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Ramanujan-like Series for $1/\pi^2$ and String Theory

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Using the machinery from the theory of Calabi–Yau differential equations, we find formulas for $1/\pi^2$ of hypergeometric and nonhypergeometric types.

1. INTRODUCTION

Almost 100 years ago, Ramanujan found 17 formulas for $1/\pi$. The most spectacular was

$$\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!^3} (26390n+1103) \frac{1}{99^{4n+2}} = \frac{\sqrt{2}}{4\pi},$$

where $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ for $n > 1$ is the Pochhammer symbol. The formulas were not proved until the 1980s by the Borwein brothers using modular forms (see [Borwein and Borwein 87] and the recent surveys [Baruah et al. 09, Zudilin 08]).

In 2002, the second author found seven similar formulas for $1/\pi^2$. Three of them were proved using the WZ-method (see [Guillera 02, Guillera 06, Guillera 07]). Others, like

$$\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n}{n!^5} (1920n^2+304n+15) \frac{1}{7^{4n}} = \frac{56\sqrt{7}}{\pi^2}$$

(see [Guillera 03, Guillera 07]), were found using PSLQ to find the triple $(1920, 304, 15)$ after guessing $z = 7^{-4}$. This was inspired by a similar formula for $1/\pi$, namely

$$\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!^3} (40n+3) \frac{1}{7^{4n}} = \frac{49\sqrt{3}}{9\pi}.$$

To avoid guessing z, the second author, using 5×5 matrices, developed a technique to find z, while instead guessing a rather small rational number k. To that purpose, one had to solve an equation of type

$$\frac{1}{6}\log^3(q) - \nu_1 \log(q) - \nu_2 - T(q) = 0,$$

where ν_1 depends on k linearly, ν_2 is a constant, and $T(q)$ is a certain power series (see [Guillera 10]). It was suggested by Wadim Zudilin that $T(q)$ had to do with the Yukawa coupling $K(q)$ of the fourth-order pullback of the fifth-order differential equation satisfied by the sum for general z. The exact relation is

$$\left(q\frac{d}{dq}\right)^3 T(q) = 1 - K(q).$$

This is explained and proved here. The reason that it works is that all differential equations involved are Calabi–Yau. The theory results in a simplified and very fast Maple program to find z. As a result, we mention the new formula

$$\frac{1}{\pi^2} = 32\sum_{n=0}^{\infty}\frac{(6n)!}{3\cdot n!^6}(532n^2+126n+9)\frac{1}{10^{6n+3}}, \quad (1\text{–}1)$$

where the summands contain no infinite decimal fractions. However, this is not a BBP-type (Bailey–Borwein–Plouffe) series [Bailey 11], and due to the factorials, it is not useful to extract individual decimal digits of $1/\pi^2$. (The manner in which we have written the formula above is due to Pigulla).

2. CALABI–YAU DIFFERENTIAL EQUATIONS

2.1. Formal Definitions

A Calabi–Yau differential equation is a fourth-order differential equation with rational coefficients,

$$y^{(4)} + c_3(z)y''' + c_2(z)y'' + c_1(z)y' + c_0(z)y = 0,$$

satisfying the following conditions.

1. It is MUM (maximal unipotent monodromy), i.e., the indicial equation at $z = 0$ has zero as a root of order 4. It means that there is a Frobenius solution of the following form:

$$\begin{aligned}
y_0 &= 1 + A_1z + A_2z^2 + \cdots ,\\
y_1 &= y_0\log(z) + B_1z + B_2z^2 + \cdots ,\\
y_2 &= \frac{1}{2}y_0\log^2(z) + (B_1z + B_2z^2 + \cdots)\log(z) + C_1z\\
&\quad + C_2z^2 + \cdots ,\\
y_3 &= \frac{1}{6}y_0\log^3(z) + \frac{1}{2}(B_1z + B_2z^2 + \cdots)\log^2(z)\\
&\quad + (C_1z + C_2z^2 + \cdots)\log(z) + D_1z + D_2z^2 + \cdots .
\end{aligned}$$

It is very useful that Maple's `formal_sol` produces the four solutions in exactly this form (though labeled 1 to 4).

2. The coefficients of the equation satisfy the identity

$$c_1 = \frac{1}{2}c_2c_3 - \frac{1}{8}c_3^3 + c_2' - \frac{3}{4}c_3c_3' - \frac{1}{2}c_3''.$$

3. Let $t = y_1/y_0$. Then

$$q = \exp(t) = z + e_2z^2 + \cdots$$

can be solved as

$$z = z(q) = q - e_2q^2 + \cdots ,$$

which is called the "mirror map." We also construct the "Yukawa coupling" defined by

$$K(q) = \frac{d^2}{dt^2}\left(\frac{y_2}{y_0}\right).$$

This can be expanded in a Lambert series

$$K(q) = 1 + \sum_{d=1}^{\infty} n_d\frac{d^3q^d}{1-q^d},$$

where the n_d are called "instanton numbers." For small d, the n_d are conjectured to count rational curves of degree d on the corresponding Calabi–Yau manifold. Then the third condition is

(a) y_0 has integer coefficients.

(b) q has integer coefficients.

(c) There is a fixed integer N_0 such that all N_0n_d are integers.

In [Almkvist 10] the first author showed how to discover Calabi–Yau differential equations.

2.2. Pullbacks of Fifth-Order Equations

Condition **2** is equivalent to

$$\mathbf{2'}. \qquad \begin{vmatrix} y_0 & y_3 \\ y_0' & y_3' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

This means that the six Wronskians formed by the four solutions to our Calabi–Yau equation reduce to five. Hence they satisfy a fifth-order differential equation

$$w^{(5)} + d_4w^{(4)} + d_3w''' + d_2w'' + d_1w' + d_0w = 0.$$

Condition **2** for the fourth-order equation leads to a corresponding condition for the fifth-order equation:

$$\mathbf{2_5}. \qquad d_2 = \frac{3}{5}d_3d_4 - \frac{4}{25}d_4^3 + \frac{3}{2}d_3' - \frac{6}{5}d_4d_4' - d_4''.$$

Conversely, given a fifth-order equation satisfying $\mathbf{2_5}$ with solution w_0, we can find a pullback, i.e., a fourth-order equation with solutions $y_0, y_1, \ldots$ such that $w_0 = z(y_0y_1' - y_0'y_1)$. There is another pullback, $\widehat{y}$, which often cuts the degree in half. It was discovered by Yifan Yang, and it is simply a multiple $\widehat{y} = gy$ of the ordinary pullback, where

$$g = z^{-1/2} \exp\left(\frac{3}{10}\int d_4 dz\right).$$

In the proof below, all formulas contain only quotients of solutions, so the factor g cancels. Hence it is irrelevant whether we use ordinary or YY-pullbacks. Since the $q(z)$ are the same, so are the inverse functions $z(q)$.

2.3. The Proof

Consider

$$w_0(z) = \sum_{n=0}^{\infty} \frac{(1/2)_n (s_1)_n (1-s_1)_n (s_2)_n (1-s_2)_n}{n!^5} (\rho z)^n = \sum_{n=0}^{\infty} A_n z^n,$$

which satisfies the differential equation

$$\left\{\theta^5 - \rho z\left(\theta+\frac{1}{2}\right)(\theta+s_1) \times (\theta+1-s_1)(\theta+s_2)(\theta+1-s_2)\right\} w_0 = 0,$$

where $\theta = z\frac{d}{dz}$. The equation satisfies $\mathbf{2'}$, so

$$w_0 = z(y_0 y_1' - y_0' y_1),$$

where y_0 and y_1 satisfy a fourth-order differential equation (the ordinary pullback). We will consider the following 14 cases (compare the 14 hypergeometric Calabi–Yau equations in the "Big Table" (see [Almkvist et al. 05] and Table 1).

#	s_1	s_2	ρ	A_n
$\widetilde{1}$	1/5	2/5	$4\cdot 5^5$	$\binom{2n}{n}^3\binom{3n}{n}\binom{5n}{2n}$
$\widetilde{2}$	1/10	3/10	$4\cdot 8\cdot 10^5$	$\binom{2n}{n}^2\binom{3n}{n}\binom{5n}{2n}\binom{10n}{5n}$
$\widetilde{3}$	1/2	1/2	$4\cdot 2^8$	$\binom{2n}{n}^5$
$\widetilde{4}$	1/3	1/3	$4\cdot 3^6$	$\binom{2n}{n}^3\binom{3n}{n}^2$
$\widetilde{5}$	1/2	1/3	$4\cdot 2^4\cdot 3^3$	$\binom{2n}{n}^4\binom{3n}{n}$
$\widetilde{6}$	1/2	1/4	$4\cdot 2^{10}$	$\binom{2n}{n}^4\binom{4n}{2n}$
$\widetilde{7}$	1/8	3/8	$4\cdot 2^{16}$	$\binom{2n}{n}^3\binom{4n}{2n}\binom{8n}{4n}$
$\widetilde{8}$	1/6	1/3	$4\cdot 2^4\cdot 3^6$	$\binom{2n}{n}^3\binom{4n}{2n}\binom{6n}{2n}$
$\widetilde{9}$	1/12	5/12	$4\cdot 12^6$	$\binom{2n}{n}^3\binom{6n}{2n}\binom{12n}{6n}$
$\widetilde{10}$	1/4	1/4	$4\cdot 2^{12}$	$\binom{2n}{n}^3\binom{4n}{2n}^2$
$\widetilde{11}$	1/4	1/3	$4\cdot 12^3$	$\binom{2n}{n}^3\binom{3n}{n}\binom{4n}{2n}$
$\widetilde{12}$	1/6	1/4	$4\cdot 2^{10}\cdot 3^3$	$\binom{2n}{n}^2\binom{3n}{n}\binom{4n}{2n}\binom{6n}{3n}$
$\widetilde{13}$	1/6	1/6	$4\cdot 2^8\cdot 3^6$	$\binom{2n}{n}\binom{3n}{n}^2\binom{6n}{3n}^2$
$\widetilde{14}$	1/2	1/6	$4\cdot 2^8\cdot 3^3$	$\binom{2n}{n}^3\binom{3n}{n}\binom{6n}{3n}$

TABLE 1. Hypergeometric cases.

Assume that the formula

$$\sum_{n=0}^{\infty} A_n(a+bn+cn^2)z^n = \frac{1}{\pi^2}$$

is a Ramanujan-like one; that is, the numbers a, b, c, and z are algebraic. Then in [Guillera 10], it is conjectured that we have an expansion

$$\sum_{n=0}^{\infty} A_{n+x}\left(a+b(n+x)+c(n+x)^2\right)z^{n+x} = \frac{1}{\pi^2} - \frac{k}{2}x^2 + \frac{j}{24}\pi^2 x^4 + O\left(x^5\right), \tag{2–1}$$

where k and j are rational numbers. This holds in all known examples. However, there is a better argument to support the conjecture. It consists in comparing ${}_5F_4$ with the cases ${}_3F_2$ of Ramanujan-type series for $1/\pi$, for which the second author proved in [Guillera 10] that k must be rational.

In A_x, we replace $x!$ by $\Gamma(x+1)$ (Maple does this automatically). Later, we use the harmonic number $H_n = 1 + 1/2 + \cdots + 1/n$, which is replaced by $H_x = \psi(x+1) - \gamma$, where $\psi(x) = \Gamma'(x)/\Gamma(x)$ and γ is Euler's constant.

The expansion (2–1) can be reformulated as

$$\sum_{n=0}^{\infty} \frac{A_{n+x}}{A_x}(a+b(n+x)+c(n+x)^2)z^n = \frac{1}{z^x A_x}\left(\frac{1}{\pi^2} - \frac{k}{2}x^2 + \frac{j}{24}\pi^2 x^4 + \cdots\right). \tag{2–2}$$

Write

$$\sum_{n=0}^{\infty} \frac{A_{n+x}}{A_x} z^n = \sum_{i=0}^{\infty} a_i x^i,$$

$$\sum_{n=0}^{\infty} \frac{A_{n+x}}{A_x}(n+x) z^n = \sum_{i=0}^{\infty} b_i x^i,$$

$$\sum_{n=0}^{\infty} \frac{A_{n+x}}{A_x}(n+x)^2 z^n = \sum_{i=0}^{\infty} c_i x^i,$$

where a_i, b_i, c_i are power series in z with rational coefficients. They are related to the solutions w_0, w_1, w_2, w_3, w_4 of the fifth-order differential equation

$$w_0 = a_0$$

$$w_1 = a_0 \log(z) + a_1$$

$$w_2 = a_0 \frac{\log^2(z)}{2} + a_1 \log(z) + a_2$$

$$w_3 = a_0 \frac{\log^3(z)}{6} + a_1 \frac{\log^2(z)}{2} + a_2 \log(z) + a_3$$

$$w_4 = a_0 \frac{\log^4(z)}{24} + a_1 \frac{\log^3(z)}{6} + a_2 \frac{\log^2(z)}{2} + a_3 \log(z) + a_4.$$

We also have $b_0 = za_0'$ and $b_k = a_{k-1} + za_k'$ for $k = 1, 2, 3, 4$.

If we write the expansion of A_x in the form

$$A_x = 1 + \frac{e}{2}\pi^2 x^2 - h\zeta(3)x^3 + \left(\frac{3e^2}{8} - \frac{f}{2}\right)\pi^4 x^4 + O\left(x^5\right), \tag{2–3}$$

then for the right-hand side M of (2–2), we have

$$\begin{aligned} M &= \frac{1}{z^x A_x}\left(\frac{1}{\pi^2} - \frac{k}{2}x^2 + \frac{j}{24}\pi^2 x^4 + \cdots\right) \\ &= m_0 + m_1 x + m_2 x^2 + m_3 x^3 + m_4 x^4 + \cdots, \end{aligned}$$

where

$$\begin{aligned} m_0 &= \frac{1}{\pi^2}, \\ m_1 &= -\frac{1}{\pi^2}\log(z), \\ m_2 &= \frac{1}{\pi^2}\left\{\frac{1}{2}\log^2(z) - \frac{\pi^2}{2}(k+e)\right\}, \\ m_3 &= \frac{1}{\pi^2}\left\{-\frac{1}{6}\log^3(z) + \frac{\pi^2}{2}(k+e)\log(z) + h\zeta(3)\right\}, \end{aligned}$$

and

$$2m_0 m_4 - 2m_1 m_3 + m_2^2 = \frac{j}{12} + \frac{k^2}{4} + ek + f.$$

Here

$$\begin{aligned} e &= \frac{5}{3} + \cot^2(\pi s_1) + \cot^2(\pi s_2), \\ f &= \frac{1}{\sin^2(\pi s_1)\sin^2(\pi s_2)}, \end{aligned}$$

and

$$\begin{aligned} h = \frac{2}{\zeta(3)}\Big\{&\zeta\left(3, \frac{1}{2}\right) + \zeta(3, s_1) + \zeta(3, 1-s_1) + \zeta(3, s_2) \\ &+ \zeta(3, 1-s_2)\Big\}, \end{aligned}$$

where

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$$

is the Hurwitz ζ-function. If one uses A_n defined by binomial coefficients, Maple finds the values of e and h directly. We conjecture that the 14 pairs (s_1, s_2) given in Table 2 are the only rational (s_1, s_2) between 0 and 1 making h an integer. Note that the same (s_1, s_2) give the only hypergeometric Calabi–Yau differential equations (see [Almkvist 06, Almkvist 07]).

Now we want to use many of the identities for the Wronskians in [Almkvist 06, pp. 4–5]. Therefore we invert the formulas

$$\begin{aligned} a_0 &= w_0, \\ a_1 &= w_1 - w_0\log(z), \\ a_2 &= w_2 - w_1\log(z) + w_0\frac{\log^2(z)}{2}, \\ a_3 &= w_3 - w_2\log(z) + w_1\frac{\log^2(z)}{2} - w_0\frac{\log^3(z)}{6}, \\ a_4 &= w_4 - w_3\log(z) + w_2\frac{\log^2(z)}{2} - w_1\frac{\log^3(z)}{6} \\ &\quad + w_0\frac{\log^4(z)}{24} \end{aligned}$$

and

$$\begin{aligned} b_0 &= zw_0', \\ b_1 &= z\big(w_1' - w_0'\log(z)\big), \\ b_2 &= z\left(w_2' - w_1'\log(z) + w_0'\frac{\log^2(z)}{2}\right), \\ b_3 &= z\left(w_3' - w_2'\log(z) + w_1'\frac{\log^2(z)}{2} - w_0'\frac{\log^3(z)}{6}\right), \\ b_4 &= z\left(w_4' - w_3'\log(z) + w_2'\frac{\log^2(z)}{2}\right. \\ &\quad \left. - w_1'\frac{\log^3(z)}{6} + w_0'\frac{\log^4(z)}{24}\right). \end{aligned}$$

The key equation in [Guillera 10] is

$$m_3 = H_0 m_0 - H_1 m_1 + H_2 m_2, \tag{2–4}$$

where

$$H_0 = \frac{a_0 b_4 - a_4 b_0}{a_0 b_1 - a_1 b_0}, \quad H_1 = \frac{a_0 b_3 - a_3 b_0}{a_0 b_1 - a_1 b_0}, \quad H_2 = \frac{a_0 b_2 - a_2 b_0}{a_0 b_1 - a_1 b_0}.$$

We get (g is a multiplicative factor defined in [Almkvist 06, p. 5]; it will cancel out)

$$a_0 b_1 - a_1 b_0 = z\begin{vmatrix} w_0 & w_1 \\ w_0' & w_1' \end{vmatrix} = z^3 g\, y_0^2.$$

#	$\widetilde{1}$	$\widetilde{2}$	$\widetilde{3}$	$\widetilde{4}$	$\widetilde{5}$	$\widetilde{6}$	$\widetilde{7}$	$\widetilde{8}$	$\widetilde{9}$	$\widetilde{10}$	$\widetilde{11}$	$\widetilde{12}$	$\widetilde{13}$	$\widetilde{14}$
e	11/3	35/3	5/3	7/3	2	8/3	23/3	5	47/3	11/3	3	17/3	23/3	14/3
h	42	290	10	18	14	24	150	70	486	38	28	80	122	66
f	16/5	16	1	16/9	4/3	2	8	16/3	16	4	8/3	8	16	4

TABLE 2. Values of e, h, f.

The double Wronskian is "almost the square"

$$a_0b_2 - a_2b_0 = z\begin{vmatrix} w_0 & w_2 \\ w_0' & w_2' \end{vmatrix} - z\log(z)\begin{vmatrix} w_0 & w_1 \\ w_0' & w_1' \end{vmatrix} = z^3 g\left\{y_0y_1 - y_0^2\log(z)\right\}.$$

It follows that

$$H_2 = \frac{z^3 g\left\{y_0y_1 - y_0^2\log(z)\right\}}{z^3 g\, y_0^2} = \frac{y_1}{y_0} - \log(z) = \log(q) - \log(z) = \log\left(\frac{q}{z}\right).$$

Furthermore,

$$a_0b_3 - a_3b_0 = z\begin{vmatrix} w_0 & w_3 \\ w_0' & w_3' \end{vmatrix} - z\log(z)\begin{vmatrix} w_0 & w_2 \\ w_0' & w_2' \end{vmatrix} + z\frac{\log^2(z)}{2}\begin{vmatrix} w_0 & w_1 \\ w_0' & w_1' \end{vmatrix} = z^3 g\left\{\frac{1}{2}y_1^2 - y_0y_1\log(z) + y_0^2\frac{\log^2(z)}{2}\right\}$$

and

$$H_1 = \frac{1}{2}\left(\frac{y_1}{y_0}\right)^2 - \frac{y_1}{y_0}\log(z) + \frac{\log^2(z)}{2} = \frac{1}{2}\log^2\left(\frac{q}{z}\right).$$

Finally, we have that

$$a_0b_4 - a_4b_0 = z\begin{vmatrix} w_0 & w_4 \\ w_0' & w_4' \end{vmatrix} - z\log(z)\begin{vmatrix} w_0 & w_3 \\ w_0' & w_3' \end{vmatrix} + z\frac{\log^2(z)}{2}\begin{vmatrix} w_0 & w_2 \\ w_0' & w_2' \end{vmatrix} - z\frac{\log^3(z)}{6}\begin{vmatrix} w_0 & w_1 \\ w_0' & w_1' \end{vmatrix} = z^3 g\left\{\frac{1}{2}(y_1y_2 - y_0y_3) - \frac{1}{2}y_1^2\log(z) + y_0y_1\frac{\log^2(z)}{2} - y_0^2\frac{\log^3(z)}{6}\right\}$$

and

$$H_0 = \frac{1}{2}\left(\frac{y_1}{y_0}\frac{y_2}{y_0} - \frac{y_3}{y_0}\right) - \frac{1}{2}t^2\log(z) + t\frac{\log^2(z)}{2} - \frac{\log^3(z)}{6}.$$

Substituting these formulas into (2–4), we obtain

$$\begin{aligned}\frac{1}{\pi^2}&\left\{-\frac{1}{6}\log^3(z) + \frac{\pi^2}{2}(k+e)\log(z) + h\zeta(3)\right\} \\ &= \frac{1}{\pi^2}\left\{\frac{1}{2}\left(\frac{y_1}{y_0}\frac{y_2}{y_0} - \frac{y_3}{y_0}\right) - \frac{1}{2}t^2\log(z) + t\frac{\log^2(z)}{2} - \frac{\log^3(z)}{6}\right\} \\ &\quad + \frac{1}{\pi^2}\log(z)\left\{\frac{t^2}{2} - t\log(z) + \frac{\log^2(z)}{2}\right\} \\ &\quad + \frac{1}{\pi^2}(t-\log(z))\left(\frac{\log^2(z)}{2} - \frac{\pi^2}{2}(k+e)\right),\end{aligned}$$

which simplifies to

$$\frac{1}{2}\left(\frac{y_1}{y_0}\frac{y_2}{y_0} - \frac{y_3}{y_0}\right) - \frac{\pi^2}{2}(k+e)\log(q) - h\zeta(3) = 0.$$

Here

$$\Phi = \frac{1}{2}\left(\frac{y_1}{y_0}\frac{y_2}{y_0} - \frac{y_3}{y_0}\right)$$

is well known in string theory and is called the Gromov–Witten potential (up to a multiplicative constant); see [Cox and Katz 99, p. 28]. It is connected to the Yukawa coupling $K(q)$ by

$$\left(q\frac{d}{dq}\right)^3\Phi = K(q).$$

Writing $\Phi = \frac{1}{6}\log^3(q) - T(q)$ (see Lemma 2.1), we get the following equation for finding q and hence z for given k:

$$\frac{1}{6}t^3 - \frac{\pi^2}{2}(k+e)t - h\zeta(3) - T(q) = 0, \quad q = \exp(t). \tag{2–5}$$

We look for real algebraic solutions of z. To look for alternating series, that is, if $z < 0$, all we need to do is to replace $q = \exp(t)$ with $q = -\exp(t)$ in (2–5). In order to make a quick sieve of the solutions, once we get q, we compute j and see whether it is an integer (or rational with small denominator). Using the formulas [Guillera 10, equations 3.48, 3.50], we obtain

$$j = 12\left\{\frac{1}{\pi^4}\left(\frac{1}{2}t^2 - q\frac{d}{dq}T(q) - \frac{\pi^2}{2}(k+e)\right)^2 - \frac{k^2}{4} - ek - f\right\}. \tag{2–6}$$

Lemma 2.1. *The function $T(q)$ is a power series with $T(0) = 0$.*

Proof. We have

$$y_1 = y_0 \log(z) + \alpha_1,$$

which implies

$$\frac{y_1}{y_0} = \log(q) = \log(z) + \frac{\alpha_1}{y_0} = \log(z) + \beta_1,$$

and hence

$$\log(z) = \log(q) - \beta_1,$$

where α_1 and $\beta_1 = \alpha_1/y_0$ are power series without a constant term. Furthermore,

$$y_2 = y_0 \frac{\log^2(z)}{2} + \alpha_1 \log(z) + \alpha_2$$

leads to

$$\begin{aligned}\frac{y_2}{y_0} &= \frac{1}{2}(\log(q) - \beta_1)^2 + \beta_1(\log(q) - \beta_1) + \beta_2 \\ &= \frac{1}{2}\log^2(q) + \beta_2 - \frac{1}{2}\beta_1^2,\end{aligned}$$

where $\beta_2 = \alpha_2/y_0$ with $\beta_2(0) = 0$. Finally,

$$y_3 = y_0 \frac{\log^3(z)}{6} + \alpha_1 \frac{\log^2(z)}{2} + \alpha_2 \log(z) + \alpha_3$$

and

$$\begin{aligned}\frac{y_3}{y_0} &= \frac{1}{6}(\log(q) - \beta_1)^3 + \frac{1}{2}\beta_1(\log(q) - \beta_1)^2 \\ &\quad + \beta_2(\log(q) - \beta_1) + \beta_3,\end{aligned}$$

where $\beta_3 = \alpha_3/y_0$ with $\beta_3(0) = 0$. Collecting like terms, we have

$$\frac{1}{2}\left(\frac{y_1}{y_0}\frac{y_2}{y_0} - \frac{y_3}{y_0}\right) = \frac{1}{6}\log^3(q) - \frac{1}{2}\left(\beta_3 - \beta_1\beta_2 + \frac{1}{3}\beta_1^3\right),$$

which proves the lemma. □

3. COMPUTATIONS

3.1. Hypergeometric Differential Equations

In only half of the 14 cases have we found solutions to (2–5) in which the indicator j is an integer. Using [Guillera 10, (3.47), (3.48)], we have the following formula for computing c:

$$\tau = \frac{c}{\sqrt{1-\rho z}}, \tag{3–1}$$

where

$$\tau^2 = \frac{j}{12} + \frac{k^2}{4} + ek + f.$$

Then a and b can be computed by [Guillera 10, (3.45)] or by PSLQ. Our results where the series converges are given in Table 3.

In all the hypergeometric cases, there is a singular solution when $k = j = 0$ (it does not have a corresponding Ramanujan-like series). For that solution we have $z = 1/\rho$, $a = b = c = 0$.

In addition, we have found the solutions $\widetilde{3}$: $k = 0$, $j = 3$, $z = -2^{-8}$, $a = 1/4$, $b = 3/2$, $c = 5/2$ and $\widetilde{11}$: $k = 1/3$, $j = 13$, $z = -2^{-12}$, $a = 3/16$, $b = 25/16$, $c = 43/12$, for which the corresponding series are "divergent" [Guillera and Zudilin 12].

Although our new program, which evaluates the function $T(q)$ much faster, has allowed us to try all rational values of k of the form $k = i/60$ with $0 \le i \le 1200$, the only new series that we have found is for $\widetilde{8}$ with $k = 8/3$, and it is

$$\sum_{n=0}^{\infty} \frac{(6n)!}{n!^6}(532n^2 + 126n + 9)\frac{1}{1000000^n} = \frac{375}{4\pi^2},$$

that is, (1–1). The other formulas in Table 3 were first discovered by the second author [Guillera 03, 12,--].

#	k	j	z_0	τ^2	a	b	c
$\widetilde{3}$	1	25	$-\frac{1}{2^{12}}$	5	$1/8$	1	$5/2$
$\widetilde{3}$	5	305	$-1/2^{20}$	41	$13/128$	$45/32$	$205/32$
$\widetilde{5}$	$2/3$	16	$1/2^{12}$	$37/9$	$1/16$	$9/16$	$37/24$
$\widetilde{5}$	$8/3$	112	$\left(\left(5\sqrt{5}-11\right)/8\right)^3$	$160/9$	$56-25\sqrt{5}$	$303-135\sqrt{5}$	$1220/3-180\sqrt{5}$
$\widetilde{6}$	2	80	$1/2^{16}$	15	$3/32$	$17/16$	$15/4$
$\widetilde{7}$	8	992	$1/2^{18}7^4$	168	$15\sqrt{7}/392$	$38\sqrt{7}/49$	$240\sqrt{7}/49$
$\widetilde{8}$	$5/3$	85	$-1/2^{18}$	$193/9$	$15/128$	$183/128$	$965/192$
$\widetilde{8}$	15	2661	$-1/2^{18}3^6 5^3$	$1075/3$	$29\sqrt{5}/640$	$693\sqrt{5}/640$	$2709\sqrt{5}/320$
$\widetilde{8}$	$8/3$	160	$1/2^6 5^6$	$304/9$	$36/375$	$504/375$	$2128/375$
$\widetilde{11}$	3	157	$-1/2^{12}3^4$	27	$5/48$	$21/16$	$21/4$
$\widetilde{12}$	7	757	$-1/2^{22}3^3$	123	$15/768\sqrt{3}$	$278\sqrt{3}/768$	$205\sqrt{3}/96$

TABLE 3. Convergent hypergeometric Ramanujan-like series for $1/\pi^2$.

Finally, we give a hypergeometric example of a different nature in the case $\widetilde{3}$. Taking $z_0 = -2^{-10}$, $q_0 = q(z_0)$, $t_0 = \log|q_0|$, and $T(q)$ of $\widetilde{3}$, we find using PSLQ, among the quantities $T(q_0)$, t_0^3, $t_0^2\,\pi$, $t_0\,\pi^2$, π^3, $\zeta(3)$, the following remarkable relation:

$$\frac{1}{6}(t_0+\pi)^3 - \frac{5}{6}\pi^2(t_0+\pi) - \frac{\pi^3}{3} - 10\zeta(3) - T(q_0) = 0.$$

The theory we have developed allows us to understand that the last relation has to do with the following formula, proved by Ramanujan [Berndt 89, p. 41]:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10n}} \binom{2n}{n}^5 (4n+1) = \frac{2}{\Gamma^4(\frac{3}{4})}.$$

To see why, we guess that

$$\frac{\Gamma^4(\frac{3}{4})}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10(n+x)}} \binom{2n+2x}{n+x}^5 [4(n+x)+1] = 1 - \pi x + \frac{\pi^2}{2}x^2 + \frac{\pi^3}{6}x^3 - \frac{19\pi^4}{24}x^4 + O(x^5)$$

by expanding the first side numerically. Hence

$$2^{10x}\binom{2x}{x}^{-5} \frac{2}{\Gamma^4\left(\frac{3}{4}\right)} \left(1 - \pi x + \frac{\pi^2}{2}x^2 + \frac{\pi^3}{6}x^3\right) = m_0 + m_1 x + m_2 x^2 + m_3 x^3 + O(x^4),$$

and we get m_0, m_1, m_2, m_3. Finally, we use identity (2–4), replacing $\log(z)$ with $\log 2^{-10}$.

3.2. Nonhypergeometric Differential Equations

If we write the ordinary pullback in the form

$$\theta_z^4 y = \left[e_3(z)\theta_z^3 + e_2(z)\theta_z^2 + e_1(z)\theta_z + e_0(z)\right] y, \qquad \theta_z = z\frac{d}{dz},$$

then the generalization of relation (3–1) is

$$\tau = c\left(\exp \int \frac{e_3(z)}{2z} dz\right), \quad \tau^2 = \frac{j}{12} + \frac{k^2}{4} + ek + f. \tag{3–2}$$

We say that a solution is singular if it does not have a corresponding Ramanujan-like series. We conjecture that h is the unique rational number such that singular solutions exist. The numbers e and f are not so important, because they can be absorbed in k and j respectively. However, to agree with the hypergeometric cases, we will choose e and f in such a way that a singular solution takes place at $k = j = 0$. This fact allows us to determine the values of the numbers e, h, and f from (2–5) and (2–6) using the PSLQ algorithm. For many sequences $A(n)$ there exists a finite value of z that is singular. Then we can get this value by solving the equation

$$\frac{dz(q)}{dq} = 0.$$

	e	h	f
$\#39 = A*\alpha$	1	14/3	1/3
$\#61 = B*\alpha$	4/3	26/3	4/9
$\#37 = C*\alpha$	2	56/3	2/3
$\#66 = D*\alpha$	4	182/3	4/3

TABLE 4. Values of e, h, f for $*\alpha$

Tables 4–8 present the rational values of the invariants e, h, and f followed by the series found.

For $A*\alpha$, taking $k = 1/3$, we get $j = 5$, and we discover the series

$$\sum_{n=0}^{\infty} \binom{2n}{n}^2 \sum_{i=0}^{n} \binom{n}{i}^2 \binom{2i}{i} \binom{2n-2i}{n-i} \times \frac{(-1)^n}{2^{8n}}(40n^2+26n+5) = \frac{24}{\pi^2}.$$

This series was first conjectured in [Sun 11], inspired by p-adic congruences.

For $B*\epsilon$, taking $k=1$, we get $j = 22$, and we obtain the formula

$$\sum_{n=0}^{\infty} \binom{2n}{n}\binom{3n}{n} \sum_{i=0}^{n} \binom{n}{i}^2 \binom{2i}{n}^2 \times \frac{1}{2^{7n}3^{3n}}(1071n^2+399n+46) = \frac{576}{\pi^2}.$$

For $A*\beta$, taking $k=1$, we get $j=13$, and we have the series

$$\sum_{n=0}^{\infty} \binom{2n}{n}^2 \sum_{i=0}^{n} \binom{2i}{i}^2 \binom{2n-2i}{n-i}^2 \frac{1}{2^{10n}}(36n^2+12n+1) = \frac{32}{\pi^2}.$$

For $B*\beta$, taking $k = 1/3$, we get $j = 1$, and we obtain

$$\sum_{n=0}^{\infty} \frac{(3n)!}{n!^3} \sum_{i=0}^{n} \binom{2i}{i}^2 \binom{2n-2i}{n-i}^2 \frac{1}{2^{9n}}(25n^2-15n-6) = \frac{192}{\pi^2}.$$

	e	h	f
$\#122 = A*\epsilon$	7/6	45/8	1/2
$\#170 = B*\epsilon$	3/2	77/8	2/3
$C*\epsilon$	13/6	157/8	1
$D*\epsilon$	25/6	493/8	2

TABLE 5. Values of e, h, f for $*\epsilon$

	e	h	f
$\#40 = A * \beta$	$2/3$	3	$1/4$
$\#49 = B * \beta$	1	7	$1/4$
$\#43 = C * \beta$	$5/3$	17	$1/4$
$\#67 = D * \beta$	$11/3$	59	$1/4$

TABLE 6. Values of e, h, f for $*\beta$

	e	h	f
$A * \theta$	$2/3$	-4	1
$B * \theta$	1	0	1
$C * \theta$	$5/3$	10	1
$D * \theta$	$11/3$	52	1

TABLE 8. Values of e, h, f for $*\theta$

For $A * \delta$, taking $k = 2/3$, we get $j = 28/3$, and we have

$$\sum_{n=0}^{\infty} \binom{2n}{n}^2 \sum_{i=0}^{n} \frac{(-1)^i 3^{n-3i}(3i)!}{i!^3} \binom{n}{3i}\binom{n+i}{i} \times \frac{(-1)^n}{3^{6n}}(803n^2 + 416n + 68) = \frac{486}{\pi^2}.$$

For $A * \theta$, taking $k = 2$, we get $j = 56$, and we discover the series

$$\sum_{n=0}^{\infty} \binom{2n}{n}^2 \sum_{i=0}^{n} 16^{n-i} \binom{2i}{i}^3 \binom{2n-2i}{n-i} \times \frac{(-1)^n}{2^{13n}}(18n^2 + 7n + 1) = \frac{4\sqrt{2}}{\pi^2}.$$

This series was first presented in [Sun 11], inspired by p-adic congruences.

For $B * \theta$, we get $T(q) = 0$, and from the equations we see that for every rational k the value of j is rational as well. Hence for every rational value of k we get a Ramanujan-like series for $1/\pi^2$. For example, for $k = 160/3$, we have

$$\sum_{n=0}^{\infty} \frac{3n!}{n!^3} \sum_{i=0}^{n} 16^{n-i} \binom{2i}{i}^3 \binom{2n-2i}{n-i} P(n) \frac{(-1)^n}{640320^{3n}} = \frac{(2^4 \cdot 3 \cdot 5 \cdot 23 \cdot 29)^3}{\pi^2},$$

where

$$P(n) = 22288332473153467n^2 + 16670750677895547n + 415634396862086,$$

	e	h	f
$A * \delta$	1	$9/2$	$17/36$
$B * \delta$	$4/3$	$17/2$	$7/12$
$C * \delta$	2	$37/2$	$29/36$
$D * \delta$	4	$121/2$	$53/36$

TABLE 7. Values of e, h, f for $*\delta$

which is the "square" [Zudilin 07b] of the Chudnovsky brothers' formula [Baruah et al. 09]

$$\sum_{n=0}^{\infty} (-1)^n \frac{(6n)!}{(3n)!n!^3}(545140134n + 13591409)\frac{1}{640320^{3n}} = \frac{53360\sqrt{640320}}{\pi}.$$

For $C * \theta$, taking $k = 1$, we get $j = 25$, and taking $k = 5$, we get $= 305$, and we recover the two series proved in [Zudilin 07a] by doing a quadratic transformation of case $\widetilde{3}$.

In [Almkvist 09], the first author, by transforming known formulas given by the second author, found formulas for $1/\pi^2$, where the coefficients belong to the Calabi–Yau equations $\widehat{3}$, $\widehat{5}$, $\widehat{6}$, $\widehat{7}$, $\widehat{8}$, $\widehat{11}$, $\widehat{12}$. Here we list some new ones for the cases $\widehat{3}$, $\widehat{5}$, $\widehat{8}$, $\widehat{11}$, and #77, some found by solving equation (2–5).

Transformation $\widehat{5}$. Here

$$A_n = \sum_{i=0}^{n} (-1)^i 1728^{n-i} \binom{n}{i}\binom{2i}{i}^4 \binom{3i}{i}.$$

Using $e = 2$, $h = 14$, $f = 4/3$, we find for $k = 8/3$ that $j = 112$ and $z_0 = -[320(131 + 61\sqrt{5})]^{-1}$. To find the coefficients we had to use the formulas in [Almkvist 09]. The resulting formula is

$$\sum_{n=0}^{\infty} A_n \Big(\big(28765285482\sqrt{5} - 64321133730\big) + \big(10068363 - 4502709\sqrt{5}\big)\, n + \big(54\sqrt{5} - 122\big)\, n^2 \Big) \frac{(-1)^n}{\big(320\,(131 + 61\sqrt{5})\big)^n} = \frac{300(1170059408\sqrt{5} - 24977012149)}{\pi^2}.$$

By the PSLQ algorithm, trying products of powers of 2 and 7 in the denominator of z, we see that

$$\sum_{n=0}^{\infty} A_n (n^2 - 63n + 300) \frac{1}{1792^n} = \frac{4704}{\pi^2},$$

but we cannot find the pair (k, j) with our program because the convergence in this case is too slow.

Transformation $\widehat{\mathbf{8}}$. Here

$$A_n = \sum_{i=0}^{n} (-1)^k 6^{6n-6i} \binom{n}{i} \binom{2i}{i}^3 \binom{4i}{2i} \cdot \binom{6i}{2i}.$$

Using $e = 5$, $h = 70$, $f = 16/3$, we find for $k = 5/3$ that $j = 85$ and $z_0 = 308800^{-1}$. This allows us to get the formula

$$\sum_{n=0}^{\infty} A_n \left(16777216n^2 - 3336192n - 2912283\right) \frac{1}{308800^n} = \frac{3 \cdot 5^5 \cdot 193^2}{5^5 \pi^2}.$$

For $k = 8/3$, we get $j = 160$ and the formula

$$\sum_{n=0}^{\infty} (-1)^n A_n \left(48828125n^2 + 17859375n + 3649554\right) \times \frac{1}{953344^n} = \frac{2^8 \cdot 3 \cdot 7^5 \cdot 19^2}{5^4 \pi^2}.$$

Transformation $\widehat{\mathbf{11}}$. Here

$$A_n = \sum_{i=0}^{n} (-1)^i 6912^{n-i} \binom{n}{i} \binom{2i}{i}^3 \binom{3i}{i} \binom{4i}{2i}.$$

Using $e = 3$, $h = 28$, $f = 8/3$, we find for $k = 1/3$ that $j = 13$, and we get the formula

$$\sum_{n=0}^{\infty} A_n (512n^2 - 1992n - 225) \frac{1}{11008^n} = \frac{3 \cdot 43^2}{2\pi^2}.$$

Transformation $\widehat{\mathbf{3}}$. Here

$$A_n = \sum_{i=0}^{n} (-1)^i 1024^{n-i} \binom{n}{i} \binom{2i}{i}^5.$$

Transforming two divergent series in [Guillera and Zudilin 12] with $z_0 = -2^{-8}$ and $z_0 = -1$ respectively (the second one given only implicitly), we obtain two (slowly) convergent formulas,

$$\sum_{n=0}^{\infty} A_n (2n^2 - 18n + 5) \frac{1}{1280^n} = \frac{100}{\pi^2}$$

and

$$\sum_{n=0}^{\infty} A_n (n^2 - 2272n + 392352) \frac{1}{1025^n} = \frac{16 \cdot 5253125}{\pi^2}.$$

This last identity converges so slowly that the power of our computers seems insufficient to check it numerically.

Transformation #77. Here

$$A_n = \binom{2n}{n} \sum_{i=0}^{n} \binom{n}{i} \binom{2i}{i}^3 \binom{4i}{2i}.$$

The pullback is equivalent to $\widetilde{6}$ (i.e., it has the same $K(q)$), so we try the same parameters: $e = 8/3$, $h = 24$, $f = 2$. For $k = 2$, we get $j = 80$ and $z_0 = 1/65540$. To find a, b, c we have to find the transformation between $\widetilde{6}$ and #77. Indeed,

$$\sum_{n=0}^{\infty} \binom{2n}{n}^4 \binom{4n}{2n} z^n = \frac{1}{\sqrt{1-4z}} \sum_{n=0}^{\infty} A_n \left(\frac{z}{1+4z}\right)^n$$

(the sequence of numbers $\widetilde{6}$ is in the left side), and using the method in [Almkvist 09] (see also [Almkvist et al. 09]), we obtain

$$\sum_{n=0}^{\infty} A_n \left(402653184n^2 + 114042880n + 10051789\right) \frac{1}{65540^n} = \frac{5^2 \cdot 29^3 \cdot 113^3}{2^6 \pi^2 \sqrt{16385}}.$$

4. SUPERCONGRUENCES

It was observed in [Zudilin 09] that the hypergeometric formulas for $1/\pi^2$ lead to supercongruences of the form

$$\sum_{n=0}^{p-1} A_n (a + bn + cn^2) z^n \equiv a \left(\frac{d}{p}\right) p^2 \pmod{p^5},$$

```
with(combinat):
p(1):=expand(-36*(2592*n^4+5184*n^3+6066*n^2+3474*n+755)):
p(2):=expand(2^4*3^10*(4*n+3)*(4*n+5)*(12*n+11)*(12*n+13)):
V:=proc(n) local j:
if n=0 then 1; else sum(stirling2(n,j)*z^j*Dz^j,j=1..n); fi; end:
L:=collect(V(4)+add(add(z^m*coeff(p(m),n,k)*V(k),m=1..2),k=0..4),Dz):
Order:=51:
with(DEtools):
r:=formal_sol(L,[Dz,z],z=0):
y0:=r[4]: y1:=r[3]: y2:=r[2]: y3:=r[1]:
q:=series(exp(y1/y0),z=0,51): m:=solve(series(q,z)=s,z):
convert(simplify(series(subs(z=m,1/2*(y1*y2/y0^2-y3/y0)),s=0,51)),polynom):
T:=coeff(-%,ln(s),0):
e:=5: h:=70: f:=16/3:

H:=proc(u) local k,y,z0,q0,j,y0,yy,i,jj; y0:=-10;
for i from 0 to 60 do
k:=i/3;
Digits:=50;
yy:=fsolve(y^3/6-Pi^2/2*(k+e)*y-h*Zeta(3)-subs(s=u*exp(y),T),y=y0);
q0:=exp(%); y0:=yy;
z0:=evalf(subs(s=u*q0,convert(m,polynom)));
j:=evalf(12*(1/Pi^4*(1/2*log(q0)^2-subs(s=u*q0,s*diff(T,s))-Pi^2/2*(k+e))^2-k^2/4-e*k-f));
jj:=convert(j,fraction,12);
if denom(jj)<30 then print([k,1/z0,j]); fi; od; end:
```

TABLE 9. Maple program for the case $\widetilde{\mathbf{8}}$.

where the notation $\left(\frac{d}{p}\right)$ stands for the Legendre symbol. Our computations show that for our new Ramanujan-like series for $1/\pi^2$, (1–1), we have again a supercongruence following Zudilin's pattern, namely

$$\sum_{n=0}^{p-1} \binom{2n}{n}^3 \binom{4n}{2n}\binom{6n}{2n} \left(532n^2+126n+9\right) \frac{1}{1000000^n} \equiv 9p^2 \pmod{p^5},$$

valid for primes $p \geq 7$.

For superconguences for $\widetilde{5}$ and $k=8/3$, which involve algebraic numbers, see [Guillera 12]. For the nonhypergeometric formulas, the best one can hope for is a congruence modulo p^3. We give some new ones that agree with Zudilin's observations in [Zudilin 09, eq. 35].

With Hadamard product #170 $= B * \epsilon$, we have

$$\sum_{n=0}^{p-1} \binom{2n}{n}\binom{3n}{n} \sum_{i=0}^{n} \binom{n}{i}^2 \binom{2i}{n}^2 \frac{1}{2^{7n}3^{3n}} \times \left(1071n^2+399n+46\right) \equiv 46p^2 \pmod{p^3},$$

for primes $p \geq 5$.

With Hadamard product #49 $B * \beta$, we have

$$\sum_{n=0}^{p-1}\sum_{i=0}^{n} \binom{2n}{n}\binom{3n}{n}\binom{2i}{i}^2\binom{2n-2i}{n-i}^2 \times (25n^2-15n-6)\frac{1}{512^n} \equiv -6p^2 \pmod{p^3},$$

for primes $p \geq 7$.

$$\left[\frac{8}{3}, 1.0000000000000000000000000000000000000000000000000\,10^6, \; 160.00000000000000000000000000000000000000000000007\right]$$

$$\left[\frac{5}{3}, -2.6214399999999999999999999999999999999999999999996\,10^5, \; 85.000000000000000000000000000000000000000000000000\right]$$

$$\left[15, -2.3887871999999999999999999999999999999999999999995\,10^{10}, \; 2660.9999999999999999999999999999999999999999999996\right]$$

TABLE 10. Solutions found with Maple program in Table 9.

With Hadamard product $A * \delta$, we have

$$\sum_{n=0}^{p-1} \binom{2n}{n}^2 \sum_{i=0}^{n} \frac{(-1)^i 3^{n-3i}(3i)!}{i!^3} \binom{n}{3i}\binom{n+i}{i} \frac{(-1)^n}{3^{6n}} \times \left(803n^2+416n+68\right) \equiv 68p^2 \pmod{p^3},$$

for primes $p \geq 5$.

With Hadamard product $C * \theta$, we have

$$\sum_{n=0}^{p-1}\sum_{i=0}^{n} 16^{n-i}\binom{2n}{n}\binom{4n}{2n}\binom{2i}{i}^3\binom{2n-2i}{n-i} \times \frac{18n^2-10n-3}{80^{2n}} \equiv -3\left(\frac{5}{p}\right)p^2 \pmod{p^3},$$

for primes $p \geq 5$ and

$$\sum_{n=0}^{p-1}\sum_{i=0}^{n} 16^{n-i}\binom{2n}{n}\binom{4n}{2n}\binom{2i}{i}^3\binom{2n-2i}{n-i} \times \frac{1046529n^2+227104n+16032}{1050625^n} \equiv 16032\left(\frac{41}{p}\right)p^2 \pmod{p^3},$$

for primes $p \geq 7$ and $p \neq 41$.

5. CONCLUSION

We have recovered the ten hypergeometric Ramanujan series in [Guillera --] and found a new one that the second author missed. But more important, finding the relation among the function $T(q)$ and the Gromov–Witten potential has allowed us to generalize the conjectures of the second author in [Guillera 10] to the case of nonhypergeometric Ramanujan–Sato-like series. Then, by getting e, h, and f from a singular solution (it always exists), we have solved our equations, finding several nice nonhypergeometric series for $1/\pi^2$. Finally, we have checked the corresponding supercongruences of Zudilin type.

6. APPENDIX: A MAPLE PROGRAM FOR THE CASE $\widetilde{8}$

We use the YY-pullback found in [Almkvist 06]. In order to treat as well the case in which z and q are negative, we introduce a sign $u = \pm 1$. The Maple program is given in Table 9.

Copy and paste the program into Maple and execute $H(1)$ and $H(-1)$. You will get the results shown in Table 10.

To use the program with other cases, one has to change the values of e, h, f and replace the polynomials $p(1)$, $p(2)$, etc. with those corresponding to the new pullback and the number 2 in $m = 1..2$, with the total number of polynomials.

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