in [**11**], we obtain some results relating $\operatorname{Ext}_\Lambda(C,G)$ and $E_\Lambda(C,G)$. Besides, we relate $\operatorname{Ext}_\Lambda(C,G)$ with Inassaridze extensions of C by G [**4**], and also with $S^1 \operatorname{Hom}_{K(\Lambda)}(-,G)(K(C))$, where $K(C)$ denotes the Grothendieck $K(\Lambda)$-module of C.

First of all, observe that the Baer addition of extensions, $E + E' = \triangledown_G(E \oplus E')\triangle_C$, makes $E_\Lambda(C,G)$ an abelian monoid [**4**]. In addition the $\equiv$-equivalence class of $0 : G \overset{i_G}{\rightarrowtail} G \oplus C \overset{\pi_C}{\twoheadrightarrow} C$ serves as a neutral element. Furthermore, a straightforward verification shows that $\alpha(E+E') \equiv \alpha E + \alpha E'$, $(E+E')\gamma \equiv E\gamma + E'\gamma$, $\alpha 0 \equiv 0$ and $0\gamma \equiv 0$. Thus $E_\Lambda(C,G)$ is in fact a functor from $(\Lambda\text{-}\mathbf{SMod})^{\mathrm{op}} \times (\Lambda\text{-}\mathbf{Mod})$ to $\mathbf{Abm}$.

We call a Takahashi extension $E : G \overset{\lambda}{\rightarrowtail} B \overset{\tau}{\twoheadrightarrow} C$ of a Λ-semimodule C by a Λ-module G an *Inassaridze extension* if E is an extension of C by G in the sense of H. Inassaridze [**4**]: whenever the equality $\lambda(g) + b = b$ holds for some $g \in G$, $b \in B$, then $g = 0$. Let $\operatorname{Ext}_\Lambda I(C,G)$ denote the set of $\equiv$-equivalence classes of Inassaridze extension of C by G. It is shown in [**4**] that $\operatorname{Ext}_\Lambda I(C,G) = U(E_\Lambda(C,G))$, the group of units of $E_\Lambda(C,G)$. This in particular means that $\operatorname{Ext}_\Lambda I(C,G)$ is an abelian group-valued subfunctor of $E_\Lambda(C,G)$. Moreover, $\operatorname{Ext}_\Lambda I(C,G)$ is additive in both its arguments.

A short exact sequence of Λ-semimodules $E : A \overset{\lambda}{\rightarrowtail} B \overset{\tau}{\twoheadrightarrow} C$ is said to be *split* if there exists a Λ-homomorphism $\nu : C \longrightarrow B$ such that $\tau\nu = 1$. Let $E_\Lambda S(C,G)$ denote the set of $\equiv$-equivalence classes of split Takahashi extensions of a Λ-semimodule C by a Λ-module G. It is easy to see that αE, $E\gamma$ and $E \oplus E'$ are split whenever E and E' are split. Consequently, $E_\Lambda S(C,G)$ is another subfunctor of the functor $E_\Lambda(C,G)$.

We shall need the following four facts.

Proposition 4.1 ([4]). *Suppose given a morphism of the form*

$$\begin{tikzcd} E: & G \arrow[r, tail, "\lambda"] \arrow[d, equal] & B \arrow[r, two heads, "\tau"] \arrow[d, "\beta"'] & C \arrow[d, equal] \\ E': & G \arrow[r, tail, "\lambda'"] & B' \arrow[r, two heads, "\tau'"] & C \end{tikzcd}$$

of Takahashi extensions of a Λ-semimodule C by a Λ-module G. If E' is an Inassaridze extension, then β is an isomorphism.

Theorem 4.2 ([4,7]). *Let $K(C)$ be the Grothendieck $K(\Lambda)$-module of a Λ-semimodule C, and $k_C : C \longrightarrow K(C)$ the canonical Λ-homomorphism. Then the natural homomorphism*

$$\mathrm{Ext}^1_{K(\Lambda)}(K(C), G) = \mathrm{Ext}_\Lambda I(K(C), G) \xrightarrow{\mathrm{Ext}_\Lambda I(k_C, G)} \mathrm{Ext}_\Lambda I(C, G)$$

is an isomorphism.

Proposition 4.3 ([7]). *An Inassaridze extension $E : G \rightarrowtail B \twoheadrightarrow C$ is split if and only if $E \equiv 0 : G \overset{i_G}{\rightarrowtail} G \oplus C \overset{\pi_C}{\twoheadrightarrow} C$.*

Proposition 4.4. *Let $E : A \overset{\lambda}{\rightarrowtail} B \overset{\tau}{\twoheadrightarrow} C$ be a split short exact sequence of Λ-semimodules, and $\nu : C \longrightarrow B$ a splitting Λ-homomorphism, i.e., $\tau\nu = 1$. If C is a Λ-module, then the map*

$$q : A \oplus C \longrightarrow B\ , \quad q(a, c) = \lambda(a) + \nu(c)$$

is a Λ-isomorphism.

Proof. Define $q' : B \longrightarrow A \oplus C$ by $q'(b) = (\lambda^{-1}(b - \nu\tau(b)), \tau(b))$. Then $q'q = 1$ and $qq' = 1$. □

For any Λ-module G and any normal Λ-semimodule C, M. Takahashi has defined a pair of maps

$$E_\Lambda(C, G) \underset{\zeta(C,G)}{\overset{\eta(C,G)}{\rightleftarrows}} \mathrm{Ext}_\Lambda(C, G)$$

as follows. $\zeta(C, G)([f]) = cl(f\mathbb{P})$, where $\mathbb{P} : R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} C$ is the chosen projective presentation of C. Next, take $cl(E) \in E_\Lambda(C, G)$. There is a morphism $(\varphi, \psi, 1) : \mathbb{P} \longrightarrow E$. And define $\eta(C, G)(cl(E)) = [\varphi]$. It is shown in [**11**] that $\eta(C, G)$ and $\zeta(C, G)$ are well defined, and $\eta(C, G)$ is natural in each of its arguments, and $\eta(C, G)\zeta(C, G) = 1$.

Observe that the surjection $\eta(C, G)$ is in fact a homomorphism. In order to see this, consider the diagram

$$\begin{tikzcd}
E_\Lambda(C,G) \arrow[rr, "\eta(C,G)"] \arrow[dr, "m(C,G)"'] & & \mathrm{Ext}_\Lambda(C,G) \arrow[dl, "w(C,G)"] \\
& \mathrm{Ext}_\Lambda T(C,G) &
\end{tikzcd}$$

where $m(C,G)$ is defined by $m(C,G)(cl(E)) = cl_{\langle\succ\rangle}(E)$, and $w(C,G)$ by $w(C,G)([\varphi]) = cl_{\langle\succ\rangle}(\varphi\mathbb{P})$ (see Theorem 3.15). One has $w(C,G)\eta(C,G)\ (cl(E)) = w(C,G)([\varphi]) = cl_{\langle\succ\rangle}(\varphi\mathbb{P})$. But, by Lemma 3.8, $\varphi\mathbb{P}\succ E$, that is, $cl_{\langle\succ\rangle}(\varphi\mathbb{P}) = cl_{\langle\succ\rangle}(E)$. Hence the diagram is commutative. Therefore, since $m(C,G)$ is a homomorphism and $w(C,G)$ an isomorphism, $\eta(C,G)$ is a homomorphism.

Proposition 4.5. *For any Λ-module G and any normal Λ-semimodule C,*

$$\mathrm{Ker}(\mu(C,G)) = E_\Lambda S(C,G),$$

that is, the sequence

$$E_\Lambda S(C,G) \xrightarrow{j(C,G)} E_\Lambda(C,G) \xrightarrow{\eta(C,G)} \mathrm{Ext}_\Lambda(C,G)\ , \tag{4.6}$$

where $j(C,G)$ denotes the inclusion, is a short exact sequence of abelian monoids.

Proof. Let $cl(\ E : G \rightarrowtail^{\lambda} B \xrightarrow{\tau} C\ ) \in E_\Lambda S(C,G)$. If $\nu : C \longrightarrow B$ is a splitting Λ-homomorphism, then the diagram

$$\begin{tikzcd}
\mathbb{P}: & R \arrow[r, tail, "\mu"] \arrow[d, "0"'] & P \arrow[r, two heads, "\varepsilon"] \arrow[d, "\nu\varepsilon"'] & C \arrow[d, equal] \\
E: & G \arrow[r, tail, "\lambda"] & B \arrow[r, two heads, "\tau"] & C
\end{tikzcd}$$

is commutative. Whence, by definition of $\eta(C,G)$, $\eta(C,G)(cl(E)) = [0] = 0$. Conversely, suppose $cl(\ E : G \rightarrowtail^{\lambda} B \xrightarrow{\tau} C\ ) \in \mathrm{Ker}(\eta(C,G))$. Assume $(\varphi, \psi, 1)$ is a morphism from $\mathbb{P}$ to E. Then $\eta(C,G)(cl(E)) = [\varphi] = 0$, i.e., there exists a Λ-homomorphism $g : P \longrightarrow G$ such that $\varphi = g\mu$. From this it follows that the diagram

$$\begin{tikzcd}
\mathbb{P}: & R \arrow[r, tail, "\mu"] \arrow[d, "\varphi"'] & P \arrow[r, two heads, "\varepsilon"] \arrow[d, "\beta"'] & C \arrow[d, equal] \\
0: & G \arrow[r, tail, "i_G"] & G \oplus C \arrow[r, two heads, "\pi_C"] & C,
\end{tikzcd}$$

where $\beta = i_G g + i_C \varepsilon$, is commutative. By Lemma 3.8, this commutative diagram and the morphism $(\varphi, \psi, 1) : \mathbb{P} \longrightarrow E$ yield the commutative diagrams

$$\begin{tikzcd} \varphi\mathbb{P}: & G \arrow[r, tail, "\mu_\varphi"] \arrow[d, equal] & P_\varphi \arrow[r, two heads, "\varepsilon_\varphi"] \arrow[d, "\beta_1"'] & C \arrow[d, equal] \\ 0: & G \arrow[r, tail, "i_G"] & G \oplus C \arrow[r, two heads, "\pi_C"] & C \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \varphi\mathbb{P}: & G \arrow[r, tail, "\mu_\varphi"] \arrow[d, equal] & P_\varphi \arrow[r, two heads, "\varepsilon_\varphi"] \arrow[d, "\beta_2"'] & C \arrow[d, equal] \\ E: & G \arrow[r, tail, "\lambda"] & B \arrow[r, two heads, "\tau"] & C, \end{tikzcd}$$

respectively. Moreover, by Proposition 4.1, β_1 is a Λ-isomorphism. Then one can write $\tau\beta_2\beta_1^{-1}i_C = \varepsilon_\varphi\beta_1^{-1}i_C = \pi_C i_C = 1_C$. Hence $\beta_2\beta_1^{-1}i_C : C \longrightarrow B$ is a splitting Λ-homomorphism for E. Consequently, $cl(E) \in E_\Lambda S(C,G)$. ☐

Corollary 4.7. *The restriction of $\eta(C,G)$ to $\mathrm{Ext}_\Lambda I(C,G)$, that is, the natural homomorphism*

$$\eta(C,G) : \mathrm{Ext}_\Lambda I(C,G) \longrightarrow \mathrm{Ext}_\Lambda(C,G)$$

of abelian groups is one-to-one.

Proof. Suppose $cl(E)$ is contained in the kernel of this homomorphism. It then follows from Proposition 4.5 that E is split. Hence, by Proposition 4.3, $cl(E) = 0$. ☐

This corollary allows to relate $S^1 \mathrm{Hom}_{K(\Lambda)}(-,G)(K(C))$ and $\mathrm{Ext}_\Lambda(C,G)$ (cf. (1.10)) as follows. Let $\mathbb{F} : L \rightarrowtail F \twoheadrightarrow K(C)$ be the chosen $K(\Lambda)$-projective presentation of $K(C)$. Define

$$S(C,G) : S^1 \mathrm{Hom}_{K(\Lambda)}(-,G)(K(C)) \longrightarrow \mathrm{Ext}_\Lambda(C,G)$$

by $S(C,G)\big([\, f : L \longrightarrow G\,]\big) = [f\varphi]$, where $\varphi : R \longrightarrow L$ is any Λ-homomorphism such that (φ, ψ, k_C) is a morphism from the Λ-projective presentation $\mathbb{P} : R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} C$ to $\mathbb{F}$. By Proposition 1.9, $S(C,G)$ is well defined. Clearly $S(C,G) = \mathrm{Ext}_\Lambda(k_C, G)(K(K(C),G))^{-1}$ (see (1.11)).

Corollary 4.8. *For any Λ-module G and any normal Λ-semimodule C, $S(C,G)$ is an injective natural homomorphism.*

Proof. Consider the diagram

$$\begin{tikzcd} S^1 \mathrm{Hom}_{K(\Lambda)}(-,G)(K(C)) \arrow[r, "S(C,G)"] \arrow[d, "r(K(C),G)"'] & \mathrm{Ext}_\Lambda(C,G) \\ \mathrm{Ext}_\Lambda I(K(C),G) = \mathrm{Ext}^1_{K(\Lambda)}(K(C),G) \arrow[r, "\mathrm{Ext}_\Lambda I(k_C,G)"] & \mathrm{Ext}_\Lambda I(C,G), \arrow[u, "\eta(C,G)"'] \end{tikzcd} \tag{4.9}$$

where $r(K(C),G)$, defined by $r(K(C),G)\big([\, g : L \longrightarrow G\,]\big) = cl(g\mathbb{F})$, is a well-known natural isomorphism. Let $f : L \longrightarrow G$ be a homomorphism of Λ-modules and (φ, ψ, k_C) a morphism from $\mathbb{P}$ to $\mathbb{F}$. By (3.4), $(f\mathbb{F})k_C \equiv f(\mathbb{F}k_C)$. Besides, it is easy to see that (φ, ψ, k_C) and f give the morphism $(f\varphi, \psi', 1_C) : \mathbb{P} \longrightarrow f(\mathbb{F}k_C)$, where $\psi' : P \longrightarrow (F^{k_C})_f$ is defined by $\psi'(p) = [0, (\psi(p), \varepsilon(p))]$. Consequently, one can write

$$\begin{aligned}
&\eta(C,G)\,\mathrm{Ext}_\Lambda\, I(k_C, G)\, r(K(C), G)\big([\, f : L \longrightarrow G\,]\big)\\
&\quad = \eta(C,G)\,\mathrm{Ext}_\Lambda\, I(k_C, G)(cl(f\mathbb{F})) = \eta(C,G)\big(cl((f\mathbb{F})k_C)\big)\\
&\quad = \eta(C,G)\big(cl((f(\mathbb{F}k_C)))\big) = [f\varphi] = S(C,G)\big([\, f : L \longrightarrow G\,]\big).
\end{aligned}$$

That is, the diagram is commutative. It then follows from Theorem 4.2 and Corollary 4.7 that $S(C,G)$ is an injective natural homomorphism. □

Before discussing the following results, recall that a semiring $\Lambda = (\Lambda, +, 0, ., 1)$ is called additively cancellative if $(\Lambda, +, 0)$ is cancellative (e.g., N, the semiring of non-negative integers). In this case every projective Λ-semimodule is obviously cancellative. Moreover, if Λ is additively cancellative, then every normal Λ-semimodule C is cancellative. (Indeed, let $M \rightarrowtail^{\varkappa} Q \xrightarrow{\sigma} C$ be a projective presentation of C. Suppose $c_1 + c = c_2 + c$, $c, c_1, c_2 \in C$. Take $q, q_1, q_2 \in Q$ so that $\sigma(q) = c$, $\sigma(q_1) = c_1$ and $\sigma(q_2) = c_2$. Then $\sigma(q_1 + q) = \sigma(q_2 + q)$. Hence $\varkappa(m_1) + q_1 + q = \varkappa(m_2) + q_2 + q$ for some m_1, $m_2 \in M$. Whence, since Q is cancellative, $\varkappa(m_1) + q_1 = \varkappa(m_2) + q_2$. Therefore $\sigma(q_1) = \sigma(q_2)$, i.e., $c_1 = c_2$.)

Theorem 4.10. *If Λ is an additively cancellative semiring, then the natural map*

$$\eta(C,G) : \mathrm{Ext}_\Lambda\, I(C,G) \longrightarrow \mathrm{Ext}_\Lambda(C,G)$$

is an isomorphism for any Λ-module G and any normal Λ-semimodule C.

Proof. The chosen projective presentation $\mathbb{P} : R \rightarrowtail^{\mu} P \xrightarrow{\varepsilon} C$ of C and a Λ-homomorphism $\varphi : R \longrightarrow G$ give the Takahashi extension $\varphi\mathbb{P} : G \rightarrowtail^{\mu_\varphi} P_\varphi \xrightarrow{\varepsilon_\varphi} C$ and the morphism $(\varphi, \xi_\varphi, 1_C) : \mathbb{P} \longrightarrow \varphi\mathbb{P}$ (see Proposition 3.1). Suppose $\mu_\varphi(g) + [h,p] = [h,p]$, $g, h \in G$, $p \in P$, i.e., $[g + h, p] = [h,p]$. Then $\mu(r_1) + p = \mu(r_2) + p$ and $\varphi(r_2) + g + h = \varphi(r_1) + h$ for some $r_1, r_2 \in R$. These two equations imply $g = 0$ since P is cancellative. Hence $\varphi\mathbb{P}$ is an Inassaridze extension. On the other hand, by definition of $\eta(C,G)$, one has $\eta(C,G)(cl(\varphi\mathbb{P})) = [\varphi]$. Thus $\eta(C,G) : \mathrm{Ext}_\Lambda\, I(C,G) \longrightarrow \mathrm{Ext}_\Lambda(C,G)$ is surjective. This together with Corollary 4.7 gives the desired result. □

Theorems 4.2 and 4.10 and the commutative diagram (4.9) (see the proof of Corollary 4.8) yield

Theorem 4.11. *If Λ is an additively cancellative semiring, then the natural map*

$$S(C,G) : S^1\,\mathrm{Hom}_{K(\Lambda)}(-,G)(K(C)) \longrightarrow \mathrm{Ext}_\Lambda(C,G)$$

is an isomorphism for any Λ-module G and any normal Λ-semimodule C.

We have already mentioned that $\zeta(C,G) : \mathrm{Ext}_\Lambda(C,G) \longrightarrow E_\Lambda(C,G)$ is defined by $\zeta(C,G)([\varphi]) = cl(\varphi\mathbb{P})$ and that $\eta(C,G)\zeta(C,G) = 1$. On the other hand, the proof of Theorem 4.10 shows that if Λ is an additively cancellative semiring, then $\zeta(C,G)$ maps $\mathrm{Ext}_\Lambda(C,G)$ into $\mathrm{Ext}_\Lambda I(C,G)$. Hence, by Theorem 4.10, $\zeta(C,G)$ is the two-sided inverse for $\eta(C,G)\colon \mathrm{Ext}_\Lambda I(C,G) \longrightarrow \mathrm{Ext}_\Lambda(C,G)$ whenever Λ is additively cancellative. Consequently, under the given hypothesis, $\zeta(C,G)$ is a splitting homomorphism for the short exact sequence (4.6). Thus, by Proposition 4.4, one has

Theorem 4.12. *Let Λ be an additively cancellative semiring. Suppose G is a Λ-module and C a normal Λ-semimodule. Then the map*

$$E_\Lambda S(C,G) \oplus \mathrm{Ext}_\Lambda(C,G) \longrightarrow E_\Lambda(C,G)\ , \quad (cl(E),[\varphi]) \longmapsto cl(E) + cl(\varphi\mathbb{P})$$

is a natural isomorphism of abelian monoids.

This theorem together with Theorems 4.10 and 4.2 gives

Corollary 4.13. *Let Λ be an additively cancellative semiring. Assume G is a Λ-module and C a normal Λ-semimodule. Then the maps*

$$E_\Lambda S(C,G) \oplus \mathrm{Ext}_\Lambda I(C,G) \longrightarrow E_\Lambda(C,G)\ , \quad (cl(E),cl(E')) \longmapsto cl(E) + cl(E')$$

and

$$E_\Lambda S(C,G) \oplus \mathrm{Ext}^1_{K(\Lambda)}(K(C),G) \longrightarrow E_\Lambda(C,G), \quad (cl(E),cl(T)) \mapsto cl(E) + cl(Tk_C)$$

are natural isomorphisms of abelian monoids.

References

[**1**] H. M. J. Al-Thani, *A note on projective semimodules*, Kobe J. Math., **12**(1995), 89–94.

[**2**] J. S. Golan, *The Theory of Semirings with Applications in Mathematics and Theoretical Computer Science*, Pitman Monographs and Surveys in Pure and Appl. Maths., **54**, Longman, New York, 1992.

[**3**] U. Hebisch and H. J. Weinert, *Semirings, Algebraic Theory and Applications in Computer Science*, Series in Algebra, **5**, World Scientific Publishing, Singapore, 1998.

[**4**] H. Inassaridze, *Some Questions of Homological and Homotopical Algebra and Their Applications*, "Metsniereba", Tbilisi, 1975 (Russian).

[**5**] G. Janelidze, *Satellites and Galois Extensions of Commutative Rings*, Ph. D. Thesis, Tbilisi State University, Tbilisi, 1978 (Russian).

[**6**] G. Janelidze, *Computation of Kan extensions by means of injective objects and functors $Ext^n_{\mathscr{C}}$ in nonadditive categories*, Trudy Tbiliss. Mat. Inst. Razmadze, **70**(1982), 42–51 (Russian).

[**7**] A. Patchkoria (A. M. Pachkoria), *On Schreier Extensions of Semimodules*, Ph. D. Thesis, Tbilisi state University, Tbilisi, 1979 (Russian).

[**8**] A. Patchkoria (A. M. Pachkoria), *On derived functors of semimodule-valued functors*, Trudy Tbiliss. Mat. Inst. Razmadze, **83**(1986), 60–75 (Russian).

[**9**] M. Takahashi, *On the bordism categories* II—*Elementary properties of semimodules*—, Math. Sem. Notes, Kobe Univ., **9**(1981), 495–530.

[**10**] M. Takahashi, *Extensions of semimodules* I, Math. Sem. Notes, Kobe Univ., **10**(1982), 563–592.

[**11**] M. Takahashi, *Extensions of semimodules* II, Math. Sem. Notes, Kobe Univ., **11**(1983), 83–118.

[**12**] N. Yoneda, *On Ext and exact sequences*, J. Fac. Sci. Univ. Tokyo, Sec. I, **8**(1961), 507–576.

This article may be accessed via WWW at http://www.rmi.acnet.ge/hha/
or by anonymous ftp at
ftp://ftp.rmi.acnet.ge/pub/hha/volumes/2003/n1a16/v5n1a16.(dvi,ps,pdf)

Alex Patchkoria `lexo@rmi.acnet.ge`

A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 0193
Georgia