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Generalized Gorshkov–Wirsing Polynomials and the Integer Chebyshev Problem

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CONTENTS

The integer Chebyshev problem is the problem of finding an integer polynomial of degree n such that the supremum norm on $[0, 1]$ is minimized. The most common technique used to find upper bounds is by explicit construction of an example. This is often (although not always) done by heavy computational use of the LLL algorithm and simplex method. One of the first methods developed to find lower bounds employed a sequence of polynomials known as the Gorshkov–Wirsing polynomials.

This paper studies properties of the Gorshkov–Wirsing polynomials. It is shown how to construct generalized Gorshkov–Wirsing polynomials on any interval $[a, b]$, with $a, b \in \mathbb{Q}$. An extensive search for generalized Gorshkov–Wirsing polynomials is carried out for a large family of $[a, b]$. Using generalized Gorshkov–Wirsing polynomials, LLL, and the simplex method, upper and lower bounds for the integer Chebyshev constant on intervals other than $[0, 1]$ are calculated. These methods are compared with other existing methods.

1. INTRODUCTION

We define the supremum norm on an interval $I = [a, b]$ as

$$\|p\|_I := \sup_{z \in I} |p(z)|.$$

For the purposes of this paper, we assume that I is a rational interval on the real line, but the integer Chebyshev problem can be extended to any compact set of the complex numbers. The case of finding monic polynomials with real coefficients and minimal supremum norm on I is related to the logarithmic capacity of the set I, and leads to the study of Chebyshev polynomials [Goluzin 69, Ransford 95]. The integer Chebyshev problem is the problem of finding an integer polynomial of degree n with minimal supremum norm on the interval $[0, 1]$. This supremum is normalized by taking the nth root. In more general terms, we define this value on any interval $[a, b]$ as

$$t_{\mathbb{Z},n}[a, b] = \inf\Big\{ \|p(x)\|_{[a,b]}^{1/n} \mid p(x) \in \mathbb{Z}[x], \deg(p(x)) \le n, p(x) \ne 0 \Big\}.$$

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Degree n	Polynomial p(x)	$t_{\mathbb{Z}n}[0,1]$
2	$x(1-x)$	$1/2$
3	$x(1-x)(2x-1)$	$1/2.18\cdots$
4	$x^2(1-x)^2$ or $x(1-x)(2x-1)^2$	$1/2$
5	$x^2(1-x)^2(2x-1)$	$1/2.23\cdots$

TABLE 1. Small-degree integer Chebyshev polynomials.

By noticing that

$$t_{\mathbb{Z},n+m}[a,b]^{n+m} \leq t_{\mathbb{Z},n}[a,b]^n t_{\mathbb{Z},m}[a,b]^m,$$

we see that the limit

$$t_{\mathbb{Z}}[a,b] = \lim_{n\to\infty} t_{\mathbb{Z},n}[a,b]$$

is well defined.

The first few known values of the integer Chebyshev problem on the interval $[0,1]$ are given in Table 1.

The paper [Habsieger and Salvy 97] determines a complete list of polynomials up to and including degree 75. This was extended in [Meichsner 09] to degree 230. For a good survey of results concerning the integer Chebyshev problem, see [Borwein 98, Borwein 02, Pritsker 05].

In [Borwein and Erdélyi 96], it is shown that $t_{\mathbb{Z}}[0,x]$ is continuous and constant near $x=1$. The authors also show that

$$\left(m+2-\frac{1}{4(m+1)}\right)^{-1} \leq t_{\mathbb{Z}}\left[0,\frac{1}{m}\right].$$

A study of the intervals $[r/s, r/s+\delta]$ as $\delta \to 0$ can be found in [Flammang et al. 97]. The multidimensional case is considered in [Chudnovsky 83, Ferguson 80]. The integer Chebyshev problem on a compact set in $\mathbb{C}$ and its relationship to Mahler measure are studied in [Flammang et al. 06]. The study of the related case, that of minimal monic integer polynomials, was pioneered in [Borwein et al. 03]. This work has been followed up by [Hare and Smyth 06, Hare and Smyth 08, Hilmar 08].

Bounds have been given on the frequency of certain factors for large-degree integer Chebyshev polynomials. For example, it is shown in [Pritsker 99] that $x(1-x)$ must show up as a factor between 0.2961 and 0.3634 of the time (normalized by the degree).

Observe that

$$t_{\mathbb{Z}}[a,b] \leq t_{\mathbb{Z}n}[a,b].$$

Thus, techniques to find an upper bound for $t_{\mathbb{Z}}[a,b]$ center on finding good examples of $t_{\mathbb{Z}n}[a,b]$ for large values of n. For example, it is shown in [Borwein and Erdélyi 96] that

$$t_{\mathbb{Z}}[0,1] \leq \frac{1}{2.3543\ldots}$$

by considering a degree-210 polynomial. In the same paper, the authors show how this bound could be improved to $1/2.3605\ldots$. This is further refined in [Habsieger and Salvy 97] to yield

$$t_{\mathbb{Z}}[0,1] \leq \frac{1}{2.3612\ldots}.$$

Recently, both the lower and upper bounds were improved in [Pritsker 05] by use of weighted potential theory to get

$$\frac{1}{2.3736} \leq t_{\mathbb{Z}}([0,1]) \leq \frac{1}{2.3629}.$$

This was again refined in [Flammang 09] to give

$$t_{\mathbb{Z}}[0,1] \leq \frac{1}{2.3646\ldots}.$$

The method that Borwein and Erdélyi, and later Habsieger and Salvy, used to find these bounds was to consider a set of polynomials $p_i(x)$ that are factors of a large integer Chebyshev polynomial and consider the problem of minimizing ℓ, where

$$\sum r_i \frac{\log(|p_i(x)|)}{\deg(p_i)} \leq \ell,$$

for all $x \in [a,b]$ with $\sum r_i = 1,\ 0 \leq r_i$. This is the logarithm of an integer Chebyshev problem. By choosing a large number of points $x \in [0,1]$, instead of the entire interval, they obtain a system of linear equations on which the simplex method can be used to get a good estimate [Borwein and Erdélyi 96, Habsieger and Salvy 97, Schrijver 86]. The initial set of polynomials p_i can be found using LLL (Lenstra, Lenstra, Lovász) on the basis $1, x, x^2, \ldots, x^n$ and using the inner product

$$\langle p(x), q(x)\rangle = inside_a^b p(x)q(x)\,dx$$

[Hare 02, Lenstra et al. 82]. Other techniques are discussed later, in Section 2.

The refinement to this upper bound in [Flammang 09] was obtained in much the same way, but instead of discretizing the linear programming problem, the author used a method of semi-infinite programming introduced in [Smyth 84]. All of these methods are discussed in Section 2, and improvements to the simplex method are discussed there as well.

Numerous methods have been proposed for lower bounds as well. The method on which we will focus in this paper is that of Gorshkov–Wirsing polynomials.

Other methods (and their limitations) are discussed in Section 3.

The following result, which is a simple consequence of properties of resultants, is needed before we begin our discussion of Gorshkov–Wirsing polynomials. For more discussion, see [Borwein 02].

Lemma 1.1. *Suppose* $q(x) \in \mathbb{Z}[x]$ *and* $\deg(q(x)) = n$, *and suppose that* $p(x) = a_k x^k + \cdots + a_0 \in \mathbb{Z}[x]$, $a_k > 0$, *has all of its roots in the interval* $[a, b]$. *If* $\gcd(p(x), q(x)) = 1$, *then*

$$\left(\|q(x)\|_{[a,b]}\right)^{1/n} \geq a_k^{-1/k}.$$

This lemma says that if $p(x)$ has all roots in I and its leading coefficient is relatively small, then we will need $p(x)$ as a factor for large-degree integer Chebyshev polynomials. We formalize this by calling such $p(x)$ *critical*.

Definition 1.2. We say that an irreducible polynomial $p(x) = a_k x^k + \cdots + a_0$ is *critical* for an interval I if all roots of p are in the interval I and $|a_k|^{-1/k} \geq t_{\mathbb{Z}}(I)$.

Notice that in the calculation of $t_{\mathbb{Z}n}(I)$, we can have only a finite number of critical polynomials. This gives us a simple corollary to Lemma 1.1 that is the basis of the study of Gorshkov–Wirsing polynomials.

Corollary 1.3. *If there exists an infinite family* $p_1, p_2, \ldots$ *such that all roots of* $p_i(x) = a_{n_i} x^{n_i} + \cdots + a_0$ *are in an interval* I, *then*

$$t_{\mathbb{Z}}(I) \geq \liminf_{i\to\infty} |a_{n_i}|^{-1/n_i}.$$

With Corollary 1.3 in mind, we can now discuss how one would find such an infinite family of polynomials. Define

$$U(x) := \frac{x(1-x)}{1-3x(1-x)},$$

and further define

$$p_0(x) := 2x - 1.$$

Define the sequence of polynomials $p_i(x)$ recursively by

$$\begin{aligned} p_i(x) :&= \mathrm{numer}(p_{i-1}(U(x)) \\ &= \left(1 - 3x + 3x^2\right)^{\deg(p_{i-1})} p_{i-1}(U(x)), \end{aligned}$$

normalized to have integer coefficients, no integer content, and positive leading coefficient. Here numer$(\cdot)$ is the numerator of this normalized rational function. The first few polynomials in this sequence are

$$\begin{aligned} p_0(x) &= 2x - 1, \\ p_1(x) &= 5x^2 - 5x + 1, \\ p_2(x) &= 29x^4 - 58x^3 + 40x^2 - 11x + 1, \\ p_3(x) &= 941x^8 - 3764x^7 + 6349x^6 - 5873x^5 + 3243x^4 \\ &\quad - 1089x^3 + 216x^2 - 23x + 1. \end{aligned}$$

This defines an infinite sequence of polynomials $p_i(x)$, known as the Gorshkov–Wirsing polynomials [Borwein 02, Lorentz et al. 96, Montgomery 94], with leading coefficient a_{n_i}. This sequence has the following nice properties:

- All roots are in $[0, 1]$.
- All polynomials are irreducible.
- Polynomials are of degree 2^i with leading coefficient a_{n_i}.
- $\lim_{i\to\infty} |a_{n_i}|^{-1/2^i} = 1/2.3768417062\ldots$.

Combining these properties with Corollary 1.3 gives us the lower bound

$$t_{\mathbb{Z}}[0, 1] \geq \frac{1}{2.3768417062}.$$

It is shown in [Borwein and Erdélyi 96] that this bound is not tight, and that there exists an $\epsilon > 0$ such that

$$t_{\mathbb{Z}}[0, 1] \geq \frac{1}{2.3768417062} + \epsilon.$$

The authors' argument relies on the fact that the endpoints of the interval are the roots of critical polynomials on the interval.

In this paper we show how to generalize the definition of Gorshkov–Wirsing polynomials to give different sequences of polynomials, and derive different bounds for different intervals.

2. UPPER BOUND TECHNIQUES

In this section we give a review of some of the methods for finding upper bounds for $t_{\mathbb{Z}}(I)$. The first two methods are those of [Amoroso 90] and [Habsieger and Salvy 97]. For a proof of correctness, see the original articles. The next two methods involve LLL and the simplex method, and use as a guiding principle that a good example of a polynomial with small norm gives a good upper bound. These upper bounds are compared in Tables 2, 3, 4, and

Interval	LLL	Simplex	HS	Amoroso	Lower	# CP
[-1, 1]	1/1.5314	1/1.5334	1/1.4772	1/1.4520	1/1.5417	8
[-1/2, 1/2]	1/2.3559	1/2.3619	1/2.1822	1/1.4520	1/2.3768	9
[-1/3, 1/3]	1/3.2522	1/3.2617	1/3.0000	1/1.3887	1/3.2842	7
[-2/3, 2/3]	1/1.8820	1/1.8883	1/1.7237	1/1.3887	1/1.9845	5
[-1/4, 1/4]	1/4.1921	1/4.2025	1/4.0000	1/1.1097	1/4.2260	6
[-3/4, 3/4]	1/1.7897	1/1.7935	1/1.7237	1/1.1097	1/1.9653	3

TABLE 2. Upper bounds for $t_{\mathbb{Z}}(I)$, for $0 \in \operatorname{int}(I)$, I symmetric.

5. For comparison, the best known lower bound is also given.

2.1. Amoroso

For a more complete discussion of this technique, see [Amoroso 90].

Define

$$\begin{aligned}\rho(r_1,r_2) = &-\frac{1}{1-r_1-r_2}\\ &\times\Bigg((r_1+r_2)\log(2) - \frac{(1-r_1-r_2)^2}{4}\log(1-r_1-r_2)\\ &-\frac{(1+r_1-r_2)^2}{4}\log(1+r_1-r_2)\\ &-\frac{(1-r_1+r_2)^2}{4}\log(1-r_1+r_2)\\ &-\frac{(1+r_1+r_2)^2}{4}\log(1+r_1+r_2) + r_1^2\log(2r_1)\\ &+r_2^2\log(2r_2)\Bigg)\end{aligned}$$

and define

$$\begin{aligned}f^+(r_1,r_2) = \log\left(\sqrt{|I|/4}\right) &+ r_1\log(b_1\sqrt{\delta})\\ &+ r_2\log(b_2\sqrt{\delta}) + \rho(r_1,r_2).\end{aligned}$$

Then

$$t_{\mathbb{Z}}(I) \le \exp\min_T f^+(r_1,r_2),$$

where T is the simplex $0 \le r_1, r_2$ and $r_1 + r_2 \le 1$.

As can be seen, the upper bound attained is very much dependent on the denominators b_1 and b_2. In fact, it is sometimes advantageous to consider an upper bound based on a slightly larger interval, along with the observation that if $I \subset J$, then $t_{\mathbb{Z}}(I) \le t_{\mathbb{Z}}(J)$. Such an example can be seen by $I = [1/24, 1 - 1/24]$ and $J = [0, 1]$.

Amoroso's lower-bound estimate based on this I is then $1/29.2213$, whereas on J it is $1/2.4141$. This is taken into account in compiling the data in the tables. Table 6 shows what happens if this is not taken into account.

2.2. Habsieger and Salvy

The method of Habsieger and Salvy is used to find explicitly the best polynomial of lower degree. For degree n, we first find a reasonably good polynomial, using LLL, for example, and hence a reasonably good upper bound ℓ. We next use this bound for good polynomials to find required factors of the best polynomial of degree n. This is done by noticing that for any polynomial P, we have from Markov's inequality that

$$\begin{aligned}&\max_{a\le x\le b}\left|P^{(r)}(x)\right|\\ &\quad\le \frac{2^r}{(b-a)^r}\cdot\frac{n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-(r-1)^2)}{(2r-1)!!}\\ &\quad\times\max_{a\le x\le b}|P(x)|,\end{aligned}$$

Interval	LLL	Simplex	HS	Amoroso	Lower	# CP
[-1/2, 1]	1/1.8133	1/1.8190	1/1.6055	1/1.4520	1/1.8743	3
[-1/3, 1]	1/2.0248	1/2.0309	1/1.8899	1/1.3887	1/2.0617	8
[-2/3, 1]	1/1.6560	1/1.6657	1/1.4142	1/1.3887	1/1.7410	3
[-1/3, 1/2]	1/2.6978	1/2.7094	1/2.1743	1/1.3887	1/2.7788	4
[-1/2, 2/3]	1/2.0303	1/2.0443	1/1.7411	1/1.3887	1/2.1865	4
[-1/3, 2/3]	1/2.2740	1/2.2801	1/1.9332	1/1.3887	1/2.4537	2

TABLE 3. Upper bounds for $t_{\mathbb{Z}}(I)$, for $0 \in \operatorname{int}(I)$, I nonsymmetric.

Interval	LLL	Simplex	HS	Amoroso	Lower	# CP
[0, 1]	1/2.3546	1/2.3613	1/2.2361	1/2.3264	1/2.3768	10
[0, 1/2]	1/3.4689	1/3.4813	1/3.1923	1/2.3264	1/3.5132	8
[1/3, 1/2]	1/5.8364	1/5.8614	1/5.3126	1/2.3264	1/5.9112	7
[0, 1/3]	1/4.5235	1/4.5444	1/4.1930	1/2.3264	1/4.5940	7

TABLE 4. Upper bounds for $t_{\mathbb{Z}}(I)$, for $0 \notin \operatorname{int}(I)$, I Farey.

where $(2i+1)!! = 1 \cdot 3 \cdot 5 \cdots (2i+1)$. Hence, if we consider an irreducible factor $p(x)$ with all its roots in an interval $[a, b]$, and consider the product of the left-hand side over these roots, then the result is 0 if $P^{(r)}$ has $p(x)$ as a factor, or is greater than or equal to $1/a_k$, where a_k is the leading coefficient of $p(x)$.

The right-hand side is explicitly computable, based on the good upper bound ℓ, found by LLL. This gives us a bound for the right-hand side of the equation, which when combined with the restrictions on the left-hand side may imply that the left-hand side of the equation is 0. Hence for some values of r, we see that the left-hand side must then equal 0, which in turn implies a multiplicity of the factor $p(x)$.

Then, using the product of all required factors, say $Q(x)$, of degree k, this good bound, and a random selection of points in the interval, say x_i, we notice that the best polynomial will satisfy

$$-\ell \leq \left(a_{n-k} x_i^{n-k} + \cdots + a_0\right) Q(x) \leq \ell$$

for all x_i, and further that $a_{n-k} \geq 1$. This is a system of linear equations. We can solve for all integer solutions of the a_i's exhaustively, and then select the best example(s) from this list.

One nice benefit is that factors of the best polynomials are useful to add to the basis of the simplex method.

This is done up to degree 3, which although not large, is good enough for our purposes.

2.3. LLL

The use of LLL tends to give a very crude estimate of an upper bound. One side benefit of this method, though, is that it tends to give a very good set of polynomials for which to start the simplex method.

Initially, we consider a basis $1, x, \ldots, x^n$ and an inner product

$$\langle p(x), q(x) \rangle = \text{int}_a^b \, p(x) q(x) \, dx.$$

Small elements in this basis have small 2-norm, which tends to mean small sup-norm. So using LLL, we get an element with small norm, say $p_1(x)$. We then repeat this process with a basis $p_1(x), x \cdot p_1(x), \ldots, x^n \cdot p_1(x)$, and iterate the process. This is done up to $n = 20$.

2.4. The Simplex Method

Consider a set of polynomials $p_i(x)$ that are factors of a large integer Chebyshev polynomial, and consider the problem of minimizing ℓ, where

$$\sum r_i \frac{\log(|p_i(x)|)}{\deg(p_i)} \leq \ell,$$

for all $x \in [a, b]$ with $\sum r_i = 1$, $0 \leq r_i$. This is the logarithm of an integer Chebyshev problem. By choosing a large number of points $x \in [0, 1]$, instead of all of them, this becomes a linear programming problem.

Now, one area that can be influenced is a careful choice of the x_j. Initially, we choose 50 points in I, uniformly distributed. We then consider the resulting object

$$\sum r_i \frac{\log(|p_i(x)|)}{\deg(p_i)}$$

and find its local maxima. We add these local maxima to our set of x_j and iterate. This has the advantage that each iteration focuses more and more attention on the "problem" spots.

If the factors exist, then this is done using up to 20 factors. Factors were chosen such that the following conditions were satisfied:

- All known critical polynomials were included.

Interval	LLL	Simplex	HS	Amoroso	Lower	# CP
[0, 2/3]	1/2.7940	1/2.8056	1/2.3811	1/2.3264	1/2.8804	5
[1/3, 2/3]	1/3.8577	1/3.8707	1/3.4641	1/2.3264	1/3.8920	6

TABLE 5. Upper bounds for $t_{\mathbb{Z}}(I)$, for $0 \notin \operatorname{int}(I)$, I non-Farey.

Interval	LLL	Simplex	HS	Amoroso	Lower	# CP
[1/5, 4/5]	1/2.6391	1/2.6426	1/2.2134	1/2.0166	1/2.8590	3
[1/10, 9/10]	1/2.3532	1/2.3623	1/2.1822	1/1.0630	1/2.7502	8
[1/20, 19/20]	1/2.3532	1/2.3622	1/2.1822	1/.66772	1/2.8949	8
[-1/5, 6/5]	1/2.0441	1/2.0486	1/2.0000	1/1.0257	1/2.2502	3
[-1/10, 11/10]	1/2.2965	1/2.2971	1/2.0000	1/.78152	1/2.5474	4
[-1/20, 21/20]	1/2.3543	1/2.3619	1/2.1822	1/.57228	1/2.6246	10

TABLE 6. Upper bounds for $t_{\mathbb{Z}}(I)$, for $I \subset [0,1]$ and $I \supset [0,1]$, exact computation.

- Good factors found through LLL or through Habsieger and Salvy were included.
- Factors that proved useful for smaller examples were included.
- Polynomials found with small $a_k^{-1/k}$ values and all with roots in the interval were included. These were normally found through searches for Gorshkov–Wirsing pairs.

3. LOWER BOUND TECHNIQUES

There are many ways to estimate lower bounds on the Integer Chebyshev Problem. We will outline some of the methods below. For a proof of correctness of the method, the reader is encouraged to read the original article. These methods are summarized in Tables 7, 8, 9, and 10.

3.1. Amoroso

For a more complete discussion, see [Amoroso 90].

Let $I = [a_1/b_1, a_2/b_2]$ be a rational interval. Let $\delta = a_2/b_2 - a_1/b_1$ be its diameter. Define

$$\begin{aligned} h(r_1, r_2) = {} & \frac{1}{2}(1 - r_1 - r_2)\log(1 - r_1 - r_2) \\ & - \frac{1}{2}(1 + r_1 + r_2)\log(1 + r_1 + r_2) \\ & - \frac{1}{2}(1 + r_1 - r_2)\log(1 + r_1 - r_2) \\ & \times \frac{1}{2}(1 - r_1 - r_2)\log(1 - r_1 - r_2) \\ & + 2r_0\log(2r_0) \end{aligned}$$

and define

$$\begin{aligned} f^-(r_1, r_2) = \log(\delta) + \max\big(& (r_1 - 1)\log(b_1\delta) + h(r_1, r_2), \\ & (r_2 - 1)\log(b_2\delta) + h(r_2, r_1)\big). \end{aligned}$$

Then

$$t_{\mathbb{Z}}(I) \geq \exp\left(\inf_T f^-(r_1, r_2)\right).$$

As before, the size of the denominator plays a crucial role in this estimate, and often it is advantageous to consider a smaller interval with a small denominator.

3.2. Flammang, Rhin, Smyth

For a more complete description, see [Flammang et al. 97]. In this method, the authors restricted their attention to Farey intervals $[p/q, r/s]$, that is, intervals where $qr - ps = 1$. They define

$$U_0 = z, \quad V_0 = 1, \quad U_{k+1} = U_k^2 + V_k^2, \quad V_{k+1} = U_k V_k,$$
$$x_k = U_k/V_k.$$

Further, take

$$g_-(z) = \prod x_k^{-1/2^k}.$$

Then

$$t_{\mathbb{Z}}(I) \geq \frac{1}{\sqrt{qs}} g_-\left(\sqrt{q/s}\right).$$

Using the fact that if $I \subset J$, then $t_{\mathbb{Z}}(I) \leq t_{\mathbb{Z}}(J)$, and the fact that every interval has a maximal Farey subinterval, we can extend this lower bound to all intervals

Interval	Upper	GW	FRS	Flammang	Amoroso
[-1, 1]	1/1.5334	1/1.5417	1/2.3768	1/2.3768	1/1.7024
[-1/2, 1/2]	1/2.3619	1/2.3768	1/3.5132	1/3.5132	1/3.4048
[-1/3, 1/3]	1/3.2617	1/3.2842	1/4.5940	1/4.5940	1/4.7345
[-2/3, 2/3]	1/1.8883	1/1.9845	1/3.5132	1/3.5132	1/3.1860
[-1/4, 1/4]	1/4.2025	1/4.2260	1/5.6494	1/5.6494	1/5.7853
[-3/4, 3/4]	1/1.7935	1/1.9653	1/3.5132	1/3.5132	1/3.0000

TABLE 7. Lower bounds for $t_{\mathbb{Z}}(I)$, for $0 \in \mathrm{int}(I)$, I symmetric.

Interval	Upper	GW	FRS	Flammang	Amoroso
[-1/2, 1]	1/1.8190	1/1.8743	1/2.3768	1/2.3768	1/2.2151
[-1/3, 1]	1/2.0309	1/2.0617	1/2.3768	1/2.3768	1/2.3187
[-2/3, 1]	1/1.6657	1/1.7410	1/2.3768	1/2.3768	1/2.1307
[-1/3, 1/2]	1/2.7094	1/2.7788	1/3.5132	1/3.5132	1/3.6580
[-1/2, 2/3]	1/2.0443	1/2.1865	1/3.5132	1/3.5132	1/3.1860
[-1/3, 2/3]	1/2.2801	1/2.4537	1/3.5132	1/3.5132	1/3.1860

TABLE 8. Lower bounds for $t_{\mathbb{Z}}(I)$, for $0 \in \mathrm{int}(I)$, I nonsymmetric.

(although, sometimes quite badly). Table 11 shows what happens if this is not taken into account.

3.3. Flammang

For a more complete description, see [Flammang 95].

Again, focus is given to Farey Intervals. Let $I = [p/q, r/s]$. Let

$$\lambda_0 = \frac{qs}{q+s} \quad \text{and} \quad \lambda_{k+1} = \frac{\lambda_k}{(1+\lambda_k)^2}.$$

Then

$$t_{\mathbb{Z}}(I) \geq \frac{1}{q+s} \prod (1+\lambda_i)^{-1/2^{i+1}}.$$

The same comments as before with respect to Farey intervals hold in this case as well.

4. GENERALIZED GORSHKOV–WIRSING RATIONAL FUNCTIONS

In Section 1, we considered the function

$$U(x) = \frac{x(1-x)}{1-3x(1-x)}.$$

The property of this function that makes it useful for finding lower bounds is that it is of degree 2, and it maps $[0,1]$ to itself twice. Hence, if $p(x)$ is a polynomial with all of its roots in $[0,1]$, then numer$(p(U(x))$ has all of its roots in $[0,1]$.

We extend this concept to give the following definition:

Definition 4.1. A *generalized Gorshkov–Wirsing rational function* on $[a,b]$ is a rational function

$$U(x) = \frac{r(x)}{s(x)}$$

mapping the interval $[a,b]$ to itself d times, where $\deg(b(x)) \leq \deg(t(x)) = d$ and $b(x), t(x) \in \mathbb{Z}[x]$. Denote the set of all such $U(x)$ by $\mathcal{U}[a,b]$.

We now give a complete description of $\mathcal{U}[a,b]$ if $a, b \in \mathbb{Q}$.

Theorem 4.2. *Let $a, b \in \mathbb{Q}$, and let $p(x)$ and $q(x)$ be integer polynomials, nonnegative on $[a,b]$, with the following properties:*

- $\deg(p) = \deg(q)$.
- *Both $p(x)$ and $q(x)$ are totally real, that is, all of their roots are real.*
- *The polynomial $p(x)$ has a single root at a and double roots at $\alpha_1 < \alpha_2 < \cdots$. If $\deg(p)$ is even, then $p(x)$ also has a single root at b.*
- *The polynomial $q(x)$ has double roots at $\beta_1 < \beta_2 < \cdots$. If $\deg(q)$ is odd, then $q(x)$ also has a single root at b.*
- *The roots interlace, that is,*

$$a < \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots < b.$$

Interval	Upper	GW	FRS	Flammang	Amoroso	BE
[0, 1]	1/2.3613	1/2.3768	1/2.3768	1/2.3768	1/2.4142	1/2.8750
[0, 1/2]	1/3.4813	1/3.5132	1/3.5132	1/3.5132	1/3.6580	1/3.9167
[1/3, 1/2]	1/5.8614	1/5.9112	1/5.9112	1/5.9112	1/8.3648	
[0, 1/3]	1/4.5444	1/4.5940	1/4.5940	1/4.5940	1/4.7345	1/4.9375

TABLE 9. Lower bounds for $t_{\mathbb{Z}}(I)$, for $0 \notin \mathrm{int}(I)$, I Farey.

Interval	Upper	GW	FRS	Flammang	Amoroso
[0, 2/3]	$\frac{1}{2.8056}$	$\frac{1}{2.8804}$	$\frac{1}{3.5132}$	$\frac{1}{3.5132}$	$\frac{1}{3.1860}$
[1/3, 2/3]	$\frac{1}{3.8707}$	$\frac{1}{3.8920}$	$\frac{1}{5.9112}$	$\frac{1}{5.9112}$	$\frac{1}{7.2426}$

TABLE 10. Lower bounds for $t_{\mathbb{Z}}(I)$, for $0 \notin \operatorname{int}(I)$, I non-Farey.

Then for all $e, f \in \mathbb{N}$,

$$\frac{a \cdot e \cdot p(x) + b \cdot f \cdot q(x)}{e \cdot p(x) + f \cdot q(x)} \quad \textit{and} \quad \frac{b \cdot e \cdot p(x) + a \cdot f \cdot q(x)}{e \cdot p(x) + f \cdot q(x)} \tag{4–1}$$

are members of $\mathcal{U}[a, b]$. *Furthermore, all* $U(x) \in \mathcal{U}[a, b]$ *can be written in this form.*

Such pairs of polynomials p and q play an important role in this study, and we will call them *Gorshkov–Wirsing pairs.*

Examining equation (4–1) of Theorem 4.2 gives the following examples:

$$\frac{1 - 5x^2}{6x^2 - 2} = \frac{\frac{1}{2}(1 - 4x^2) - \frac{1}{2}x^2}{(1 - 4x^2) + x^2} \in \mathcal{U}\left[-\frac{1}{2}, \frac{1}{2}\right],$$

$$\frac{x(2x - 1)}{-2 + 10x - 14x^2} = \frac{\frac{1}{2}x(1 - 2x) + 0(3x - 1)^2}{(x(1 - 2x)) + (3x - 1)^2} \in \mathcal{U}\left[0, \frac{1}{2}\right],$$

$$2x^2 - 1 = \frac{x^2 - (1 - x^2)}{x^2 + (1 - x^2)} \in \mathcal{U}[-1, 1].$$

Proof of Theorem 4.2: Denote (4–1) by $U(x)$. We see that $U(x) = a$ between a and b precisely when $U(x) - a = 0$. Upon simplification, we see that this is precisely when $p(x) = 0$. Similarly, we see that $U(x) = b$ precisely when $q(x) = 0$. Since $p(x)$ and $q(x)$ are nonnegative and do not share any roots on the interval $[a, b]$, we have that $p(x) + q(x)$ is positive on $[a, b]$. By considering the degrees of $p(x)$ and $q(x)$ and the fact that the roots interlace, we get that $U(x)$ maps $[a, b]$ to itself d times, where $\deg(p) = \deg(q) = d$. Hence $U(x) \in \mathcal{U}[a, b]$.

To see that all $U(x)$ must be of this form, we consider two cases:

Case 1. Assume that the degree of the numerator is odd, say $2n + 1$. We notice that either $U(a) = a$ and $U(b) = b$, or $U(a) = b$ and $U(b) = a$. By noticing that

$$\frac{bq(x) + ap(x)}{p(x) + q(x)} = b + a - \frac{aq(x) + bp(x)}{p(x) + q(x)},$$

it suffices to prove the result for the first situation. So we can assume that $U(a) = a$ and $U(b) = b$. Notice that $U(x) - a$ has $n + 1$ roots, say $a < \alpha_1 < \alpha_2 < \cdots < \alpha_n$. Further, we see that $U(x) - a$ must have double roots at $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, for otherwise $U(x) < a$ for some $x \in [a, b]$. Write

$$p_1(x) = (x - a)(x - \alpha_1)^2 \cdots (x - \alpha_n)^2.$$

Similarly, we see that $U(x) - b$ has $n + 1$ roots, which interlace with the α_i, and that all but b must be double roots. Call these roots $\beta_1 < \cdots < \beta_n < b$. Define

$$q_1(x) = (b - x)(x - \beta_1)^2 \cdots (x - \beta_n)^2.$$

Writing $U(x) = r(x)/s(x)$, we notice that

$$U(x) - a = \frac{e \cdot p_1(x)}{s(x)}$$

and further that

$$U(x) - b = \frac{-f \cdot q_1(x)}{s(x)}$$

for $e, f > 0$. This gives

$$(b - a)s(x) = e \cdot p_1(x) + f \cdot q_1(x).$$

Thus we can write

$$U(x) = \frac{b \cdot e \cdot p_1(x) + a \cdot f \cdot q_1(x)}{e \cdot p_1(x) + f \cdot q_1(x)}.$$

By multiplying the numerator and denominator by the appropriate integer, we may assume that e and f are integers.

Interval	Upper	GW	Amoroso
[1/5, 4/5]	1/2.6426	1/2.8590	1/8.0189
[1/10, 9/10]	1/2.3623	1/2.7502	1/13.953
[1/20, 19/20]	1/2.3622	1/2.8949	1/24.906
[-1/5, 6/5]	1/2.0486	1/2.2502	1/7.1367
[-1/10, 11/10]	1/2.2971	1/2.5474	1/13.117
[-1/20, 21/20]	1/2.3619	1/2.6246	1/24.351

TABLE 11. Lower bounds for $t_{\mathbb{Z}}(I)$, for $I \subset [0, 1]$ and $I \supset [0, 1]$, exact computation.

Case 2. The case in which the degree of the numerator is even is similar. □

By simple algebra, we get the following lemma:

Lemma 4.3. *Let (P, Q) and (p_1, q_1) be Gorshkov–Wirsing pairs satisfying the conditions of Theorem 4.2. Let $e, f \in \mathbb{N}$. Define inductively the generalized Gorshkov–Wirsing rational function*

$$\begin{aligned} U_i(x) &= \frac{a \cdot e \cdot p_i(x) + b \cdot f \cdot q_i(x)}{e \cdot p_i(x) + f \cdot q_i(x)}, \\ p_{i+1}(x) &= \text{numer}(P(U_i(x))), \\ q_{i+1}(x) &= \text{numer}(Q(U_i(x))), \end{aligned}$$

normalized so that p_{i+1} and q_{i+1} are integer polynomials with no integer content, and are positive on $[a, b]$. Then the following hold:

- *p_i and q_i satisfy the conditions of Theorem 4.2.*
- $\deg(p_i(x)) = \deg(q_i(x)) = \deg(p_1(x))^{\deg P(x)}$.

In practice, we take e and f such that $e \mid \text{denom}(a)$ and $f \mid \text{denom}(b)$.

Consider, for example, $[a, b] = [0, 1]$, and take $p_1(x) = P(x) = (2x - 1)^2$ and $q_1(x) = Q(x) = x(1 - x)$. Take $e = f = 1$. This gives us

$$\begin{aligned} U(x) &= \frac{x(1-x)}{x(1-x) + (2x-1)^2} = \frac{x(1-x)}{1 - 3x(1-x)}, \\ p_2(x) &= (5x^2 - 5x + 1)^2, \\ q_2(x) &= x(1-x)(2x-1)^2. \end{aligned}$$

In fact, this is the start of the classic Gorshkov–Wirsing sequence (squared).

This leads to the following definition:

Definition 4.4. A set of polynomials (P, Q), (p_i, q_i), and integers e, f that satisfy the conditions of Lemma 4.3 are called *generalized Gorshkov–Wirsing polynomials*.

It is worth noting that given two pairs of Gorshkov–Wirsing pairs, there are eight different ways that they can be combined to give a Gorshkov–Wirsing sequence.

At this point we need some discussion on how to ensure that p_{n+1} and q_{n+1} have no integer content. We will do this by way of an example.

Example 4.5. Consider $I = [-1/2, 1/2]$. Let $p_1(x) = P(x) = x^2$, $q_1(x) = Q(x) = 1 - 4x^2$, and $e = f = 1$. We see then that p_1, q_1 and P, Q are both Gorshkov–Wirsing pairs. Consider $P(U_n(x))$ for a generic p and q. We see that

$$P(U_n(x)) = \frac{1}{4} \times \frac{(p_n - q_n)^2}{(p_n + q_n)^2}.$$

This gives us $p_{n+1} = (p_n - q_n)^2$.

Next consider $Q(U_n(x))$ for a generic p_n and q_n. We get

$$Q(U_n(x)) = 1 - \left(\frac{p-q}{p+q}\right)^2 = \frac{4p_n q_n}{(p_n + q_n)^2},$$

which gives us $q_{n+1} = p_n q_n$. So we see that we can remove integer content in most cases in advance. In the second case, there will be no other integer content that may creep into the calculation by accident. If there is integer content that should be removed but is not, then we still have a valid Gorshkov–Wirsing sequence, although not generally as good.

Given such a sequence, it is important to consider how one computes $\lim a_n^{-1/\deg p_n}$. To simplify the discussion, we will assume that $e = f = 1$, although the proof is equally valid for arbitrary positive integers e and f. Let p_1, q_1 and P, Q be Gorshkov–Wirsing pairs. Let $P(x, y)$ and $Q(x, y)$ be the homogenizations of P and Q, and let C_p and C_Q be constants such that $P(a \cdot x + b \cdot y, x + y)c_P$ and $Q(a \cdot x + b \cdot y, x + y)c_Q$ have no integer content. That is, for the example above we have $P(x, y) = (x - y)^2$ and $Q(x, y) = x \cdot y$.

Then we see that

$$p_{n+1} = P(a \cdot p_n + b \cdot q_n, p_n + q_n)c_P,$$

and similarly for q_{n+1}. This method of computing p_{n+1} and q_{n+1} allows us to compute the leading coefficient of p_{n+1} and q_{n+1}. Namely, we have

$$a_{n+1} = P(a \cdot a_n + b \cdot b_n, a_n + b_n)c_P$$

and

$$b_{n+1} = Q(a \cdot a_n + b \cdot b_n, a_n + b_n)c_Q.$$

Theorem 4.6. *With a_n and b_n defined as above and $d_n = \deg(p_n) = \deg(q_n)$, then $d_n = \deg(p_1)\deg(P)^{n-1}$. Further, $\lim |a_n|^{-1/d_n}$ and $\lim |b_n|^{-1/d_n}$ are both well defined and are equal.*

Proof: Let p_1 and q_1 be of degree k, and P and Q of degree m. We easily see by induction that

$$\deg p_n = \deg q_n = \deg p_1 (\deg P)^{n-1} = k \cdot m^{n-1}.$$

For now, assume that $|a_n| \le |b_n|$. Then we get

$$\begin{aligned} a_{n+1} &= P\left(a \cdot a_n + b \cdot b_n, b_n + a_n\right) c_P \\ &= P\left(a \cdot \frac{a_n}{b_n} + b, 1 + \frac{a_n}{b_n}\right) c_P b_n^m, \\ b_{n+1} &= Q\left(a \cdot a_n + b \cdot b_n, b_n + a_n\right) c_Q \\ &= Q\left(a \cdot \frac{a_n}{b_n} + b, 1 + \frac{a_n}{b_n}\right) c_Q b_n^m. \end{aligned}$$

We see that $P(ax+b, 1+x)$ and $Q(ax+b, 1+x)$ are bounded for $x \in [-1,1]$, say by M. Then $a_{n+1} \le M b_n^m$ and $b_{n+1} \le M b_n^m$. We see that the degree of p_n and q_n is $d_n = k \cdot m^{n-1}$. Now

$$|a_{n+1}|^{-1/d_{n+1}} \le \prod M^{-1/(k \cdot m^i)} |b_0|,$$

and similarly for b_{n+1}. This is easily seen to converge by taking logarithms of the right-hand side. Since the $|a_n|^{-1/d_n}$ are increasing and bounded, they converge.

To see that they are equal, we observe that

$$\begin{aligned} a_{n+1}^{-1/d_{n+1}} &= \left(P\left(a \cdot \frac{a_n}{b_n} + b, 1 + \frac{a_n}{b_n}\right) c_P b_n^m\right)^{-1/d_{n+1}} \\ &= \left(P\left(a \cdot \frac{a_n}{b_n} + b, 1 + \frac{a_n}{b_n}\right) c_P\right)^{-1/d_{n+1}} b_n^{-m/d_{n+1}} \\ &= \left(P\left(a \cdot \frac{a_n}{b_n} + b, 1 + \frac{a_n}{b_n}\right) c_P\right)^{-1/d_{n+1}} b_n^{-1/d_n}. \end{aligned}$$

Here

$$\left(P\left(a \cdot \frac{a_n}{b_n} + b, 1 + \frac{a_n}{b_n}\right) c_P\right)^{-1/d_{n+1}}$$

goes to 1 as n goes to infinity, from which the result follows.

We note that a similar argument holds if $|a_n| > |b_n|$ for some or all of the n. $\square$

5. SEARCHING FOR GENERALIZED GORSHKOV–WIRSING POLYNOMIALS

The next two lemmas combine to give us an algorithm to find $U(x) \in \mathcal{U}[a,b]$. This is a heuristic technique only, and is not proven to get all good values in $\mathcal{U}[a,b]$. An additional benefit of this technique is that it finds many "good" polynomials for an interval, in the sense that $a_k^{-1/k}$ is small and all of its roots are in the interval in question.

Lemma 5.1. *If $p(x)$ has all of its roots in $[a,b]$ and $U(x) \in \mathcal{U}[a,b]$, then* numer$(p(U(x))$ *has all of its roots in $[a,b]$.*

Lemma 5.2. *If $U(x) \in \mathcal{U}[a,b]$, then* numer$(U(x) - x)$ *has all of its roots in $[a,b]$.*

To find $U(x) \in \mathcal{U}[a,b]$, we first let $\mathcal{Q}$ be a set of irreducible polynomials with all of their roots in the interval $[a,b]$. Included in this set are the two linear polynomials with roots at a and b. One easy way to derive a starting set of polynomials is using LLL. From this set, we find the set of all pairs (p,q) that satisfy the conditions of Theorem 4.2, and call this set $\mathcal{P}$.

We use the sets $\mathcal{P}$ and $\mathcal{Q}$ and Lemma 5.1 to find new polynomials with all of their roots in $[a,b]$ and add these to the set $\mathcal{Q}$. We use the set $\mathcal{P}$ and Lemma 5.2 to find new polynomials with all of their roots in $[a,b]$ and add these to the set $\mathcal{Q}$. Normally, we limit the set $\mathcal{Q}$ in some way, based on Lemma 1.1 and on the degree of the polynomials. Using this new set $\mathcal{Q}$ we derive a new set $\mathcal{P}$. We repeat this procedure until we no longer find any new polynomials or pairs given some set of restrictions.

Example 5.3. Consider the interval $[-1/2, 1/2]$ and the set

$$\mathcal{Q} = \{2x-1, x, 2x+1\}.$$

From this we derive the pairs

$$\mathcal{P} = \left\{\left[x^2, (1-2x)(2x+1)\right]\right\}.$$

From Lemma 5.1, this gives us the new polynomial

$$\left\{5x^2 - 1\right\}.$$

From Lemma 5.2, we get the new polynomials

$$\left\{5x^2 - 1, 3x - 1\right\}.$$

So our new set $\mathcal{Q}$ becomes

$$\mathcal{Q} = \{x, 2x-1, 2x+1, 5x^2-1, 3x-1\}.$$

From this we derive the pairs

$$\begin{aligned} \mathcal{P} = \Big\{ &\left[x^2, (1-2x)(2x+1)\right], \\ &\left[(5x^2-1)^2, (1-2x)(2x+1)(3x-1)^2\right], \\ &\left[(5x^2-1)^2, (1-2x)(2x+1)x^2\right], \\ &\left[(3x-1)^2, (1-2x)(2x+1)\right], \\ &\left[(1-2x)x^2, (2x+1)(3x-1)^2\right]\Big\}. \end{aligned}$$

From here, we can keep repeating the process, eliminating pairs and polynomials that fail to meet some sort of criteria, and eventually decide that we are done.

6. SOME UPPER AND LOWER BOUNDS FOR INTERVALS

Using the techniques of Section 5, we compute upper and lower bounds for a number of sets.[1] The purpose of these tables is to compare Gorshkov–Wirsing pairs with other methods. The Gorshkov–Wirsing pairs were found up to degree $n = 11$, and beyond whenever possible.

We looked at five different types of intervals. Very little will be said about Tables 2, 3, 4, and 5. These were included for reference only.

We see from Tables 7, 8, and 10 that the Gorshkov–Wirsing pairs give a tighter lower bound than that given by the other methods. This is not surprising, given that these other methods are designed for Farey intervals. We see in Table 9 that the Gorshkov–Wirsing pairs give the same values as those of Flammang, Rhin, and Smyth and that of Flammang. In these cases, they are just alternative ways of computing the same limit point.

Tables 6 and 11 show what happens for intervals close to $[0, 1]$ for which the actual interval has very high denominator. We see that LLL, simplex, and the method of Habsieger and Salvy do not suffer any ill effects from this. The Gorshkov–Wirsing method does suffer some effects, but not as severe as those suffered by Amoroso. The methods of Flammang and those of Flammang, Rhin, and Smyth were not included, since they are relevant only for Farey intervals.

7. CONCLUSIONS

It is shown in [Borwein and Erdélyi 96] that $[0, x]$ is continuous and constant for x near 1. Further, the authors showed that there exists a δ such that for all $0 \leq a < \delta$, $t_{\mathbb{Z}}[-a, 1 + a] = t_{\mathbb{Z}}[0, 1]$. Unfortunately, their method does not easily allow an explicit computation of δ. It would be interesting to use these Gorshkov–Wirsing pairs for $a \in \mathbb{Q}$ with $a < \delta$ to see whether better lower bounds could be found for $t_{\mathbb{Z}}[0, 1]$.

In this same paper, Borwein and Erdélyi also showed that the limit coming from these Gorshkov–Wirsing pairs cannot be tight for the interval $[0, 1]$. This argument, in fact, holds when the endpoints of the interval are roots of critical polynomials. This is not always the case for some of the problems looked at in this paper, and it is possible (although the author does not consider it likely) that the limit coming from these Gorshkov–Wirsing pairs is tight in these cases.

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[1]The precise pairs, as well as more complete tables (up to denominator 5) can be found at the author's home page (http://www.math.uwaterloo.ca/~kghare).

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