

HOMOTOPY LIE ALGEBRA OF CLASSIFYING SPACES FOR HYPERBOLIC COFORMAL 2-CONES

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(communicated by Johannes Huebschmann)

Abstract

In this paper, we show that the rational homotopy Lie algebra of classifying spaces for certain types of hyperbolic coformal 2-cones is not nilpotent.

1. Introduction

A simply connected space X is called an n -cone if it is built up by a sequence of cofibrations

$$Y_k \xrightarrow{f} X_{k-1} \xrightarrow{j_k} X_k$$

with $X_0 = *$ and $X_n \simeq X$. One can further assume that $Y_k \simeq \Sigma^{k-1}W_k$ is a $(k-1)$ -fold suspension of a connected space W_k [3]. In particular a 2-cone X is the cofibre of a map between two suspensions

$$\Sigma A \xrightarrow{f} \Sigma B \rightarrow X. \quad (1)$$

Spaces under consideration are assumed to be 1-connected and of finite type, that is, $H^i(X; \mathbb{Q})$ is a finite-dimensional $\mathbb{Q}$ -vector space. To every space X corresponds a free chain Lie algebra of the form $(\mathbb{L}(V), \delta)$ [2], called a Quillen model of X . It is an algebraic model of the rational homotopy type of X . In particular, one has an isomorphism of Lie algebras $H_*(\mathbb{L}(V), \delta) \cong \pi_*(\Omega X) \otimes \mathbb{Q}$. The model is called minimal if $\delta V \subset \mathbb{L}^{\geq 2}(V)$. A space X is called coformal if there is a map of differential Lie algebras $(\mathbb{L}(V), \delta) \rightarrow (\pi_*(\Omega X) \otimes \mathbb{Q}, 0)$ that induces an isomorphism in homology. Any continuous map $f : X \rightarrow Y$ has a Lie representative $\tilde{f} : (\mathbb{L}(W), \delta') \rightarrow (\mathbb{L}(V), \delta)$ between respective models of X and Y .

If X is a 2-cone as defined by (1) and $\tilde{f} : \mathbb{L}(W) \rightarrow \mathbb{L}(V)$ is a model of f , then a Quillen model of the cofibre X of f is obtained as the push out of the following diagram:

$$\begin{array}{ccc} (\mathbb{L}(W), 0) & \xrightarrow{\tilde{f}} & (\mathbb{L}(V), 0) \\ \downarrow \iota & & \downarrow \bar{\iota} \\ (\mathbb{L}(W \oplus sW), d) & \xrightarrow{\tilde{f}} & (\mathbb{L}(V \oplus sW), \delta) \end{array}$$

This work was supported by the Abdus Salam ICTP in cooperation with SIDA.

Received November 14, 2003, revised April 21, 2004; published on May 3, 2004.

2000 Mathematics Subject Classification: Primary 55P62; Secondary 55M30.

Key words and phrases: rational homotopy, coformal spaces, 2-cones, differential Ext.

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where $(\mathbb{L}(W \oplus sW), d)$ is acyclic. Moreover the differential on $\mathbb{L}(V \oplus sW)$ verifies $\delta sW \subset \mathbb{L}(V)$. Hence a 2-cone X has a Quillen model of the form $(\mathbb{L}(V_1 \oplus V_2), \delta)$ such that $\delta V_1 = 0$ and $\delta V_2 \subset \mathbb{L}(V_1)$.

A Sullivan model of a space X is a cochain algebra $(\wedge Z, d)$ that algebraically models the rational homotopy type of X . In particular, one has an isomorphism of graded algebras $H^*(\wedge Z, d) \cong H^*(X; \mathbb{Q})$. The model is called minimal if $dZ \subset \wedge^{\geq 2} Z$. In this case the vector spaces Z^n and $\text{Hom}(\pi_n(X), \mathbb{Q})$ are isomorphic. If X has the rational homotopy type of a finite CW-complex, we say that X is elliptic if Z is finite dimensional, otherwise X is called hyperbolic.

2. Models of classifying spaces

Henceforth X will denote a simply connected finite CW-complex and $\mathcal{L}_X$ its homotopy Lie algebra. Let $\text{aut } X$ denote the space of free self homotopy equivalences of X , $\text{aut}_1(X)$ the path component of $\text{aut } X$ containing the identity map of X . The space $\text{Baut}_1(X)$ classifies fibrations with fibre X over simply connected base spaces [4].

The Schlessinger-Stasheff model for $\text{Baut}_1(X)$ is defined as follows [12]. If $(\mathbb{L}(V), \delta)$ is a Quillen model of X , we define a differential Lie algebra $\text{Der } \mathbb{L}(V) = \bigoplus_{k \geq 1} \text{Der}_k \mathbb{L}(V)$ where $\text{Der}_k \mathbb{L}(V)$ is the vector space of derivations of $\mathbb{L}(V)$ which increase the degree by k , with the restriction that $\text{Der}_1 \mathbb{L}(V)$ is the vector space of derivations of degree 1 that commute with the differential δ .

Define the differential Lie algebra $(s\mathbb{L}(V) \oplus \text{Der } \mathbb{L}(V), D)$ as follows:

- The graded vector space $s\mathbb{L}(V) \oplus \text{Der } \mathbb{L}(V)$ is isomorphic to $s\mathbb{L}(V) \oplus \text{Der } \mathbb{L}(V)$,
- If $\theta, \gamma \in \text{Der } \mathbb{L}(V)$ and $sx, sy \in s\mathbb{L}(V)$, then $[\theta, \gamma] = \theta\gamma - (-1)^{|\theta||\gamma|}\gamma\theta$, $[\theta, sx] = (-1)^{|\theta|}s\theta(x)$ and $[sx, sy] = 0$,
- The differential D is defined by $D\theta = [\delta, \theta]$, $D(sx) = -s\delta x + ad x$, where $ad x$ is the inner derivation determined by x .

From the Sullivan minimal model $(\wedge Z, d)$, Sullivan defines the graded differential Lie algebra $(\text{Der } \wedge Z, D)$ as follows [13]. For $k > 1$, the vector space $(\text{Der } \wedge Z)_k$ consists of the derivations on $\wedge Z$ that decrease the degree by k and $(\text{Der } \wedge Z)_1$ is the vector space of derivations of degree 1 verifying $d\theta + \theta d = 0$. For $\theta, \gamma \in \text{Der } \wedge Z$, the Lie bracket is defined by $[\theta, \gamma] = \theta\gamma - (-1)^{|\theta||\gamma|}\gamma\theta$ and the differential D is defined by $D\theta = [d, \theta]$.

We have the following result:

Theorem 1. [13, 12, 14] *The differential Lie algebras $(\text{Der } \wedge Z, D)$ and $(s\mathbb{L}(V) \oplus \text{Der } \mathbb{L}(V), D)$ are models of the classifying space $\text{Baut}_1(X)$.*

An indirect proof of the Schlessinger-Stasheff model is given in [8, Theorem 2].

3. The classifying space spectral sequence

Recall that if (L, δ) is a graded differential Lie algebra, then L becomes an UL module by the adjoint representation $ad : L \rightarrow \text{Hom}(L, L)$. In the sequel all Lie

algebras are endowed with the above module structure.

Let $(\mathbb{L}(V), \delta)$ be a Quillen model of a finite CW-complex and (TV, d) its enveloping algebra. On the TV -module $TV \otimes (\mathbb{Q} \oplus sV)$, define a $\mathbb{Q}$ -linear map

$$S : TV \otimes (\mathbb{Q} \oplus sV) \rightarrow TV \otimes (\mathbb{Q} \oplus sV)$$

as follows:

- $S(1 \otimes x) = 0$ for all $x \in \mathbb{Q} \oplus sV$,
- $S(v \otimes 1) = 1 \otimes sv$ for all $v \in V$,
- If $a \in TV$ and $x \in TV \otimes (\mathbb{Q} \oplus sV)$ with $|x| > 0$, then $S(ax) = (-1)^{|a|} a.S(x)$.

The differential on the TV -module $TV \otimes (\mathbb{Q} \oplus sV)$ is defined by

$$D(1 \otimes sv) = v \otimes 1 - S(dv \otimes 1) \text{ for } v \in V \text{ and } D(1 \otimes 1) = 0.$$

It follows from [1] that $(TV \otimes (\mathbb{Q} \oplus sV), D)$ is acyclic, hence it is a semifree resolution of $\mathbb{Q}$ as a (TV, d) -module [6, §6].

Using the Schlessinger-Stasheff model of the classifying space, the author proved the following:

Theorem 2. [8] *The differential graded vector spaces $\text{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(V))$ and $s\mathbb{L}(V) \oplus \text{Der } \mathbb{L}(V)$ are isomorphic. Moreover, for $n \geq 0$, the $\mathbb{Q}$ -vector spaces $\text{Ext}_{TV}^n(\mathbb{Q}, \mathbb{L}(V))$ and $\pi_{n+1}(\Omega \text{Baut}_1 X) \otimes \mathbb{Q}$ are isomorphic.*

In particular if X is a coformal space, one has an isomorphism $\pi_n(\text{Baut}_1 X) \otimes \mathbb{Q} \cong \text{Ext}_{U\mathcal{L}_X}^n(\mathbb{Q}, \mathcal{L}_X)$. Therefore $\pi_*(\text{Baut}_1 X) \otimes \mathbb{Q}$ can be computed by the means of a projective resolution of $\mathbb{Q}$ as an $U\mathcal{L}_X$ -module.

Consider the complex $\mathcal{P} = \text{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(V))$. Filter V as follows

$$F_0 V = 0, \quad F_{p+1} V = \{x \in V : dx \in \mathbb{L}(F_p V)\}.$$

We will denote $V_p = F_p V / F_{p-1} V$. If $F_{n-1} V \neq F_n V = V$, following Lemaire [10] we say that V is of length n . We will restrict to spaces with a Quillen model of length n .

Define a filtration on $P = TV \otimes (\mathbb{Q} \oplus sV)$ as follows:

$$P_0 = TV \otimes \mathbb{Q}, \quad P_1 = TV \otimes (\mathbb{Q} \oplus sV_1), \dots, \quad P_n = TV \otimes (\mathbb{Q} \oplus sV_{\leq n}).$$

We filter the complex

$$\text{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(V))$$

by

$$F_k = \{f : f(P_{k-1}) = 0\}.$$

This yields a spectral sequence E_r such that $E_1^{p,q} = \text{Hom}_{\mathbb{Q}}(sV_p, \mathcal{L}_X)$ for $p > 1$, $E_1^{0,q} = \text{Hom}_{\mathbb{Q}}(\mathbb{Q}, \mathcal{L}_X)$ and that converges to $\text{Ext}_{TV}^*(\mathbb{Q}, \mathbb{L}(V))$. This sequence will be called the *classifying space spectral sequence* of X .

Now assume that X is coformal and let $A = U\mathcal{L}_X$. If $\mathbb{L}(V_1)/I$ is a minimal presentation of $\mathcal{L}_X$, then there is a quasi-isomorphism $(\mathbb{L}(V_1 \oplus V_2 \oplus \dots \oplus V_n), \delta) \rightarrow \mathcal{L}_X$ which extends to $p : (TV, d) \xrightarrow{\simeq} (A, 0)$. The (E_1, d) term provides a resolution

$$\dots \rightarrow A \otimes sV_n \rightarrow A \otimes sV_{n-1} \rightarrow \dots \rightarrow A \otimes sV_1 \rightarrow A \rightarrow \mathbb{Q}$$

of $\mathbb{Q}$ as an A -module. Here the differential is given by the composition

$$sV_n \xrightarrow{D} TV \otimes (\mathbb{Q} \oplus sV_{n-1}) \xrightarrow{p \otimes id} A \otimes (\mathbb{Q} \oplus sV_{n-1}).$$

The spectral sequence will therefore collapse at E_2 level. Moreover $\text{Ext}_A^*(\mathbb{Q}, \mathcal{L}_X)$ is endowed with a Lie algebra structure verifying

$$[\text{Ext}^{p,*}, \text{Ext}^{q,*}] \subset \text{Ext}^{p+q-1,*}. \quad (2)$$

The Lie bracket can be defined using the bijection between the Koszul complex $C^*(\mathcal{L}_X, \mathcal{L}_X)$ and derivations on the Sullivan model $C^*(\mathcal{L}_X, \mathbb{Q})$ of X [9, Proposition 4] (see also [7] for a direct definition of the Lie bracket on $C^*(\mathcal{L}_X, \mathcal{L}_X)$). Alternatively one may use the bijection

$$\text{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(V)) \cong s\mathbb{L}(V) \oplus \text{Der } \mathbb{L}(V)$$

to transfer a Lie algebra structure on $\text{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(V))$ from $s\mathbb{L}(V) \oplus \text{Der } \mathbb{L}(V)$.

Definition 3. Let L be a Lie algebra. An element $x \in L$ is called locally nilpotent if for every $y \in L$, there is a positive integer k such that $(ad x)^k(y) = 0$. A subset $K \subset L$ is called locally nilpotent if each element of K is locally nilpotent.

We deduce from Equation (2) the following

Proposition 4. Let X be a coformal space of homotopy Lie algebra denoted $\mathcal{L}_X$. If X has a Quillen model $(\mathbb{L}(V), \delta)$, of length n , one has:

1. For $k \neq 1$, $\text{Ext}_A^k(\mathbb{Q}, \mathcal{L}_X)$ is locally nilpotent,
2. $\text{Ext}_A^1(\mathbb{Q}, \mathcal{L}_X)$ is a subalgebra of $\text{Ext}_A(\mathbb{Q}, \mathcal{L}_X)$,
3. If $\text{Ext}_A^0(\mathbb{Q}, \mathcal{L}_X) = 0$, then $\oplus_{i \geq i_0} \text{Ext}_A^i(\mathbb{Q}, \mathcal{L}_X)$ is an ideal of $\text{Ext}_A(\mathbb{Q}, \mathcal{L}_X)$, for $i_0 \geq 1$.

We will now assume that X is a coformal 2-cone. Recall that X has a Quillen minimal model of the form $(\mathbb{L}(V_1 \oplus V_2), \delta)$, with $\delta V_1 = 0$ and $\delta V_2 \subset \mathbb{L}(V_1)$. Moreover $\pi_*(\Omega X) \otimes \mathbb{Q} = H_*(\mathbb{L}(V_1 \oplus V_2), \delta) = \mathbb{L}(V_1)/I$, where I is the ideal of $\mathbb{L}(V_1)$ generated by δV_2 .

Definition 5. Let $\mathbb{L}(V)$ be a free Lie algebra where $\{a, b, c, \dots\}$ is a basis of V . Denote $\mathbb{L}^n(V)$ the subspace of $\mathbb{L}(V)$ consisting of Lie brackets of length n . Consider a basis $\{u_1, u_2, \dots\}$ of $\mathbb{L}^n(V)$ where each u_i is a Lie monomial. If $x \in \{a, b, c, \dots\}$, we define the length of u_i in the variable x , $l_x(u_i)$, as the number of occurrences of the letter x in u_i . If $u = \sum r_i u_i \in \mathbb{L}^n(V)$, define $l_x(u) = \min\{l_x(u_i)\}$ and if $v = \sum v_i$ where $v_i \in \mathbb{L}^i(V)$, $l_x(v) = \min\{l_x(v_i)\}$.

It is straightforward that the above definition extends to the enveloping algebra $T(V)$.

Theorem 6. Let X be a coformal 2-cone and $(\mathbb{L}(V_1 \oplus V_2), \delta)$ be its Quillen minimal model. Choose a basis $\{x_1, x_2, \dots\}$ for V_1 and a basis $\{y_1, y_2, \dots\}$ for V_2 . If for some $x_k \in \{x_1, x_2, \dots\}$, $l_{x_k}(\delta y_j) \geq 2$ for all $y_j \in \{y_1, y_2, \dots\}$, then $\text{Ext}_A^{2,*}(\mathbb{Q}, \mathcal{L}_X)$ is infinite dimensional.

Proof. Note that for $i \neq k$ the element $(ad x_i)^n(x_k)$ is a nonzero homology class in $H_*(\mathbb{L}(V_1 \oplus V_2), \delta)$ as it contains only one occurrence of x_k . Take $y_t \in \{y_1, y_2, \dots\}$ and $x_m \in \{x_1, x_2, \dots\}$ with $m \neq k$. For each $n \geq 1$, define $f_n \in \text{Hom}_A(A \otimes sV_2, \mathcal{L}_X)$ by $f_n(sy_t) = (ad x_m)^n(x_k)$ and $f_n(sy_j) = 0$ for $j \neq t$. Obviously $f_n \in \text{Hom}_A(A \otimes sV_2, \mathcal{L}_X)$ is a cocycle. Suppose that f_n is a coboundary. There exists $g_n \in \text{Hom}_A(A \otimes sV_1, \mathcal{L}_X)$ such that $f_n(sy_t) = g_n(dsy_t)$. From the definition of the differential d , one has $dsy_t = \sum_i p_i s x_i$, where the p_i 's are polynomials in the variables $x_1, x_2, \dots$. From the hypothesis on the differential dy_t one knows that $l_{x_k}(p_i) \geq 2$ for $i \neq k$ and $l_{x_k}(p_k) \geq 1$. By using the number of occurrences of the variable x_k , one deduces from the previous equalities that $(ad x_m)^n(x_k)$ equals the component of length 1 in x_k of $p_k g_n(sx_k)$. Therefore, in the monomial decomposition of $g_n(sx_k)$ (resp. p_k) there must exist $(ad x_m)^{n-s}(x_k)$ (resp. x_m^s). We obtain a contradiction with $l_{x_k}(p_k) \geq 1$.

The cocycles f_n create an infinite number of non zero classes (of distinct degrees) and the space $\text{Ext}_A^{2,*}(\mathbb{Q}, \mathcal{L}_X)$ is infinite dimensional. $\square$

Corollary 7. *If hypotheses of the above theorem are satisfied, then $\text{cat}(Baut_1(X)) = \infty$.*

Proof. If $sx \in \text{Ext}^{0,*} \subset \mathbb{L}(V_1)/I$ and $f \in \text{Ext}^{2,*}$ then $[f, sx] = \pm sf(x)$. As elements of $\text{Ext}^{2,*}$ vanish on V_1 , we deduce that $[\text{Ext}^{2,*}, \text{Ext}^{0,*}] = 0$. It follows from Theorem 6 that $J = \text{Ext}_{U\mathcal{L}_X}^2(\mathbb{Q}, \mathcal{L}_X)$ is an infinite dimensional ideal of $\pi_*(\Omega Baut_1(X))$. Moreover it follows from Equation (2) that J is abelian. We deduce that the category of $Baut_1(X)$ is infinite [5, Theorem 12.2]. $\square$

If X is an elliptic space of Sullivan minimal model $(\wedge Z, d)$ then $Der \wedge Z$ is a finite dimensional $\mathbb{Q}$ -vector space. Hence the homotopy Lie algebra of $Baut_1(X)$ is finite dimensional, therefore $\pi_*(\Omega Baut_1(X)) \otimes \mathbb{Q}$ is nilpotent. In [11], P. Salvatore proved that if $X = S^{2n+1} \vee S^{2n+1}$, then $\pi_*(\Omega Baut_1(X)) \otimes \mathbb{Q}$ contains an element α that is not locally nilpotent. The proof consists in the construction of two outer derivations α and β of the free Lie algebra $\mathbb{L}(a, b)$, where $|a| = |b| = 2n$, such that $(ad \alpha)^i(\beta) \neq 0$, for every integer $i > 0$. The technique can be applied to any free Lie algebra with at least two generators. Therefore $\pi_*(\Omega Baut_1(X)) \otimes \mathbb{Q}$ contains an element α that is not locally nilpotent if X is a wedge of two spheres or more.

P. Salvatore asked if $\pi_*(\Omega Baut_1(X)) \otimes \mathbb{Q}$ has always such a property for every hyperbolic space X . A positive answer to this question would provide another characterization of the elliptic-hyperbolic dichotomy [5].

For a product space we have the following

Proposition 8. *If $X = Y \times Z$ is a product space such that the Lie algebra $\pi_*(\Omega Baut_1(Y)) \otimes \mathbb{Q}$ is not nilpotent, then $\pi_*(\Omega Baut_1(X)) \otimes \mathbb{Q}$ is not nilpotent.*

Proof. Let $(\wedge V, d)$ and $(\wedge W, d')$ be Sullivan models of Y and Z respectively. Therefore $(\wedge V \otimes \wedge W, d \otimes d')$ is a Sullivan model of X . It follows from [12] that

$$H_*(Der(\wedge V \otimes \wedge W)) \cong H_*(Der \wedge V) \otimes H^*(\wedge W) \oplus H^*(\wedge V) \otimes H_*(Der \wedge W).$$

Therefore $\pi_*(\Omega Baut_1(Y)) \otimes \mathbb{Q}$ is a subalgebra of $\pi_*(\Omega Baut_1(X)) \otimes \mathbb{Q}$. $\square$

In particular if Y is a wedge of at least two spheres, then the Lie algebra $\pi_*(\Omega B \operatorname{aut}_1(Y)) \otimes \mathbb{Q}$ is not nilpotent and so is $\pi_*(\Omega B \operatorname{aut}_1(X)) \otimes \mathbb{Q}$.

We can extend Salvatore's result to some certain types of coformal hyperbolic 2-cones.

Theorem 9. *Under the hypotheses of Theorem 6, the rational homotopy Lie algebra of $B \operatorname{aut}_1(X)$ is not nilpotent.*

Proof. For $i \neq k$, let $(ad x_k)^n(x_i)$ be a nonzero element of $\mathcal{L}_X$. Define $\alpha_n \in \operatorname{Ext}_A^1(\mathbb{Q}, \mathcal{L}_X)$ by $\alpha_n(sx_i) = (ad x_k)^n(x_i)$ and zero on the other generators of $\mathcal{L}_X$. Take $w \in V_2$ and define $\beta_m \in \operatorname{Ext}_A^2(\mathbb{Q}, \mathcal{L}_X)$ by $\beta_m(sw) = (ad x_k)^m(x_i)$ and zero elsewhere. A short computation shows that $[\alpha_n, \beta_m] = \pm \beta_{m+n}$. Hence $(ad \alpha_n)^l(\beta_m) \neq 0$ for all $l \geq 1$. Therefore $\pi_*(\Omega B \operatorname{aut}_1(X)) \otimes \mathbb{Q}$ is not nilpotent. $\square$

Example 10. Consider the space X for which the Quillen minimal model is $(\mathbb{L}(a, b, c), d)$ with $da = db = 0$ and $dc = [b, [b, a]]$. The space X satisfies the hypothesis of Theorem 6. Therefore $\operatorname{cat}(B \operatorname{aut}_1(X))$ is infinite. Moreover the homotopy Lie algebra of $B \operatorname{aut}_1(X) \otimes \mathbb{Q}$ is not nilpotent.

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