On the Number of Daubechies Scaling Functions and a Conjecture of Chyzak et al.

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Using a result on Riesz factorizations, we show that there are at most $2^{N-1}$ and at least $2^{\left\lceil \frac{N}{2} \right\rceil}$ distinct Daubechies scaling functions with support in $[1-N, N]$.

We define a Daubechies scaling function to be a function $\varphi \in L^2(\mathbb{R})$, with support in $[1-N, N]$ and satisfying the dilation equation

$$\varphi(x) = \sum_{k \in \mathbb{Z}} h_k \varphi(2x-k),$$

where $N$ is a positive integer and the sequence $\{h_k\}$, known as the scaling function’s filter sequence, satisfies the conditions $h_k = 0$ for $k \notin [1-N, N]$ and

$$\sum_{k=1-N}^{N} h_k = 2,$$

$$\sum_{k=1-N}^{N} h_k h_{k-2l} = 2\delta_{0,l}, \quad \text{for } l = 0, \ldots, N-1,$$

$$\sum_{k=1-N}^{N} (-1)^k h_{1-k} k^l = 0, \quad \text{for } l = 0, \ldots, N-1.$$

These conditions are motivated by the use of scaling functions and filter sequences in wavelet analysis [Daubechies and Lagarias 1991; 1992]; see also [Chyzak et al. 2001] in this issue, where the equations above are summarized (Section 2) and where it is conjectured (page 75) that there are at most $2^{N-1}$ solutions to the system (1).

Here we prove that there are at most $2^{N-1}$ real solutions to the system (1), this being the case of consequence in wavelet analysis.

**Theorem 1.** For a fixed $N > 0$, the system (1) in $h_{1-N}, \ldots, h_N$ has at most $2^{N-1}$ real solutions. In other words, there are at most $2^{N-1}$ distinct Daubechies scaling functions with support in $[1-N, N]$.

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The proof uses a lemma on Riesz factorizations. Let $G(z)$ be a Laurent polynomial, that is,
$$G(z) = \sum_{k \in \mathbb{Z}} c_k z^k,$$
where finitely many $c_k \neq 0$. We consider the question: How may real Laurent polynomials $f(z)$ are there such that $f(z)f(z^{-1}) = G(z)$ (Riesz factorization)? Obviously if $f(z)$ is a solution then so is $\pm z^m f(z)$ for any $m \in \mathbb{Z}$. Call two Laurent polynomials $f(z)$ and $g(z)$ equivalent if $g(z) = z^m f(z)$ or $g(z) = -z^m f(z)$ for some $m \in \mathbb{Z}$. So our question concerns the number of inequivalent solutions.

Not every Laurent polynomial $G(z)$ has a Riesz factorization $G(z) = f(z)f(z^{-1})$ for some real Laurent polynomial $f(z)$. If it does we call $G(z)$ Riesz factorizable. It is well known [Daubechies 1992] that $G(z)$ is Riesz factorizable if and only if $G(z) = G(z^{-1})$ and $G(z) \geq 0$ on the unit circle $|z| = 1$.

**Lemma 2.** Let $G(z) = \sum_{k=-M}^{M} c_k z^k$ be real and Riesz factorizable. Then the number of inequivalent real Laurent polynomials $f(z)$ satisfying
$$f(z)f(z^{-1}) = G(z)$$
is at most $2^{r+s}$, where $2r$ and $4s$ denote the number of real and complex roots of $G(z)$ (counting multiplicity) not on the unit circle. In particular, it is at most $2^M$.

*Proof.* For any two roots $z_1$ and $z_2$ of $G(z)$ write $z_1 \sim z_2$ if $z_2$ is one of $z_1, z_1^{-1}, \overline{z_1},$ or $\overline{z_1}^{-1}$. It follows from $G(z) = f(z)f(z^{-1})$ that if $z_2$ is a root of $G$ then so is every $w \sim z_2$ and with the same multiplicity. We partition the roots of $G$ not on the unit circle into equivalent classes of the relation $\sim$, and label them (counting multiplicity)
$$\mathcal{R}_1, \ldots, \mathcal{R}_r, \mathcal{C}_1, \ldots, \mathcal{C}_s,$$
where each $\mathcal{R}_r$ and $\mathcal{C}_j$ contain real and complex roots of $G$ not on the unit circle, respectively. Clearly $|\mathcal{R}_r| = 2$ and $|\mathcal{C}_j| = 4$. Let $\mathcal{U}$ denote the roots of $G$ that are on the unit circle.

Observe that up to equivalence a Riesz factorization $G(z) = f(z)f(z^{-1})$ is completely determined by the roots of $f(z)$. Furthermore, if $z_0$ is a root of $f(z)$ with $|z_0| = 1$ then so is $\overline{z_0} = z_0^{-1}$. It follows that $z_0$ must also be a root of $f(z^{-1})$. Hence all roots in $\mathcal{U}$ have even multiplicities and they split evenly between $f(z)$ and $f(z^{-1})$. This fact implies that the Riesz factorization $G(z) = f(z)f(z^{-1})$ is determined completely by the roots of $f(z)$ that do not lie on the unit circle.

To count the number of different factors $f(z)$, note that if $z_i \in \mathcal{R}_r$ is a root of $f(z)$ then the other element $z_i^{-1}$ in $\mathcal{R}_r$ must be a root of $f(z^{-1})$. Similarly if $z_j \in \mathcal{C}_j$ is a root of $f(z)$ then so is $\overline{z_j}$, while the other two elements in $\mathcal{C}_j$ will be roots of $f(z^{-1})$. So there are two ways to select roots for $f(z)$ from each of $\mathcal{R}_r$ and $\mathcal{C}_j$.

The number of different factors $f(z)$ such that $G(z) = f(z)f(z^{-1})$ is therefore at most $2^{r+s}$. Finally, $G(z)$ has at most $2M$ roots. Hence $2^{r+s} \leq 2^M$. \hfill $\square$

**Remark.** The number of Riesz factorizations of $G(z)$ is exactly $2^{r+s}$ if all roots of $G$ not on the unit circle are distinct. Otherwise it is strictly less. The exact number is not hard to compute, following the proof of the lemma. Let $\mathcal{R}_1, \ldots, \mathcal{R}_r$, and $\mathcal{C}_1, \ldots, \mathcal{C}_s$ be the distinct equivalent classes of $\mathcal{R}_1, \ldots, \mathcal{R}_r$ and $\mathcal{C}_1, \ldots, \mathcal{C}_s$, respectively. Let $m_i$ and $n_j$ denote the multiplicity of the roots of $\mathcal{R}_r$ and $\mathcal{C}_s$, respectively. Then the number of inequivalent Riesz factorizations of $G(z)$ is
$$(m_1+1) \cdots (m_r+1)(n_1+1) \cdots (n_s+1).$$

*Proof of Theorem 1.* Suppose that the sequence of real numbers $\{h_k : 1-N \leq k \leq N\}$ satisfies (1). Let
$$H(z) = \frac{1}{2} \sum_{k=1-N}^{N} h_k z^k.$$

Recall (from [Daubechies 1992], for example) that the third set of equations in (1) is equivalent to
$$H(z) = \left(\frac{1+z}{2}\right)^N f(z)$$
for some real Laurent polynomial $f(z)$, whereas the middle set of equations is equivalent to
$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1. \quad (4)$$

Furthermore $H(z)$ satisfies (4) if and only if
$$f(e^{i\theta})f(e^{-i\theta}) = \sum_{k=0}^{N-1} \binom{N+k-1}{k}(1-\cos \theta)^k,$$
which is equivalent to
$$f(z)f(z^{-1}) = \sum_{k=0}^{N-1} \binom{N+k-1}{k}\left(\frac{2-z-z^{-1}}{2}\right)^k. \quad (5)$$
Expanding the right hand side of (5) yields
\[ G(z) = \sum_{k=1-N}^{N-1} c_k z^k \]
for some \(c_k\). By Lemma 2 there are at most \(2^{N-1}\) inequivalent solutions \(f(z)\) satisfying (5).

We now need only show that any two equivalent solutions \(f_1(z)\) and \(f_2(z)\) of (3) and (5) must be identical. First, it follows from \(H(1) = 1\) that \(f_1(1) = f_2(1) = 1\). Hence \(f_2(z) = z^m f_1(z)\) for some \(m \in \mathbb{Z}\). Next, any solution \(f(z)\) to (3) and (5) must have the form
\[ f(z) = \sum_{k=1-N}^{0} c_k z^k \]
with \(c_{1-N} \neq 0\) by (5). Hence \(m = 0\), and so \(f_1(z) = f_2(z)\). \(\Box\)

**Remark.** This also shows that there are at least \(2^{\lfloor N/2 \rfloor}\) distinct Daubechies scaling functions with support in \([1-N, N]\), where \(\lfloor N/2 \rfloor\) denotes the largest integer not exceeding \(N/2\). This is because the function in (5) cannot have zeros on the unit disk.

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**REFERENCES**


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