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Combinatorial Properties of the $K3$ Surface: Simplicial Blowups and Slicings

Jonathan Spreer and Wolfgang Kühnel

CONTENTS

The 4-dimensional abstract Kummer variety K^4 with 16 nodes leads to the $K3$ surface by resolving the 16 singularities. Here we present a simplicial realization of this minimal resolution. Starting with a minimal 16-vertex triangulation of K^4, we resolve its 16 isolated singularities—step by step—by simplicial blowups. As a result we obtain a 17-vertex triangulation of the standard PL $K3$ surface. A key step is the construction of a triangulated version of the mapping cylinder of the Hopf map from real projective 3-space onto the 2-sphere with the minimum number of vertices. Moreover, we study simplicial Morse functions and the changes of their levels between the critical points. In this way we obtain slicings through the $K3$ surface of various topological types.

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1. INTRODUCTION

Triangulations of manifolds with few vertices have been a growing subject of research during the last few years. This is due to new computer facilities that allow calculations and even computer experiments with a list of simplices on, say, up to 50 vertices or more. Here we are dealing with *combinatorial d-manifolds*, which are d-dimensional simplicial complexes such that the link of every i-simplex is a triangulated $(d-i-1)$-dimensional standard PL-sphere. For a *combinatorial d-pseudomanifold with isolated singularities*, we require that the link of each vertex be a combinatorial $(d-1)$-manifold, not necessarily a sphere. Not all triangulated pseudomanifolds satisfy this property. It turns out that there is a triangulated 5-sphere with only 20 vertices that is not combinatorial [Björner and Lutz 00]. This example is not even a combinatorial pseudomanifold.

The problem of finding a combinatorial version of an abstract d-pseudomanifold is not trivial, especially if some additional properties such as vertex minimality are required. It is well known that one has the following operations in the class of combinatorial manifolds for solving this problem: *products* and *connected sums*. The products require a simplicial subdivision of prisms, but that is available. In algebraic geometry there is a third operation on a certain type of pseudomanifold, namely, the *resolution of singularities*. A fourth operation would be a combinatorial version of *Dehn twists*. If these could be applied to simply connected combinatorial 4-manifolds, we could make progress toward a solution of some interesting problems:

Problem 1.1. Find a pair of orientable PL d-manifolds (M_1, M_2) such that

(i) M_1 and M_2 are not homeomorphic,

(ii) there are combinatorial triangulations of M_1 and M_2 with n vertices but not with $n-1$ vertices,

(iii) the *f-vector* of such an n-vertex triangulation is unique for both M_1 and M_2.

The entries of the f-vector are defined as the numbers f_i of i-dimensional simplices of the triangulation.

Problem 1.2. Find two concrete combinatorial triangulations of a 4-manifold such that the underlying PL manifolds are homeomorphic but not PL-homeomorphic. It is known that some compact topological 4-manifolds admit exotic PL structures. Furthermore, any combinatorial triangulation induces a unique PL structure and thus a unique smooth structure.

Concerning Problem 1.1, there are pairs of nonorientable and orientable surfaces with the same minimum number of vertices, e.g., the two surfaces with $\chi = -10$ admit triangulations with the f-vector $(12, 66, 44)$ but no smaller triangulations. Moreover, the existence of pairs of nonhomeomorphic lens spaces with the same minimum number of vertices is known due to [Brehm and Swiatkowsk 93]. However, so far, no such pair of concrete combinatorial manifolds has been constructed.

Concerning Problem 1.2, it is well known that in a topological classification of simply connected 4-manifolds, the relevant pieces are $\mathbb{C}P^2$ (with two orientations), $S^2 \times S^2$, and the $K3$ surface (with two orientations). However, for topological 4-manifolds it can happen that there are possibly many distinct PL structures. There is a method to construct exotic PL structures on 4-manifolds using *Akbulut corks*: Akbulut and Yasui investigated bounded submanifolds of a 4-manifold M. These so-called corks can be cut out and glued back into the original manifold, thus changing the PL type of M (see [Akbulut 91] and [Akbulut and Yasui 08]).

However, applying Akbulut corks to combinatorial manifolds requires more experiments. For $\mathbb{C}P^2$ and $S^2 \times S^2$ we have standard triangulations. For the $K3$ surface we have one optimal triangulation with the minimum number of 16 vertices [Casella and Kühnel 01], but so far, the PL type has not been precisely identified. Presumably, it is the standard structure of the classical $K3$ surface.

In this article we describe a purely combinatorial version of resolving ordinary nodes or double points in real dimension 4. In particular, we describe this procedure for the $K3$ surface as a resolution of the Kummer variety with 16 nodes. "Purely combinatorial" here means that we are dealing with simplicial complexes (or subdivisions of such) with a relatively small number of vertices such that topological properties or modifications can be recognized or carried out by an efficient computer algorithm.

The construction itself is fairly general. We are going to illustrate it for the example of the $K3$ surface as a desingularization of what we call a *Kummer variety*, following [Spanier 56]. In particular, we describe a straightforward and "canonical" procedure to obtain a concrete triangulation of the $K3$ surface with a small number of vertices and with the classical PL structure. As we will see in Section 6, this procedure also gives some insights into Problem 1.1. In principle, such a procedure seems to be possible in any even dimension.

For all this, computer algorithms are employed and implemented in the GAP system. Here, a key operation is the concept of *bistellar flips* due to [Pachner 87] that establishes a PL homeomorphism on a combinatorial level. A GAP program due to [Björner and Lutz 00] implements a heuristic algorithm that reduces the number of vertices of a given combinatorial manifold without changing its PL type. Since this process is not deterministic, its character is rather experimental and needs a great deal of computer calculation. Nonetheless, we can use this concept together with some theoretical lower bounds to get closer to a solution of the following problem:

Problem 1.3. For any given abstract compact PL d-manifold, find the minimum number of vertices for a combinatorial triangulation of it, and find out which topological invariants are related to this number.

For pseudomanifolds admitting some combinatorial triangulation we have the same problem.

2. THE KUMMER VARIETY AND THE $K3$ SURFACE

An abstract d-dimensional Kummer variety $K^d = \mathbb{T}^d/_{x \sim -x}$ can be interpreted as the d-dimensional torus modulo involution [Spanier 56]. It is a d-dimensional *flat orbifold* in the sense that the neighborhood of any point of K^d is a quotient of Euclidean d-space by an orthogonal group. Topologically, K^d can be seen as a pseudomanifold with 2^d isolated singularities that are the fixed points of the involution. A typical neighborhood of a singularity is a cone over a real projective $(d-1)$-space whose apex represents the singularity. Thus, any combinatorial triangulation of K^d needs at least 2^d vertices as a kind of *absolute vertices* [Fáry 77]. In more concrete terms, a series of minimal triangulations of K^d for any $d \geq 3$ has been given in [Kühnel 86].

These combinatorial pseudomanifolds are 2-neighborly (i.e., any two vertices are joined by an edge) and highly symmetric, with a transitive automorphism group of order $(d+1)! \cdot 2^d$. Moreover, they contain a specific combinatorial real projective space $\mathbb{R}P^{d-1}$ with $2^d - 1$ vertices as each vertex link. This vertex link happens to coincide with a 2-fold nonbranched quotient of the vertex link of a series of combinatorial d-tori with $2^{d+1} - 1$ vertices [Kühnel and Lassmann 88], which presumably has the minimum possible number of vertices among all combinatorial d-tori.

In particular, we have a minimal 2-neighborly 16-vertex triangulation of the 4-dimensional Kummer variety, which will be denoted by $(K^4)_{16}$. A few of its properties are the following: The f-vector is given by $f = (16, 120, 400, 480, 192)$, the Euler characteristic is $\chi(K^4) = 8$, and the integral homology groups are

$$H_*(K^4) = (\mathbb{Z}, 0, \mathbb{Z}^6 \oplus (\mathbb{Z}_2)^5, 0, \mathbb{Z}). \tag{2–1}$$

Its intersection form is even of rank 6 and signature 0. We use an integer vertex labeling ranging from 1 to 16. The automorphism group of order $5! \cdot 2^4 = 1920$ is generated by two permutations as follows:

$$\langle\, (1,7,12)(2,8,11)(3,10,16)(4,9,15),\\ (1,9,10,14,16,8,7,3)(2,13,12,6,15,4,5,11)\,\rangle.$$

The complex coincides with the orbit $(1,2,4,8,16)_{192}$ (cf. [Kühnel 86], where the labeling is chosen as $0,1,2,\ldots,15$ instead of $1,2,3,\ldots,16$).

The $K3$ surface, on the other hand, is a prime (i.e., indecomposable by nontrivial connected sums; see [Donaldson 86]) compact oriented connected and simply connected 4-manifold admitting a unique smooth or PL structure. By Freedman's theorem [Freedman 82], it is uniquely determined up to homeomorphism by its intersection form. The Euler characteristic is $\chi(K3) = 24$, the integral homology groups are

$$H_*(K3) = (\mathbb{Z}, 0, \mathbb{Z}^{22}, 0, \mathbb{Z}),$$

and the intersection form is even of rank 22 and signature 16. In a suitable basis it is represented by the unimodular matrix

$$\mathbb{E}_8 \oplus \mathbb{E}_8 \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\mathbb{E}_8$ is given by

$$\mathbb{E}_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

This makes the $K3$ surface distinguished from the topological point of view. Also, from the combinatorial point of view this 4-manifold is fairly special, since the data $n = 16$, $\chi = 24$ coincide with the case of equality in the generalized Heawood inequality

$$\binom{n-4}{3} \geq 10(\chi(M) - 2), \tag{2–2}$$

which holds for any combinatorial n-vertex triangulation of any compact 4-manifold M; see [Kühnel 94], [Kühnel 95, 4B].

Inequality (2–2) is also a partial solution to Problem 1.3 in the introduction. Equality can occur only for 3-neighborly triangulations, i.e., for which any triple of vertices determines a triangle of the triangulation. Consequently, the f-vector has to start with $\left(n, \binom{n}{2}, \binom{n}{3}\right)$ in this case. In other words, any combinatorial triangulation of the $K3$ surface has at least 16 vertices (the same number as required for the Kummer variety K^4), and one with precisely 16 vertices must necessarily be 3-neighborly (or *superneighborly*). Such a 3-neighborly vertex-minimal 16-vertex triangulation of a PL manifold homeomorphic to the $K3$ surface $(K3)_{16}$ was published in [Casella and Kühnel 01]. The f-vector is $f = (16, 120, 560, 720, 288)$, and observe the 3-neighborliness $f_2 = 560 = \binom{16}{3}$. Its automorphism group is isomorphic to the affine linear group $\mathrm{AGL}(1, 16)$ and is generated by two permutations as follows:

$$\langle\, (1,3,8,4,9,16,15,2,14,12,6,7,13,5,10),\\ (1,11,16)(2,10,14)(3,12,13)(4,9,15)(5,7,8)\,\rangle.$$

This group of order $16 \cdot 15 = 240$ acts 2-transitively on the set of vertices $(1,\dots,16)$ of $(K3)_{16}$. The triangulation $(K3)_{16}$ itself is defined as the union of the orbits $(1,2,3,8,12)_{240}$ and $(1,2,5,8,14)_{48}$ under this permutation group; see [Casella and Kühnel 01], where the labeling is chosen as $0,1,2,\dots,15$ instead of $1,2,3,\dots,16$.

3. THE HOPF σ-PROCESS

By the *Hopf σ-process* we mean the process of blowing up a point, and simultaneously, the resolution of nodes or ordinary double points of a complex algebraic variety. This was described in [Hopf 51]; compare [Hirzebruch 53] and [Hauser 00]. From the topological point of view the process consists in cutting out some subspace and gluing in some other subspace. In complex algebraic geometry, one point is replaced by the projective line $\mathbb{C}P^1 \cong S^2$ of all complex lines through that point. This is often called *blowing up* the point. In general, the process can be applied to nonsingular 4-manifolds, and it yields a transformation of a manifold M to $M\#(+\mathbb{C}P^2)$ or $M\#(-\mathbb{C}P^2)$, depending on the choice of orientation.

The same construction is possible for nodes or ordinary double points (a special type of singularity), and also the ambiguity of the orientation is the same for the blowup process of a node. Similarly, it has been used in arbitrary even dimension in [Spanier 56] as a so-called *dilatation process*.

In the particular case of the 4-dimensional Kummer variety with 16 nodes, a result of [Hironaka 64] states that the singularities of a 4-dimensional Kummer variety K^4 can be resolved into a smooth manifold, birationally equivalent to K^4. It is also well known that the minimal resolution of the 4-dimensional Kummer variety is a $K3$ surface. This raises the question whether it is possible to carry out the Hopf σ-process in the purely combinatorial category. In this case, one would have to cut out a certain neighborhood A of each of the singularities and to glue in an appropriate simplicial complex B.

The spaces A_i that have to be cut out are the following: The Kummer variety K^4 is the quotient of a 4-dimensional torus $\mathbb{T}^4 = \mathbb{R}^4/_{\mathbb{Z}^4}$ by the central involution $\sigma : x \mapsto -x$ with precisely 16 fixed points x_i, $1 \le i \le 16$. Let X_i be a suitable neighborhood of x_i. Then σ acts on $X = \mathbb{T}^4 \setminus \bigcup X_i$ without fixed points. The involution σ acts as the antipodal map on each connected component of ∂X.

Therefore, the quotient of ∂X_i is a projective space $\mathbb{R}P^3$ of dimension 3 for each $1 \le i \le 16$, and the quotient of X_i itself is a cone over it, which we denote by A_i. Thus the quotient $\widetilde{X} = X/_\sigma$ is a manifold having 16 disjoint copies of $\mathbb{R}P^3$ as its boundary, and the quotient $K^4 = \mathbb{T}^4/_\sigma$ contains the disjoint subsets $A_1,\dots,A_{16}$ as neighborhoods of the 16 singularities.

The spaces B_i that have to be glued in are the following: The Hopf map $h : S^3 \to \mathbb{C}P^1$ induces a map $\tilde{h} : \mathbb{R}P^3 \to \mathbb{C}P^1$, since the Hopf map identifies antipodal pairs of points. We consider the cylinder $C = \mathbb{R}P^3 \times [0,1]$ with the identification along the bottom of the cylinder by an equivalence relation $\sim$ defined by $(x,0) \sim (\tilde{h}(x),0)$. The quotient $\tilde{C} = C/_\sim$ is a manifold with boundary $\mathbb{R}P^3$. If we identify the boundary of $\widetilde{X}$ with the union of the boundaries of 16 copies $B_1,\dots,B_{16}$ of $\widetilde{C}$, we get a closed manifold S. Alternatively, each B_i can be seen as a copy of $(\mathbb{C}P^2 \setminus B^4)/_\sigma$, where the involution $\tilde{\sigma} : \mathbb{C}P^2 \to \mathbb{C}P^2$ is defined by

$$\tilde{\sigma}[z_0,z_1,z_2] = [-z_0,z_1,z_2]$$

with a fixed-point set consisting of the point $[1,0,0]$ at the center of the ball B^4 and the polar projective line $z_0 = 0$.

It is proved in [Spanier 56] that S is in fact a $K3$ surface. Our main result is a simplicial realization of this construction. In principle, one can expect that such a combinatorial construction is possible, but there are a number of technical difficulties to overcome. One of the problems is to make the procedure efficient and to keep the number of vertices sufficiently small at each intermediate step.

4. FROM K^4 TO $K3$: COMBINATORIAL RESOLUTION OF THE 16 SINGULARITIES

Our goal is to construct a simplicial version of the $K3$ surface out of $(K^4)_{16}$ by a combinatorial version of Spanier's dilatation process. More precisely, we find a way to cut out a certain simplicial version of A_i and to glue in a simplicial version of B_i. We prefer a description of B_i as the mapping cylinder of the Hopf map $\tilde{h}$, defined on $\mathbb{R}P^3 = \partial B^4/_{\tilde{\sigma}}$. In the combinatorial setting this is possible if the corresponding boundaries are combinatorially isomorphic, i.e., if they are equal up to a relabeling of the vertices. However, in general, the boundaries are PL-homeomorphic but not combinatorially isomorphic. This

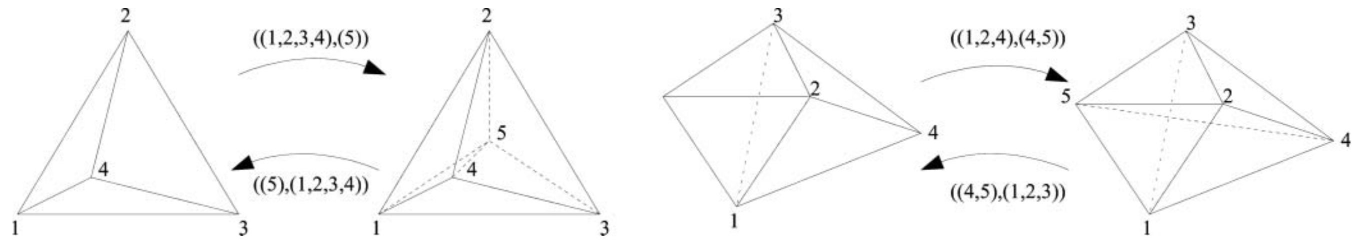

FIGURE 1. 3-dimensional bistellar moves.

is the main difficulty here. Therefore, we need an efficient procedure to change the combinatorial type while preserving the PL-homeomorphism type of the manifold. One possibility of such a procedure is the well-established concept of bistellar moves. Therefore, we start with a short review of bistellar moves; see [Pachner 87].

Definition 4.1. (Pachner's bistellar moves.) Let M be a d-dimensional simplicial complex, and let A be a $(d-i)$-face of M, where $0 \leq i \leq d$. If $\mathrm{lk}_M(A)$ is the boundary complex ∂B of an i-simplex B that is not a face of M, the operation Φ_A on M defined by

$$\Phi_A(M) := (M \setminus (A * \partial B)) \cup (\partial A * B)$$

is called a *bistellar i-move* or *bistellar i-flip*. Similarly, we have the *reverse bistellar i-flip* Φ_A^{-1}, which can also be interpreted as a $(d-i)$-flip.

For $d = 3$ all flips and reverse flips are shown in Figure 1. Two simplicial complexes K and L are called *combinatorially isomorphic*, or just *isomorphic*, if they are equal up to a relabeling of the vertices. Two simplicial complexes K and L are called *bistellarly equivalent* if there exists a sequence of bistellar flips from K to a complex K' such that K' is isomorphic to L. This concept of bistellar flips has been a useful tool in several ways:

1. By a theorem of [Pachner 87], two combinatorial manifolds are PL-homeomorphic if and only if the triangulations are bistellarly equivalent.
2. From a practical point of view bistellar moves allow a reduction in the number of vertices of a given triangulation without changing its PL-homeomorphism type. Many examples have been investigated, and many small triangulations of 3- and 4-manifolds have been found using this technique; see [Lutz 99] or [Lutz et al. 08].
3. It is possible to decide whether two given complexes are PL-homeomorphic by finding a connecting sequence of bistellar flips. This has been successful in many cases even if it cannot be excluded that the algorithm does not terminate.
4. In particular, it is possible to decide whether a given simplicial complex is a combinatorial manifold: One just has to examine the PL-homeomorphism types of all links.
5. These algorithms are implemented in a GAP program; see [Lutz 08].

4.1. A Triangulated Mapping Cylinder Of The Hopf Map $\tilde{h} : \mathbb{R}P^3 \to \mathbb{C}P^1$ with the Minimum Number of Vertices

From the topology of the complex projective plane it is fairly clear that one can construct a triangulation of $\mathbb{C}P^2$ from a triangulated version of the Hopf map $h : S^3 \to S^2$. Conversely, every triangulation of $\mathbb{C}P^2$ contains implicitly a triangulation of the Hopf map (possibly with collapsing of certain simplices) by considering a neighborhood of a triangulated $\mathbb{C}P^1$ inside the triangulation.

Theorem 4.2. [Madahar and Sarkaria 00] *There is a simplicial version of the Hopf map $h : S^3 \to S^2$ with the minimum number 12 of vertices for S^3 that are mapped in triplets onto the 4-vertex S^2. From this simplicial Hopf map one can reconstruct the unique 9-vertex triangulation of $\mathbb{C}P^2$, which was known before; see [Kühnel 95].*

Roughly, the procedure for the construction of a triangulated $\mathbb{C}P^2$ is the following:

1. Find a simplicial subdivision of the mapping cylinder of the Hopf map that is a triangulated $\mathbb{C}P^2$ minus an open 4-ball.
2. Close it up on top by a suitable simplicial 4-ball.
3. Finally, reduce the number of vertices by bistellar flips as far as possible.

For our purpose here, we can follow an analogous procedure:

1. Find a simplicial version of the Hopf map $\tilde{h} : \mathbb{R}P^3 \to S^2$.
2. Find a simplicial subdivision of the mapping cylinder $\widetilde{C}$, which is nothing but a triangulated complex projective plane with one hole modulo the involution $\tilde{\sigma}$. There is one boundary component, which is homeomorphic to $\mathbb{R}P^3$.
3. Finally, reduce the number of vertices by bistellar flips as far as possible.It is well known that any combinatorial triangulation of $\mathbb{R}P^3$ has at least 11 vertices [Walkup 70]. Therefore, 11 is the minimum also for the space we are looking for.

Theorem 4.3. *There is an* 11*-vertex triangulation of the mapping cylinder of the Hopf map* $\tilde{h} : \mathbb{R}P^3 \to S^2$ *such that all vertices and edges are contained in the boundary. This is the minimum possible number of vertices, since it is the minimum already for the boundary.*

Proof. On the boundary of $\mathbb{C}P^2 \setminus B^4$, the involution σ coincides with $\tilde{\sigma}$ and leads to a twofold quotient map $S^3 \to \mathbb{R}P^3$. From this it is clear that a triangulated version of the Hopf map from $\mathbb{R}P^3$ onto S^2 requires a simplicial Hopf map $h : S^3 \to S^2$ that is centrally symmetric on S^3, i.e., that is invariant under σ. Therefore we need to construct a centrally symmetric triangulation of S^3 first. This should allow a simplicial fibration by Hopf fibers.

For the construction, we start with two regular hexagons (2-polytopes) P_1, P_2 in the plane and take the product polytope [Ziegler 95, p. 10] $P := P_1 \times P_2$. The vertices will be denoted by a_{ij}, where i, j range from 1 to 6. The facets of P are $6 + 6$ hexagonal prisms, one of them having vertices $a_{11}, \dots, a_{16}$ on top and $a_{21}, \dots, a_{26}$ on the bottom. The subcomplex $\partial P_1 \times \partial P_2 \subset \partial P$ is the standard (6×6)-grid torus as a subcomplex decomposing ∂P into two solid tori, one on each side of the torus. One of the squares has vertices $a_{11}, a_{12}, a_{21}, a_{22}$, see Figure 2, where the labeling is simply ij instead of a_{ij}. For a simplicial version we need to subdivide the prisms. In a first step, we subdivide each square in the torus by the main diagonal, as indicated in Figure 2. Next we introduce one extra vertex b_i at the center of the six prisms on one side and c_i at the center of the six prisms on the other side, $i = 1, \dots, 6$. That is to say, $b_1, \dots, b_6$ represent the core of one solid torus, and $c_1, \dots, c_6$ the core of the other.

Furthermore, we introduce the pyramids from each b_i and c_i to the 12 triangles of each corresponding prism. Finally, the remaining holes are closed by copies of the join of the edge between two adjacent center vertices and the edge of a hexagon. Typical tetrahedra of this type are $\langle b_1 b_2 a_{11} a_{12} \rangle$ and $\langle c_1 c_2 a_{11} a_{21} \rangle$. This procedure is carried out for each of the two solid tori; see Figure 3.

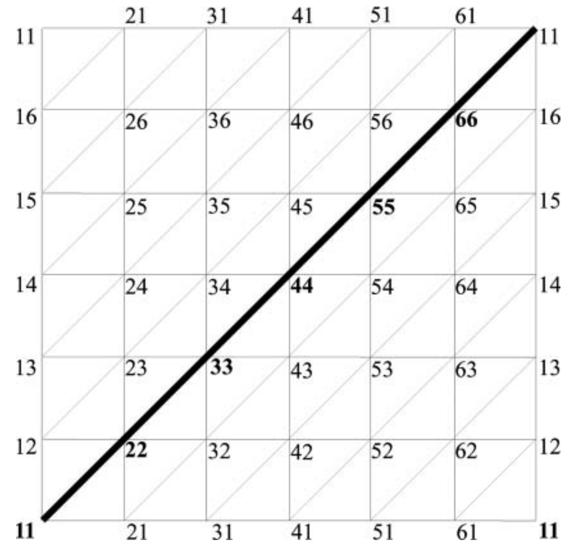

FIGURE 2. Combinatorial (6×6)-grid torus with Hopf fibres.

Thus, we get a centrally symmetric triangulation S^3_{cs} of the 3-sphere with 48 vertices, with $2 \cdot (6 \cdot (12 + 6)) = 216$ tetrahedra and with an automorphism group G of order 144.

On this triangulation of S^3 we define the simplicial Hopf map $h_{\text{cs}} : S^3_{\text{cs}} \to S^2$ by the following identifications:

$$\begin{aligned}
\{a_{ij} \mid j - i \equiv 1 \ (6)\} &\mapsto a_1, \\
\{a_{ij} \mid j - i \equiv 2 \ (6)\} &\mapsto a_2, \\
\{a_{ij} \mid j - i \equiv 3 \ (6)\} &\mapsto a_3, \\
\{a_{ij} \mid j - i \equiv 4 \ (6)\} &\mapsto a_4, \\
\{a_{ij} \mid j - i \equiv 5 \ (6)\} &\mapsto a_5, \\
\{a_{ij} \mid j - i = 0\} &\mapsto a_6, \\
\{b_i\} &\mapsto b, \\
\{c_i\} &\mapsto c.
\end{aligned}$$

The image is a simplicial 2-sphere with eight vertices, namely, a double pyramid from b and c over the hexagon $a_1, a_2, a_3, a_4, a_5, a_6$. Note that the antipodal map

$$\sigma : a_{ij} \mapsto a_{i+3,j+3}, \quad b_i \mapsto b_{i+3}, \quad c_i \mapsto c_{i+3}$$

(all indices taken modulo 6) is compatible with the simplicial Hopf map. By construction, the quotient $\mathbf{P} = S^3_{\text{cs}}/_{\sigma}$ is a 24-vertex triangulation of $\mathbb{R}P^3$. The automorphism group is a normal subgroup of G of index 2. It

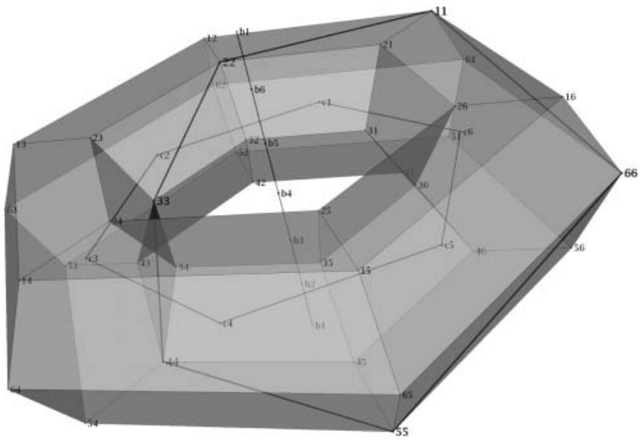

FIGURE 3. A solid torus as half of S^3_{cs} with two Hopf fibers.

follows that this triangulated $\mathbb{R}P^3$ again allows a simplicial version of the Hopf map $\tilde{h} : \mathbf{P} \to S^2$.

The image of the torus in Figure 3 under h forms the hexagon $(a_1, a_2, a_3, a_4, a_5, a_6)$, and each of the solid tori on each side gets mapped to a cone over it. A suitable simplicial decomposition C of the cylinder $\mathbf{P} \times [0,1]$ is compatible with the projection map

$$\tilde{h} \times \{0\} : \mathbf{P} \times \{0\} \to S^2 \times \{0\},$$
$$(x, 0) \mapsto (\tilde{h}(x), 0),$$

on the bottom of C and leads to a triangulated mapping cylinder $C/_\sim$. Its boundary is PL-homeomorphic to the link of any vertex of $(K^4)_{16}$.

For better handling of the blowup process, we computed a reduced version of $\mathbf{C} \cong_{PL} C/_\sim$ by bistellar flips. In this reduced version, the boundary is isomorphic to a vertex-minimal triangulation of $\mathbb{R}P^3$ with the f-vector $(11, 51, 80, 40)$. Moreover, the boundary $\partial\mathbf{C}$ is bistellarly equivalent to $\text{lk}_{(K^4)_{16}}(v)$ for any vertex v, which will be needed for the construction of a triangulated $K3$ surface. On 11 vertices $1, 2, \ldots, 11$ this complex is the following:

$$\begin{aligned}\mathbf{C} = \big\langle\, &(1,3,5,6,11), (2,3,5,6,11), (2,4,5,6,11),\\ &(2,3,6,9,11), (3,6,7,9,11), (1,3,6,7,11),\\ &(6,7,8,10,11), (1,6,7,10,11), (4,6,8,9,11),\\ &(6,7,8,9,11), (2,4,6,9,11), (1,2,3,5,8),\\ &(1,2,3,5,11), (1,5,7,8,9), (1,2,5,7,8),\\ &(1,4,5,9,11), (4,5,7,9,11), (1,4,5,7,9),\\ &(1,2,4,5,11), (1,2,4,5,7), (3,4,7,8,11),\\ &(1,3,4,7,8), (1,2,3,8,11), (1,3,7,8,11),\\ &(1,2,8,10,11), (1,7,8,10,11), (1,2,7,8,10),\\ &(4,7,8,9,11), (1,4,7,8,9), (1,2,4,9,11)\,\big\rangle.\end{aligned}$$

It has the f-vector $(11, 51, 107, 95, 30)$, and its boundary $\partial\mathbf{C}$ contains the complete 1-skeleton of $\mathbf{C}$. In particular, $\mathbf{C}$ is vertex minimal, since the boundary $\mathbb{R}P^3$ requires at least 11 vertices for any simplicial triangulation [Walkup 70]. □

Remark 4.4. By starting with the product polytope of two $3k$-gons containing a $3k \times 3k$-grid torus, one can similarly obtain a simplicial version of the Hopf map from the lens space $L(k, 1)$ to S^2. Furthermore, the same procedure as above can be carried out for the corresponding mapping cylinder.

4.2. Simplicial Blowups

A PL version of the Hopf σ-process from Section 2 is the following: We cut out the star of one of the singular vertices, which is nothing but a cone over a triangulated $\mathbb{R}P^3$. This corresponds to the space A_i above. The boundary of the resulting space is this triangulated $\mathbb{R}P^3$ and is therefore PL-homeomorphic to the boundary of the triangulated mapping cylinder $\mathbf{C}$ from Section 3, which corresponds to the space B_i. Then we cut out A_i and glue in B_i by an appropriate PL homeomorphism, as indicated in Section 2.

For a combinatorial version with concrete triangulations, however, we face the problem that these two triangulations are not isomorphic. This implies that before cutting out and gluing in, we have to modify the triangulations by bistellar flips until they coincide. This computation is provided by the GAP program `bistellar`, which is available from [Lutz 08].

Definition 4.5. (Resolution of singularities in the PL topology.) Let v be a singular vertex of a PL 4-pseudomanifold M with a compact neighborhood A of the type "cone over an $\mathbb{R}P^3$" and let $\phi : \partial A \to \partial\mathbf{C}$ be a PL homeomorphism. A *PL resolution of the singularity* v is given by the following construction:

$$M \mapsto \widetilde{M} := (M \setminus A^\circ) \cup_\phi \mathbf{C}.$$

We will refer to this operation as a *PL blowup* of v.

Definition 4.6. (Simplicial resolution of singularities.) Let v be a vertex of a 4-pseudomanifold M whose link is isomorphic to the particular 11-vertex triangulation of $\mathbb{R}P^3$ that is given by the boundary complex of the triangulated $\mathbf{C}$ above. Let $\psi : \text{lk}(v) \to \partial\mathbf{C}$ denote such an isomorphism. A *simplicial resolution of the singularity* v is given by the following construction:

$$M \mapsto \widetilde{M} := (M \setminus \text{star}(v)^\circ) \cup_\psi \mathbf{C}.$$

We will refer to this operation as a *simplicial blowup*, or just a *blowup* of v.

Since in either case both parts are glued together along their PL-homeomorphic boundaries, the resulting complex is closed, and the construction of $\widetilde{M}$ is well defined. The pseudomanifold $\widetilde{M}$ is closed, and the number of singular points in $\widetilde{M}$ is the number of singular points in M minus one. In particular, we can apply this to $M = (K^4)_{16}$ and then repeat the procedure for the resulting spaces until the last singularity disappears. We can now prove the following main result:

Theorem 4.7. *There is a* 17*-vertex triangulation of the* $K3$ *surface* $(K3)_{17}$ *with the standard PL structure, which can be constructed from* $(K^4)_{16}$ *by a sequence of bistellar flips and in between by* 16 *simplicial blowups.*

The proof of the theorem is constructive and will be given in the form of an algorithm. From the construction, it is clear that the resulting PL manifold is PL-homeomorphic to the classical $K3$ surface, not only homeomorphic.

Let $\widetilde{K}_i$, $0 \leq i \leq 16$, be the 4-dimensional Kummer variety after the ith blowup. Since we have to modify its combinatorial type repeatedly, our notation will not distinguish here between two different complexes after bistellar flips. Furthermore, let $\mathbf{C}$ be the bounded complex from Section 4.1 and Q_i the intersection form of $\widetilde{K}_i$.

We start with a singular vertex $v \in \widetilde{K}_i$. In general, its link is not isomorphic to $\partial\mathbf{C}$. Thus, we have to modify $\widetilde{K}_i \setminus \mathrm{star}(v)$ in a suitable way to yield a complex that allows a simplicial blowup with the space $\mathbf{C}$. This is accomplished by modifying $\partial(\widetilde{K}_i \setminus \mathrm{star}(v)) = \mathrm{lk}(v)$ with respect to the combinatorial structure of the complex. Even though in general we cannot claim that this must always be possible, in this particular case we were able to find fairly short sequences of bistellar moves realizing this modification at any of the 16 steps. The sequences were found using the approach from [Lutz 08].

Once $\partial(\widetilde{K}_i \setminus \mathrm{star}(v))$ is isomorphic to $\partial\mathbf{C}$, we can perform the simplicial blowup and gain $\widetilde{K}_{i+1}$, as indicated above. Note that in each step we can perform the blowup in two significantly different ways, corresponding to the choice of orientation of $\mathbf{C}$. For the verification of the choice of the right orientation we compute the intersection form Q_{i+1} and check that

$$|\,\mathrm{sign}(Q_{i+1})| = |\,\mathrm{sign}(Q_i) + 1|$$

holds (note that $Q(K^4) = 0$ and $Q(K3) = \pm 16$).

Due to the various modifications above, the resulting complex $\widetilde{K}_{i+1}$ will be considerably bigger than $\widetilde{K}_i$. Thus we use bistellar flips to reduce it before repeating the same operation for the remaining singularities. At every step, the signature of the intersection form and the second Betti number will increase by one, and the number of singularities will decrease by one. Also, the torsion part of $H_2(K^4)$ will gradually decline. This, however, depends on the order of the blowing-up process of the singularities. It follows that the resulting complex is a triangulation of the $K3$ surface with the expected intersection form and with the standard PL structure. The algorithm is written in GAP. For the computation of the intersection form we use polymake [Gawrilow and Joswig 00].

The smallest complex (with respect to the f-vector) that we were able to obtain by bistellar moves is a 17-vertex version of the $K3$-surface, which will be denoted by $(K3)_{17}$. Its facets as well as some basic properties are listed in Table 1.

Further data as well as all 16 steps of the dilatation process are available from the web page of the first author, given at the end of this article. The source code itself is available upon request.

So far, we have not been able to prove PL-equivalence to $(K3)_{16}$. However, since the given complex has only 17 vertices, this is most likely to be true, and further experiments will probably prove the following conjecture.

Conjecture 4.8. $(K3)_{17}$ *is PL-homeomorphic to* $(K3)_{16}$.

Remark 4.9. If Conjecture 4.8 is false, this would imply that $(K3)_{16}$ is exotic. In this case, to our knowledge, $(K3)_{16}$ would be the first explicit triangulation with few vertices of a nonstandard combinatorial 4-manifold.

5. CRITICAL POINT THEORY AND SLICINGS

The Morse theory for smooth functions defined on smooth manifolds is an important tool in topology. Similarly, the PL structure of a d-dimensional combinatorial (pseudo)manifold M can be examined using an analogous concept of critical points of functions, defined on a triangulation of a manifold M; compare [Kühnel 90, Kühnel 95]. In this section we describe a few computer experiments on triangulations of the $K3$ surface and the Kummer variety. As a result, we obtain a picture of slices through these spaces by levels of perfect Morse functions in a PL version.

⟨1 2 3 8 13⟩,	⟨1 2 3 8 14⟩,	⟨1 2 3 12 13⟩,	⟨1 2 3 12 15⟩,	⟨1 2 3 14 15⟩,	⟨1 2 4 7 13⟩,
⟨1 2 4 7 15⟩,	⟨1 2 4 13 15⟩,	⟨1 2 5 6 9⟩,	⟨1 2 5 6 14⟩,	⟨1 2 5 9 17⟩,	⟨1 2 5 14 15⟩,
⟨1 2 5 15 17⟩,	⟨1 2 6 8 14⟩,	⟨1 2 6 8 15⟩,	⟨1 2 6 9 16⟩,	⟨1 2 6 15 17⟩,	⟨1 2 6 16 17⟩,
⟨1 2 7 8 11⟩,	⟨1 2 7 8 15⟩,	⟨1 2 7 11 13⟩,	⟨1 2 8 10 11⟩,	⟨1 2 8 10 13⟩,	⟨1 2 9 16 17⟩,
⟨1 2 10 11 13⟩,	⟨1 2 12 13 15⟩,	⟨1 3 4 5 8⟩,	⟨1 3 4 5 17⟩,	⟨1 3 4 6 10⟩,	⟨1 3 4 6 12⟩,
⟨1 3 4 8 9⟩,	⟨1 3 4 9 10⟩,	⟨1 3 4 12 17⟩,	⟨1 3 5 7 11⟩,	⟨1 3 5 7 14⟩,	⟨1 3 5 8 16⟩,
⟨1 3 5 11 16⟩,	⟨1 3 5 14 15⟩,	⟨1 3 5 15 17⟩,	⟨1 3 6 10 12⟩,	⟨1 3 7 8 11⟩,	⟨1 3 7 8 14⟩,
⟨1 3 8 9 12⟩,	⟨1 3 8 11 16⟩,	⟨1 3 8 12 13⟩,	⟨1 3 9 10 12⟩,	⟨1 3 12 15 17⟩,	⟨1 4 5 8 16⟩,
⟨1 4 5 11 16⟩,	⟨1 4 5 11 17⟩,	⟨1 4 6 7 12⟩,	⟨1 4 6 7 15⟩,	⟨1 4 6 10 15⟩,	⟨1 4 7 12 13⟩,
⟨1 4 8 9 16⟩,	⟨1 4 9 10 14⟩,	⟨1 4 9 14 16⟩,	⟨1 4 10 14 16⟩,	⟨1 4 10 15 16⟩,	⟨1 4 11 16 17⟩,
⟨1 4 12 13 17⟩,	⟨1 4 13 15 16⟩,	⟨1 4 13 16 17⟩,	⟨1 5 6 9 13⟩,	⟨1 5 6 13 14⟩,	⟨1 5 7 10 12⟩,
⟨1 5 7 10 14⟩,	⟨1 5 7 11 12⟩,	⟨1 5 9 11 13⟩,	⟨1 5 9 11 17⟩,	⟨1 5 10 12 14⟩,	⟨1 5 11 12 13⟩,
⟨1 5 12 13 14⟩,	⟨1 6 7 8 14⟩,	⟨1 6 7 8 15⟩,	⟨1 6 7 10 12⟩,	⟨1 6 7 10 16⟩,	⟨1 6 7 14 16⟩,
⟨1 6 9 11 13⟩,	⟨1 6 9 11 14⟩,	⟨1 6 9 14 16⟩,	⟨1 6 10 15 16⟩,	⟨1 6 11 13 14⟩,	⟨1 6 15 16 17⟩,
⟨1 7 10 14 16⟩,	⟨1 7 11 12 13⟩,	⟨1 8 9 10 11⟩,	⟨1 8 9 10 12⟩,	⟨1 8 9 11 16⟩,	⟨1 8 10 12 13⟩,
⟨1 9 10 11 14⟩,	⟨1 9 11 16 17⟩,	⟨1 10 11 13 14⟩,	⟨1 10 12 13 14⟩,	⟨1 12 13 15 16⟩,	⟨1 12 13 16 17⟩,
⟨1 12 15 16 17⟩,	⟨2 3 4 6 12⟩,	⟨2 3 4 6 16⟩,	⟨2 3 4 7 14⟩,	⟨2 3 4 7 16⟩,	⟨2 3 4 12 14⟩,
⟨2 3 5 6 12⟩,	⟨2 3 5 6 16⟩,	⟨2 3 5 12 16⟩,	⟨2 3 7 14 16⟩,	⟨2 3 8 13 14⟩,	⟨2 3 9 10 13⟩,
⟨2 3 9 10 14⟩,	⟨2 3 9 11 14⟩,	⟨2 3 9 11 16⟩,	⟨2 3 9 12 13⟩,	⟨2 3 9 12 16⟩,	⟨2 3 10 13 14⟩,
⟨2 3 11 14 16⟩,	⟨2 3 12 14 15⟩,	⟨2 4 5 7 10⟩,	⟨2 4 5 7 11⟩,	⟨2 4 5 8 10⟩,	⟨2 4 5 8 12⟩,
⟨2 4 5 11 12⟩,	⟨2 4 6 11 12⟩,	⟨2 4 6 11 16⟩,	⟨2 4 7 9 14⟩,	⟨2 4 7 9 15⟩,	⟨2 4 7 10 13⟩,
⟨2 4 7 11 16⟩,	⟨2 4 8 10 12⟩,	⟨2 4 9 10 13⟩,	⟨2 4 9 10 14⟩,	⟨2 4 9 13 15⟩,	⟨2 4 10 12 14⟩,
⟨2 5 6 7 10⟩,	⟨2 5 6 7 12⟩,	⟨2 5 6 8 10⟩,	⟨2 5 6 8 14⟩,	⟨2 5 6 9 16⟩,	⟨2 5 7 11 12⟩,
⟨2 5 8 12 14⟩,	⟨2 5 9 15 16⟩,	⟨2 5 9 15 17⟩,	⟨2 5 12 14 15⟩,	⟨2 5 12 15 16⟩,	⟨2 6 7 10 13⟩,
⟨2 6 7 11 12⟩,	⟨2 6 7 11 13⟩,	⟨2 6 8 10 15⟩,	⟨2 6 10 11 13⟩,	⟨2 6 10 11 17⟩,	⟨2 6 10 15 17⟩,
⟨2 6 11 16 17⟩,	⟨2 7 8 11 15⟩,	⟨2 7 9 11 14⟩,	⟨2 7 9 11 15⟩,	⟨2 7 11 14 16⟩,	⟨2 8 10 11 15⟩,
⟨2 8 10 12 14⟩,	⟨2 8 10 13 14⟩,	⟨2 9 11 15 17⟩,	⟨2 9 11 16 17⟩,	⟨2 9 12 13 16⟩,	⟨2 9 13 15 16⟩,
⟨2 10 11 15 17⟩,	⟨2 12 13 15 16⟩,	⟨3 4 5 8 17⟩,	⟨3 4 6 8 9⟩,	⟨3 4 6 8 11⟩,	⟨3 4 6 9 15⟩,
⟨3 4 6 10 15⟩,	⟨3 4 6 11 13⟩,	⟨3 4 6 13 16⟩,	⟨3 4 7 12 14⟩,	⟨3 4 7 12 17⟩,	⟨3 4 7 13 16⟩,
⟨3 4 7 13 17⟩,	⟨3 4 8 11 17⟩,	⟨3 4 9 10 15⟩,	⟨3 4 11 13 17⟩,	⟨3 5 6 10 12⟩,	⟨3 5 6 10 15⟩,
⟨3 5 6 13 15⟩,	⟨3 5 6 13 16⟩,	⟨3 5 7 11 15⟩,	⟨3 5 7 14 15⟩,	⟨3 5 8 16 17⟩,	⟨3 5 9 10 15⟩,
⟨3 5 9 10 17⟩,	⟨3 5 9 15 17⟩,	⟨3 5 10 12 16⟩,	⟨3 5 10 16 17⟩,	⟨3 5 11 15 16⟩,	⟨3 5 13 15 16⟩,
⟨3 6 7 8 9⟩,	⟨3 6 7 8 15⟩,	⟨3 6 7 9 15⟩,	⟨3 6 8 11 15⟩,	⟨3 6 11 13 15⟩,	⟨3 7 8 9 13⟩,
⟨3 7 8 11 15⟩,	⟨3 7 8 13 14⟩,	⟨3 7 9 13 17⟩,	⟨3 7 9 15 17⟩,	⟨3 7 12 14 15⟩,	⟨3 7 12 15 17⟩,
⟨3 7 13 14 16⟩,	⟨3 8 9 12 13⟩,	⟨3 8 10 11 16⟩,	⟨3 8 10 11 17⟩,	⟨3 8 10 16 17⟩,	⟨3 9 10 11 14⟩,
⟨3 9 10 11 16⟩,	⟨3 9 10 12 16⟩,	⟨3 9 10 13 17⟩,	⟨3 10 11 13 14⟩,	⟨3 10 11 13 17⟩,	⟨3 11 13 14 16⟩,
⟨3 11 13 15 16⟩,	⟨4 5 7 8 13⟩,	⟨4 5 7 8 16⟩,	⟨4 5 7 10 13⟩,	⟨4 5 7 11 16⟩,	⟨4 5 8 10 15⟩,
⟨4 5 8 11 12⟩,	⟨4 5 8 11 17⟩,	⟨4 5 8 13 15⟩,	⟨4 5 9 10 13⟩,	⟨4 5 9 10 15⟩,	⟨4 5 9 13 15⟩,
⟨4 6 7 9 12⟩,	⟨4 6 7 9 15⟩,	⟨4 6 8 9 14⟩,	⟨4 6 8 11 14⟩,	⟨4 6 9 12 14⟩,	⟨4 6 11 12 14⟩,
⟨4 6 11 13 17⟩,	⟨4 6 11 16 17⟩,	⟨4 6 13 16 17⟩,	⟨4 7 8 13 16⟩,	⟨4 7 9 12 14⟩,	⟨4 7 12 13 17⟩,

(Continued on next page)

TABLE 1. 17-vertex triangulation of the $K3$ surface $(K3)_{17}$ with the standard PL structure. Its f-vector is $f(K3) = (17, 135, 610, 780, 312)$. Note that the complex is not 2-neighborly.

$\langle 4\,8\,9\,14\,16\rangle$,	$\langle 4\,8\,10\,11\,12\rangle$,	$\langle 4\,8\,10\,11\,15\rangle$,	$\langle 4\,8\,11\,14\,15\rangle$,	$\langle 4\,8\,13\,15\,16\rangle$,	$\langle 4\,8\,14\,15\,16\rangle$,
$\langle 4\,10\,11\,12\,15\rangle$,	$\langle 4\,10\,12\,14\,16\rangle$,	$\langle 4\,10\,12\,15\,16\rangle$,	$\langle 4\,11\,12\,14\,15\rangle$,	$\langle 4\,12\,14\,15\,16\rangle$,	$\langle 5\,6\,7\,10\,12\rangle$,
$\langle 5\,6\,8\,10\,15\rangle$,	$\langle 5\,6\,8\,13\,14\rangle$,	$\langle 5\,6\,8\,13\,15\rangle$,	$\langle 5\,6\,9\,13\,16\rangle$,	$\langle 5\,7\,8\,9\,13\rangle$,	$\langle 5\,7\,8\,9\,17\rangle$,
$\langle 5\,7\,8\,16\,17\rangle$,	$\langle 5\,7\,9\,10\,13\rangle$,	$\langle 5\,7\,9\,10\,17\rangle$,	$\langle 5\,7\,10\,14\,16\rangle$,	$\langle 5\,7\,10\,16\,17\rangle$,	$\langle 5\,7\,11\,15\,16\rangle$,
$\langle 5\,7\,14\,15\,16\rangle$,	$\langle 5\,8\,9\,12\,13\rangle$,	$\langle 5\,8\,9\,12\,17\rangle$,	$\langle 5\,8\,11\,12\,17\rangle$,	$\langle 5\,8\,12\,13\,14\rangle$,	$\langle 5\,9\,11\,12\,13\rangle$,
$\langle 5\,9\,11\,12\,17\rangle$,	$\langle 5\,9\,13\,15\,16\rangle$,	$\langle 5\,10\,12\,14\,16\rangle$,	$\langle 5\,12\,14\,15\,16\rangle$,	$\langle 6\,7\,8\,9\,16\rangle$,	$\langle 6\,7\,8\,14\,16\rangle$,
$\langle 6\,7\,9\,12\,16\rangle$,	$\langle 6\,7\,10\,13\,17\rangle$,	$\langle 6\,7\,10\,16\,17\rangle$,	$\langle 6\,7\,11\,12\,13\rangle$,	$\langle 6\,7\,12\,13\,17\rangle$,	$\langle 6\,7\,12\,16\,17\rangle$,
$\langle 6\,8\,9\,14\,16\rangle$,	$\langle 6\,8\,11\,13\,14\rangle$,	$\langle 6\,8\,11\,13\,15\rangle$,	$\langle 6\,9\,11\,12\,13\rangle$,	$\langle 6\,9\,11\,12\,14\rangle$,	$\langle 6\,9\,12\,13\,16\rangle$,
$\langle 6\,10\,11\,13\,17\rangle$,	$\langle 6\,10\,15\,16\,17\rangle$,	$\langle 6\,12\,13\,16\,17\rangle$,	$\langle 7\,8\,9\,12\,16\rangle$,	$\langle 7\,8\,9\,12\,17\rangle$,	$\langle 7\,8\,12\,16\,17\rangle$,
$\langle 7\,8\,13\,14\,16\rangle$,	$\langle 7\,9\,10\,13\,17\rangle$,	$\langle 7\,9\,11\,14\,15\rangle$,	$\langle 7\,9\,12\,14\,15\rangle$,	$\langle 7\,9\,12\,15\,17\rangle$,	$\langle 7\,11\,14\,15\,16\rangle$,
$\langle 8\,9\,10\,11\,16\rangle$,	$\langle 8\,9\,10\,12\,16\rangle$,	$\langle 8\,10\,11\,12\,17\rangle$,	$\langle 8\,10\,12\,13\,14\rangle$,	$\langle 8\,10\,12\,16\,17\rangle$,	$\langle 8\,11\,13\,14\,15\rangle$,
$\langle 8\,13\,14\,15\,16\rangle$,	$\langle 9\,11\,12\,14\,15\rangle$,	$\langle 9\,11\,12\,15\,17\rangle$,	$\langle 10\,11\,12\,15\,17\rangle$,	$\langle 10\,12\,15\,16\,17\rangle$,	$\langle 11\,13\,14\,15\,16\rangle$.

TABLE 1. 17-vertex triangulation of the $K3$ surface $(K3)_{17}$ with the standard PL structure. Its f-vector is $f(K3) = (17, 135, 610, 780, 312)$. Note that the complex is not 2-neighborly *(Continued).*

Definition 5.1. Let M^d be a combinatorial manifold. A function $f : M \to \mathbb{R}$ is called *regular simplexwise linear (rsl)* if $f(v) \neq f(w)$ for any two vertices $w \neq v$ and if f is linear when restricted to an arbitrary simplex of the triangulation.

A point $x \in M$ is said to be *critical* for an rsl function $f : M \to \mathbb{R}$ if

$$H_\star(M_x, M_x \setminus \{x\}, F) \neq 0,$$

where $M_x := \{y \in M | f(y) \leq f(x)\}$ and F is a field.

It follows that no point of M can be critical except possibly for the vertices. If we fix an rsl function f and a vertex v of M, we can define the *multiplicity vector* $\mathbf{m}(v, F)$ as the following $(d+1)$-tuple of integers:

$$\mathbf{m}(v, F) := \big(\dim H_0(M_v, M_v \setminus \{v\}, F), \ldots, \dim H_d(M_v, M_v \setminus \{v\}, F)\big).$$

The vertex v is said to be *critical of index i and multiplicity m* if $\dim H_i(M_v, M_v \setminus \{v\}, F) = m > 0$;

$$\sum_{i=0}^{d} \dim H_i(M_v, M_v \setminus \{v\}, F)$$

is called the *total multiplicity* of v;

$$\mu_i(f, F) := \sum_{v \in V} \dim H_i(M_v, M_v \setminus \{v\}, F)$$

is referred to as the *number of critical points of index i* in M (where V denotes the set of vertices of M); and

$$\mu(f, F) := \sum_{i=0}^{d} \mu_i(f, F)$$

is said to be the *number of critical points of M*. The multiplicity vector together with the number of critical points has to be considered for appropriately encoding the relevant information about the PL structure of M, since in contrast to the smooth case, higher multiplicities cannot be avoided in general.

The classical Morse relation $\mu_i(f, F) \geq \mathrm{b}_i(M, F)$ is still true for any rsl function f on M, where $\mathrm{b}_i(M, F) = \dim H_i(M; F)$ denotes its ith Betti number. Equality refers to the case of a tight or perfect function. If any rsl function has this property, we call the triangulation of M a *tight triangulation*; cf. [Kühnel 95]. In particular, μ_i does not depend on f in the tight case. For some examples of multiplicities on $(K3)_{16}$, see Table 2.

Definition 5.2. Let M be an orientable combinatorial d-(pseudo)manifold and let $f : M \to \mathbb{R}$ be an rsl function. Then we call the preimage $f^{-1}(x)$ a *slicing* of M whenever $x \neq f(v)$ for any vertex $v \in M$.

By construction, a slicing is a PL $(d-1)$-manifold, and we have $f^{-1}(x) \cong f^{-1}(y)$ whenever $f^{-1}[f(x), f(y)]$ contains no vertex, i.e., if no vertex is mapped into the interval $[f(x), f(y)]$. Note that any partition of the set of vertices $V = V_1 \dot\cup V_2$ of M already determines a slicing: Just define an rsl function g with $g(v) < g(w)$ for all $v \in V_1$ and $w \in V_2$ and look at a suitable preimage. For an example of a slicing of a 3-manifold and a 3-pseudomanifold see [Kühnel 95, Figures 9, 11].

	$f_{\{1,\dots,5,7,6,8,9,11,10,12,\dots,16\}}$		$f_{\{2,\dots,7,1,8,9,16,10,\dots,15\}}$		$f_{\{1,\dots,5,7,6,8,9,11,10,12,\dots,16\}}$	
$f_i(v)$	v	$\mathbf{m}(v,\mathbb{F}_2)$	v	$\mathbf{m}(v,\mathbb{F}_2)$	v	$\mathbf{m}(v,\mathbb{F}_2)$
0	1	$(1,0,0,0,0)$	2	$(1,0,0,0,0)$	1	$(1,0,0,0,0)$
$\frac{1}{15}$	2	$(0,0,0,0,0)$	3	$(0,0,0,0,0)$	2	$(0,0,0,0,0)$
$\frac{2}{15}$	3	$(0,0,0,0,0)$	4	$(0,0,0,0,0)$	3	$(0,0,0,0,0)$
$\frac{3}{15}$	4	$(0,0,1,0,0)$	5	$(0,0,0,0,0)$	4	$(0,0,1,0,0)$
$\frac{4}{15}$	5	$(0,0,2,0,0)$	6	$(0,0,1,0,0)$	5	$(0,0,1,0,0)$
$\frac{5}{15}$	7	$(0,0,3,0,0)$	7	$(0,0,3,0,0)$	7	$(0,0,2,0,0)$
$\frac{6}{15}$	6	$(0,0,2,0,0)$	1	$(0,0,4,0,0)$	6	$(0,0,4,0,0)$
$\frac{7}{15}$	8	$(0,0,3,0,0)$	8	$(0,0,3,0,0)$	8	$(0,0,3,0,0)$
$\frac{8}{15}$	9	$(0,0,3,0,0)$	9	$(0,0,3,0,0)$	9	$(0,0,3,0,0)$
$\frac{9}{15}$	11	$(0,0,2,0,0)$	16	$(0,0,4,0,0)$	11	$(0,0,4,0,0)$
$\frac{10}{15}$	10	$(0,0,3,0,0)$	10	$(0,0,3,0,0)$	10	$(0,0,2,0,0)$
$\frac{11}{15}$	12	$(0,0,2,0,0)$	11	$(0,0,1,0,0)$	12	$(0,0,1,0,0)$
$\frac{12}{15}$	13	$(0,0,1,0,0)$	12	$(0,0,0,0,0)$	13	$(0,0,1,0,0)$
$\frac{13}{15}$	14	$(0,0,0,0,0)$	13	$(0,0,0,0,0)$	14	$(0,0,0,0,0)$
$\frac{14}{15}$	15	$(0,0,0,0,0)$	14	$(0,0,0,0,0)$	15	$(0,0,0,0,0)$
1	16	$(0,0,0,0,1)$	15	$(0,0,0,0,1)$	16	$(0,0,0,0,1)$

TABLE 2. Multiplicity vectors of the critical points of $f_{\{1,\dots,5,7,6,8,9,11,10,12,\dots,16\}}$, $f_{\{2,\dots,7,1,8,9,16,10,\dots,15\}}$, $f_{\{1,\dots,5,7,6,8,9,11,10,12,\dots,16\}} : (K3)_{16} \to [0,1]$

Since every combinatorial (pseudo)manifold has a finite number of vertices, there exists only a finite number of slicings. Hence, if f is chosen carefully, the induced slicings admit a useful visualization of M.

5.1. Combinatorial Morse analysis on $(K3)_{16}$

The 16-vertex triangulation of the $K3$ surface is a very special object in combinatorial topology:

1. It satisfies equality in the generalized Heawood inequality (2–2) for the number n of vertices of a 4-manifold M with Euler characteristic $\chi(M)$.
2. It is 3-neighborly, or superneighborly, meaning that $f_2 = \binom{n}{3}$. See [Spanier 09, Theorem 5.8] for a characterization of all possible g-vectors of a triangulated $K3$ surface, starting with the minimum $(g_0, g_1, g_2) = (1, 10, 55)$.
3. It is the only known triangulation of a 4-manifold admitting an automorphism group acting 2-*transitively* on the set of vertices (besides the trivial case of the 6-vertex 4-sphere).

From the viewpoint of Morse theory, this has the following consequence:

Proposition 5.3. *Any rsl function f defined on $(K3)_{16}$ is a perfect function in the sense that the total number of critical points is* 24. *More precisely, we have $\mu_0(f) = \mu_4(f) = 1$, $\mu_1(f) = \mu_3(f) = 0$, and $\mu_2(f) = 22$. This holds for any choice of a field F.*

Proof. This follows from the fact that the 16-vertex triangulation of the $K3$ surface is a tight triangulation in the sense of [Kühnel 95, Kühnel and Lutz 99]. The reason is that the triangulation is 3-neighborly, which implies that there are no critical points of index 1: Any subset of vertices spans a connected and simply connected subset. The 2-neighborliness implies that every rsl function has exactly one critical point of index 0. The rest follows from the duality $\mu_i(f) = \mu_{4-i}(f)$ and the Poincaré relation $\mu_0 - \mu_1 + \mu_2 - \mu_3 + \mu_4 = \chi(K3) = 24$. □

Corollary 5.4. *Any rsl function f defined on $(K3)_{16}$ has a critical point of index* 2 *with a multiplicity greater than* 2. *More precisely, ten possible critical vertices have to build up the second Betti number* 22. *This holds for any choice of field F.*

Moreover, any slicing of an rsl function on $(K3)_{16}$ is a connected 3*-manifold.*

Proof. The first part is obvious from the Morse inequality $\mu_2(f) \geq b_2(M) = 22$ and the fact that by the 3-neighborliness, only the middle vertices (i.e., all but the three on top and the three on bottom) can be critical of index 2. For examples of multiplicity vectors see Table 2. The second part follows from the fact that there is no critical point of index 1. If there were a disconnected level, it would have to be modified into a connected level later, and this procedure requires a critical point of index 1 in between. □

It may be interesting to see how the levels of such a function change in passing through a critical level. It does not seem to be known from differential topology what the possible levels can be for smooth perfect functions on the $K3$ surface. The standard embedding $(z_0, z_1, z_2, z_3) \mapsto (z_i\bar{z}_j)_{ij}$ of a quartic surface in projective 3-space $K3 \to \mathbb{C}P^3 \to S^{14} \to \mathbb{R}^{15}$ induces smooth Morse functions by linear projections from 15-space to $\mathbb{R}$. However, in general, these won't be perfect. It is well known that there is no tight smooth embedding or immersion of the $K3$ surface into any Euclidean space [Thorbergsson 86]. Not too much seems to be known about possible slicings of perfect smooth Morse functions defined on the $K3$ surface. In the PL case, we have the following feature: An rsl function

$$f_\Omega : (K3)_{16} \to [0, 1]$$

on $(K3)_{16}$ is essentially determined by a fixed ordering on the set of vertices $\Omega := \{v_1, \ldots, v_{16}\}$ determining the function f_Ω by the condition $0 = f_\Omega(v_1) < \cdots < f_\Omega(v_{16}) = 1$. Any slicing $f_\Omega^{-1}(\alpha)$ of a 4-manifold consists only of tetrahedra and 3-dimensional prisms of type $\Delta^2 \times [0, 1]$ (where Δ^2 denotes a triangle), induced by proper sections with the 4-simplices of $(K3)_{16}$. In many cases, the topological type of $f_\Omega^{-1}(\alpha)$ can be identified using standard techniques. Some of the slicings can be seen in advance:

The 3-torus. Obviously, there is a 3-torus as a slicing of the 4-torus. It can be arranged that this avoids all the 16 fixed points of the involution. Hence we have the same slicing in the Kummer variety and, by the purely local resolution procedure, also in the $K3$ surface.

Real projective 3-space. The link of any singular point in K^4 is a real projective 3-space. By resolving the singularities, we change only a neighborhood of these points. Thus, there are slicings in $(K3)_{16}$ separating such a neighborhood. These are homeomorphic to $\mathbb{R}P^3$.

The Poincaré homology sphere Σ^3. There is a surgery description of the $K3$ surface showing the Poincaré homology sphere as a possible slicing (see [Saveliev 99] for details). Even though this does not tell about the number of vertices that will be needed, it turns out that a certain slicing of the 16-vertex triangulation is this manifold Σ^3; see below.

Proposition 5.5. *As slicings of $(K3)_{16}$ we obtain at least the manifolds*

$$S^3, \ \mathbb{R}P^3, \ L(3,1), \ L(4,1), \ L(5,1), \ \Sigma^3,$$

and a number of other space forms: the 3*-torus, the cube space, the octahedron space, the truncated cube space, and the prism space $P(3)$. Here Σ^3 denotes the Poincaré homology sphere with a fundamental group of order* 120.

Proof. The permutation

$$(1,16)(2,15)(3,14)(4,13)(5,12)(6,11)(7,10)(8,9)$$

on the 16 vertices is an automorphism of $(K3)_{16}$, and we have $f^{-1}_{\{1,\ldots,16\}}(\frac{1}{2}) \cong \mathbb{T}^3$. Hence, we use this slicing as a starting point and analyze all possible slicings of $(K3)_{16}$ around this 3-torus in the middle.

Since $(K3)_{16}$ is 3-neighborly, all slicings with three or fewer vertices on one side are trivial (i.e., the slicing is a 3-sphere). With four vertices on one side we have two possible situations. Either the tetrahedron formed by the four vertices is contained in the complex (in this case the slicing is clearly trivial), or it is not. In the latter case, we have a slicing behind an empty tetrahedron. This type of slicing is a real projective 3-space. Therefore, a simplicial decomposition of the set

$$|f_\Omega^{-1}([0,\alpha])|, \quad \frac{3}{15} < \alpha < \frac{4}{15},$$

is PL-homeomorphic to the mapping cylinder $\mathbf{C}$ of the Hopf map $\tilde{h}$ in Section 4. Hence, we can find topological copies of $\mathbf{C}$ in $(K3)_{16}$ (which is not surprising).

Neither the span of $\{1,\ldots,8\}$ nor the span of $\{9,\ldots,16\}$ contains a 4-simplex of $(K3)_{16}$. Thus, five vertices on one side cannot induce a trivial slicing with a sphere but can do so with a lens space of type $\mathrm{L}(4,1)$, $\mathrm{L}(3,1)$, or $\mathrm{L}(2,1) = \mathbb{R}P^3$. In the case of six or seven vertices we have the cube space, the octahedron space, or the Poincaré homology sphere. Eight vertices on each side result in the 3-torus, the only nonspherical 3-manifold in this series.

For a complete list of the topological types of these slicings see Table 3.

Besides the symmetrical slicings of Table 3 we found a number of other 3-dimensional spherical space forms such as the truncated cube space, the prism space $P(3)$, and the lens space $\mathrm{L}(5,1)$, as well as some orientable flat manifolds. Triangulations of such spaces were found in [Lutz 99]. These can be used for comparison by bistellar flips. □

α	$f_\Omega^{-1}(\alpha)$	Slicing in between
$\frac{1}{30}$	S^3	$\{1\}$ and $\{2,\ldots,16\}$
$\frac{1}{10}$	S^3	$\{1,2\}$ and $\{3,\ldots,16\}$
$\frac{1}{6}$	S^3	$\{1,2,3\}$ and $\{4,\ldots,16\}$
$\frac{7}{30}$	S^3	$\{2,3,4,5\}$ and $\{1,6,\ldots,16\}$
	$\mathbb{R}P^3$	$\{1,2,3,4\}$ and $\{5,\ldots,16\}$
$\frac{3}{10}$	$\mathrm{L}(4,1)$	$\{1,\ldots,5\}$ and $\{6,\ldots,16\}$
	$\mathrm{L}(3,1)$	$\{2,\ldots,6\}$ and $\{1,7,\ldots,16\}$
	$\mathbb{R}P^3$	$\{1,2,3,5,6\}$ and $\{4,7,\ldots,16\}$
$\frac{11}{30}$	$\mathbf{C}^3$	$\{2,\ldots,7\}$ and $\{1,8,\ldots,16\}$
	$\mathbf{O}^3$	$\{1,\ldots,5,7\}$ and $\{6,8,\ldots,16\}$
$\frac{13}{30}$	Σ^3	$\{1,\ldots,7\}$ and $\{8,\ldots,16\}$
$\frac{1}{2}$	$\mathbb{T}^3$	$\{1,\ldots,8\}$ and $\{9,\ldots,16\}$

TABLE 3. Topological types of slicings of $(K3)_{16}$. Here Σ^3 denotes the Poincaré homology sphere, $\mathbf{C}^3$ the cube space and $\mathbf{O}^3$ the octahedron space.

5.2. Combinatorial Morse Analysis on $(K^4)_{16}$

In this section we will use the field $F := \mathbb{F}_2$ because the Kummer variety has 2-torsion in the homology; see equation (2–1). Since $(K^4)_{16}$ is not a combinatorial manifold, we cannot apply critical point theory as easily as for the $K3$ surface. The reason is that now all vertex links are distinct from combinatorial 3-spheres. This implies that duality no longer holds. Moreover, it has the following consequence, somehow against our intuition about Morse theory: Slicings below a noncritical point do not necessarily have to be homeomorphic to those above the same noncritical point. Moreover, $(K^4)_{16}$ is not a tight triangulation. A tight triangulation of a simply connected space (manifold or not) must be 3-neighborly, but $(K^4)_{16}$ is not, because of $f_2 = 400 < \binom{16}{3}$. In particular, not all rsl functions are perfect functions.

From the $\mathbb{F}_2$-Betti numbers $b_0 = 1$, $b_1 = 0$, $b_2 = 11$, $b_3 = 5$, $b_4 = 1$ of the Kummer variety, we expect that any rsl function has 18 or more critical points, counted with multiplicity. The question is whether there is a perfect rsl function on this triangulation that in addition fits the symmetry of the complex. This would be an excellent candidate for visualizing the space $(K^4)_{16}$ by various 3-dimensional slicings.

Proposition 5.6. *As slicings associated with perfect functions on $(K^4)_{16}$ we obtain at least the manifolds*

$$\mathbb{R}P^3, \quad \mathbb{R}P^3\#\mathbb{R}P^3, \quad \mathbb{R}P^3\#\mathbb{R}P^3\#\mathbb{R}P^3,$$
$$S^2\times S^1\#\mathbb{R}P^3\#\mathbb{R}P^3,$$

and the 3-torus.

Proof. There is a perfect rsl function $f_{\{1,\ldots,16\}}$ given by

$$f_{\{1,\ldots,16\}} : (K^4)_{16} \to [0,1]\,; \quad i \mapsto \frac{i-1}{15}.$$

level of $f_{\{1,\ldots,16\}}$	type	slicing in between
$\frac{1}{30}$	$\mathbb{R}P^3$	$\{1\}$ and $\{2,\ldots,16\}$
$\frac{1}{10}$	$\mathbb{R}P^3\#\mathbb{R}P^3$	$\{1,2\}$ and $\{3,\ldots,16\}$
$\frac{1}{6}$	$\mathbb{R}P^3\#\mathbb{R}P^3\#\mathbb{R}P^3$	$\{1,\ldots,3\}$ and $\{4,\ldots,16\}$
$\frac{7}{30}$	$(S^2\times S^1)\#2(\mathbb{R}P^3)$	$\{1,\ldots,4\}$ and $\{5,\ldots,16\}$
$\frac{1}{2}$	$\mathbb{T}^3$	$\{1,\ldots,8\}$ and $\{9,\ldots,16\}$

TABLE 4. Slicings of $(K^4)_{16}$ by the perfect and symmetric rsl function $f_{\{1,\ldots,16\}}$

As we already know, the first and the last slicings represent the link of the vertex 1 and 16 and are therefore combinatorial real projective 3-spaces. Furthermore, the middle slicing $f^{-1}_{\{1,\ldots,16\}}(\frac{1}{2})$ is homeomorphic to the 3-torus, which is more or less immediate from the construction of $(K^4)_{16}$ as the 4-torus modulo the central involution. The other slicings are connected sums of $\mathbb{R}P^3$ and $S^2\times S^1$. They are listed in Table 4. The multiplicity vectors are shown in Table 5.

An example of an rsl function that is not a perfect function is the function

$$f_{\{1,4,6,2,3,5,7,\ldots,16\}} : (K^4)_{16} \to [0,1]\,.$$

This admits an empty triangle on one side, leading to a critical point of index 1. In fact, $f_{\{1,4,6,2,3,5,7,\ldots,16\}}$ has precisely 20 critical points, counted with multiplicity; see Table 5. □

	$f_{\{1,\ldots,16\}}$		$f_{\{1,4,6,2,3,5,7,\ldots,16\}}$	
level	v	$\mathbf{m}(v,\mathbb{F}_2)$	v	$\mathbf{m}(v,\mathbb{F}_2)$
0	1	$(1,0,0,0,0)$	1	$(1,0,0,0,0)$
$\frac{1}{15}$	2	$(0,0,0,0,0)$	4	$(0,0,0,0,0)$
$\frac{2}{15}$	3	$(0,0,0,0,0)$	6	$(0,1,0,0,0)$
$\frac{3}{15}$	4	$(0,0,1,0,0)$	2	$(0,0,1,0,0)$
$\frac{4}{15}$	5	$(0,0,0,0,0)$	3	$(0,0,1,0,0)$
$\frac{5}{15}$	6	$(0,0,1,0,0)$	5	$(0,0,1,0,0)$
$\frac{6}{15}$	7	$(0,0,1,0,0)$	7	$(0,0,1,0,0)$
$\frac{7}{15}$	8	$(0,0,1,1,0)$	8	$(0,0,1,1,0)$
$\frac{8}{15}$	9	$(0,0,0,0,0)$	9	$(0,0,0,0,0)$
$\frac{9}{15}$	10	$(0,0,1,0,0)$	10	$(0,0,1,0,0)$
$\frac{10}{15}$	11	$(0,0,1,0,0)$	11	$(0,0,1,0,0)$
$\frac{11}{15}$	12	$(0,0,1,1,0)$	12	$(0,0,1,1,0)$
$\frac{12}{15}$	13	$(0,0,1,0,0)$	13	$(0,0,1,0,0)$
$\frac{13}{15}$	14	$(0,0,1,1,0)$	14	$(0,0,1,1,0)$
$\frac{14}{15}$	15	$(0,0,1,1,0)$	15	$(0,0,1,1,0)$
1	16	$(0,0,1,1,1)$	16	$(0,0,1,1,1)$

TABLE 5. Multiplicity vectors of two rsl functions $f_{\{1,\ldots,16\}}$ and $f_{\{1,4,6,2,3,5,7,\ldots,16\}}$ on $(K^4)_{16}$.

6. FURTHER RESULTS

In each of the 16 steps of the dilatation process for the Kummer variety we have the choice between two orientations. Consequently, for the resulting nonsingular manifold at the end there are a number of possible topological types. One can describe these by the intersection form.

Proposition 6.1. *One can construct some combinatorial* 4*-manifolds realizing any of the intersection forms of rank* 22 *and signature* $2n$, $n\in\{0,\ldots,8\}$, *from the triangulated* 4*-dimensional Kummer variety* K^4 *by* 16 *simplicial blowups, except for* $19(\mathbb{C}P^2)\#3(-\mathbb{C}P^2)$ *and possibly* $11(S^2\times S^2)$.

Proof. The case $n=8$ was already treated in Section 4. In this case, the orientation was uniquely determined at every step by that in the first step. Therefore, the construction is essentially unique (up to the orientation in the first blowup) and leads to the $K3$ surface. In particular, the manifold $19(\mathbb{C}P^2)\#3(-\mathbb{C}P^2)$ cannot be obtained in this way.

The signature of an even intersection form of a simply connected PL 4-manifold is divisible by 16 by Rohlin's theorem; cf. [Freedman and Kirby 76]. It follows that for $n \in \{1, \dots, 7\}$, we have an odd intersection form. In these cases the manifold is homeomorphic to

$$k(\mathbb{C}P^2)\#l(-\mathbb{C}P^2),$$

where $k - l = \pm 2n$, $n \in \{1, \dots, 7\}$, and $k + l = 22$. In the case $n = 0$, the construction is not unique: The pattern of the orientations of all 16 blowups is not determined, since there are eight positive and eight negative blowups distributed arbitrarily in K^4. An odd intersection form was obtained by one particular sequence. This leads to the manifold $11(\mathbb{C}P^2)\#11(-\mathbb{C}P^2)$. □

The question whether the manifold $11(S^2 \times S^2)$ can also be obtained by this construction remains open at this point. It must also be left open whether any of the other manifolds with a 22-dimensional second homology admits a triangulation with only 16 vertices. By [Kühnel 95, Theorem 4.9], such a 16-vertex triangulation would have to be 3-neighborly and would, by the Dehn–Sommerville equations, have the same f-vector as $(K3)_{16}$ and thus would give a solution to Problem 1.1 in the introduction. Further experiments in this direction could possibly produce such an example. This is still work in progress.

In the case of ten vertices and $\chi = 4$, the combinatorial data correspond to three topological types of simply connected 4-manifolds, namely $S^2 \times S^2$, $\mathbb{C}P^2\#\mathbb{C}P^2$, and $\mathbb{C}P^2\#(-\mathbb{C}P^2)$. These would be candidates for a solution to Problem 1.2. However, it was shown in [Kühnel and Lassmann 83] that in fact, none of the topological manifolds above has a combinatorial triangulation with only ten vertices.

More details of the combinatorial processes described above are available from the first author's web page, http://www.igt.uni-stuttgart.de/LstDiffgeo/Spreer/k3. Moreover, most of the algorithms used to compute simplicial blowups and multiplicity vectors of rsl functions are planned to be available soon within the GAP package `simpcomp` [Effenberger and Spreer 11], maintained by Effenberger and the first author.

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Jonathan Spreer, Institut für Geometrie und Topologie, Universität Stuttgart, 70550 Stuttgart, Germany (spreer@mathematik.uni-stuttgart.de)

Wolfgang Kühnel, Institut für Geometrie und Topologie, Universität Stuttgart, 70550 Stuttgart, Germany (kuehnel@mathematik.uni-stuttgart.de)

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