On the Number of Daubechies Scaling Functions and a Conjecture of Chyzak et al.

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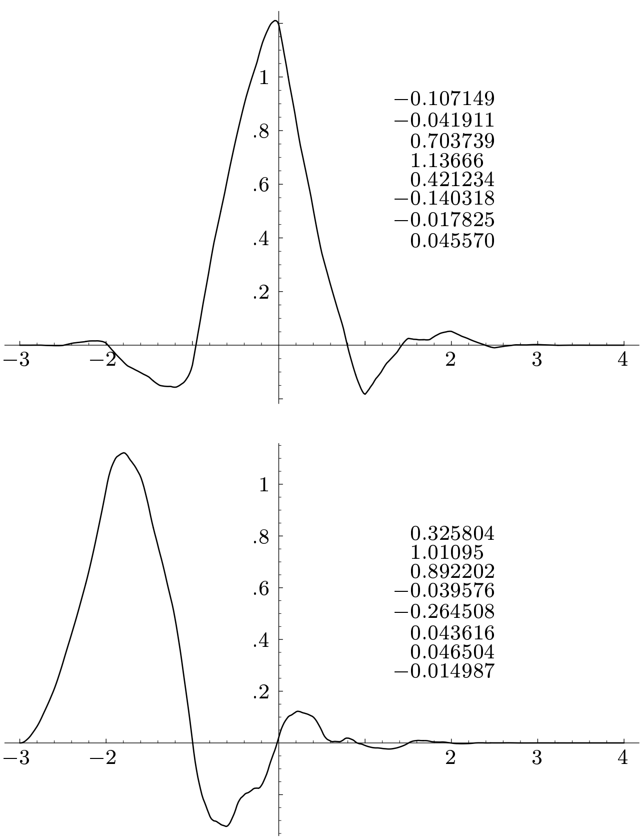
Using a result on Riesz factorizations, we show that there are at most 2^{N-1} and at least $2^{\lfloor N/2 \rfloor}$ distinct Daubechies scaling functions with support in $[1-N, N]$.



Two of the four distinct Daubechies scaling functions for $N = 4$. The numbers indicate the coefficients $h_{-3}, \dots, h_4$. The other two scaling functions can be obtained by reversing the coefficients h_k.

We define a *Daubechies scaling function* to be a function $\varphi \in L^2(\mathbb{R})$, with support in $[1-N, N]$ and satisfying the *dilation equation*

$$\varphi(x) = \sum_{k \in \mathbb{Z}} h_k \varphi(2x-k),$$

where N is a positive integer and the sequence $\{h_k\}$, known as the scaling function's *filter sequence*, satisfies the conditions $h_k = 0$ for $k \notin [1-N, N]$ and

$$\left.\begin{aligned} &\sum_{k=1-N}^{N} h_k = 2, \\ &\sum_{k=1-N}^{N} h_k h_{k-2l} = 2\delta_{0,l}, \qquad \text{for } l = 0, \dots, N-1, \\ &\sum_{k=1-N}^{N} (-1)^k h_{1-k} k^l = 0, \quad \text{for } l = 0, \dots, N-1. \end{aligned}\right\} \quad (1)$$

These conditions are motivated by the use of scaling functions and filter sequences in wavelet analysis [Daubechies and Lagarias 1991; 1992]; see also [Chyzak et al. 2001] in this issue, where the equations above are summarized (Section 2) and where it is conjectured (page 75) that there are at most 2^{N-1} solutions to the system (1).

Here we prove that there are at most 2^{N-1} *real* solutions to the system (1), this being the case of consequence in wavelet analysis.

Theorem 1. *For a fixed* $N > 0$, *the system* (1) *in* $h_{1-N}, \dots, h_N$ *has at most* 2^{N-1} *real solutions. In other words, there are at most* 2^{N-1} *distinct Daubechies scaling functions with support in* $[1-N, N]$.

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The proof uses a lemma on Riesz factorizations. Let $G(z)$ be a Laurent polynomial, that is,

$$G(z) = \sum_{k\in\mathbb{Z}} c_k z^k,$$

where finitely many $c_k \neq 0$. We consider the question: How may real Laurent polynomials $f(z)$ are there such that $f(z)f(z^{-1}) = G(z)$ (Riesz factorization)? Obviously if $f(z)$ is a solution then so is $\pm z^m f(z)$ for any $m \in \mathbb{Z}$. Call two Laurent polynomials $f(z)$ and $g(z)$ *equivalent* if $g(z) = z^m f(z)$ or $g(z) = -z^m f(z)$ for some $m \in \mathbb{Z}$. So our question concerns the number of inequivalent solutions.

Not every Laurent polynomial $G(z)$ has a Riesz factorization $G(z) = f(z)f(z^{-1})$ for some real Laurent polynomial $f(z)$. If it does we call $G(z)$ *Riesz factorizable*. It is well known [Daubechies 1992] that $G(z)$ is Riesz factorizable if and only if $G(z) = G(z^{-1})$ and $G(z) \geq 0$ on the unit circle $|z| = 1$.

Lemma 2. *Let $G(z) = \sum_{k=-M}^{M} c_k z^k$ be real and Riesz factorizable. Then the number of inequivalent real Laurent polynomials $f(z)$ satisfying*

$$f(z)f(z^{-1}) = G(z)$$

is at most 2^{r+s}, where $2r$ and $4s$ denote the number of real and complex roots of $G(z)$ (counting multiplicity) not on the unit circle. In particular, it is at most 2^M.

Proof. For any two roots z_1 and z_2 of $G(z)$ write $z_1 \sim z_2$ if z_2 is one of z_1, z_1^{-1}, $\bar{z}_1$, or $\bar{z}_1^{-1}$. It follows from $G(z) = f(z)f(z^{-1})$ that if z_* is a root of G then so is every $w \sim z_*$ and with the same multiplicity. We partition the roots of G not on the unit circle into equivalent classes of the relation $\sim$, and label them (counting multiplicity)

$$\mathcal{R}_1, \dots, \mathcal{R}_r, \mathcal{C}_1, \dots, \mathcal{C}_s,$$

where each $\mathcal{R}_i$ and $\mathcal{C}_j$ contain real and complex roots of G not on the unit circle, respectively. Clearly $|\mathcal{R}_i| = 2$ and $|\mathcal{C}_j| = 4$. Let $\mathcal{U}$ denote the roots of G that are on the unit circle.

Observe that up to equivalence a Riesz factorization $G(z) = f(z)f(z^{-1})$ is completely determined by the roots of $f(z)$. Furthermore, if z_0 is a root of $f(z)$ with $|z_0| = 1$ then so is $\bar{z}_0 = z_0^{-1}$. It follows that z_0 must also be a root of $f(z^{-1})$. Hence all roots in $\mathcal{U}$ have even multiplicities and they split evenly between $f(z)$ and $f(z^{-1})$. This fact implies that the Riesz factorization $G(z) = f(z)f(z^{-1})$ is determined completely by the roots of $f(z)$ that do not lie on the unit circle.

To count the number of different factors $f(z)$, note that if $z_i \in \mathcal{R}_i$ is a root of $f(z)$ then the other element z_i^{-1} in $\mathcal{R}_i$ must be a root of $f(z^{-1})$. Similarly if $z_j \in \mathcal{C}_j$ is a root of $f(z)$ then so is $\bar{z}_j$, while the other two elements in $\mathcal{C}_j$ will be roots of $f(z^{-1})$. So there are two ways to select roots for $f(z)$ from each of $\mathcal{R}_i$ and $\mathcal{C}_j$. The number of different factors $f(z)$ such that $G(z) = f(z)f(z^{-1})$ is therefore at most 2^{r+s}. Finally, $G(z)$ has at most $2M$ roots. Hence $2^{r+s} \leq 2^M$. □

Remark. The number of Riesz factorizations of $G(z)$ is exactly 2^{r+s} if all roots of G not on the unit circle are distinct. Otherwise it is strictly less. The exact number is not hard to compute, following the proof of the lemma. Let $\mathcal{R}'_1, \dots \mathcal{R}'_{r'}$ and $\mathcal{C}'_1, \dots, \mathcal{C}'_{s'}$ be the distinct equivalent classes of $\mathcal{R}_1, \dots \mathcal{R}_r$ and $\mathcal{C}_1, \dots, \mathcal{C}_s$, respectively. Let m_i and n_j denote the multiplicity of the roots of $\mathcal{R}'_i$ and $\mathcal{C}'_j$, respectively. Then the number of inequivalent Riesz factorizations of $G(z)$ is

$$(m_1{+}1)\cdots(m_{r'}{+}1)(n_1{+}1)\cdots(n_{s'}{+}1). \tag{2}$$

Proof of Theorem 1. Suppose that the sequence of real numbers $\{h_k : 1{-}N \leq k \leq N\}$ satisfies (1). Let

$$H(z) = \frac{1}{2}\sum_{k=1-N}^{N} h_k z^k.$$

Recall (from [Daubechies 1992], for example) that the third set of equations in (1) is equivalent to

$$H(z) = \Big(\frac{1+z}{2}\Big)^N f(z) \tag{3}$$

for some real Laurent polynomial $f(z)$, whereas the middle set of equations is equivalent to

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1. \tag{4}$$

Furthermore $H(z)$ satisfies (4) if and only if

$$f(e^{i\theta})f(e^{-i\theta}) = \sum_{k=0}^{N-1}\binom{N{+}k{-}1}{k}(1-\cos\theta)^k,$$

which is equivalent to

$$f(z)f(z^{-1}) = \sum_{k=0}^{N-1}\binom{N{+}k{-}1}{k}\Big(\frac{2-z-z^{-1}}{2}\Big)^k. \tag{5}$$

Expanding the right hand side of (5) yields

$$G(z) = \sum_{k=1-N}^{N-1} c_k z^k$$

for some c_k. By Lemma 2 there are at most 2^{N-1} inequivalent solutions $f(z)$ satisfying (5).

We now need only show that any two equivalent solutions $f_1(z)$ and $f_2(z)$ of (3) and (5) must be identical. First, it follows from $H(1) = 1$ that $f_1(1) = f_2(1) = 1$. Hence $f_2(z) = z^m f_1(z)$ for some $m \in \mathbb{Z}$. Next, any solution $f(z)$ to (3) and (5) must have the form

$$f(z) = \sum_{k=1-N}^{0} c_k z^k$$

with $c_{1-N} \neq 0$ by (5). Hence $m = 0$, and so $f_1(z) = f_2(z)$. □

Remark. This also shows that there are at least $2^{\lfloor N/2 \rfloor}$ distinct Daubechies scaling functions with support in $[1-N, N]$, where $\lfloor N/2 \rfloor$ denotes the largest integer not exceeding $N/2$. This is because the function in (5) cannot have zeros on the unit disk.

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