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Numerical Evidence Toward a 2-adic Equivariant "Main Conjecture"

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CONTENTS

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We test a conjectural nonabelian refinement of the classical 2-adic Main Conjecture of Iwasawa theory. In the first part, we show how, in the special case that we study, the validity of this refinement is equivalent to a congruence condition on the coefficients of some power series. Then, in the second part, we explain how to compute the first coefficients of this power series and thus numerically check the conjecture in that setting.

1. THE CONJECTURE

Let K be a totally real finite Galois extension of $\mathbb{Q}$ with Galois group G dihedral of order 8, and suppose that $\sqrt{2}$ is not in K. Fix a finite set S of primes of $\mathbb{Q}$ including 2, ∞, and all primes that ramify in K. Let C be the cyclic subgroup of G of order 4 and F the fixed field of C acting on K. Fix a 2-adic unit $u \equiv 5 \bmod 8\mathbb{Z}_2$.

Write $L_F(s,\chi)$ for the 2-adic L-functions, normalized as in [Wiles 90], of the 2-adic characters χ of C, or equivalently, by class field theory, of the corresponding 2-adic primitive ray class characters. We always work with their S-*truncated* forms

$$L_{F,S}(s,\chi) = L_F(s,\chi)\prod_{\mathfrak{p}}\left(1 - \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})}\langle N(\mathfrak{p})\rangle^{1-s}\right),$$

where $\mathfrak{p}$ runs through all primes of F above $S\backslash\{2,\infty\}$, and $\langle\ \rangle : \mathbb{Z}_2^\times \to 1+4\mathbb{Z}_2$ is the unique function with $\langle x\rangle x^{-1} \in \{-1,1\}$ for all x.

Now our interest is in the 2-adic function

$$f_1(s) = \frac{\rho_{F,S}\log(u)}{8(u^{1-s}-1)} + \frac{1}{8}\left(L_{F,S}(s,1) + L_{F,S}(s,\beta^2) - 2L_{F,S}(s,\beta)\right),$$

where β is a faithful irreducible 2-adic character of C and

$$\rho_{F,S} = \lim_{s\to 1}(s-1)L_{F,S}(s,1).$$

It follows from known results that $\frac{1}{2}\rho_{F,S} \in \mathbb{Z}_2$ and that $f_1(s)$ is an *Iwasawa analytic* function of $s \in \mathbb{Z}_2$, in the sense of [Ribet 79]. This means that there is a unique power series $F_1(T) \in \mathbb{Z}_2[[T]]$ such that

$$F_1(u^n - 1) = f_1(1-n) \quad \text{for} \quad n = 1, 2, 3, \dots .$$

We want to test the following conjecture:

Conjecture 1.1.

$$\frac{1}{2}\rho_{F,S} \in 4\mathbb{Z}_2 \quad \textit{and} \quad F_1(T) \in 4\mathbb{Z}_2[[T]].$$

Testing the conjecture amounts to calculating $\frac{1}{2}\rho_{F,S}$ and (many of) the power series coefficients of

$$F_1(T) = \sum_{j=1}^{\infty} x_j T^{j-1}$$

modulo $4\mathbb{Z}_2$. Were the conjecture false, we would expect to find a counterexample in this way.

The idea of the calculation is, roughly, to express the coefficients of the power series $F_1(T)$ as integrals over suitable 2-adic continuous functions with respect to the measures used to construct the 2-adic L-functions.

The conjecture has been tested for 60 fields K determined by the size of their discriminant and the splitting of 2 in the field F. For this purpose, it is convenient to replace the datum K by F together with the ray class characters of F that determine K (see Section 5). A description of the results appears in Section 6. They are affirmative.

Where does $f_1(s)$ come from? It is an example that arises from an attempt to refine the main conjecture of Iwasawa theory. This connection will be discussed next in order to prove that $F_1(T)$ is in $\mathbb{Z}_2[[T]]$.

2. THE MOTIVATION

The main conjecture of classical Iwasawa theory was proved by Wiles [Wiles 90] for odd prime numbers ℓ. More recently, an equivariant "main conjecture" has been proposed [Ritter and Weiss 04] that would both generalize and refine the classical one for the same ℓ. When a certain μ-invariant vanishes, as is expected for odd ℓ (by a conjecture of Iwasawa), this equivariant "main conjecture," up to its uniqueness assertion, depends only on properties of ℓ-adic L-functions, by [Ritter and Weiss 06, Theorem A].

The point is that it is possible to test numerically this Theorem A property of ℓ-adic L-functions, at least in simple special cases in which it may be expressed in terms of congruences and the special values of these L-functions can be computed. Conjecture 1.1 is perhaps the simplest nonabelian example in which this happens, but with the price of taking $\ell = 2$. Although there are some uncertainties about the formulation of the "main conjecture" for $\ell = 2$, partly because [Wiles 90] applies only in the cyclotomic case, it seems clearer what the 2-adic analogue of the Theorem A properties of L-functions should be, in view of their "extra" 2-power divisibilities [Deligne and Ribet 80].

More precisely, let

$$L_{k,S} \in \mathrm{Hom}^*\left(R_\ell(G_\infty), \mathcal{Q}^c(\Gamma_k)^\times\right)$$

be the "power series"-valued function of ℓ-adic characters χ of $G_\infty = \mathrm{Gal}(K_\infty/k)$ defined in [Ritter and Weiss 04, Section 4]. This is made from the values of ℓ-adic L-functions by viewing them as a quotient of Iwasawa analytic functions, by the proof of Proposition 11 in [Ritter and Weiss 04]. When $\ell \neq 2$, the vanishing of the μ-invariant mentioned above means precisely that the coefficients of these power series have no nontrivial common divisor; and the Theorem A property of L-functions is then that $L_{k,S}$ is in $\mathrm{Det}\,(K_1(\Lambda(G_\infty)_\bullet))$ (see the next section for precise definitions).

When $\ell = 2$, we can still form $L_{k,S}$, but now its values at characters χ of degree 1 have numerators divisible by $2^{[k:\mathbb{Q}]}$, because of [Ribet 79, (4.8), (4.9)]. Define

$$\widetilde{L}_{k,S}(\chi) = 2^{-[k:\mathbb{Q}]\chi(1)} L_{k,S}(\chi)$$

for all 2-adic characters χ of G_∞, so that the deflation and restriction properties of [Ritter and Weiss 04, Proposition 12] are maintained. Then the analogous coprimality condition on the coefficients of the "power series" values $\widetilde{L}_{k,S}(\chi)$, for all characters χ, will be referred to as vanishing of the $\widetilde{\mu}$-invariant of K_∞/k: The Theorem A property we want to test is therefore contained in the following conjecture:

Conjecture 2.1. $\widetilde{L}_{k,S}$ *is in* $\mathrm{Det}\left(K_1\left(\Lambda(G_\infty)_\bullet\right)\right)$.

Remark 2.2. (a) When the assertion of Conjecture 2.1 holds, then $\widetilde{L}_{k,S}(\chi)$ is in $\Lambda^c(\Gamma_k)^\times_\bullet$ for all $\chi \in R_2(G_\infty)$, implying the vanishing of the $\widetilde{\mu}$-invariant of K_∞/k.

(b) For $\ell \neq 2$, some cases of the equivariant "main conjecture" have recently been proved [Ritter and Weiss 08].

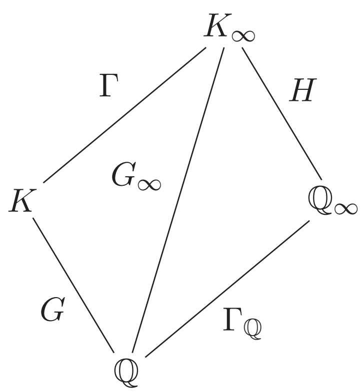

FIGURE 1. Lattice of field extensions considered.

3. INTERPRETING CONJECTURE 2.1 AS A CONGRUENCE

We now specialize to the situation of Section 1, so we use the notation of its first paragraph in order to exhibit a congruence equivalent to Conjecture 2.1 (see Figure 1).

Let $\mathbb{Q}_\infty$ be the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{Q}$, i.e., the maximal totally real subfield of the field obtained from $\mathbb{Q}$ by adjoining all 2-power roots of unity, and set $\Gamma_\mathbb{Q} = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \simeq \mathbb{Z}_2$. Let $K_\infty = K\mathbb{Q}_\infty$, noting that $K \cap \mathbb{Q}_\infty = \mathbb{Q}$ follows from $\sqrt{2} \notin K$, and set $G_\infty = \mathrm{Gal}(K_\infty/\mathbb{Q})$. Defining $\Gamma = \ker(G_\infty \to G)$, $H = \ker(G_\infty \to \Gamma_\mathbb{Q})$, we now have $H \hookrightarrow G_\infty \twoheadrightarrow \Gamma_\mathbb{Q}$, in the notation of [Ritter and Weiss 04].

Since $G_\infty = \Gamma \times H$ with $\Gamma \simeq \Gamma_\mathbb{Q}$ and $H \simeq G$ is dihedral of order 8, we can understand the structure of

$$\Lambda(G_\infty)_\bullet = \Lambda(\Gamma)_\bullet \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[H] = \Lambda(\Gamma)_\bullet[H],$$

where $\bullet$ means "invert all elements of $\Lambda(\Gamma)\backslash 2\Lambda(\Gamma)$."

Namely, choose σ, τ in G such that $C = \langle\tau\rangle$ with $\sigma^2 = 1$, $\sigma\tau\sigma^{-1} = \tau^{-1}$, and extend them to K_∞, with trivial action on $\mathbb{Q}_\infty$, to get s, t respectively. Then the abelianization of H is $H^{\mathrm{ab}} = H/\langle t^2\rangle$, and we get a pullback diagram

$$\begin{array}{ccc} \Lambda(G_\infty)_\bullet = \Lambda(\Gamma)_\bullet[H] & \longrightarrow & (\Lambda(\Gamma)_\bullet(\zeta_4)) * \langle s\rangle \\ \downarrow & & \downarrow \\ \Gamma(G_\infty^{\mathrm{ab}})_\bullet = \Lambda(\Gamma)_\bullet[H^{\mathrm{ab}}] & \longrightarrow & \Lambda(\Gamma)_\bullet[H^{\mathrm{ab}}]/2\Lambda(\Gamma)_\bullet[H^{\mathrm{ab}}] \end{array}$$

where the upper-right-hand term is the crossed product order with $\Lambda(\Gamma)_\bullet$-basis $1, \zeta_4, \widetilde{s}, \zeta_4\widetilde{s}$ with $\zeta_4^2 = -1$, $\widetilde{s}^{\,2} = 1$, $\widetilde{s}\zeta_4 = \zeta_4^{-1}\widetilde{s} = -\zeta_4\widetilde{s}$, and the top map takes t, s to $\zeta_4, \widetilde{s}$ respectively, while the right map takes $\zeta_4, \widetilde{s}$ to $t^{\mathrm{ab}}, s^{\mathrm{ab}}$. This diagram originates in the pullback diagram for the cyclic group $\langle t\rangle$ of order 4, then goes to the dihedral group ring $\mathbb{Z}_2[H]$ by incorporating the action of s, and finally ends by an application of $\Lambda(\Gamma)_\bullet \otimes_{\mathbb{Z}_2} -$.

We now turn to getting the first version of our congruence in terms of the pullback diagram above. This is possible, since $R^\times \to K_1(R)$ is surjective for all the rings considered there. We also simplify notation a little by setting $\mathfrak{A} = (\Lambda(\Gamma)_\bullet(\zeta_4)) * \langle s\rangle$ and writing $\widetilde{L}_{k,S}$ as $\widetilde{L}_{K_\infty/k}$, because we will now have to vary the fields and S is fixed anyway. The dihedral group G has four degree-1 irreducible characters $1, \eta, \nu, \eta\nu$, with $\eta(\tau) = 1$, $\nu(\sigma) = 1$, and a unique degree-2 irreducible α, which we view as characters of G_∞ by inflation.

Proposition 3.1. *Let K_∞^{ab} be the fixed field of $\langle t^2\rangle$; hence $\mathrm{Gal}(K_\infty^{\mathrm{ab}}/\mathbb{Q}) = G_\infty^{\mathrm{ab}}$. Then:*

(a) $\widetilde{L}_{K_\infty^{\mathrm{ab}}/\mathbb{Q}} = \mathrm{Det}(\widetilde{\Theta}^{\mathrm{ab}})$ *for some* $\widetilde{\Theta}^{\mathrm{ab}} \in \Lambda(G_\infty^{\mathrm{ab}})_\bullet^\times$*;*

(b) $\widetilde{L}_{K_\infty/\mathbb{Q}} \in \mathrm{Det}(K_1(\Lambda(G_\infty)_\bullet))$ *if and only if any* $y \in \mathfrak{A}$ *mapping to* $\widetilde{\Theta}^{\mathrm{ab}} \bmod 2$ *in* $\Lambda(G_\infty^{\mathrm{ab}})_\bullet/2\Lambda(G_\infty^{\mathrm{ab}})_\bullet$ *has*

$$\mathrm{nr}(y) \equiv \widetilde{L}_{K_\infty/\mathbb{Q}}(\alpha) \bmod 4\Lambda(\Gamma_\mathbb{Q})_\bullet.$$

Here nr *is the reduced norm of* (*the total ring of fractions of*) $\mathfrak{A}$ *to its center* $\Lambda(\Gamma)_\bullet$*, and we identify* $\Lambda(\Gamma)_\bullet$ *with* $\Lambda(\Gamma_\mathbb{Q})_\bullet$ *via* $\Gamma \xrightarrow{\simeq} \Gamma_\mathbb{Q}$*.*

Proof. (a) The vanishing of $\widetilde{\mu}$ for $K_\infty/\mathbb{Q}$, in the sense of Section 2, is known by [Ferrero and Washington 79], i.e., $\widetilde{L}_{K_\infty^{\mathrm{ab}}/\mathbb{Q}}(\chi)$ is a unit in $\Lambda(\Gamma_\mathbb{Q})_\bullet$ for all 2-adic characters χ of G_∞^{ab}. By the proof of Theorem 9 in [Ritter and Weiss 06], we have $L_{K_\infty^{\mathrm{ab}}/\mathbb{Q}} = \mathrm{Det}(\lambda)$, with $\lambda \in \Lambda(G_\infty^{\mathrm{ab}})_\bullet$ the pseudomeasure of Serre. The point is then that $\lambda = 2\widetilde{\Theta}^{\mathrm{ab}}$ with $\widetilde{\Theta}^{\mathrm{ab}} \in \Lambda(G_\infty^{\mathrm{ab}})_\bullet$, which follows from [Ribet 79, Theorem 3.1b], because of [Ribet 79, Theorem 4.1] and the relation between λ and μ_c discussed just after it. Then $\widetilde{L}_{K_\infty^{\mathrm{ab}}/\mathbb{Q}} = \mathrm{Det}(\widetilde{\Theta}^{\mathrm{ab}})$, and now the proof of the corollary to Theorem 9 in [Ritter and Weiss 06] shows that $\widetilde{\Theta}^{\mathrm{ab}}$ is a unit of $\Lambda(G_\infty^{\mathrm{ab}})$.

(b) We make the following claim.

Claim 3.2. $\mathrm{nr}(1 + 2\,\mathfrak{A}) = 1 + 4\Lambda(\Gamma)_\bullet$.

Proof of Claim 3.2. If $x = a1 + b\zeta_4 + c\widetilde{s} + d\zeta_4\widetilde{s}$ with $a, b, c, d \in \Lambda(\Gamma)_\bullet$, one computes $\mathrm{nr}(x) = (a^2 + b^2) - (c^2 + d^2)$, from which $\mathrm{nr}(1 + 2\,\mathfrak{A}) \subseteq 1 + 4\Lambda(\Gamma)_\bullet$; equality follows from

$$\mathrm{nr}\left((1 + 2a) + 2a\widetilde{s}\right) = (1 + 2a)^2 - (2a)^2 = 1 + 4a$$

for $a \in \Lambda(\Gamma)_\bullet$ $\square$

Suppose first that the congruence for $\widetilde{L}_{K_\infty/\mathbb{Q}}(\alpha)$ holds. Start with $\widetilde{\Theta}^{\mathrm{ab}}$ from (a) in the lower left corner of the pullback square and map it to $\widetilde{\Theta}^{\mathrm{ab}} \bmod 2$ in the lower right corner. Choosing any $y_0 \in \mathfrak{A}$ mapping to $\widetilde{\Theta}^{\mathrm{ab}} \bmod 2$, we note that $y_0 \in \mathfrak{A}^\times$, because the maps in the pullback diagram are ring homomorphisms and the kernel $2\mathfrak{A}$ of the right one is contained in the radical of $\mathfrak{A}$. Thus $\mathrm{nr}(y_0) \in \Lambda(\Gamma)_\bullet^\times$ has $\mathrm{nr}(y_0)^{-1}\widetilde{L}_{K_\infty/\mathbb{Q}}(\alpha) \in 1 + 4\Lambda(\Gamma_\mathbb{Q})_\bullet$ by the congruence, and hence, by the claim, $\mathrm{nr}(y_0)^{-1}\widetilde{L}_{K_\infty/\mathbb{Q}}(\alpha) = \mathrm{nr}(z)$, $z \in 1 + 2\mathfrak{A}$. So $y_1 = y_0 z$ is another lift of $\widetilde{\Theta}^{\mathrm{ab}} \bmod 2$ and $\mathrm{nr}(y_1) = \widetilde{L}_{K_\infty/\mathbb{Q}}(\alpha)$. By the pullback diagram we get $Y \in \Lambda(G_\infty)_\bullet^\times$, which maps to $\widetilde{\Theta}^{\mathrm{ab}}$ and y_1, where $\mathrm{nr}(y_1) = \widetilde{L}_{K_\infty/\mathbb{Q}}(\alpha)$.

It follows that $\mathrm{Det}\, Y = \widetilde{L}_{K_\infty/\mathbb{Q}}$. To see this, we check that their values agree at every irreducible character χ of G_∞; it even suffices to check it on the characters $1, \eta, \nu, \eta\nu, \alpha$ of G by [Ritter and Weiss 04, Theorem 8 and Proposition 11], because every irreducible character of G_∞ is obtained from these by multiplying by a character of type W. It works for the characters $1, \eta, \nu, \eta\nu$ of G_∞^{ab} by [Ritter and Weiss 04, Proposition 12(1b)], since the deflation of Y equals $\widetilde{\Theta}^{\mathrm{ab}}$, and $\mathrm{Det}\,\widetilde{\Theta}^{\mathrm{ab}} = \widetilde{L}_{K_\infty^{\mathrm{ab}}/\mathbb{Q}}$ by (a). Finally, $(\mathrm{Det}\, Y)(\alpha) = j_\alpha\big(\mathrm{nr}(Y)\big) = \mathrm{nr}(y_1) = \widetilde{L}_{K_\infty/\mathbb{Q}}(\alpha)$ by the commutative triangle before [Ritter and Weiss 04, Theorem 8], the definition of j_α, and $G_\infty = \Gamma \times H$.

The converse depends on related ingredients. More precisely, $\widetilde{L}_{K_\infty/\mathbb{Q}} \in \mathrm{Det}\, K_1\big((\Lambda G_\infty)_\bullet\big)$ implies $\widetilde{L}_{K_\infty/\mathbb{Q}} = \mathrm{Det}\, Y$ with $Y \in (\Lambda G_\infty)_\bullet^\times$ by surjectivity of $(\Lambda G_\infty)_\bullet^\times \to K_1\big((\Lambda G_\infty)_\bullet\big)$. Since $(\Lambda G_\infty^{\mathrm{ab}})_\bullet^\times \to K_1\big((\Lambda G_\infty^{\mathrm{ab}})_\bullet\big)$ is an isomorphism, we get that the deflation of Y equals $\widetilde{\Theta}^{\mathrm{ab}}$ in $\Lambda(G_\infty^{\mathrm{ab}})^\times$. Letting $y_1 \in \mathfrak{A}^\times$ be the image of Y in the pullback diagram, it follows that $\mathrm{nr}(y_1) = \widetilde{L}_{K_\infty/\mathbb{Q}}(\alpha)$ and that y_1 maps to $\widetilde{\Theta}^{\mathrm{ab}} \bmod 2$ in $\Lambda(G_\infty^{\mathrm{ab}})_\bullet/2\Lambda(G_\infty^{\mathrm{ab}})_\bullet$. Given any y as in (b), then $y_1^{-1}y$ maps to 1, hence is in $1 + 2\mathfrak{A}$, and our congruence follows from the claim on applying nr. $\square$

4. REWRITING THE CONGRUENCE IN TESTABLE FORM

Set $F_0 = \dfrac{\widetilde{L}_{K_\infty/F,S}(1) + \widetilde{L}_{K_\infty/F,S}(\beta^2)}{2} - \widetilde{L}_{K_\infty/F,S}(\beta)$.

Proposition 4.1.

(a) F_0 *is in* $\Lambda(\Gamma_\mathbb{Q})_\bullet$;

(b) $\widetilde{L}_{K_\infty/\mathbb{Q}} \in \mathrm{Det}\, K_1\big(\Lambda(G_\infty)_\bullet\big)$ *if and only if* $F_0 \in 4\Lambda(\Gamma_\mathbb{Q})_\bullet$.

Proof. Note that $\mathrm{ind}_C^G 1_C = 1_G + \eta$, $\mathrm{ind}_C^G \beta^2 = \nu + \eta\nu$, $\mathrm{ind}_C^G \beta = \alpha$. When we inflate β to a character of $\mathrm{Gal}(K_\infty/F)$, then $\mathrm{ind}_{\mathrm{Gal}(K_\infty/F)}^{G_\infty} \beta = \alpha$ with α inflated to G_∞, etc.

By Proposition 3.1, we can write $\widetilde{L}_{K_\infty^{\mathrm{ab}}/\mathbb{Q}} = \mathrm{Det}(\widetilde{\Theta}^{\mathrm{ab}})$ with

$$\widetilde{\Theta}^{\mathrm{ab}} = a + bt^{\mathrm{ab}} + cs^{\mathrm{ab}} + ds^{\mathrm{ab}}t^{\mathrm{ab}}$$

for some a, b, c, d in $\Lambda(\Gamma)_\bullet$. It follows that

$$\begin{aligned}
\widetilde{L}_{K_\infty/\mathbb{Q}}(1) &= a + b + c + d,\\
\widetilde{L}_{K_\infty/\mathbb{Q}}(\eta) &= a + b - c - d,\\
\widetilde{L}_{K_\infty/\mathbb{Q}}(\nu) &= a - b + c - d,\\
\widetilde{L}_{K_\infty/\mathbb{Q}}(\eta\nu) &= a - b - c + d.
\end{aligned}$$

Form $y = a + b\zeta_4 + c\widetilde{s} + d\zeta_4\widetilde{s}$ in $\big(\Lambda(\Gamma)_\bullet(\zeta_4)\big) * \langle s\rangle$. By the computation in Claim 3.2, we have

$$\begin{aligned}
\mathrm{nr}(y) &= (a+c)(a-c) + (b+d)(b-d)\\
&= \frac{\widetilde{L}_\mathbb{Q}(1) + \widetilde{L}_\mathbb{Q}(\nu)}{2}\,\frac{\widetilde{L}_\mathbb{Q}(\eta) + \widetilde{L}_\mathbb{Q}(\eta\nu)}{2}\\
&\quad + \frac{\widetilde{L}_\mathbb{Q}(1) - \widetilde{L}_\mathbb{Q}(\nu)}{2}\,\frac{\widetilde{L}_\mathbb{Q}(\eta) - \widetilde{L}_\mathbb{Q}(\eta\nu)}{2}\\
&= \frac{1}{4}\big(\widetilde{L}_\mathbb{Q}(1+\eta) + \widetilde{L}_\mathbb{Q}(1+\eta\nu) + \widetilde{L}_\mathbb{Q}(\nu+\eta)\\
&\qquad + \widetilde{L}_\mathbb{Q}(\nu+\eta\nu)\big)\\
&\quad + \frac{1}{4}\big(\widetilde{L}_\mathbb{Q}(1+\eta) - \widetilde{L}_\mathbb{Q}(1+\eta\nu) - \widetilde{L}_\mathbb{Q}(\nu+\eta)\\
&\qquad + \widetilde{L}_\mathbb{Q}(\nu+\eta\nu)\big)\\
&= \frac{\widetilde{L}_\mathbb{Q}(1+\eta) + \widetilde{L}_\mathbb{Q}(\nu+\eta\nu)}{2} = \frac{\widetilde{L}_F(1) + \widetilde{L}_F(\beta^2)}{2},
\end{aligned}$$

because

$$\widetilde{L}_{K_\infty/\mathbb{Q}}\left(\mathrm{ind}_{\mathrm{Gal}(K_\infty/F)}^{G_\infty} \chi\right) = \widetilde{L}_{K_\infty/F}(\chi)$$

for all characters χ of $\mathrm{Gal}(K_\infty/F)$. Thus also $\widetilde{L}_{K_\infty/\mathbb{Q}}(\alpha) = \widetilde{L}_{K_\infty/F}(\beta)$, so we have now shown that

$$F_0 = \mathrm{nr}(y) - \widetilde{L}_{K_\infty/\mathbb{Q}}(\alpha),$$

proving (a), since $\widetilde{L}_{K_\infty/F}(\beta) \in (\Lambda\Gamma_F)_\bullet$ by Section 2, since β has degree 1.

Moreover, the image of y under the right arrow of the pullback diagram of Section 3 equals $\widetilde{\Theta}^{\mathrm{ab}} \bmod 2$, by construction; hence (b) follows directly from Proposition 3.1(b). $\square$

Remark 4.2. Considering F_0 in $\Lambda(\Gamma_\mathbb{Q})_\bullet$, instead of its natural home $\Lambda(\Gamma_F)_\bullet$, is done to be consistent with the identification in (b) of Proposition 3.1, via the natural isomorphisms $\Gamma \to \Gamma_F \to \Gamma_\mathbb{Q}$: this is the sense in which $L_{K_\infty/\mathbb{Q}}(\alpha) = L_{K_\infty/F}(\beta)$.

The congruence $F_0 \equiv 0 \bmod 4\Lambda(\Gamma_{\mathbb{Q}})_\bullet$ can now be put in the more testable form of Conjecture 1.1. Let $\gamma_{\mathbb{Q}}$ be the generator of $\Gamma_{\mathbb{Q}}$ that when extended to $\mathbb{Q}(\sqrt{-1})$ as the identity acts on all 2-power roots of unity in $\mathbb{Q}_\infty(\sqrt{-1})$ by raising them to the uth power, where $u \equiv 5 \bmod 8\mathbb{Z}_2$ as fixed before. Then the Iwasawa isomorphism $\Lambda(\Gamma_{\mathbb{Q}}) \simeq \mathbb{Z}_2\left[[T]\right]$, under which $\gamma_{\mathbb{Q}} - 1$ corresponds to T, makes $F_0 \in \Lambda(\Gamma_{\mathbb{Q}})_\bullet$ correspond to some $F_0(T) \in \mathbb{Z}_1\left[[T]\right]_\bullet$ and the congruence of Proposition 4.1(b) to

$$F_0(T) \equiv 0 \bmod 4\mathbb{Z}_2\left[[T]\right]_\bullet.$$

Since β is an abelian character, we know that $\widetilde{L}_{F,S}(\beta^2)$, $\widetilde{L}_{F,S}(\beta)$ correspond to elements of $\mathbb{Z}_2\left[[T]\right]$, not just $\mathbb{Z}_2\left[[T]\right]_\bullet$ (cf. [Ritter and Weiss 04, Section 4]), and $\widetilde{L}_F(1)$ to one of $T^{-1}\mathbb{Z}_2\left[[T]\right]$. We thus have

$$F_0(T) = \frac{x_0}{T} + \sum_{j=1}^{\infty} x_j T^{j-1}$$

with $x_j \in \mathbb{Z}_2$ for all $j \geq 0$.

By the interpolation definition of $\left(\widetilde{L}_{F,S}(\beta^i)\right)(T)$ (cf [Ribet 79, Section 4]), it follows that

$$F_0(u^s - 1) = \frac{1}{2}\Big(\frac{L_{F,S}(1-s,1)}{4} + \frac{L_{F,S}(1-s,\beta^2)}{4} - 2\frac{L_{F,S}(1-s,\beta)}{4}\Big).$$

We abbreviate the right-hand side of the equality as $f_0(1-s)$. This implies

$$x_0 = -\frac{\rho_{F,S}\log(u)}{8},$$

because the left side is

$$\begin{aligned}\lim_{T\to 0} T F_0(T) &= \lim_{s\to 1}\frac{u^{1-s}-1}{s-1}(s-1)f_0(s)\\ &= -\log(u)\lim_{s\to 1}(s-1)\frac{L_{F,S}(s,1)}{8},\end{aligned}$$

as required. Note that $u \equiv 5 \bmod 8$ implies that $\log(u)/4$ is a 2-adic unit; hence $\rho_{F,S}/2 \in \mathbb{Z}_2$ is in $4\mathbb{Z}_2$ if and only if $x_0 \in 4\mathbb{Z}_2$. Define

$$F_1(T) = F_0(T) - x_0 T^{-1} = \sum_{j=1}^{\infty} x_j T^{j-1} \in \mathbb{Z}_2\left[[T]\right].$$

It follows that

$$\begin{aligned}F_1(u^s - 1) &= -\frac{x_0}{u^s - 1} + F_0(u^s - 1)\\ &= \frac{\rho_{F,S}\log(u)}{8(u^s-1)} + f_0(1-s),\end{aligned}$$

which is $f_1(1-s)$, with f_1 as in Section 1; hence our present $F_1(T)$ is also the same as in Section 1. Thus Conjecture 1.1 is equivalent to Conjecture 2.1 for the special case $K_\infty/\mathbb{Q}$ of Section 1.

5. TESTING CONJECTURE 1.1

Let χ be a 2-adic character of the Galois group C of K/F and let $\mathfrak{f}$ be the conductor of K/F. By class field theory, we view χ as a map on the group of ideals relatively prime to $\mathfrak{f}$. Fix a prime ideal $\mathfrak{c}$ not dividing $\mathfrak{f}$. For $\mathfrak{a}$, a fractional ideal relatively prime to $\mathfrak{c}$ and $\mathfrak{f}$, let $\mathcal{Z}_{\mathfrak{f}}(\mathfrak{a},\mathfrak{c};s)$ denote the associated 2-adic twisted partial zeta function [Cassou-Noguès 79]. Thus, we have

$$\begin{aligned}L_{F,S}(s,\chi) = &\frac{1}{\chi(\mathfrak{c})\langle N\mathfrak{c}\rangle^{1-s} - 1}\prod_{\mathfrak{p}}\left(1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}}\langle N\mathfrak{p}\rangle^{1-s}\right)\\ &\times \sum_{\sigma\in G}\chi(\sigma)^{-1}\mathcal{Z}_{\mathfrak{f}}(\mathfrak{a}_\sigma^{-1},\mathfrak{c};s),\end{aligned}$$

where $\mathfrak{p}$ runs through the prime ideals of F in S not dividing $2\,\mathfrak{f}$, and $\mathfrak{a}_\sigma$ is a (fixed) integral ideal coprime to $2\,\mathfrak{f}\mathfrak{c}$ whose Artin symbol is σ.

Denote the ring of integers of F by $\mathcal{O}_F$ and let $\gamma \in \mathcal{O}_F$ be such that $\mathcal{O}_F = \mathbb{Z} + \gamma\mathbb{Z}$. In [Roblot 11] (see also [Besser et al. 09] for a slightly different presentation), it is shown that the function $\mathcal{Z}_{\mathfrak{f}}(\mathfrak{a},\mathfrak{c};s)$ is defined by the following integral:

$$\mathcal{Z}_{\mathfrak{f}}(\mathfrak{a},\mathfrak{c};s) = \int \frac{\langle N\mathfrak{a}N(x_1 + x_2\gamma)\rangle^{1-s}}{N\mathfrak{a}N(x_1 + x_2\gamma)}\,d\mu_{\mathfrak{a}}(x_1,x_2),$$

where the domain of integration is $\mathbb{Z}_2^2$, $\langle\;\rangle$ is extended to $\mathbb{Z}_2$ by $\langle x\rangle = 0$ if $x \in 2\mathbb{Z}_2$, and the measure $\mu_{\mathfrak{a}}$ is a measure of norm 1 (depending also on γ, $\mathfrak{f}$, and $\mathfrak{c}$).

Assume now, as we may without loss of generality, that the ideal $\mathfrak{c}$ is such that $\langle N\mathfrak{c}\rangle \equiv 5 \pmod{8\mathbb{Z}_2}$ and take $u = \langle N\mathfrak{c}\rangle$. For $s \in \mathbb{Z}_2$, we let $t = t(s) = u^s - 1 \in 4\mathbb{Z}_2$, so that $s = \log(1+t)/\log(u)$. For $x \in \mathbb{Z}_2^\times$, one can check readily that

$$\langle x\rangle^s = \left(u^{\mathcal{L}(x)}\right)^s = (1 + u^s - 1)^{\mathcal{L}(x)} = \sum_{n\geq 0}\binom{\mathcal{L}(x)}{n}t^n,$$

where $\mathcal{L}(x) = \log\langle x\rangle/\log u \in \mathbb{Z}_2$. For $x \in \mathbb{Z}_2^\times$, we set

$$L(x;T) = \sum_{n\geq 0}\binom{\mathcal{L}(x)}{n}T^n \in \mathbb{Z}_2[[T]]$$

and $L(x;T)=0$ if $x\in 2\mathbb{Z}_2$. Now we define

$$R(\mathfrak{a},\mathfrak{c};T)=\int \frac{L\big(N\mathfrak{a}\,N(x_1+x_2\gamma);T\big)}{N\mathfrak{a}\,N(x_1+x_2\gamma)}\,d\mu_{\mathfrak{a}}(x_1,x_2)\in \mathbb{Z}_2[[T]],$$

$$B(\chi;T)=\chi(\mathfrak{c})(T+1)-1\in\mathbb{Z}_2[\chi][T],$$

$$A(\chi;T)=\prod_{\mathfrak{p}}\left(1-\frac{\chi(\mathfrak{p})}{N\mathfrak{p}}L(N\mathfrak{p};T)\right)\times\sum_{\sigma\in G}\chi(\sigma)^{-1}R(\mathfrak{a}_\sigma^{-1},\mathfrak{c};T)\in\mathbb{Z}_2[\chi][[T]],$$

where $\mathfrak{p}$ runs through the prime ideals of F in S not dividing $2\mathfrak{f}$.

Proposition 5.1. *We have, for all $s\in\mathbb{Z}_2$,*

$$L_{F,S}(1-s,\chi)=\frac{A(\chi;u^s-1)}{B(\chi;u^s-1)}.$$

We now specialize to our situation. For that, we need to make the additional assumption that $\beta^2(\mathfrak{c})=-1$, so $\beta(\mathfrak{c})$ is a fourth root of unity in $\mathbb{Q}_2^c$, which we will denote by i. Thus, we have

$$B(1;T)=T,\quad B(\beta;T)=i(T+1)-1,$$
$$B(\beta^2;T)=-T-2,\quad B(\beta^3;T)=-i(T+1)-1.$$

Let $x\mapsto\bar{x}$ be the $\mathbb{Q}_2$-automorphism of $\mathbb{Q}_2(i)$ sending i to $-i$. Then we have $\overline{L_{F,S}(1-s,\beta)}=L_{F,S}(1-s,\beta^3)$ by the expression of $L_{F,S}(s,\chi)$ given at the beginning of the section, since the twisted partial zeta functions have values in $\mathbb{Q}_2$ and $\bar{\beta}=\beta^3$. And furthermore,

$$L_{F,S}(s,\beta^3)=L_{\mathbb{Q},S}(s,\mathrm{Ind}_C^G(\beta^3))=L_{\mathbb{Q},S}(s,\mathrm{Ind}_C^G(\beta))=L_{F,S}(s,\beta).$$

Therefore, by Proposition 5.1, we deduce that

$$\begin{aligned}A(\beta;u^s-1)+\bar{A}(\beta;u^s-1)&=\big(B(\beta;T)+B(\beta^3;T)\big)L_{F,S}(1-s,\beta)\\&=-2L_{F,S}(1-s,\beta).\end{aligned}$$

Since

$$f_1(s)=\frac{\rho_{F,S}\log u}{8(u^{1-s}-1)}+\frac{1}{8}\left(L_{F,S}(s,1)+L_{F,S}(s,\beta^2)-2L_{F,S}(s,\beta)\right),$$

we find that

$$F_1(T)=\frac{\rho_{F,S}\log u}{8T}+\frac{1}{8}\left(\frac{A(1;T)}{T}-\frac{A(\beta^2;T)}{T+2}+A(\beta;T)+\bar{A}(\beta,T)\right)$$

is such that $F_1(u^n-1)=f_1(1-n)$ for $n=1,2,3,\ldots$.

The conjecture that we wish to check states that

$$\frac{1}{2}\rho_{F,S}\in 4\mathbb{Z}_2\quad\text{and}\quad F_1(T)\in 4\mathbb{Z}_2[[T]].$$

Now define $D(T)=8T(T+2)F_1(T)$, so that

$$\begin{aligned}D(T)=&(T+2)\left(\rho_{F,S}\log u+A(1;T)\right)\\&-TA(\beta^2;T)+T(T+2)\left(A(\beta;T)+\bar{A}(\beta,T)\right).\end{aligned}$$

We can now give a final reformulation of the conjecture that is the one that we actually tested.

Conjecture 5.2.

$$\rho_{F,S}\in 8\mathbb{Z}_2\quad\textit{and}\quad D(T)\in 32\mathbb{Z}_2[[T]].$$

The computation of $\rho_{F,S}$ is done using the following formula [Colmez 88]:

$$\rho_{F,S}=2h_F R_F d_F^{-1/2}\prod_{\mathfrak{p}}\left(1-1/N(\mathfrak{p})\right),$$

where h_F, R_F, d_F are respectively the class number and 2-adic regulator and discriminant of F, and $\mathfrak{p}$ runs through all primes of F above 2. Note that although R_F and $d_F^{-1/2}$ are defined only up to sign, the quantity $R_F d_F^{-1/2}$ is uniquely determined in the following way: Let ι be the embedding of F into $\mathbb{R}$ for which $\sqrt{d_F}$ is positive and let ε be the fundamental unit of F such that $\iota(\varepsilon)>1$. Then for any embedding g of F into $\mathbb{Q}_2^c$, we have

$$R_F d_F^{-1/2}=\frac{\log_2 g(\varepsilon)}{g(\sqrt{d})}.$$

Now for the computation of $D(T)$, the only difficult part is the computations of the $R(\mathfrak{a},\mathfrak{c};T)$. The measures $\mu_{\mathfrak{a}}$ are computed explicitly using the methods of [Roblot 11] (see also [Besser et al. 09]), that is, we construct a power series $M_{\mathfrak{a}}(X_1,X_2)$ in $\mathbb{Q}_2[X_1,X_2]$ with integral coefficients, such that

$$\int(1+t_1)^{x_1}(1+t_2)^{x_2}\,d\mu_{\mathfrak{A}}(x_1,x_2)=M_{\mathfrak{A}}(t_1,t_2)$$

for all $t_1,t_2\in 2\mathbb{Z}_2$. In particular, if f is a continuous function on $\mathbb{Z}_2^2$ with values in $\mathbb{C}_2$ and Mahler expansion

$$f(x_1,x_2)=\sum_{n_1,n_2\geq 0}f_{n_1,n_2}\binom{x_1}{n_1}\binom{x_2}{n_2},$$

then we have

$$\int f(x_1,x_2)\,d\mu_{\mathfrak{A}}(x_1,x_2)=\sum_{n_1,n_2\geq 0}f_{n_1,n_2}m_{n_1,n_2},$$

where $M_{\mathfrak{A}}(X_1,X_2)=\sum_{n_1,n_2\geq 0}m_{n_1,n_2}X_1^{n_1}X_2^{n_2}$.

We compute in this way the first few coefficients of the power series $A(\chi;T)$, for $\chi=\beta^j$, $j=0,1,2,3$, and then deduce the first coefficients of $D(T)$ to see whether they

2 ramified in F			2 inert in F			2 split in F		
d_F	$\mathfrak{f}$	d_K	d_F	$\mathfrak{f}$	d_K	d_F	$\mathfrak{f}$	d_K
44	3	2 732 361 984	445	1	39 213 900 625	145	1	442 050 625
156	2	9 475 854 336	5	21	53 603 825 625	41	5	44 152 515 625
220	2	37 480 960 000	205	3	143 054 150 625	505	1	65 037 750 625
12	14	39 033 114 624	221	3	193 220 905 761	689	1	225 360 027 841
156	4	151 613 669 376	61	5	216 341 265 625	777	1	364 488 705 441
380	2	333 621 760 000	205	4	452 121 760 000	793	1	395 451 064 801
152	3	389 136 420 864	221	4	610 673 479 936	17	13	403 139 914 489
24	11	587 761 422 336	901	1	659 020 863 601	897	1	647 395 642 881
876	1	588 865 925 376	29	15	895 152 515 625	905	1	670 801 950 625
220	4	599 695 360 000	1 045	1	1 192 518 600 625	305	3	700 945 700 625
444	2	621 801 639 936	5	16	1 911 029 760 000	377	3	1 636 252 863 921
12	28	624 529 833 984	109	5	2 205 596 265 625	1 145	1	1 718 786 550 625
44	12	699 484 667 904	1 221	1	2 222 606 887 281	145	8	1 810 639 360 000
92	6	835 600 748 544	29	20	2 829 124 000 000	305	4	2 215 334 560 000
60	8	849 346 560 000	29	13	3 413 910 296 329	1 313	1	2 972 069 112 961
44	10	937 024 000 000	205	7	4 240 407 600 625	377	4	5 171 367 076 096
12	19	975 543 388 416	149	5	7 701 318 765 625	545	3	7 146 131 900 625
12	26	1 601 419 382 784	1 677	1	7 909 194 404 241	17	21	7 163 272 192 041
44	15	1 707 726 240 000	21	19	9 149 529 982 761	1 705	1	8 450 794 350 625
1 164	1	1 835 743 170 816	341	3	9 857 006 530 569	329	3	8 541 047 165 049

TABLE 1. Examples tested.

indeed belong to $32\,\mathbb{Z}_2[[T]]$. We found that this was indeed always the case; see the next section for more details.

To conclude this section, we remark that in fact, we do not need the above formula to compute $\rho_{F,S}$, since the constant coefficient of $A(1;T)$ is $-\rho_{F,S}\log u$. (This can be seen directly from the expression of x_0 given at the end of Section 4 or using the fact that $D(T)$ has zero constant coefficient, since $F_1(T) \in \mathbb{Z}_2[[T]]$.) However, we did compute it using this formula, since it then provides a neat way to check that (at least one coefficient of) $A(1;T)$ is correct.

6. THE NUMERICAL VERIFICATIONS

We have tested the conjecture on 60 examples. The examples are separated into three subcases of 20 examples each according to the way 2 decomposes in the quadratic subfield F: ramified, split, or inert. In each subcase, the examples are actually the first 20 extensions $K/\mathbb{Q}$ of the suitable form of the smallest discriminant. These are given in Table 1, where the entries are the discriminant d_F of F, the conductor $\mathfrak{f}$ of K/F (which is always a rational integer), and the discriminant d_K of K. In each example, we have computed $\rho_{F,S}$ and the first 30 coefficients of $D(T)$ to a precision of at least 2^8 and checked that they satisfy the conjecture.

We now give an example, namely the smallest example for the discriminant of K. We have $F = \mathbb{Q}(\sqrt{145})$ and that K is the Hilbert class field of F. The prime 2 is split in $F/\mathbb{Q}$, and the primes above 2 in F are inert in K/F. We compute $\rho_{F,S}$ and find that

$$\rho_{F,S} \equiv 2^7 \pmod{2^8}.$$

Using the method of the previous section, we compute the first 30 coefficients of the power series $A(\,\cdot\,;T)$ to a 2-adic precision of 2^8. We get

$$\begin{aligned}
&A(1;T)\\
&\quad\equiv 2^2\big(16T + 57T^3 + 44T^4 + 8T^5 + 40T^6 + 21T^7\\
&\qquad + 40T^8 + 30T^9 + 16T^{10} + 49T^{11} + 56T^{12} + 29T^{13}\\
&\qquad + 32T^{14} + 50T^{15} + 62T^{16} + 47T^{17} + 48T^{18}\\
&\qquad + 60T^{19} + 32T^{20} + 16T^{21} + 8T^{22} + 21T^{23} + 30T^{24}\\
&\qquad + 26T^{25} + 2T^{26} + 9T^{27} + 56T^{28} + 34T^{29}\big)\\
&\qquad + O(T^{30}) \pmod{2^8},\\
&A(\beta;T)\\
&\quad\equiv 2^2\big((28 + 1124i) + (36 + 1728i)T + (47 + 45i)T^2\\
&\qquad + (56 + 153i)T^3 + (46 + 154i)T^4 + (56 + 282i)T^5
\end{aligned}$$

$$
\begin{aligned}
&+ (55+433i)T^6 + (54+435i)T^7 + (40+386i)T^8 \\
&+ (48+392i)T^9 + (63+65i)T^{10} + (48+257i)T^{11} \\
&+ (63+161i)T^{12} + (20+477i)T^{13} \\
&+ (38+182i)T^{14} + (56+66i)T^{15} + (37+35i)T^{16} \\
&+ (6+341i)T^{17} + (20+446i)T^{18} + (40+412i)T^{19} \\
&+ 368iT^{20} + (56+336i)T^{21} + (61+291i)T^{22} \\
&+ (40+427i)T^{23} + (34+38i)T^{24} + (48+94i)T^{25} \\
&+ (9+47i)T^{26} + (6+497i)T^{27} + (40+42i)T^{28} \\
&+ (44+52i)T^{29}\big) + O(T^{30}) \pmod{2^8},
\end{aligned}
$$

$$
\begin{aligned}
&A(\beta^2;T) \\
&\quad \equiv 2^2\big(32 + 32T + 22T^2 + 39T^3 + 36T^4 + 20T^5 \\
&\qquad + 62T^6 + 27T^7 + 16T^8 + 62T^9 + 46T^{10} + 23T^{11} \\
&\qquad + 30T^{12} + 51T^{13} + 4T^{14} + 2T^{15} + 56T^{16} + 33T^{17} \\
&\qquad + 44T^{18} + 12T^{19} + 40T^{20} + 8T^{21} + 54T^{22} + 11T^{23} \\
&\qquad + 34T^{24} + 42T^{25} + 43T^{27} + 56T^{28} + 46T^{29}\big) \\
&\qquad + O(T^{30}) \pmod{2^8}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&D(T) \\
&\quad \equiv 2^5\big(6T + 7T^2 + 4T^3 + 5T^4 + 4T^7 + 2T^8 + 4T^9 \\
&\qquad + 2T^{10} + 4T^{11} + T^{12} + 6T^{13} + 7T^{14} + 3T^{16} \\
&\qquad + 5T^{17} + 2T^{18} + 3T^{19} + 7T^{20} + 5T^{21} + 7T^{22} \\
&\qquad + 4T^{23} + 4T^{24} + T^{25} + 7T^{26} + 3T^{27} + 7T^{28} \\
&\qquad + 6T^{29}\big) + O(T^{30}) \pmod{2^8},
\end{aligned}
$$

and the conjecture is satisfied by the first 30 coefficients of the series D associated with the extension.

Note, as a final remark, that we have tested the conjecture in the same way for 30 additional examples in which F is real quadratic, K/F is cyclic of order 4, but K is not a dihedral extension of $\mathbb{Q}$ (either $K/\mathbb{Q}$ is not Galois or its Galois group is not the dihedral group of order 8). In all of these examples, we found that the conjecture was not satisfied, that is, either $\rho_{F,S}$ did not belong to $8\mathbb{Z}_2$ or one of the first 30 coefficients of the associated power series D did not belong to $32\mathbb{Z}_2$.

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