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Finite Symplectic Matrix Groups

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CONTENTS

This paper classifies the maximal finite subgroups of $\mathrm{Sp}_{2n}(\mathbb{Q})$ for $1 \leq n \leq 11$ up to conjugacy in $\mathrm{GL}_{2n}(\mathbb{Q})$.

1. INTRODUCTION

The maximal finite subgroups of $\mathrm{Sp}_{2n}(\mathbb{Q})$ correspond to maximal finite subgroups of $\mathrm{GL}_m(K)$ for minimal totally complex fields K with $2n = m \cdot [K : \mathbb{Q}]$. Moreover, the maximal finite subgroups of $\mathrm{Sp}_{2n}(\mathbb{Q})$ are full automorphism groups of Euclidean lattices fixing an additional nondegenerate symplectic form. So the classification yields highly symmetric symplectic structures on interesting Euclidean lattices.

The (conjugacy classes of) maximal finite subgroups of $\mathrm{GL}_m(\mathbb{Q})$ have been classified up to $m = 31$ in a series of papers [Brown et al. 77, Plesken 91, Nebe and Plesken 95, Nebe 96a, Nebe 96b].

The strategy for these classifications is as follows. First, it suffices to classify only the maximal finite (symplectic) matrix groups whose natural representation is irreducible over $\mathbb{Q}$. Then one reduces the problem to the so-called (symplectic) primitive matrix groups. The concept of primitivity is the key ingredient for these classifications, since it has important consequences for normal subgroups. Most notably, the restriction of the natural representation of an irreducible (symplectic) primitive matrix group $G < \mathrm{GL}_m(\mathbb{Q})$ onto a normal subgroup N splits into copies of a single irreducible representation of N. In particular, each abelian characteristic subgroup of the p-core $\mathrm{O}_p(G)$ must be cyclic. By a theorem of Philip Hall, this restricts the Fitting subgroup $F(G)$ of G to a finite number of candidates depending only on m.

The layer $E(G)$ of G is the central product of all subnormal quasisimple subgroups of G. Thus the possible candidates for $E(G)$ can be obtained from the *Atlas of Finite Simple Groups* or the classification of [Hiss and Malle 01] if m is not too large.

Hence there are only finitely many candidates for the generalized Fitting subgroup $F^*(G)$, which is the central

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2n	2	4	6	8	10	12	14	16	18	20	22
# classes (total)	2	6	4	28	5	32	5	91	10	36	6
# primitive classes	2	5	2	21	3	23	3	63	6	26	4
# almost r.i.m.f.	2	0	1	2	2	8	2	8	5	10	2

TABLE 1. Subgroups of $\mathrm{Sp}_{2n}(\mathbb{Q})$ (up to conjugacy in $\mathrm{GL}_{2n}(\mathbb{Q})$). The third row lists the number of primitive classes that are contained in rationally irreducible maximal finite (r.i.m.f.) subgroups of $\mathrm{GL}_{2n}(\mathbb{Q})$ with index at most 2.

product of $F(G)$ and $E(G)$. Then G can be constructed from $F^*(G)$ using group theory, cohomology, and the fact that $G/F^*(G) \leq \mathrm{Out}(F^*(G))$.

However, these constructions are quite cumbersome and error-prone. Moreover, they do not use the fact that G is a maximal finite (symplectic) subgroup of $\mathrm{GL}_m(\mathbb{Q})$. One very useful algorithmic tool is the so-called generalized Bravais group [Nebe and Plesken 95]. It is used to replace a normal subgroup N of G by some (hopefully) larger normal subgroup $\mathcal{B}^\circ(N)$.

The main result of this classification is the following theorem.

Theorem 1.1. *The number of rationally irreducible maximal finite subgroups of* $\mathrm{Sp}_{2n}(\mathbb{Q})$ *(up to conjugacy in* $\mathrm{GL}_{2n}(\mathbb{Q})$*) are given by Table 2.*

Another class of large symplectic groups is that of groups whose commuting algebras are imaginary quadratic number fields. If n is an odd prime, then every irreducible maximal finite subgroup of $\mathrm{Sp}_{2n}(\mathbb{Q})$ is of this type (see Section 5.4). For $2n = 4, 8, 12, 16, 18, 20$, the numbers of classes whose commuting algebras are not imaginary quadratic are 1, 4, 3, 29, 0, and 3 respectively.[1] A proof of completeness is given in the thesis [Kirschmer 09].

The methods used are explained here, since they might be of independent interest and can be applied to classify symplectic matrix groups also in higher degree.

The existence of the famous Leech lattice makes dimension 24 very interesting. Though to construct a full classification of all symplectic irreducible maximal finite (s.i.m.f.) subgroups of $\mathrm{Sp}_{24}(\mathbb{Q})$ is possible with the methods developed in this paper, it would be quite tedious to do so due to the large number of possible Fitting subgroups.

[1] Representatives of all conjugacy classes are available from http://www.math.rwth-aachen.de/~Markus.Kirschmer/symplectic/, and they are also included in MAGMA 2.16 [Bosma et al. 97].

The author believes that the methods described in this paper are the most appropriate ones for obtaining a full classification of s.i.m.f. subgroups. However, partial classifications (for instance classifying only the irreducible symplectic structures of the Leech lattice) are possible with more direct approaches.

This paper is organized as follows. Section 2 contains some basic definitions for matrix groups. It is shown that it suffices to classify only the s.i.m.f. matrix groups and that they can be characterized by automorphism groups of lattices.

Section 3 shows that it suffices to classify only the so-called symplectic primitive matrix groups, and it contains the general outline of the classification. The definition and some properties of the generalized Bravais groups are also recalled.

Section 4 introduces some infinite families of maximal finite symplectic matrix groups. In particular, all maximal finite symplectic subgroups of $\mathrm{GL}_{p-1}(\mathbb{Q})$ and $\mathrm{GL}_{p+1}(\mathbb{Q})$ whose orders are divisible by some prime p are determined.

Section 5 describes some shortcuts that are used frequently in the classification. It gives two results that can be used to rule out some candidates for normal subgroups of index 2^k. In this section, the m-parameter argument is recalled, which can be used to find all s.i.m.f. matrix groups that contain a given matrix group whose commuting algebra is a field. The section also explains how to find all s.i.m.f. matrix groups that contain a given normal subgroup whose commuting algebra is a quaternion algebra. Finally, these methods are applied to $\mathrm{Sp}_{2p}(\mathbb{Q})$ for primes $p \geq 5$.

2. SOME DEFINITIONS AND BASIC PROPERTIES

Definition 2.1. Let $G < \mathrm{GL}_m(\mathbb{Q})$.

1. $\mathcal{F}(G) = \{F \in \mathbb{Q}^{m\times m} \mid gFg^{\mathrm{tr}} = F \text{ for all } g \in G\}$ is called the *form space* of G. Further, $\mathcal{F}_{\mathrm{sym}}$, $\mathcal{F}_{>0}$,

and $\mathcal{F}_{\mathrm{skew}}$ denote the subsets of symmetric, positive definite symmetric, and skew-symmetric forms.

2. The group G is called *symplectic* if $\mathcal{F}_{\mathrm{skew}}(G)$ contains an invertible element.
3. The set

$$\mathcal{Z}(G) = \{L \subset \mathbb{Q}^{1\times m} \mid L \text{ a rank-}m \text{ lattice with } Lg = L \text{ for all } g \in G\}$$

comprises the (full) *G-invariant lattices*.

4. $C_{\mathbb{Q}^{m\times m}}(G) = \{c \in \mathbb{Q}^{m\times m} \mid cg = gc \text{ for all } g \in G\}$ is the *commuting algebra* of G.
5. Let L be a rank-m lattice in $\mathbb{Q}^{1\times m}$ and suppose $F \in \mathbb{Q}^{m\times m}$ is symmetric and positive definite. If K is a subfield of $\mathbb{Q}^{m\times m}$, then

$$\mathrm{Aut}_K(L,F) = \{g \in \mathbb{Q}^{m\times m} \mid Lg = L,\ gFg^{\mathrm{tr}} = F,\ gc = cg \text{ for all } c \in K\}$$

denotes the group of K-linear automorphisms of (L, F). If $K \simeq \mathbb{Q}$, the subscript K is usually omitted.

An averaging argument shows that $G < \mathrm{GL}_m(\mathbb{Q})$ is finite if and only if $\mathcal{F}_{>0}(G)$ and $\mathcal{Z}(G)$ are both nonempty. Thus, the finite subgroup $G < \mathrm{GL}_m(\mathbb{Q})$ is maximal finite if and only if $G = \mathrm{Aut}(L, F)$ for all $(L, F) \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G)$.

If $F \in \mathcal{F}(G)$ is invertible, then $C_{\mathbb{Q}^{m\times m}}(G) \to \mathcal{F}(G)$, $c \mapsto cF$, is an isomorphism of $\mathbb{Q}$-spaces. This can be used to characterize the maximal finite symplectic matrix groups as automorphism groups of lattices.

Before this result is stated, the next lemma shows that it suffices to consider only rationally irreducible matrix groups, i.e., matrix groups whose natural representation is irreducible over the rationals.

Lemma 2.2. *The natural representation $\triangle\colon G \to \mathrm{GL}_{2n}(\mathbb{Q})$, $g \mapsto g$, of a maximal finite symplectic subgroup $G < \mathrm{GL}_{2n}(\mathbb{Q})$ is the direct sum of rationally irreducible nonisomorphic representations of G.*

Proof: Suppose $\triangle$ decomposes into $\oplus_{i=1}^{s} n_i\triangle_i$, where $\triangle_1, \dots, \triangle_s$ are rationally irreducible and nonisomorphic. Since the commuting algebra and thus the form space of G decompose into direct sums, it follows that each group $n_i\triangle_i(G)$ is maximal finite symplectic. Hence it suffices to show that $n_i = 1$ for all i. Let E_i be the commuting algebra of $\triangle_i(G)$. If E_i is not a totally real number field, then $\triangle_i(G)$ is already symplectic (see Lemma 2.3), and thus $n_i\triangle_i(G)$ is contained in the wreath product $\triangle_i(G) \wr S_{n_i}$. If E_i is a totally real number field, then n_i must be even, since the form space of $n_i\triangle_i(G)$ contains an invertible skew-symmetric element. But then $n_i\triangle_i(G)$ is conjugate to a subgroup of the tensor product $\triangle_i(G) \otimes H$ for any maximal finite symplectic $H < \mathrm{GL}_{n_i}(\mathbb{Q})$. ☐

Lemma 2.3. *Let $G < \mathrm{GL}_{2n}(\mathbb{Q})$ be finite and rationally irreducible with commuting algebra C. The following statements are equivalent:*

1. *G is symplectic.*
2. *C contains some (minimal) totally complex subfield.*
3. *There exist some (minimal) totally complex field K of degree d and some representation $\triangle_K : G \to \mathrm{GL}_{2n/d}(K)$ such that the natural character of G is the trace character corresponding to $\triangle_K$.*

Proof: Let $F \in \mathcal{F}_{>0}(G)$. The space $\mathcal{F}(G)$ is closed under taking transposes; hence it decomposes into $\mathcal{F}_{\mathrm{sym}}(G) \oplus \mathcal{F}_{\mathrm{skew}}(G)$. Since C is a skew field, it follows from $\mathcal{F}(G) = C \cdot F$ that G is symplectic if and only if $\mathcal{F}(G) \neq \mathcal{F}_{\mathrm{sym}}(G)$, which is equivalent to saying that $\mathcal{F}_{\mathrm{skew}}(G) \neq \{0\}$.

If $c \in C$ is such that cF is symmetric (skew-symmetric), then c is a self-adjoint (anti-self-adjoint) endomorphism of the Euclidean space $(\mathbb{R}^n, F)$. Hence $\mathbb{Q}[c] \leq C$ is totally real (complex). Thus the first two statements are equivalent. The equivalence of the last two statements follows from ordinary representation theory. ☐

The symplectic irreducible maximal finite (s.i.m.f.) matrix groups can now be characterized as follows.

Corollary 2.4. *A finite rationally irreducible symplectic matrix group $G < \mathrm{GL}_{2n}(\mathbb{Q})$ is s.i.m.f. if and only if $G = \mathrm{Aut}_K(L, F)$ for all $(L, F) \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G)$ and all minimal totally complex subfields K of $C_{\mathbb{Q}^{2n\times 2n}}(G)$.*

It follows, for example, from [Artin 57, Theorems 3.7 and 3.25] that $\mathbb{Q}^{m\times m}$ contains an invertible skew-symmetric matrix A if and only if m is even. Moreover, if $m = 2n$, then there exists some $T \in \mathrm{GL}_{2n}(\mathbb{Q})$ such that $TAT^{\mathrm{tr}} = J_n$, where $J_n := \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$. Thus each $\mathrm{GL}_{2n}(\mathbb{Q})$-conjugacy class of symplectic subgroups of $\mathrm{GL}_{2n}(\mathbb{Q})$ contains a representative in

$$\mathrm{Sp}_{2n}(\mathbb{Q}) := \left\{g \in \mathrm{GL}_{2n}(\mathbb{Q}) \mid gJ_ng^{\mathrm{tr}} = J_n\right\}.$$

If $G < \mathrm{Sp}_{2n}(\mathbb{Q})$, one might ask how

$$\{G^x \mid x \in \mathrm{GL}_{2n}(\mathbb{Q}) \text{ and } G^x < \mathrm{Sp}_{2n}(\mathbb{Q})\}$$

decomposes into $\mathrm{Sp}_{2n}(\mathbb{Q})$-conjugacy classes. The following example shows that in general, there are infinitely many such classes.

Example 2.5. Let $g = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $G = \langle g \rangle < \mathrm{Sp}_2(\mathbb{Q})$. For $a \in \mathbb{Q}$ set $h_a = \mathrm{Diag}(1, a)$. Then $G^{h_a} < \mathrm{Sp}_2(\mathbb{Q})$ for all such a. Moreover, G^{h_a} is conjugate to G^{h_b} in $\mathrm{Sp}_2(\mathbb{Q})$ if and only if $|ab| \in \mathrm{Nr}_{\mathbb{Q}(i)/\mathbb{Q}}(\mathbb{Q}(i)^*)$.

Proof: Let $t := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then G^{h_a} is conjugate to G^{h_b} in $\mathrm{Sp}_2(\mathbb{Q})$ if and only if there exists some $x \in \mathrm{Sp}_{2n}(\mathbb{Q})$ such that $g^{h_a} = \pm g^{h_b x} = g^{t^k h_b x}$ with $k \in \{0, 1\}$. This is equivalent to saying that $\alpha := t^k h_b x h_{a^{-1}}$ is contained in the commuting algebra of G, which is isomorphic to $\mathbb{Q}(i)$. Taking determinants yields $(-1)^k \cdot b/a = \mathrm{Nr}_{\mathbb{Q}(i)/\mathbb{Q}}(\alpha) > 0$ as claimed. □

3. PRIMITIVITY

Lemma 2.2 shows that it suffices to classify only the (conjugacy classes of) rationally irreducible maximal finite symplectic matrix groups. This set can be restricted further as follows.

Definition 3.1. Let K be a field. An irreducible matrix group $G < \mathrm{GL}_m(K)$ is called *primitive* if it is not conjugate to some subgroup $H \wr S_k$ for some $H < \mathrm{GL}_d(K)$ with $kd = m$.

Similarly, an irreducible symplectic subgroup $G < \mathrm{GL}_{2n}(\mathbb{Q})$ is called *symplectic primitive* if it is not conjugate to a subgroup of $H \wr S_k$ for some $H < \mathrm{Sp}_{2d}(\mathbb{Q})$ with $kd = n$.

Clearly, the conjugacy classes of symplectic imprimitive matrix groups can be constructed from the classifications of smaller dimensions. Furthermore, an irreducible symplectic matrix group H is imprimitive if and only if there exists some invariant lattice $L \in \mathcal{Z}(H)$ that admits a nontrivial decomposition $L = \perp_i L_i$, where the components L_i are mutually perpendicular with respect to the full form space $\mathcal{F}(H)$. Thus, symplectic imprimitive groups can be recognized easily.

The concept of primitivity has some important consequences for normal subgroups.

Remark 3.2. Let $G < \mathrm{GL}_m(K)$ be primitive and $N \trianglelefteq G$. Then G acts on N by conjugation. Hence it also acts on the set of central primitive idempotents of $\langle N \rangle_K$. Thus G permutes the homogeneous components of the natural KN-module $K^{1\times m}$. But since G is primitive, there can be only one such component. Hence the natural KN-module $K^{1\times m}$ splits into a direct sum of k isomorphic KN-modules of dimension m/k.

Lemma 3.3. *Let $G < \mathrm{GL}_{2n}(\mathbb{Q})$ be irreducible and symplectic primitive.*

1. *If $N \trianglelefteq G$, then the natural character of N is a multiple of a single rationally irreducible character.*
2. *If $\mathrm{O}_p(G) \neq 1$, then $p - 1$ divides $2n$.*
3. *Every abelian characteristic subgroup of $\mathrm{O}_p(G)$ is cyclic.*

Proof: The last two statements are immediate consequences of the first one.

Let $K < C_{\mathbb{Q}^{2n\times 2n}}(G)$ be a minimal totally complex subfield of degree d. Let $\{f_1, \dots, f_r\}$ and $\{e_1, \dots, e_s\}$ be the central primitive idempotents of the enveloping algebras $\langle N \rangle_{\mathbb{Q}}$ and $\langle G \rangle_K$ respectively. Let χ_i denote the character corresponding to a simple $\langle G \rangle_K\, e_i$-module and let $L = \mathbb{Q}(\chi_1, \dots, \chi_s) \subseteq K$ be their character field. Then $L/\mathbb{Q}$ is Galois and

$$e_i = \frac{\chi_i(1)}{|G|} \sum_{g \in G} \chi_i(g^{-1}) g \in \langle G \rangle_L .$$

Since G is irreducible, $\{e_1, \dots, e_s\}$ is a Galois orbit under $\mathrm{Gal}(L/\mathbb{Q})$. For any $1 \le j \le r$ there exists some i such that $e_i f_j \neq 0$. Since $f_j \in \langle N \rangle_{\mathbb{Q}}$ is fixed under $\mathrm{Gal}(L/\mathbb{Q})$, it follows that $e_i f_j \neq 0$ for all i, j.

Denote by $\triangle_K : G \to \mathrm{GL}_{2n/d}(K)$ an irreducible representation of G as in Lemma 2.3. The enveloping algebra $\langle \triangle_K(G) \rangle_K$ is isomorphic to $\langle G \rangle_K\, e_i$ for some i. Now $\{e_i f_1, \dots, e_i f_r\}$ is a set of central idempotents of $\langle N \rangle_K\, e_i \simeq \langle \triangle_K(N) \rangle_K$. But since G is symplectic primitive, $\triangle_K(G)$ is primitive as well. Therefore $\langle \triangle_K(N) \rangle_K$ is a simple algebra by the remark above. This shows that $r = 1$, since no $e_i f_j$ vanishes. □

In particular, the candidates for $\mathrm{O}_p(G)$ are well known by a theorem of Philip Hall (see, for example, [Huppert 67, Satz 13.10]).

Theorem 3.4. (Philip Hall.) *Suppose P is a p-group such that every abelian characteristic subgroup of P is cyclic. Then P is the central product of two subgroups E_1 and E_2 satisfying one of the following conditions:*

1. *$p \neq 2$ and E_1 is extraspecial of exponent p and E_2 is cyclic.*
2. *$E_1 \simeq 2_+^{1+2m}$ and E_2 is cyclic, dihedral, quasidihedral, or a generalized quaternion 2-group.*

3. $E_1 = 2_-^{1+2m}$ *and* $E_2 = 1$.

The outline of the classification of all conjugacy classes of s.i.m.f. subgroups of $\mathrm{Sp}_{2n}(\mathbb{Q})$ is now as follows:

1. The symplectic imprimitive matrix groups come from the classifications of $\mathrm{Sp}_{2d}(\mathbb{Q})$, where d runs through all divisors of n. The wreath products $H \wr S_k$ for some maximal finite irreducible and symplectic primitive (s.p.i.m.f.) group $H < \mathrm{Sp}_{2d}(\mathbb{Q})$ with $dk = n$ are most often maximal finite. (The only exception up to dimension $2n = 22$ is $C_4 \wr S_2 < \mathrm{Sp}_4(\mathbb{Q})$.)

Suppose now $G < \mathrm{Sp}_{2n}(\mathbb{Q})$ is s.p.i.m.f.

2. There are only finitely many candidates for the Fitting subgroup $F(G)$ according to Hall's classification.

3. There are only finitely many candidates for the layer $E(G)$ (the central product of all subnormal quasisimple subgroups of G). These are described in [Hiss and Malle 01] for $2n \leq 250$, which is based on the *Atlas of Finite Groups* [Conway et al. 85]. Note that this step depends on the completeness of the classification of all finite simple groups. For small dimensions, one can also use the results of Blichfeldt, Brauer, Lindsey, Wales, and Feit (see, for example, [Feit 76]), who classified the finite (quasiprimitive) subgroups of $\mathrm{GL}_m(\mathbb{C})$ for $m \leq 10$.

4. So there are only finitely many candidates for the generalized Fitting subgroup $F^*(G)$, i.e., the central product of $E(G)$ and $F(G)$. Since $G/F^*(G)$ is isomorphic to a subgroup of $\mathrm{Out}(F^*(G))$, it remains to construct all possible extensions of $F^*(G)$ up to conjugacy in $\mathrm{GL}_{2n}(\mathbb{Q})$.

The last step is the crucial one. Theoretically, all abstract extensions G of $F^*(G)$ can be constructed by group cohomology. But for each abstract extension, a realization as symplectic matrix group has to be constructed. Thus it is advantageous to construct the matrix group G from $F^*(G)$ directly, if possible. A first step is to replace $F^*(G)$ by its generalized Bravais group, as described below.

3.1. Generalized Bravais Groups

The concept of generalized Bravais groups was introduced in [Nebe and Plesken 95, p. 82]. This gives a general construction of how to replace candidates for normal subgroups of s.p.i.m.f. matrix groups by (hopefully) larger normal subgroups.

Let $N < \mathrm{GL}_m(\mathbb{Q})$ be such that the enveloping algebra $\langle N \rangle_{\mathbb{Q}}$ is simple. (By Lemma 3.3. this holds for normal subgroups of s.p.i.m.f. matrix groups.) Then the natural $\mathbb{Q}N$-module $\mathbb{Q}^{1\times m}$ is isomorphic to a multiple of some irreducible $\mathbb{Q}N$-module V.

First, one applies the so-called radical idealizer process to the $\mathbb{Z}$-order $\Lambda_0 := \langle N \rangle_{\mathbb{Z}}$ in $\langle N \rangle_{\mathbb{Q}}$ of N. That is, one iterates the following steps:

- Let I_i be the arithmetic radical of Λ_i, that is, the intersection of all maximal right ideals of Λ_i that contain the (reduced) discriminant of Λ_i.
- Let Λ_{i+1} be the right idealizer of I_i in $\langle N \rangle_{\mathbb{Q}}$.

This process stabilizes in a necessarily hereditary order Λ_∞; see [Reiner 03, Theorems 39.11, 39.14, 40.5].

Definition 3.5. Suppose N, V, and Λ_∞ are as above. Let $L_1, \dots, L_s$ represent the isomorphism classes of Λ_∞-lattices in V and fix some $F \in \mathcal{F}_{>0}(N)$. Then

$$\mathcal{B}^\circ(N) = \{g \in \langle N \rangle_{\mathbb{Q}} \mid L_i g = L_i \text{ for all } 1 \leq i \leq s \text{ and } gFg^{\mathrm{tr}} = F\}$$

denotes the *generalized Bravais group* of N.

The group $\mathcal{B}^\circ(N)$ is finite, since it fixes a full lattice as well as a positive definite form. By the double centralizer property, $\mathcal{B}^\circ(N)$ and N have the same commuting algebras and thus the same form spaces. In particular, the definition of $\mathcal{B}^\circ(N)$ does not depend on the choice of the form F. Moreover, $\mathcal{B}^\circ(N)$ has the following properties:

Lemma 3.6. [Nebe and Plesken 95, Proposition II.10] *If G is a s.p.i.m.f. subgroup of $\mathrm{Sp}_{2n}(\mathbb{Q})$ and $N \trianglelefteq G$, then the following hold:*

1. $N \trianglelefteq \mathcal{B}^\circ(N) \trianglelefteq G$.
2. *If X is a finite subgroup of $\langle N \rangle_{\mathbb{Q}}^*$ such that $N \trianglelefteq X$, then $X \leq \mathcal{B}^\circ(N)$.*
3. $\mathcal{B}^\circ(N) = \{g \in G \mid g \text{ centralizes } C_{\mathbb{Q}^{2n\times 2n}}(N)\}$.

The candidates of $\mathrm{O}_p(G)$ for some irreducible symplectic primitive $G < \mathrm{GL}_{2n}(\mathbb{Q})$ are well known by Theorem 3.4, and their generalized Bravais groups have been determined generically.

Proposition 3.7. [Nebe 98a, Chapter 8] *The candidates for $N = \mathrm{O}_p(G)$ and their generalized Bravais groups for an irreducible symplectic primitive matrix group G are given in Table 2.*

N	$\mathcal{B}^\circ(N)$	$d = \dim_{\mathbb{Q}} \langle N \rangle_{\mathbb{Q}}$	$C_{\mathbb{Q}^{d\times d}}(N)$
C_{p^m}	$\pm$N	$p^{m-1}(p-1)$	$\mathbb{Q}(\zeta_{p^m})$
$p_+^{1+2n}, \quad (p>2)$	$\pm N.\,\mathrm{Sp}_{2n}(p)$	$p^n(p-1)$	$\mathbb{Q}(\zeta_p)$
2_+^{1+2n}	$N.\,\mathrm{O}_{2n}^+(2)$	2^n	$\mathbb{Q}$
2_-^{1+2n}	$N.\,\mathrm{O}_{2n}^-(2)$	2^{n+1}	$Q_{\infty,2}$
$p_+^{1+2n} \curlyvee C_{p^m}, \quad (m>1)$	$\pm N.\,\mathrm{Sp}_{2n}(p)$	$p^{m+n-1}(p-1)$	$\mathbb{Q}(\zeta_p)$
$2_+^{1+2n} \curlyvee D_{2^m}, \quad (m>3)$	$N.\,\mathrm{Sp}_{2n}(2)$	2^{n+m-2}	$\mathbb{Q}(\theta_{2^{m-1}})$
$2_+^{1+2n} \curlyvee Q_{2^m}, \quad (m>3)$	$N.\,\mathrm{Sp}_{2n}(2)$	2^{n+m-1}	$Q_{\theta_{2^{m-1}},\infty}$
$2_+^{1+2n} \curlyvee QD_{2^m}, \quad (m>3)$	$N.\,\mathrm{Sp}_{2n}(2)$	2^{n+m-2}	$\mathbb{Q}(\zeta_{2^{m-1}} - \zeta_{2^{m-1}}^{-1})$

TABLE 2. The candidates for $N = \mathrm{O}_p(G)$ and their generalized Bravais groups for an irreducible symplectic primitive matrix group G. Here $\theta_k = \zeta_k + \zeta_k^{-1}$ generates the maximal totally real subfield of the cyclotomic field $\mathbb{Q}(\zeta_k)$ and $Q_{\theta_{2^{m-1}},\infty} \simeq Q_{\infty,2} \underset{\mathbb{Q}}{\otimes} \mathbb{Q}(\theta_{2^{m-1}})$ denotes the quaternion algebra with center $\mathbb{Q}(\theta_{2^{m-1}})$ ramified only at the infinite places.

The following example shows how the generalized Bravais group can be used to eliminate possible candidates for normal subgroups.

Example 3.8. Let $q \equiv \pm 3 \bmod 8$ be a prime power. Suppose $N < \mathrm{GL}_q(\mathbb{Q})$ is isomorphic to $\mathrm{L}_2(q)$ such that the natural character of N is the Steinberg character. Then $\mathcal{B}^\circ(N)$ is conjugate to the automorphism group $\mathrm{Aut}(A_q) \simeq \pm S_{q+1}$ of the root lattice A_q. In particular, there exists no s.p.i.m.f. $G < \mathrm{GL}_{kq}(\mathbb{Q})$ that contains a normal subgroup isomorphic to $\mathrm{L}_2(q)$ such that its natural character is k times the Steinberg character.

Proof: The natural representation of N is the q-dimensional summand of the permutation module of $\mathrm{L}_2(q)$ on the projective space $\mathbb{P}(\mathbb{F}_q)$. Hence one can assume that N fixes the root lattice $L := A_q$.

Since $q \neq 7, 8$, the group $\mathrm{Aut}(A_q)$ is a maximal finite subgroup of $\mathrm{GL}_q(\mathbb{Q})$ (see [Burnside 12]). So it suffices to check that N and $\mathrm{Aut}(A_q)$ have the same invariant lattices. Let ℓ be a prime divisor of $|N| = \frac{1}{2}(q-1)q(q+1)$. If ℓ divides $q(q-1)$, then the decomposition matrix of $\mathrm{L}_2(q)$ [Burkhardt 76] shows that $L/\ell L$ is an irreducible $\mathbb{F}_\ell\,\mathrm{L}_2(q)$-module.

Now S_{q+1} and $\mathrm{L}_2(q)$ both act 2-transitively on $\mathbb{P}(\mathbb{F}_q)$. If $\ell \neq 2$ divides $q+1$, then the corresponding $\mathbb{F}_\ell$-modular representations have two composition factors (again from [Burkhardt 76]). Hence it follows from [Plesken 77, Theorem 5.1] that N and $\mathrm{Aut}(A_q)$ have the same invariant lattices. Thus $\mathcal{B}^\circ(N) = \mathrm{Aut}(A_q)$, as claimed. The second statement follows immediately from the definition of generalized Bravais groups and Lemma 3.6. □

4. SOME INFINITE FAMILIES

4.1. Some Subgroups of $\mathrm{Sp}_{p-1}(\mathbb{Q})$

Let $p \geq 5$ be prime. In the spirit of [Nebe and Plesken 95, Chapter V] this section describes all s.i.m.f. subgroups of $\mathrm{Sp}_{p-1}(\mathbb{Q})$ whose orders are divisible by p.

Let o be odd such that $p - 1 = 2^a \cdot o$. The group $C_2 \times (C_p : C_o)$ has only one rationally irreducible representation of degree $p-1$, which will be denoted by $\pm C_p : C_o$. This group is symplectic, since its commuting algebra contains $\mathbb{Q}(\sqrt{-p})$.

Another class of candidates are extensions of $\mathrm{L}_2(p)$. The smallest faithful irreducible complex representations of $\mathrm{L}_2(p)$ are of degree $(p-1)/2$ and algebraically conjugate. The corresponding character field is $\mathbb{Q}(\sqrt{\pm p})$, with the minus sign if and only if $p \equiv -1 \bmod 4$ (see [Schur 07]).

If $p \equiv -1 \bmod 4$, then $\mathrm{L}_2(p)$ contains a subgroup U isomorphic to $C_p : C_{(p-1)/2}$. The restriction of the natural representation of $\mathrm{L}_2(p)$ on U is irreducible and has the same character field $\mathbb{Q}(\sqrt{-p})$ [Schur 07]. By [Lorenz 71, Satz 1.2.1], the Schur index of $\mathrm{L}_2(p)$ is equal to the Schur index of U, which is 1. Thus $C_2 \times \mathrm{L}_2(p)$ has a unique $(p-1)$-dimensional rationally irreducible representation (denoted by ${}_{\sqrt{-p}}[\pm\,\mathrm{L}_2(p)]_{(p-1)/2}$ in the sequel) with commuting algebra $\mathbb{Q}(\sqrt{-p})$.

The next result shows that there are no further possibilities.

Theorem 4.1. *Let $p \geq 5$ be prime and $G < \mathrm{Sp}_{p-1}(\mathbb{Q})$ be such that p divides the order $|G|$. Write $p - 1 = 2^a \cdot o$ with o odd.*

Then G is s.i.m.f. if and only if either $p \equiv +1 \bmod 4$ and G is conjugate to $\pm C_p : C_o$, or $p \equiv -1 \bmod 4$ and G is conjugate to ${}_{\sqrt{-p}}[\pm \mathrm{L}_2(p)]_{(p-1)/2}$.

Proof: If $p < 11$ one applies the 2-parameter argument (see Corollary 5.7). Let $p \geq 11$ and $P \in \mathrm{Syl}_p(G)$. Minkowski's bound [Minkowski 87] on the order of G shows that $P \simeq C_p$. In particular, $\pm P$ is self-centralizing in G. So if $P \trianglelefteq G$, then $G \simeq \pm C_p : C_o$. Otherwise, $Z(G) = \pm I_{p-1}$, and a theorem of H. Blau [Feit 82, VIII Theorem 7.2] shows that $G/Z(G) \simeq \mathrm{L}_2(p)$. Since $G \simeq \mathrm{SL}_2(p)$ is ruled out by Schur indices (see [Schur 07]), the result follows from the discussion above. □

4.2. Some Subgroups of $\mathrm{Sp}_{p+1}(\mathbb{Q})$

Let $p \geq 5$ be a prime. If $p \equiv -1 \bmod 4$, then $G := \mathrm{SL}_2(p)$ has only two algebraically conjugate complex representations of degree $(p+1)/2$, as the generic character table [Schur 07] shows. Let χ denote one of the corresponding characters and let $P \in \mathrm{Syl}_p(G)$. An explicit calculation shows that $(1_P^G, \chi)_G = (1_P, \chi|_P)_P = 1$. Thus (by [Isaacs 94, Corollary 10.2(c)]) χ is realizable over its character field, which is $\mathbb{Q}(\sqrt{-p})$. So χ gives rise to a subgroup of $\mathrm{Sp}_{p+1}(\mathbb{Q})$, denoted by ${}_{\sqrt{-p}}[\mathrm{SL}_2(p)]_{(p+1)/2}$.

Theorem 4.2. *Let $p \geq 11$ be prime and $G < \mathrm{Sp}_{p+1}(\mathbb{Q})$ such that p divides $|G|$. Then G is s.i.m.f. if and only if $p \equiv -1 \bmod 4$ and G is conjugate to ${}_{\sqrt{-p}}[\mathrm{SL}_2(p)]_{(p+1)/2}$.*

The proof of this theorem is very similar to the proof of Theorem 4.1.

The classification of all conjugacy classes of s.i.m.f. subgroups of $\mathrm{Sp}_k(\mathbb{Q})$ for $k \in \{6, 8\}$ show that the above result also holds for $p = 5$. But the unique s.i.m.f. subgroup of $\mathrm{Sp}_8(\mathbb{Q})$ whose order is divisible by 7 is ${}_{\sqrt{-7}}[2.\mathrm{Alt}_7]_4$ (which contains a subgroup conjugate to ${}_{\sqrt{-7}}[\mathrm{SL}_2(7)]_4$).

4.3. The Group QD_{2^n}

Let $n \geq 4$. The group

$$QD_{2^n} = \left\langle x, y \mid x^{2^{n-1}}, y^2, x^y = x^{2^{n-2}-1} \right\rangle$$

has one rationally irreducible representation of degree 2^{n-2}. This representation has $\mathbb{Q}(\zeta_{2^{n-1}} - \zeta_{2^{n-1}}^{-1})$ as commuting algebra and is denoted by ${}_{\zeta_{2^{n-1}} - \zeta_{2^{n-1}}^{-1}}[QD_{2^n}]_2$.

Proposition 4.3. *If $n \geq 5$, then ${}_{\zeta_{2^{n-1}} - \zeta_{2^{n-1}}^{-1}}[QD_{2^n}]_2$ is a s.i.m.f. subgroup of $\mathrm{Sp}_{2^{n-2}}(\mathbb{Q})$.*

Proof: Since $\mathbb{Q}(\zeta_{2^{n-1}} - \zeta_{2^{n-1}}^{-1})$ is minimal totally complex, the result follows from Blichfeld's classification of the finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$ [Blichfeldt 17]. □

4.4. The Group 2_+^{1+2n}

Let $T_n < \mathrm{GL}_{2^n}(\mathbb{Q})$ be the n-fold tensor product of $\langle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \rangle \simeq D_8$. So T_n is isomorphic to the extraspecial group 2_+^{1+2n}.

This section, which is heavily based on [Nebe et al. 01, Section 5], describes the construction of $\mathcal{B}^\circ(T_n)$ and defines a maximal finite subgroup of $\mathrm{GL}_{2^n}(\mathbb{Q}(\sqrt{-2}))$ that gives rise to a s.p.i.m.f. subgroup of $\mathrm{Sp}_{2^{n+1}}(\mathbb{Q})$.

Suppose the standard basis of $\mathbb{Q}^{1\times 2^n}$ is indexed by the elements of $\mathbb{F}_2^n$. For an affine subspace U of $\mathbb{F}_2^n$ let $\chi_U = \sum_{u\in U} b_u$. Then L_n and L_n' are the $\mathbb{Z}$-lattices in $\mathbb{Q}^{1\times 2^n}$ spanned by

$$\left\{ 2^{\lfloor (n - \dim(U) + \delta)/2 \rfloor} \chi_U \mid U \text{ an affine subspace of } \mathbb{F}_2^n \right\},$$

where $\delta = 0$ for L_n and $\delta = 1$ for L_n'.

In [Wall 62, Theorem 3.2] it is shown that

$$H_n := \mathrm{Aut}(L_n, I_{2^n}) \cap \mathrm{Aut}(L_n', I_{2^n})$$

is isomorphic to $2_+^{1+2n}.\mathrm{O}_{2n}^+(2)$ and further that $\mathrm{O}_2(H_n)$ is conjugate to T_n.

It is shown in [Winter 72] that $\mathrm{Out}(2_+^{1+2n}) \simeq \mathrm{O}_{2n}^+(2) : 2 \simeq \mathrm{GO}_{2n}^+(2)$ is the full orthogonal group of a quadratic form of Witt defect 0. Conjugation by $h_n := \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right) \otimes I_{2^{n-1}}$ induces an outer automorphism on T_n that is not realized by H_n. Since $C_{\mathbb{Q}^{2^n\times 2^n}}(T_n) \simeq \mathbb{Q}$ and $h_n^2 = 2I_{2^n}$, there exists no element in $\mathrm{GL}_{2^n}(\mathbb{Q})$ of finite order that induces the same automorphism on T_n. Thus H_n is conjugate to $\mathcal{B}^\circ(T_n)$ (once it is shown that $T_n \trianglelefteq \mathcal{B}^\circ(T_n)$). Moreover, $\frac{1}{\sqrt{-2}} h_n$ normalizes T_n and therefore $\mathcal{B}^\circ(T_n)$.

A comparison of the orders of H_n and $\mathrm{Out}(T_n)$ shows that $\mathcal{H}_n := \left\langle H_n, \frac{1}{\sqrt{-2}} h_n \right\rangle$ is the unique extension of H_n by C_2 in $\mathrm{GL}_{2^n}(\mathbb{Q}(\sqrt{-2}))$. The group $\mathcal{H}_n$ gives rise to a finite symplectic matrix group in $\mathrm{Sp}_{2^{n+1}}(\mathbb{Q})$, which will be denoted by ${}_{\sqrt{-2}}[2_+^{1+2n}.(\mathrm{O}_{2n}^+(2):2)]_{2^n}$.

Theorem 4.4. *If $n \geq 2$, then ${}_{\sqrt{-2}}[2_+^{1+2n}.(\mathrm{O}_{2n}^+(2):2)]_{2^n}$ is the unique (up to conjugacy) s.p.i.m.f. subgroup of $\mathrm{Sp}_{2^{n+1}}(\mathbb{Q})$ with Fitting group 2_+^{1+2n}.*

Proof: Let $G_n < \mathrm{GL}_{2^n}(\mathbb{Z}[\sqrt{-2}])$ be the automorphism group of the $\mathbb{Z}[\sqrt{-2}]$-lattice $M_n = \sqrt{-2}L_n' + L_n$. Adapting slightly the proofs given in [Nebe et al. 01, Section 5], it follows that $\langle G_n \rangle_{\mathbb{Z}} = \mathbb{Z}[\sqrt{-2}]^{2^n\times 2^n}$.

In particular, each G_n-invariant $\mathbb{Z}[\sqrt{-2}]$-lattice is a multiple of M_n, since $\mathbb{Z}[\sqrt{-2}]$ has class number 1.

Therefore G_n is a maximal finite subgroup of $\mathrm{GL}_{2^n}(\mathbb{Q}(\sqrt{-2}))$. Hence $G := {}_{\sqrt{-2}}[2_+^{1+2n}.(\mathrm{O}_{2n}^+(2){:}2)]_{2^n} < \mathrm{Sp}_{2^{n+1}}(\mathbb{Q})$ is maximal finite as well.

Suppose now $S < \mathrm{Sp}_{2^{n+1}}(\mathbb{Q})$ is s.p.i.m.f. such that $F(S) \simeq 2_+^{1+2n}$. Then by Lemma 3.3, $F(S)$ must be conjugate to $F(G)$, so one can assume $F(G) = F(S)$. But then $\mathcal{B}^\circ(F(G)) \trianglelefteq S$. Now $|\mathrm{Out}(2^{1+2n})| = 2[\mathcal{B}^\circ(F(G)) : F(G)]$ shows that both G and S contain $\mathcal{B}^\circ(F(G))$ with index 2. Thus G and S are conjugate by Theorem 5.8. $\square$

4.5. The Group p_+^{1+2n}

Let p be an odd prime. In this section, a family of irreducible symplectic matrix groups in dimension $p^n(p-1)$ is described that will be maximal finite in the case that p is a Fermat prime, i.e., $p-1$ is a power of two.

Let $T_n^{(p)} \simeq p_+^{1+2n}$ be the n-fold tensor product of $T_1^{(p)}$, where $T_1^{(p)}$ is the subgroup of $\mathrm{GL}_p(\mathbb{Q}(\zeta_p))$ generated by the diagonal matrix $\mathrm{Diag}(1, \zeta_p, \ldots, \zeta_p^{p-1})$ and the permutation matrix corresponding to the p-cycle $(1, \ldots, p)$. Further, let $H_n^{(p)} = N_{U_{p^n}(\mathbb{Q}(\zeta_p))}(T_n^{(p)})$. As in Section 4.4, it follows from [Winter 72] and [Wall 62, Section 4] that

$$H_n^{(p)} = \mathcal{B}^\circ(T_n^{(p)}) \simeq C_2 \times p_+^{1+2n}.\mathrm{Sp}_{2n}(p).$$

The group $H_n^{(p)} < \mathrm{GL}_p(\mathbb{Q}(\zeta_p))$ gives rise to a finite subgroup of $\mathrm{Sp}_{p^n(p-1)}(\mathbb{Q})$, which will be denoted by ${}_{\zeta_p}[\pm p_+^{1+2n}.\mathrm{Sp}_{2n}(p)]_{p^n}$. Since $H_n^{(p)}$ is maximal finite in $\mathrm{GL}_{p^n}(\mathbb{Q}(\zeta_p))$ by [Nebe et al. 01, Theorem 7.3] and $\mathbb{Q}(\zeta_p)$ is minimal totally complex if p is a Fermat prime, one obtains the following theorem.

Theorem 4.5. *If p is a Fermat prime, then*

$${}_{\zeta_p}[\pm p_+^{1+2n}.\mathrm{Sp}_{2n}(p)]_{p^n}$$

is a s.i.m.f. subgroup of $\mathrm{Sp}_{p^n(p-1)}(\mathbb{Q})$.

5. SOME METHODS

5.1. Normal Subgroups of Index 2^k

If $G < \mathrm{Sp}_{2n}(\mathbb{Q})$ is maximal finite, then $G/\mathcal{B}^\circ(F^*(G))$ is very often a 2-group. Thus two criteria are given that eliminate some candidates for $F^*(G)$ in these cases.

The following lemma is an analogue of [Nebe and Plesken 95, Corollary III.4].

Lemma 5.1. *Let N be a subgroup of some s.p.i.m.f. group $G < \mathrm{Sp}_{2n}(\mathbb{Q})$ of index 2. If N is rationally reducible, then $C_{\mathbb{Q}^{2n\times 2n}}(N) \simeq L^{2\times 2}$ for some totally real number field L.*

Proof: Let $g \in G - N$. By Cifford theory, the natural representation of G splits into $\triangle_1 + \triangle_1^g$ for some irreducible $\triangle_1 \colon N \to \mathrm{GL}_n(\mathbb{Q})$. Thus G is conjugate to $\left\langle I_2 \otimes \triangle_1(N), \begin{pmatrix} 0 & I_n \\ \triangle_1(g^2) & 0 \end{pmatrix} \right\rangle \le \triangle_1(N) \wr C_2$. Since G is symplectic primitive and $\triangle_1(N)$ is rationally irreducible, this implies $\dim_{\mathbb{Q}}(\mathcal{F}_{\mathrm{skew}}(\triangle_1(N))) = 0$.

Let $\delta \colon N \to \mathrm{GL}_n(\mathbb{R}), h \mapsto \triangle_1(h)$. Then $\delta = \delta_1 + \cdots + \delta_s$ for some $\mathbb{R}$-irreducible representations $\delta_i \colon N \to \mathrm{GL}_{n_i}(\mathbb{R})$. It follows from the above that $\mathcal{F}(\delta(N)) = \mathcal{F}_{\mathrm{sym}}(\delta(N))$. In particular, this implies that the δ_i are nonisomorphic and $C_{\mathbb{R}^{n_i\times n_i}}(\delta_i(N)) \simeq \mathbb{R}$. Hence $L \otimes_{\mathbb{Q}} \mathbb{R} \simeq \oplus_{i=1}^s \mathbb{R}$, and therefore L is a totally real number field. $\square$

The smallest example of such a group N is

$$N = \langle -I_2 \rangle \trianglelefteq \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle < \mathrm{Sp}_4(\mathbb{Q}),$$

which has $\mathbb{Q}^{2\times 2}$ as commuting algebra.

Example 5.2. The Fitting subgroup of a s.p.i.m.f. group $G < \mathrm{Sp}_8(\mathbb{Q})$ is not isomorphic to Q_8.

Proof: Suppose otherwise. Then

$$C := C_{\mathbb{Q}^{8\times 8}}(F(G)) \simeq Q_{\infty,2}^{2\times 2},$$

where $Q_{\infty,2}$ denotes the quaternion algebra over $\mathbb{Q}$ ramified only at 2 and ∞. Thus $E(G) = 1$ by [Hiss and Malle 01]. Hence

$$G/F(G) \le \mathrm{Out}(F(G)) \simeq S_3.$$

Since $\mathcal{B}^\circ(F(G)) \simeq \mathrm{SL}_2(3)$, one gets $[G : \mathcal{B}^\circ(F(G))] \le 2$. This contradicts the result above, since

$$C_{\mathbb{Q}^{8\times 8}}(\mathcal{B}^\circ(F(G))) = C,$$

and the result follows. $\square$

If n is not a power of 2, the following lemma can be used to rule out several candidates for normal subgroups.

Lemma 5.3. *Let $G < \mathrm{Sp}_{2n}(\mathbb{Q})$ be rationally irreducible and symplectic primitive. Let $N \trianglelefteq G$ be such that $[G : N] = 2^k$. If $\triangle$ denotes the natural representation of N, then $\triangle = 2^\ell \cdot \delta$ for some irreducible representation δ of N and some $0 \le \ell \le k$.*

Proof: The proof given in [Nebe 96a, Lemma III.4] applies mutatis mutandis. $\square$

Example 5.4. Let $G < \mathrm{Sp}_{12}(\mathbb{Q})$ be s.p.i.m.f. If $F(G) = \mathrm{O}_2(G)$, then $\mathrm{O}_2(G)$ is isomorphic to C_2, C_4, or D_8, and $E(G) \neq 1$.

Proof: Let

$$U = \{Q_8, C_8, D_{16}, QD_{16}, D_8 \otimes C_4, 2_+^{1+4}\}.$$

Hall's theorem (see Theorem 3.4) shows that $\mathrm{O}_2(G) \in \{C_2, C_4, D_8\} \cup U$. Suppose first that $E(G) = 1$. Then $G/\mathcal{B}^\circ(F(G)) \leq \mathrm{Out}(F(G))$ is a 2-group. By the previous result, this implies that the natural representation of $\mathcal{B}^\circ(F(G))$ must be a multiple of a faithful rationally irreducible representation of degree 3, 6, or 12. But $F(G) = \mathrm{O}_2(G)$, and thus $\mathcal{B}^\circ(F(G))$ does not admit such representations. This implies $E(G) \neq 1$. Suppose now $\mathrm{O}_2(G) \in U$. Then $C_{\mathbb{Q}^{12\times 12}}(\mathrm{O}_2(G)) \simeq Q^{3\times 3}$, where Q is isomorphic to $\mathbb{Q}$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{\pm 2})$, or $Q_{\infty,2}$. But no quasisimple group embeds into $Q^{3\times 3}$ (see [Hiss and Malle 01]). Thus $E(G) = 1$ gives the desired contradiction. □

5.2. *m*-Parameter Argument

Suppose $U < \mathrm{GL}_n(\mathbb{Q})$ is finite such that $C := C_{\mathbb{Q}^{n\times n}}(U)$ is a number field of absolute degree m and let $F_0 \in \mathcal{F}_{>0}(U)$. In particular, this applies to irreducible cyclic matrix groups. If C^+ denotes the maximal totally real subfield of C, then $\mathcal{F}_{sym}(U) = C^+ \cdot F_0$. The paper [Nebe 96a] gives an algorithm that uses this parameterization and ideal arithmetic in C^+ to show that every finite supergroup $G \geq U$ fixes (up to conjugacy) one of the forms cF, where c runs through some finite set.

Two more definitions are needed to state the result.

Definition 5.5. For a totally real number field K, denote by $\Pi(K)$ a finite set of prime numbers such that the following conditions are satisfied:

- For each $x \in K$, there exists some $y \in \mathbb{Z}_K$ whose norm is supported at $\Pi(K)$ such that xy is totally positive.
- In each ideal class of $\mathbb{Z}_K$ there exists some integral ideal that contains a natural number whose prime factors come from $\Pi(K)$.

Furthermore, if $\ell \in \mathbb{Z}$, set

$$\tilde{\Pi}(\ell, K) = \{p \mid p \text{ is a prime divisor of } \ell \text{ or } p \in \Pi(k) \text{ for some subfield } k \text{ of } K\}.$$

Definition 5.6. Let L be a lattice in $\mathbb{Q}^{1\times n}$ of full rank and let $F \in \mathbb{Q}^{n\times n}$ be symmetric positive definite.

1. $L^{\#,F} = \{x \in \mathbb{Q}^{n\times 1} \mid xFy^{\mathrm{tr}} \in \mathbb{Z} \text{ for all } y \in L\}$ denotes the *dual lattice* of L with respect to F.
2. The pair (L, F) is said to be *integral* if $L \subset L^{\#,F}$.
3. The integral pair (L, F) is said to be *normalized* if the abelian group $L^{\#,F}/L$ has square-free exponent and its rank is at most $n/2$.

Theorem 5.7. [Nebe 96a, Korollar III.3] *Suppose $U < \mathrm{GL}_n(\mathbb{Q})$ is finite such that $C := C_{\mathbb{Q}^{n\times n}}(U)$ is a field and denote by C^+ its maximal totally real subfield. Let $U < G < \mathrm{GL}_n(\mathbb{Q})$ be finite and fix some $L \in \mathcal{Z}(G)$. Then there exists some $F \in \mathcal{F}_{>0}(U)$ such that (L, F) is integral and the prime divisors of $\det(L, F)$ are contained in $\tilde{\Pi}(|G|, C^+)$.*

To find all (conjugacy classes of) s.i.m.f. matrix groups G that contain a conjugate copy of U, one has to consider the groups $\mathrm{Aut}_K(L, F)$ for some minimal totally complex subfield K of C and some integral pair (L, F) where the prime divisors of $\det(L, F)$ are bounded by $\tilde{\Pi}((n+1)!, C^+)$ (see Minkowski's bound [Minkowski 87]).

The following ideas reduce the number of pairs (L, F) that one has to consider to a finite number (see [Nebe 96a, Chapter VII] and [Kirschmer 09, Remark 2.2.13] for details).

- If G fixes some integral (L, F) that is not normalized, then exists some prime p dividing $\det(L, F)$ such that $(L \cap p\ L^{\#,F}, p^{-1}F)$ is integral. Iterating this process, one finds some normalized pair $(L', F') \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G)$.
- Suppose $L_1, \ldots, L_r$ represent the isomorphism classes of $\langle U\rangle_{\mathbb{Z}}$-invariant lattices in $\mathbb{Q}^{n\times n}$. Then L' is isomorphic to some L_i, i.e., there exists some $x \in \mathrm{GL}_n(\mathbb{Q})$ such that $L' = L_i x$.
Since
$$G = \mathrm{Aut}_K(L', F) = \mathrm{Aut}_K(L_i x, F) = x^{-1}\mathrm{Aut}_{\mathrm{k}}(L_i, xFx^{tr})x$$
one has to consider only the normalized pairs (L_i, F) with $F \in \mathcal{F}_{>0}(U)$ such that the prime divisors of $\det(L_i, F)$ are contained in $\tilde{\Pi}((n+1)!, C^+)$.
- Let $c \in C$ be such that $L_i c = L_i$. Then $\mathrm{Aut}_K(L_i c, F)$ is conjugate to
$$\mathrm{Aut}_K(L_i, cFc^{\mathrm{tr}}) = \mathrm{Aut}_K(L_i, \mathrm{Nr}_{C/C^+}(c)F).$$

5.3. Quaternion Algebras as Commuting Algebras

Suppose $N < \mathrm{Sp}_{2n}(\mathbb{Q})$ is finite such that $Q := C_{\mathbb{Q}^{2n\times 2n}}(N)$ is a quaternion algebra, i.e., a central simple algebra of degree 4 over its center K. The next theorem explains how to find (conjugacy classes of) s.i.m.f. supergroups G of N such that $N \trianglelefteq G$. After replacing N by $\mathcal{B}^\circ(N)$, one may assume that $N = \mathcal{B}^\circ(N)$.

Theorem 5.8. *Suppose $x \in \mathrm{GL}_{2n}(\mathbb{Q})$ commutes with K and acts on N such that $x^2 \in N$. Then:*

1. *$C_{\mathbb{Q}^{2n\times 2n}}(\langle N, x\rangle) \simeq K[X]/(X^2 - \mathrm{nr}(a))$ for some $a \in Q$, where $\mathrm{nr}\colon Q \to K$ denotes the reduced norm.*
2. *If $y \in \mathrm{GL}_{2n}(\mathbb{Q})$ induces the same outer automorphism on N as x, then*
$$C_{\mathbb{Q}^{2n\times 2n}}(\langle N, y\rangle) \simeq K[X]/(X^2 - u\,\mathrm{nr}(a))$$
for some torsion unit $u \in K^* \cap \mathrm{nr}(Q^*)$.
3. *If $C_{\mathbb{Q}^{2n\times 2n}}(\langle N, x\rangle)$ is a field and $u \in (K^*)^2$, then $\langle N, x\rangle$ and $\langle N, y\rangle$ are conjugate.*

Proof: 1. For any $z \in Q$ one has
$$z^x \in C_{\mathbb{Q}^{2n\times 2n}}(N^x) = C_{\mathbb{Q}^{2n\times 2n}}(N) = Q.$$
Hence x induces a K-automorphism on Q. By the Skolem–Noether theorem, there exists some $a \in Q^*$ such that $z^a = z^x$ for all $z \in Q$. Since x does not centralize Q (otherwise, we would have $x \in \mathcal{B}^\circ(N) = N$ by Lemma 3.6), it follows that
$$K \subsetneq K[a] \subseteq C_Q(a) = C_{\mathbb{Q}^{2n\times 2n}}(\langle N, x\rangle)) \subsetneq Q.$$
Thus $K[a] = C_{\mathbb{Q}^{2n\times 2n}}(\langle N, x\rangle)$. Since $x^2 \in N$, it induces the identity on Q. Hence $a^2 \in K$, and therefore $K[a] \cong K[X]/(X^2 - \mathrm{nr}(a))$.

2. Since yx^{-1} induces an inner automorphism on N, it is contained in NQ^*. Say $yx^{-1} = gc$ with $g \in N$ and $c \in Q$. Then
$$\begin{aligned} C_{\mathbb{Q}^{2n\times 2n}}(\langle N, y\rangle) &= C_{\mathbb{Q}^{2n\times 2n}}(\langle N, cx\rangle) \\ &\simeq K[X]/(X^2 - \mathrm{nr}(ca)). \end{aligned}$$
Further, $cxcx^{-1} = (g^{-1}y)^2 x^{-2} \in N$ has finite order. Hence the reduced norm $\mathrm{nr}(c)^2 = \mathrm{nr}(c \cdot (xcx^{-1})) \in K^*$ also has finite order. Thus $u := \mathrm{nr}(c)$ is a torsion unit, as claimed.

3. Let $v \in K$ be such that $v^2 = u$. Since v has finite order, it is contained in $\mathcal{B}^\circ(N) = N$ by Lemma 3.6. The elements a^2 and $(v^{-1}ca)^2$ are both contained in K. So $\mathrm{nr}(v^{-1}c) = 1$ implies that a and $v^{-1}ca$ have the same minimal polynomial over K. It follows from the Skolem–Noether theorem that there exists some $t \in Q^*$ such that $a^t = v^{-1}ca$. Finally,
$$x^t = t^{-1}xtx^{-1}x = t^{-1}ata^{-1}x = v^{-1}caa^{-1}x = (gv)^{-1}y$$
shows that $\langle N, x\rangle^t = \langle N, (gv)^{-1}y\rangle = \langle N, y\rangle$. □

In most cases, the structure of Q allows some simplifications:

1. If $K \simeq \mathbb{Q}$, then obviously every $x \in \mathrm{GL}_{2n}(\mathbb{Q})$ commutes with K.
2. If Q is totally definite, then $C_{\mathbb{Q}^{2n\times 2n}}(\langle N, x\rangle)$ must be a field, and there exists no nontrivial torsion unit in $K^* \cap \mathrm{nr}(Q^*)$. That is, each class of outer automorphisms of N gives rise to at most one conjugacy class $\langle N, x\rangle$.
3. Suppose one wants to find all s.p.i.m.f. supergroups G of N satisfying $N = \mathcal{B}^\circ(F^*(G))$. If Q is totally definite and $K \simeq \mathbb{Q}$, then the proof of [Nebe 98b, Theorem 4] shows that the exponent of G/N is at most 2. Hence one gets all such groups G (up to conjugacy) with the above result and the 2-parameter argument.

Here is a simple example.

Example 5.9. Suppose $G < \mathrm{Sp}_4(\mathbb{Q})$ is s.p.i.m.f. and it contains a normal subgroup U isomorphic to Q_8. (Note that the character of U is uniquely determined by Lemma 3.3.)

Let $N := \mathcal{B}^\circ(U) \simeq \mathrm{SL}_2(3)$. Then
$$C_{\mathbb{Q}^{2n\times 2n}}(U) = C_{\mathbb{Q}^{2n\times 2n}}(N) \simeq Q_{\infty,2}$$
is the quaternion algebra over $\mathbb{Q}$ ramified only at 2 and ∞. If there exists some $x \in C_G(N) - N$, then x is contained in some maximal order of Q. These orders are all conjugate and have $\mathrm{SL}_2(3)$ as group of torsion units. Thus one can assume that the order of x equals 3 or 4. In both cases, the commuting algebra of $H := \langle N, x\rangle$ is an imaginary quadratic number field. If x has order 3, then H is s.i.m.f. In the other case, it follows from the 2-parameter argument that H is contained only in the s.i.m.f. group $\mathcal{B}^\circ(D_8 \otimes C_4) = (D_8 \otimes C_4).S_3$.

Suppose now that N is self-centralizing in G. One immediately obtains the 4-dimensional faithful representation of $\mathrm{GL}_2(3)$ with commuting algebra $\mathbb{Q}(\sqrt{-2})$ as one possibility for G. By the above result, $\mathrm{GL}_2(3)$ is (up to conjugacy) the only possibility, since $\mathrm{Out}(N) \simeq C_2$ and $\mathbb{Q}^* \cap \mathrm{nr}(Q_{\infty,2}) \subseteq \mathbb{Q}_{>0}$ contains no nontrivial torsion unit.

5.4. Application to Dimension 2*p*

Let $p \geq 5$ be prime. The structure of s.p.i.m.f. subgroups of $\mathrm{Sp}_{2p}(\mathbb{Q})$ is rather restricted.

Lemma 5.10. *If $G < \mathrm{Sp}_{2p}(\mathbb{Q})$ is s.p.i.m.f., then $F(G)$ is cyclic of order 2, 4, or 6.*

Proof: Clearly, if $q \geq 5$ is a prime such that $\mathrm{O}_q(G) \neq 1$, then $q = 2p+1$ by Lemma 3.3. In particular, $q \equiv -1 \bmod 4$ and $q \geq 11$. This contradicts Theorem 4.1. So $F(G) = \mathrm{O}_2(G)\,\mathrm{O}_3(G)$. If G is cyclic, then $G = F(G) = \mathrm{O}_2(G)\,\mathrm{O}_3(G)$ is reducible. So G embeds into $\mathrm{GL}_p(K)$ for some imaginary quadratic number field K. Hence the result follows from Hall's classification (see Theorem 3.4). □

In particular, the generalized Fitting subgroup $F^*(G)$ of a s.p.i.m.f. group $G < \mathrm{Sp}_{2p}(\mathbb{Q})$ cannot be solvable. Thus $F^*(G)$ is either reducible with $\mathbb{Q}^{2\times 2}$ as commuting algebra or irreducible and its commuting algebra is an imaginary quadratic number field. Thus G can usually be recovered from $F^*(G)$ by Theorems 5.7 and 5.8.

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