

MATRICES AND FINITE QUANDLES

BENITA HO AND SAM NELSON

(communicated by Ronald Brown)

Abstract

Finite quandles with n elements can be represented as $n \times n$ matrices. We show how to use these matrices to distinguish all isomorphism classes of finite quandles for a given cardinality n , as well as how to compute the automorphism group of each finite quandle. As an application, we classify finite quandles with up to 5 elements and compute the automorphism group for each quandle.

1. Introduction

A *quandle* is a set Q with a binary operation $\triangleright : Q \times Q \rightarrow Q$ satisfying the three axioms

- (i) for every $a \in Q$, we have $a \triangleright a = a$,
- (ii) for every pair $a, b \in Q$ there is a unique $c \in Q$ such that $a = c \triangleright b$, and
- (iii) for every $a, b, c \in Q$, we have $(a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c)$.

The uniqueness in axiom (ii) implies that the map $f_b : Q \rightarrow Q$ defined by $f_b(a) = a \triangleright b$ is a bijection; the inverse map f_b^{-1} then defines the *dual operation* $a \triangleleft b = f_b^{-1}(a)$. The set Q then forms a quandle under \triangleleft , called the *dual* of (Q, \triangleright) .

Quandle theory may be thought of as analogous to group theory. Indeed, groups are quandles with the quandle operation given by n -fold conjugation for an integer n , i.e.,

$$a \triangleright b = b^{-n} a b^n.$$

Another important example of a type of quandle structure is the category of *Alexander quandles*, i.e., modules M over the ring $\Lambda = \mathbb{Z}[t^{\pm 1}]$ of Laurent polynomials in one variable with quandle operation

$$a \triangleright b = ta + (1 - t)b.$$

The second author has written elsewhere on Alexander quandles; see [7] and [8].

Other examples of quandles include *Dehn quandles*, i.e., the set of isotopy classes of simple closed curves on a surface Σ with action given by Dehn twists, and *Coxeter*

Received May 29, 2005, revised July 9, 2005; published on Nov. 7, 2005; updated Nov. 10, 2005.

2000 Mathematics Subject Classification: 57M27.

Key words and phrases: Finite quandles.

© 2005, Benita Ho and Sam Nelson. Permission to copy for private use granted.

quandles, i. e., $\mathbb{R}^n \setminus 0$ with

$$u \triangleright v = 2 \frac{(u, v)}{(v, v)} - u$$

where $(,)$ is a symmetric bilinear form. See [11] and [3] for more.

So far, quandles have been of interest primarily to knot theorists, due to their utility in defining invariants of knots. In [5], a quandle is associated to every topological space, called the *fundamental quandle*. In particular, it is shown that isomorphisms of the *knot quandle* (definable from a knot diagram by a Wirtinger-style presentation) preserve peripheral structure, making the knot quandle a complete invariant of knot type considered up to homeomorphism of topological pairs, though not up to ambient isotopy.

Finite quandles have been used to define invariants of both knots and links in S^3 and generalizations of knots such as knotted surfaces in \mathbb{R}^4 and virtual knots. The simplest example of such an invariant is the number of homomorphisms from the knot quandle to a chosen finite quandle. One can also obtain knot invariants by counting homomorphisms with crossings weighted by quandle cocycles arising in various quandle cohomology theories. See [1] and [2] for more.

In this paper, we show how to associate to any finite quandle $Q = \{x_1, x_2, \dots, x_n\}$ an $n \times n$ matrix M_Q . We then define an equivalence relation on the set of $n \times n$ quandle matrices, which we call p -equivalence (after we decided that our initial choice of “ ρ -equivalence” would be confusing). Our main theorem then says that two such matrices represent isomorphic quandles iff they are p -equivalent.

We then give an algorithm for applying this result to determine all isomorphism classes of quandles with n elements as well as their automorphism groups, and as an application we determine all isomorphism classes and automorphism groups of quandles with up to 5 elements.

After the initial posting of the preprint of this paper to arXiv.org, a paper with similar results was posted by Lopes and Roseman [6]. We have also learned that related results were obtained by Hayley Ryder in her dissertation [10].

2. The matrix of a finite quandle

Let $Q = \{x_1, x_2, \dots, x_n\}$ be a finite quandle with n elements. We define the matrix of Q , denoted M_Q , to be the matrix whose entry in row i column j is $x_i \triangleright x_j$:

$$M_Q = \begin{bmatrix} x_1 \triangleright x_1 & x_1 \triangleright x_2 & \dots & x_1 \triangleright x_n \\ x_2 \triangleright x_1 & x_2 \triangleright x_2 & \dots & x_2 \triangleright x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n \triangleright x_1 & x_n \triangleright x_2 & \dots & x_n \triangleright x_n \end{bmatrix}.$$

The matrix M_Q is really just the quandle operation table considered as a matrix, with the columns acting on the rows. In particular, if the elements of the quandle are the numbers $Q = \{1, 2, \dots, n\}$ with $M_Q = (\alpha_{ij})$, call M_Q an *integral quandle matrix*. We may obtain an integral quandle matrix by suppressing the “ x ”s in the notation and just writing the subscripts; hence we lose no generality by restricting

our attention to integral quandle matrices. If the entries on the diagonal in an integral quandle matrix are in the usual order, i.e., $\alpha_{ii} = i$, then $i \triangleright j$ is just the entry in row i column j . An integral quandle matrix of this type is in *standard form*.

The quandle axioms place certain restrictions on what kind of matrices can arise from a quandle.

Lemma 1. Let $M = (\alpha_{ij})$ be an $n \times n$ matrix with $\alpha_{ij} \in \{1, 2, \dots, n\}$. Then $M = M_Q$ for a finite quandle Q if and only if the following conditions are satisfied:

- (i) The diagonal entries are distinct, i.e. $\alpha_{ii} = \alpha_{jj}$ implies $i = j$. If this condition is satisfied, denote the row number containing x on the diagonal by $r(x)$ and the column number containing x on the diagonal by $c(x)$.
- (ii) The entries in each column are distinct, i.e. $\alpha_{ij} = \alpha_{kj}$ implies $i = k$.
- (iii) The entries must satisfy

$$\alpha_{r(\alpha_{r(x)c(y)})c(z)} = \alpha_{r(\alpha_{r(x)c(z)})c(\alpha_{r(y)c(z)})},$$

or if we denote $\alpha_{ij} = \alpha[i, j]$,

$$\alpha[\alpha[i, j], k] = \alpha[\alpha[i, k], \alpha[j, k]].$$

Proof. Suppose Q is a quandle and consider the matrix $M_Q = (\alpha_{ij})$. Since $x \triangleright x = x$, we have

$$\alpha_{ii} = x_i \triangleright x_i = x_i.$$

Distinctness of elements on the diagonal is then equivalent to distinctness of elements of the quandle. Conversely, if x_i appears in two positions on the diagonal, say $\alpha_{ii} = x_i = \alpha_{jj}$ then we have $x_j \triangleright x_j \neq x_j$ and Q is not a quandle.

If Q is a quandle, then since $\alpha_{ij} = x_i \triangleright x_j$, column j of M_Q consists of elements of the form $x_i \triangleright x_j$. Quandle axiom (ii) says that for every $a, b \in Q$ there is a unique c such that $a = c \triangleright b$, so

$$\alpha_{ij} = x_i \triangleright x_j = x_k \triangleright x_j = \alpha_{kj}$$

implies $x_i = x_k$, which implies $i = k$. Conversely, if the entries in column $c(x_j)$ are distinct, the fact that there are n entries chosen from $\{x_1, x_2, \dots, x_n\}$ implies that every element appears in the column $c(x_j)$, that is, every element is $x_i \triangleright x_j$ for a unique x_i .

Finally, condition (iii) is simply quandle axiom (iii) rewritten with the notation $\alpha_{r(x_i)x(x_j)} = x_i \triangleright x_j$. □

Corollary 2. If M_Q is a quandle matrix, we can read the row and column labels off the diagonal: if $\alpha_{ii} = x$, then the entries in row i are of the form $x \triangleright y$ and the entries in column i are of the form $y \triangleright x$.

It is worth noting that if Q is not a quandle but a *rack*, i.e. if $\{Q, \triangleright\}$ satisfies quandle axioms (ii) and (iii) but not necessarily (i), then corollary 2 does not hold, and there is no standard form matrix presentation for non-quandle racks. In particular, to represent racks with matrices, we need to keep track of which row and column

represent which element of Q , since unlike the quandle case, in a non-quandle rack we cannot recover this information from the matrix itself.

Corollary 3. If $M = M_Q$ is an integral quandle matrix for a quandle Q then the trace of M_Q is

$$\text{tr}(M_Q) = \frac{n(n+1)}{2}.$$

Proof. By lemma 2, the diagonal is a permutation of $\{1, 2, \dots, n\}$. Then

$$\text{tr}(M_Q) = \sum_{x=1}^n x = \frac{n(n+1)}{2}.$$

□

Definition 1. Let $\rho \in \Sigma_n$ be a permutation of $\{1, 2, \dots, n\}$. Set

$$\rho(M_Q) = A_\rho^{-1}(\rho(\alpha_{ij}))A_\rho$$

where $M_Q = (\alpha_{ij})$ and A_ρ is the permutation matrix of ρ . Then we say $\rho(M_Q)$ is *p-equivalent* or *permutation-equivalent* to M_Q , and write $\rho(M_Q) \sim_p M_Q$.

The fact that *p-equivalence* is an equivalence relation follows from the fact that Σ_n is a group. We now can prove our main theorem.

Theorem 4. Two integral quandle matrices in standard form determine isomorphic quandles iff they are *p-equivalent* by a permutation $\rho \in \Sigma_n$.

Proof. Let $\rho : Q \rightarrow Q'$ be an isomorphism of finite quandles and let $M_Q, M_{Q'}$ be the standard form integral quandle matrices of Q and Q' respectively. Since ρ is a bijection $\rho : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$, we have $\rho \in \Sigma_n$.

Then $\rho(i \triangleright j) = \rho(i) \triangleright \rho(j)$ says that in the operation table of Q' , the element in row $r(\rho(i))$ and column $c(\rho(j))$ is $\rho(i \triangleright j)$; that is, we obtain an operation table for Q' by applying the permutation ρ to every element in the table, including the row and column labels (which we can recover from the diagonal). Conjugation by the permutation matrix of ρ then puts the matrix back in standard form.

Conversely, if $M_{Q'}$ is *p-equivalent* to M_Q by a permutation ρ , then the element in row $r(\rho(i))$ and column $c(\rho(j))$ in $M_{Q'}$ is $\rho(i \triangleright j)$, that is,

$$\rho(i) \triangleright \rho(j) = \rho(i \triangleright j)$$

and ρ is an isomorphism of quandles. □

Corollary 5. The automorphism group of a finite quandle Q of order n is isomorphic to the subgroup of Σ_n which fixes M_Q , i.e.,

$$\text{Aut}(Q) \cong \{\rho \in \Sigma_n \mid \rho(M_Q) = M_Q\} \subseteq \Sigma_n.$$

Proof. A quandle automorphism of Q is a quandle isomorphism $\rho : Q \rightarrow Q$. Theorem 2 then implies that $\rho \in \Sigma_n$ induces an automorphism of Q iff $\rho(M_Q) = M_Q$. □

Example 1. The trivial quandle of order n , T_n , has integral quandle matrix

$$M_{T_n} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 2 & 2 & \dots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ n & n & \dots & n \end{bmatrix}.$$

It is easy to check that $\rho(M_{T_n}) = M_{T_n}$ for all $\rho \in \Sigma_n$, and by corollary 2, $\text{Aut}(T_n) \cong \Sigma_n$.

Example 2. The quandle matrix

$$M_Q = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 2 \\ 4 & 4 & 4 & 4 \end{bmatrix} \text{ is } p\text{-equivalent to } \rho(M_Q) = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 2 & 1 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$$

with $\rho = (1432)$, since

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 2 & 1 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 1 \\ 3 & 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &= A_p^{-1}(\rho(M_Q))A_p. \end{aligned}$$

The number $N_p(Q)$ of standard form integral quandle matrices in the p -equivalence class of Q is an invariant of quandle type. A conjugation $\phi\rho\phi^{-1}$ of an automorphism $\rho \in \text{Aut}(Q)$ by an isomorphism $\phi : Q \rightarrow Q'$ yields an automorphism of Q' , so we have

$$Q \cong Q' \Rightarrow N_p(Q) = N_p(Q').$$

Then since every permutation $\rho \in \Sigma$ defines either an automorphism of Q or an isomorphism from Q to a p -equivalent quandle Q' , we have

Corollary 6. Let Q be a quandle with n elements. Then

$$|\Sigma_n| = N_p(Q)|\text{Aut}(Q)|.$$

Joyce, in [5], defined quandle to be *algebraically connected* or just *connected* if the quandle has only one orbit under the inner automorphism group, that is, if the set

$$O(a) = \{(\dots((a \diamond_1 b_1) \diamond_2 b_2) \dots \diamond_n b_n) \mid b_i \in Q, \diamond_i \in \{\triangleright, \triangleleft\}\} = Q \text{ for all } a \in Q.$$

By lemma 2, we know that the columns in an integral quandle matrix M_Q must be permutations of $\{1, 2, \dots, n\}$. If the rows in M_Q are also permutations of $\{1, 2, \dots, n\}$, then Q is connected. A matrix in which both rows and columns are permutations of $\{1, 2, \dots, n\}$ is called a *latin square*, and a quandle whose matrix is a latin square is connected. However, not every latin square is a quandle matrix;

for example, the latin square

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

fails the first condition for being a quandle matrix.

Definition 2. A quandle Q is *latin* if the matrix of Q is a latin square, that is, if every row of the matrix of Q is a permutation of $\{1, 2, \dots, n\}$.

Moreover, not every connected quandle is latin. The conditions of Q being latin and Q being connected coincide when Q is the conjugation quandle of a group, since

$$\begin{aligned} (\dots((a \diamond_1 b_1) \diamond_2 b_2) \dots \diamond_n b_n) &= b_n^{-j_n} \dots (b_2^{-j_2} (b_1^{-j_1} a b_1^{j_1}) b_2^{j_2}) \dots b_n^{j_n} \\ &= (b_n^{-j_n} \dots b_2^{-j_2} b_1^{-j_1}) a (b_1^{j_1} b_2^{j_2} \dots b_n^{j_n}) \\ &= (b_1^{j_1} b_2^{j_2} \dots b_n^{j_n})^{-1} a (b_1^{j_1} b_2^{j_2} \dots b_n^{j_n}), \end{aligned}$$

where $j_k = \pm 1$, and every element of $O(a)$ is $a \triangleright b$ for some $b \in Q$. If a quandle is isomorphic to union of a proper subset of conjugacy classes in a group, then the group elements defining some inner automorphisms may not be elements of the quandle, and we can have connected quandles which are non-latin. For example, the quandle of transpositions in Σ_6 is connected and non-latin.¹

$$M_Q = \begin{bmatrix} 1 & 4 & 5 & 2 & 3 & 1 \\ 4 & 2 & 6 & 1 & 2 & 3 \\ 5 & 6 & 3 & 3 & 1 & 2 \\ 2 & 1 & 4 & 4 & 6 & 5 \\ 3 & 5 & 1 & 6 & 5 & 4 \\ 6 & 3 & 2 & 5 & 4 & 6 \end{bmatrix}$$

Connected quandles are of prime interest since knot quandles are connected. A list of known connected quandles together with an algorithm for finding connected quandles is given in [9]. A previous computer search by S. Yamada for isomorphism classes of quandles is mentioned, though only the resulting connected quandles are listed.

3. Computational results

In this section, we describe an algorithm for determining all quandles of order n by computing all standard form integral quandle matrices of order n . We then give the results of application of this algorithm for $n = 3$, $n = 4$ and $n = 5$. We also determine the automorphism group of each quandle as well as a presentation of Q as an Alexander quandle when appropriate. The maple code used to obtain these results is available on the second author’s website at <http://www.esotericka.org/quandles>, as is some more recent and much faster C code [4].

To determine all quandles of order n , we first determine for each $i = 1, \dots, n$ a list $P_{n,i}$ of all vectors whose entries are permutations of the set $\{1, 2, \dots, n\}$

¹Thanks to Steven Wallace for bringing this example to the authors’ attention.

with entry i in the i th position. We then consider all matrices $M = (\alpha_{ij})$ with columns C_1, \dots, C_n chosen from $P_{n,i}$ respectively, since we lose no generality by considering only quandle matrices in standard form. For each matrix which satisfies this condition, we then check whether

$$\alpha_{\alpha_{ij}k} = \alpha_{\alpha_{ik}\alpha_{jk}}$$

for each triple $i, j, k = 1, 2, \dots, n$.² We then check the resulting list of quandle matrices to determine p -equivalence classes. One way to do this is to compare M and $\rho(M')$ for each $\rho \in P_n$ for every pair M, M' of quandle matrices, removing M' from the list whenever $M = \rho(M')$ for some $M \neq M'$. To compute $\text{Aut}(Q)$, we simply note which permutations fix a representative matrix M_Q of Q .

It is easy to check that there is only one quandle of order 1 and one quandle of order 2, both trivial (i.e., $x \triangleright y = x \forall x \in Q$.) Application of the above algorithm shows that there are three quandle isomorphism classes of order 3, 7 isomorphism classes of quandles of order 4 and 22 isomorphism classes of quandles of order 5. Representative quandle matrices for each of these are listed in the tables below. In general, for quandles of order n , the above algorithm requires $(n - 1)^n$ passes through the loop, each pass of which can require up to n^3 checks of the third quandle condition.

As a question for further research, we would like to know whether there are quandle invariants derivable from M_Q via linear algebra. A natural first attempt to find such an invariant is to consider the determinant of M_Q . Unfortunately, p -equivalence does not generally preserve determinants:

$$\begin{vmatrix} 1 & 4 & 5 & 2 & 3 \\ 3 & 2 & 1 & 5 & 4 \\ 4 & 5 & 3 & 1 & 2 \\ 5 & 3 & 2 & 4 & 1 \\ 2 & 1 & 4 & 3 & 5 \end{vmatrix} = -825 \neq -1875 = \begin{vmatrix} 1 & 5 & 4 & 3 & 2 \\ 3 & 2 & 1 & 5 & 4 \\ 5 & 4 & 3 & 2 & 1 \\ 2 & 1 & 5 & 4 & 3 \\ 4 & 3 & 2 & 1 & 5 \end{vmatrix},$$

but these two matrices are p -equivalent via the permutation (153)(24).

²It is helpful to make sure the program exits the loop at the first triple which does not satisfy the condition!

References

- [1] **Carter, J.S. and Saito, M.** *Generalizations of Quandle Cocycle Invariants and Alexander Modules from Quandle Modules*, arXiv.org: math.GT/0406190
- [2] **Carter, J. S., Elhamdadi, M., Graña, M., and Saito, M.** *Cocycle Knot Invariants from Quandle Modules and Generalized Quandle Cohomology*, arXiv.org: math.GT/0306068
- [3] **Fenn, R. and Rourke, C.** *Racks and links in codimension two*, J. Knot Theory Ramifications 1 (1992) 343-406.
- [4] **Henderson, R., Macedo, T. and Nelson, S.** *Symbolic computation with finite quandles*, arXiv.org:Math.GT/0508351
- [5] **Joyce, D.** *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra 23 (1982) 37-65.
- [6] **Lopes, P and Roseman, D.** *On Finite Racks and Quandles*, arXiv.org: math.GT/0412487.
- [7] **Nelson, S.** *Classification of finite Alexander quandles*, Topology Proceedings 27 (2003) 245-258.
- [8] **Murillo, G. and Nelson, S.** *Alexander quandles of order 16*, arXiv.org:Math.GT/0409460
- [9] **Ohtsuki, T. (Ed).** *Problems on invariants of knots and 3-manifolds*, Geom. Topol. Monogr. 4 (2002) 377-572.
- [10] **Ryder, H.**, *The structure of racks*, Ph.D. thesis, Warwick University, 1992.
- [11] **Yetter, D. N.** *Quandles and monodromy*, J. Knot Theory Ramifications 12 (2003) 523-541.

M_Q	Alexander presentation	$\text{Aut}(Q)$	$N_p(Q)$
$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$	$\Lambda_3/(t+2)$	Σ_3	1
$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$	$\Lambda_3/(t+1)$	Σ_3	1
$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 2 \\ 2 & 3 & 3 \end{bmatrix}$	-	\mathbb{Z}_2	3

Figure 1: Quandle matrices for quandles of order 3

M_Q	Alexander presentation	$\text{Aut}(Q)$
$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$	$\Lambda_4/(t+3)$	Σ_4
$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 2 \\ 4 & 4 & 4 & 4 \end{bmatrix}$	–	\mathbb{Z}_2
$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 1 \\ 4 & 4 & 4 & 4 \end{bmatrix}$	–	\mathbb{Z}_3
$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$	–	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 4 & 3 \\ 3 & 4 & 3 & 2 \\ 4 & 3 & 2 & 4 \end{bmatrix}$	–	Σ_3
$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 4 & 4 & 3 & 3 \\ 3 & 3 & 4 & 4 \end{bmatrix}$	$\Lambda_2/(t^2+1)$	D_8
$\begin{bmatrix} 1 & 4 & 2 & 3 \\ 3 & 2 & 4 & 1 \\ 4 & 1 & 3 & 2 \\ 2 & 3 & 1 & 4 \end{bmatrix}$	$\Lambda_2/(t^2+t+1)$	A_4

Figure 2: Quandle matrices for quandles of order 4

Q_M	Alex. pres.	Aut(Q)	Q_M	Alex. pres.	Aut(Q)
$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}$	$\Lambda_5/(t+4)$	Σ_5	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 4 & 3 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}$	-	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 4 & 2 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}$	-	\mathbb{Z}_3	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 1 \\ 3 & 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 4 & 3 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}$	-	D_8
$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 4 & 1 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}$	-	\mathbb{Z}_4	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 3 & 3 \\ 3 & 3 & 3 & 2 & 2 \\ 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}$	-	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 & 3 \\ 3 & 3 & 3 & 1 & 1 \\ 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}$	-	$\mathbb{Z}_3 \oplus \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 & 1 \\ 3 & 3 & 3 & 1 & 2 \\ 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}$	-	Σ_3
$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 5 & 4 \\ 4 & 4 & 5 & 4 & 3 \\ 5 & 5 & 4 & 3 & 5 \end{bmatrix}$	-	$\Sigma_3 \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 5 & 4 & 4 \\ 5 & 5 & 4 & 5 & 5 \end{bmatrix}$	-	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}$	-	$\Sigma_3 \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 4 & 3 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}$	-	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$

Figure 3: Quandle matrices for quandles of order 5 - part 1

Q_M	Alex. pres.	Aut(Q)	Q_M	Alex. pres.	Aut(Q)
$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 1 & 1 & 1 \\ 3 & 3 & 3 & 5 & 4 \\ 4 & 4 & 5 & 4 & 3 \\ 5 & 5 & 4 & 3 & 5 \end{bmatrix}$	-	$\Sigma_3 \times \mathbb{Z}_2$	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 5 & 5 & 5 & 4 & 4 \\ 4 & 4 & 4 & 5 & 5 \end{bmatrix}$	-	D_8
$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 5 & 3 & 4 \\ 3 & 4 & 3 & 5 & 2 \\ 4 & 5 & 2 & 4 & 3 \\ 5 & 3 & 4 & 2 & 5 \end{bmatrix}$	-	A_4	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 3 & 3 \\ 3 & 3 & 3 & 2 & 2 \\ 5 & 5 & 5 & 4 & 4 \\ 4 & 4 & 4 & 5 & 5 \end{bmatrix}$	-	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 & 3 \\ 3 & 3 & 3 & 1 & 1 \\ 5 & 5 & 5 & 4 & 4 \\ 4 & 4 & 4 & 5 & 5 \end{bmatrix}$	-	$\mathbb{Z}_3 \oplus \mathbb{Z}_2$	$\begin{bmatrix} 1 & 3 & 4 & 5 & 2 \\ 3 & 2 & 5 & 1 & 4 \\ 4 & 5 & 3 & 2 & 1 \\ 5 & 1 & 2 & 4 & 3 \\ 2 & 4 & 1 & 3 & 5 \end{bmatrix}$	$\Lambda_5/(t+2)$	D_{20}
$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 1 & 1 & 1 \\ 4 & 5 & 3 & 5 & 4 \\ 5 & 3 & 5 & 4 & 3 \\ 3 & 4 & 4 & 3 & 5 \end{bmatrix}$	-	Σ_3	$\begin{bmatrix} 1 & 4 & 5 & 3 & 2 \\ 3 & 2 & 4 & 5 & 1 \\ 2 & 5 & 3 & 1 & 4 \\ 5 & 1 & 2 & 4 & 3 \\ 4 & 3 & 1 & 2 & 5 \end{bmatrix}$	$\Lambda_5/(t+1)$	D_{20}
$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 3 & 3 \\ 3 & 3 & 3 & 2 & 2 \\ 4 & 5 & 5 & 4 & 4 \\ 5 & 4 & 4 & 5 & 5 \end{bmatrix}$	-	D_8	$\begin{bmatrix} 1 & 4 & 5 & 2 & 3 \\ 3 & 2 & 1 & 5 & 4 \\ 4 & 5 & 3 & 1 & 2 \\ 5 & 3 & 2 & 4 & 1 \\ 2 & 1 & 4 & 3 & 5 \end{bmatrix}$	$\Lambda_5/(t+3)$	D_{20}

Figure 4: Quandle matrices for quandles of order 5 - part 2

This article is available at <http://intlpress.com/HHA/v7/n1/a11/>

Benita Ho benitakho@yahoo.com

University of California, Riverside
900 University Avenue
Riverside, CA, 92521

Sam Nelson knots@esotericka.org

University of California, Riverside
900 University Avenue
Riverside, CA, 92521