

# Spot-Based Generations for Meta-Fibonacci Sequences

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For many meta-Fibonacci sequences it is possible to identify a partition of the sequence into successive intervals (sometimes called blocks) with the property that the sequence behaves “similarly” in each block. This partition provides insights into the sequence properties. To date, for any given sequence, only ad hoc methods have been available to identify this partition. We apply a new concept—the spot-based generation sequence—to derive a general methodology for identifying this partition for a large class of meta-Fibonacci sequences. This class includes the Conolly and Conway sequences and many of their well-behaved variants, and even some highly chaotic sequences, such as Hofstadter’s famous  $Q$ -sequence.

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## 1. INTRODUCTION

In this paper we explore certain properties of the solutions to the recurrence relations in two very general families, both of which have received increasing attention of late (see, for example, [Grytczuk 04, Isgur et al. 09, Ruskey and Deugau 09] and the references cited therein). The first of these recurrence families is defined as follows: for a positive integer  $k > 1$  and nonnegative integer parameters  $a_p, b_p, p = 1, \dots, k$ ,

$$C(n) = \sum_{p=1}^k C(n - a_p - C(n - b_p)). \quad (1-1)$$

We often abbreviate a recurrence in this family by  $(a_1, b_1, a_2, b_2, \dots, a_k, b_k)$ . The second family of recurrence relations is defined by

$$A(n) = A(n - A^k(n - 1)) + A(A^k(n - 1)), \quad (1-2)$$

where  $k > 0$  and  $A^k(n)$  means a  $k$ -fold composition of the function  $A$ . For convenience, in our notation for both recurrence families we suppress the parameter  $k$  from the variable name. It will be evident from the context when a specific value of  $k$  is intended.

The recurrence relations (1-1) and (1-2) are examples of meta-Fibonacci recurrences, which refers to the

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“self-referencing” nature of these recurrence relations. An integer sequence is a meta-Fibonacci sequence if it is a solution to a meta-Fibonacci recurrence. Many well-known meta-Fibonacci recurrences, with specified initial conditions, are special cases of the above two recurrence families. Examples of (1–1) include Hofstadter’s  $Q$ -recurrence  $(0, 1, 0, 2)$  [Hofstadter 79, Guy 04], the Conolly recurrence  $(0, 1, 1, 2)$  [Conolly 89, Tanny 92], and the celebrated  $V$ -recurrence  $(0, 1, 0, 4)$  [Balamohan et al. 07]. Two special cases of (1–2) have been examined in detail. For  $k = 1$  this is the meta-Fibonacci recurrence variously attributed to Conway, Hofstadter, and Newman (see [Kubo and Vakil 96, Mallows 91, Newman 88] for more on this), while the case  $k = 2$  is explored in [Grytczuk 04].

For the last four of these examples (that is, excluding the  $Q$ -recurrence), the solution with initial conditions all set to 1 is completely understood. In particular, each of the resulting sequences is monotonically increasing, with the difference between successive terms always 0 or 1. Following Ruskey, we call such a sequence slow-growing, or slow. For each of these meta-Fibonacci sequences, and indeed for many others (including the  $Q$ -sequence), it is possible to identify a partition of the domain of the sequence into successive intervals (sometimes called blocks) with the property that the sequence behaves roughly “in the same way” in each block. See, for example, [Conolly 89, Mallows 91, Tanny 92, Balamohan et al. 07], where the nature of the block structure has been characterized precisely for the slow-growing sequences mentioned above.

In each case, the approach to identifying this partition and what is meant precisely by behaves “in the same way” varies from one sequence to the next; in all cases, however, the basic idea is that there appears to be a discernible pattern in the behavior of the sequence that repeats in successive blocks. This property can also be found in meta-Fibonacci sequences with much more chaotic behavior; in [Pinn 99], the author provides considerable experimental evidence for the existence of an underlying block structure in Hofstadter’s  $Q$ -sequence.

In this paper we introduce an approach that formalizes and unifies this heuristic notion of an underlying block structure for a meta-Fibonacci sequence that is a solution to recurrence (1–1) or (1–2). In so doing we explicitly connect the block structure to the form of the recurrence and its parameters in an intuitive way. As a result, for an arbitrary sequence defined by these recurrences, we identify a partition that often appears to highlight important properties of the sequence for further consideration. Such insight into the apparent block structure of a

yet unknown sequence can provide helpful guideposts for developing conjectures and proofs.<sup>1</sup>

## 2. SPOT-BASED GENERATIONS

We define a homogeneous meta-Fibonacci recurrence to be any recurrence of the form

$$T(n) = \sum_{p=1}^k T(S_p(n, T_{<n})). \quad (2-1)$$

We refer to the function  $S_p(n, T_{<n})$  as the  $p$ th spot function; it depends on the index  $n$  and values of  $T(j)$  for  $j < n$ , which we indicate by the symbol  $T_{<n}$ . In the homogeneous recurrence (1–1), the spot functions are  $S_p(n, C_{<n}) = n - a_p - C(n - b_p)$  for  $1 \leq p \leq k$ . In the homogeneous recurrence (1–2), there are two spot functions, namely  $S_1(n, A_{<n}) = n - A^k(n - 1)$  and  $S_2(n, A_{<n}) = A^k(n - 1)$ . For convenience and when the context is clear, we often write  $S_p(n)$  in place of  $S_p(n, T_{<n})$ .

To ensure that  $T(n)$  is defined by (2–1) for all  $n$ , we require that for  $1 \leq p \leq k$ ,  $S_p(n) < n$  for all  $n$  following the initial conditions. We assume that this holds for the recurrences that we discuss.<sup>2</sup> For each spot function  $S_p(n)$ , we define a new sequence by the recurrence

$$M_p(n) = M_p(S_p(n)) + 1 \quad (2-2)$$

with initial conditions  $M_p(n) = 1$  for  $n = 1, \dots, r$ , where  $r$  is to be the same as the number of initial conditions used in the definition of  $T(n)$ .<sup>3</sup>

We call the sequence  $M_p(n)$  the generation sequence for  $T(n)$  based on spot  $p$ .<sup>4</sup> For  $g \geq 1$  we define the  $g$ th generation with respect to the  $p$ th spot function to be the set  $M_p^{-1}(\{g\})$ , which we denote by  $G_p(g)$ . For ease of notation we may omit reference to the index  $p$  in  $G_p(g)$  when the index  $p$  is clear from the context. The definition of  $M_p(n)$  is motivated by considering the index  $n$  to be in the “next generation” of its  $p$ th spot ancestor  $S_p(n)$ , which itself is a member of a previous generation

<sup>1</sup>See, for example, [Callaghan et al. 05], where block structure insights are used to help identify and formulate the appropriate approach and specific induction assumptions required to prove the behavior of a family of sequences related to (1–1).

<sup>2</sup>If it fails, then for the smallest integer  $n$  for which it fails, we say that the sequence terminates at index  $n$ .

<sup>3</sup>In general, this value of  $r$  will be greater than the minimum value that is required by the specific nature of the recurrence (2–2); further, this minimum value can differ for different values of  $p$ .

<sup>4</sup>We often refer to  $M_p(n)$  as the generation structure for  $T(n)$  based on spot  $p$ , especially when we are considering the overall characteristics of this sequence rather than the behavior of individual terms.

with generation number  $M_p(S_p(n))$ . When there are two spot functions, we call  $M_1(n)$  the mother function and  $M_2(n)$  the father function (here we adapt terminology introduced in [Pinn 99]). We call the generation sequences that result from these spot functions the maternal and paternal generation sequences, respectively. Similarly, the  $g$ th generations in this case are called the  $g$ th maternal and paternal generations, respectively.

It follows immediately from the recurrence (2-2) for  $M_p$  and the initial conditions that the generation sequence begins at 1 and is onto either all of the positive integers or an interval of the positive integers beginning at 1. For fixed  $p$ , it is often the case that the  $g$ th generation  $G_p(g)$  is a finite interval of positive integers for all  $g \geq 1$ , and the generations partition the positive integers into intervals. However, this is not always the case. We discuss this, together with a variety of other issues, in the following sections, where we apply our generation notion to specific meta-Fibonacci sequences.

At this point an example may be helpful. In the notation we introduced above, the Conolly sequence (0, 1, 1, 2) [Conolly 89] is given by

$$C(n) = C(n - C(n - 1)) + C(n - 1 - C(n - 2)),$$

with initial conditions  $C(1) = C(2) = 1$ . It is well known that  $C(n)$  is slow, and that for each  $n$ ,  $C(n)$  equals  $n$  exactly  $\nu_2(2n)$  times, where  $\nu_2(n)$  is the highest power of 2 that divides  $n$ . The behavior of  $C(n)$  between successive powers of 2 provided important insights for formulating the original induction-based proofs of the properties of  $C(n)$  (see [Tanny 92]).

The maternal generation sequence for  $C(n)$  is given as

$$M_1(n) = M_1(n - C(n - 1)) + 1$$

with  $M_1(1) = M_1(2) = 1$ . In the next section we prove that  $M_1(n)$ , the maternal generation sequence of the Conolly sequence, is slow-growing, and further, that

$$G_1(g) = [2^{g-1} + 1, 2^g]$$

for all  $g \geq 2$ . In this case the generation structure aligns at successive powers of 2, exactly where the natural division points for the “frequency” function  $\nu_2(2n)$  of  $C(n)$  are observed to occur.

For any homogeneous meta-Fibonacci recurrence, we define the beginning of the  $g$ th generation with respect to  $M_p$  by  $\alpha_p(g) = \min\{n \mid n \in G_p(g)\}$ . Similarly, we define the end of the  $g$ th generation with respect to  $M_p$  by  $\beta_p(g) = \max\{n \mid n \in G_p(g)\}$ , provided it exists. When the context is clear we will drop the subscript  $p$  from the notation. By definition,  $G_p(g) \subseteq [\alpha_p(g), \beta_p(g)]$ .

If  $G_p(g) \neq [\alpha_p(g), \beta_p(g)]$ , we say that the  $g$ th generation  $G_p(g)$  is *fragmented*. Otherwise, when  $G_p(g) = [\alpha_p(g), \beta_p(g)]$  for all  $g \geq 1$ , we say that the generational structure with respect to  $M_p$  has an *interval structure*. This is the case for the Conolly sequence above.

### 3. GENERATION SEQUENCES BASED ON SLOW-GROWING SPOT SEQUENCES

For the recurrences (1-1) and (1-2) the spot sequences are of the form  $S(n) = n - a - T^k(n - b)$  or  $S(n) = T^k(n - b)$ , respectively. These spot sequences will be slow-growing if the original sequence itself is slow-growing. For this reason we turn our attention to the situation that the spot sequence  $S_p(n)$  in (2-1) is slow-growing. In this case much can be deduced about the generational structure of  $T(n)$  based on spot  $p$ .

**Theorem 3.1.** *For the meta-Fibonacci sequence  $T(n)$  in (2-1), if the spot function  $S_p(n)$  is slow-growing, then so is the  $p$ th spot-based generation sequence  $M_p(n)$ .*

*Proof:* The proof is by induction on  $n$ . For the base case, note that by the initial conditions in (2-2),  $M(n) = 1$  for  $1 \leq n \leq r$ . Let  $\Delta M(n) = M(n + 1) - M(n)$ ; note that if  $r > 1$ , then  $\Delta M(1) = 0$ . If  $r = 1$ , then for  $T(2)$  to be well defined, we must have  $S_p(2) = 1$ , from which it follows that

$$\begin{aligned} \Delta M(1) &= M(2) - M(1) = M(S_p(2)) - M(S_p(1)) \\ &= M(1) - M(1) = 0. \end{aligned}$$

Thus, in all cases,  $\Delta M(1) \in \{0, 1\}$ .

For  $n > 1$ , assume that  $\Delta M(k) \in \{0, 1\}$  for all  $k < n$ . Since  $S_p(i)$  is slow-growing, observe that

$$M(n + 1) = M(S_p(n + 1)) + 1 = M(S_p(n) + t) + 1,$$

where  $t = 0$  or  $t = 1$ . Since  $T(n)$  is well defined, we must have  $S_p(n) + t \leq n$ . Thus, by the induction assumption,

$$M(S_p(n) + t) = M(S_p(n)) + j,$$

where  $j \in \{0, 1\}$ . It follows that

$$M(n + 1) = [M(S_p(n)) + 1] + j = M(n) + j,$$

from which we get that  $\Delta M(n) = j \in \{0, 1\}$ . This completes the induction.  $\square$

**Corollary 3.2.** *If  $S_p(n)$  is slow-growing, then the generation sequence of  $T(n)$  based on spot  $p$  is an interval structure. Further, in this case, for  $g \geq 1$ ,  $\beta_p(g) = \alpha_p(g + 1) - 1$ .*

*Proof:* Since  $M_p(n)$  is slow-growing, for any  $g \geq 1$  there are a minimum index  $\alpha_p(g)$  and a maximum index  $\beta_p(g)$  such that  $M_p(\alpha_p(g)) = M_p(\beta_p(g)) = g$ . If  $S_p(n)$  becomes constant, then for some  $g^*$  we have  $\beta_p(g^*) = \infty$ , and thus there is only a finite number of generations. Otherwise, for every  $g \geq 1$ ,  $G_p(g) = [\alpha_p(g), \beta_p(g)]$  and  $\beta_p(g) = \alpha_p(g + 1) - 1$ .  $\square$

For a slow-growing spot sequence  $S_p(n)$ , there is an elegant relation between this spot sequence and the generation sequence  $M_p(n)$  that it induces. By the definition of  $\alpha_p(g + 1)$ , we have that

$$g + 1 = M_p(\alpha_p(g + 1)) = M_p(S_p(\alpha_p(g + 1))) + 1.$$

So  $S_p(\alpha_p(g + 1)) \geq \alpha_p(g)$ , since  $\alpha_p(g)$  is the beginning of generation  $g$ . To see that equality holds, assume the contrary, namely, that  $S_p(\alpha_p(g + 1)) > \alpha_p(g)$ . Since  $S_p(n)$  is slow-growing, there exists  $\gamma < \alpha_p(g + 1)$  such that  $S_p(\gamma) = \alpha_p(g)$ . But this implies that

$$M_p(\gamma) = M_p(S_p(\gamma)) + 1 = M_p(\alpha_p(g)) + 1 = g + 1,$$

which contradicts the definition of  $\alpha_p(g + 1)$ . Thus,  $S_p(\alpha_p(g + 1)) = \alpha_p(g)$ .

Since  $S_p(n)$  is slow,  $M_p(n)$  is also slow. Thus, for every  $g \geq 1$ ,  $\beta_p(g) = \alpha_p(g + 1) - 1$ . By what we have just shown,  $S_p(\alpha_p(g + 2)) = \alpha_p(g + 1)$ , so using the fact that  $S_p(n)$  is slow, we have that

$$S_p(\beta_p(g + 1)) \in \{\alpha_p(g + 1) - 1, \alpha_p(g + 1)\}.$$

However,  $S_p(\beta_p(g + 1)) = \alpha_p(g + 1)$  would imply that

$$M_p(\beta_p(g + 1)) = M_p(\alpha_p(g + 1)) + 1 = g + 2.$$

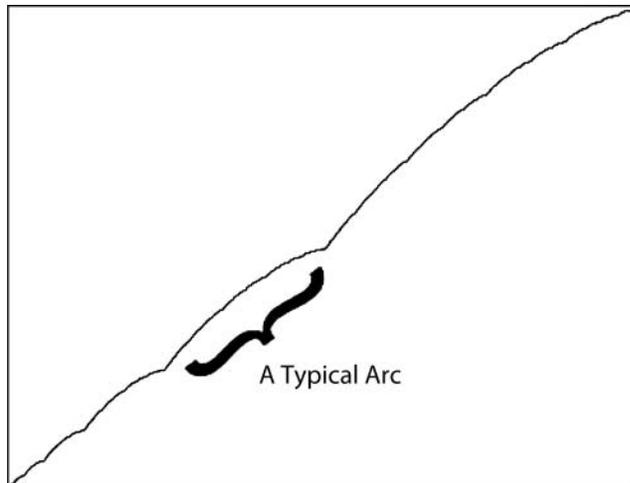
This is a contradiction to the definition of  $\beta_p(g + 1)$ . Thus,

$$S_p(\beta_p(g + 1)) = \alpha_p(g + 1) - 1 = \beta_p(g),$$

and we have proved the following result:

**Theorem 3.3.** *Suppose that  $S_p(n)$  is a slow-growing spot of  $T(n)$ . Then for every  $g > 0$ ,  $S_p(n)$  maps the  $(g + 1)$ th generation onto the  $g$ th generation. That is,  $S_p(\alpha_p(g + 1)) = \alpha_p(g)$  and  $S_p(\beta_p(g + 1)) = \beta_p(g)$ .*

Any slow-growing spot  $S_p(n)$  of  $T(n)$  specifies a generation structure that is uniquely determined by the sequence of generation interval starting points  $\alpha_p(g)$  for all positive integers  $g$ . If the recurrence for  $T(n)$  has  $r$  initial conditions, then the starting points of the generations are uniquely determined by the property that  $\alpha_p(2) = r + 1$  and for all subsequent  $\alpha_p(g)$ ,  $g > 2$ ,  $\alpha_p(g)$  is the *smallest* number with the property that  $S_p(\alpha_p(g + 1)) = \alpha_p(g)$ .



**FIGURE 1.** Graph of Conway sequence  $A(n)$  for  $1 \leq n \leq 1024$ .

We use this fact extensively in what follows, where we compute the maternal generation structure for several well-known slow-growing sequences. In so doing, we show how our spot-based maternal generations are essentially congruent to the block structures for these sequences that are identified in an ad hoc way in the literature.

We begin with the Conway sequence and several of its variants. The Conway sequence is defined by

$$A(n) = A(n - A(n - 1)) + A(A(n - 1)),$$

with  $A(1) = A(2) = 1$ . The graph of the Conway sequence in Figure 1 provides striking evidence of what is usually meant by a block structure: a series of arcs between successive powers of 2 indicating that the behavior of the sequence is essentially the same on these intervals of the domain.

The interval  $[2^m, 2^{m+1})$  has been termed the  $m$ th “octave” of the sequence [Conolly 89]. Various patterns in the Conway sequence have been shown to persist from one octave to the next. For example,  $A(2^m) = 2^{m-1}$  for  $m \geq 1$ ; that is, the beginning of each octave is mapped to the beginning of the previous one. Also, for  $m \geq 2$ ,  $A(n) = 2^{m-1}$  for exactly the last  $m$  indices in the  $m$ th octave, and  $A(n) \geq n/2$  with equality only when  $n$  is a power of 2.

It is readily confirmed that the maternal generation structure of  $A(n)$  conforms naturally to these octaves.

**Proposition 3.4.** *For the Conway sequence,  $\alpha_1(g) = 2^{g-1} + 1$  for  $g > 1$ .*

$g$	1	2	3	4	5	6	7	8	9	10
$G_1(g)$	[1, 2]	[3, 4]	[5, 8]	[9, 16]	[17, 32]	[33, 64]	[65, 128]	[129, 256]	[257, 512]	[513, 1024]

**TABLE 1.** First ten maternal generations of the Conway Sequence.

*Proof:* We proceed by induction on  $g$ . The base case is clear from Table 1. We use the fact that for  $g > 0$ ,

$$A(2^g) = A(2^g - 1) = 2^{g-1}$$

[Kubo and Vakil 96]. Assume the proposition up to generation  $g$ . For  $g + 1$  we have that

$$M(2^g + 1) = M(2^g + 1 - A(2^g)) + 1.$$

Since  $A(2^g) = 2^{g-1}$ , we get  $2^g + 1 - A(2^g) = 2^{g-1} + 1$ . By the induction hypothesis we have that  $\alpha_1(g) = 2^{g-1} + 1$ , and so

$$M(2^g + 1) = M(\alpha_1(g)) + 1 = g + 1.$$

On the other hand,  $M(2^g) = M(2^g - A(2^g - 1)) + 1$ , and since  $A(2^g - 1) = 2^{g-1}$ , this implies

$$2^g - A(2^g - 1) = 2^{g-1} = \alpha_1(g) - 1.$$

Hence,

$$M(2^g) = M(\alpha_1(g) - 1) + 1 = g - 1 + 1 = g.$$

Since  $M(n)$  is slow, it follows that  $M(2^g + 1)$  is the first occurrence of  $g + 1$ , so  $\alpha_1(g + 1) = 2^g + 1$ . This completes the induction.  $\square$

Next we show that an analogous result holds for the Newman–Conway sequences (see [Newman 88]). These sequences are defined by

$$f_r(n) = f_r(n - f_r(n - 1)) + f_r(f_r(n - 1))$$

with initial conditions  $f_r(i) = 1$  for  $1 \leq i \leq r + 1$  and  $r \geq 1$ . Note that the Conway sequence corresponds to  $r = 1$ . For any fixed  $r > 1$ , the following hold:

- (1) The sequence  $f_r(n)$  is slow-growing.
- (2) Like that of the Conway sequence, the graph of  $f_r(n)$  consists of successive arcs that begin and end at the “Fibonacci-type” numbers  $E_n$  defined by  $E_n = E_{n-1} + E_{n-r}$ , with initial conditions  $E_i = 1$  for  $1 \leq i \leq r$ .
- (3) These arcs identify a natural partition of the domain at the points  $E_n$ .
- (4) For  $n > r$ ,  $f_r(E_n) = f_r(E_n - 1) = E_{n-r}$  (see [Kleitman 91]).

As is the case for the Conway sequence, the maternal generations for the Newman–Conway sequences create essentially this same partition of the domain. More precisely, for  $g > 1$ , the maternal generation begins at  $\alpha_1(g) = E_{2r+g-2} + 1$ . Clearly this holds for the second generation, which starts at  $r + 2 = E_{2r} + 1$ . By definition,  $S_1(E_n + 1) = E_n + 1 - f_r(E_n)$ . Since  $f_r(E_n) = E_{n-r}$  for  $n > r$ , we have that

$$S_1(E_n + 1) = E_n - E_{n-r} + 1 = E_{n-1} + 1$$

by the recurrence for  $E_n$ . But  $f_r(E_n - 1) = E_{n-r}$  for  $n > r$ , so we have that

$$S_1(E_n) = E_n - f_r(E_n - 1) = E_n - E_{n-r} = E_{n-1}.$$

By Theorem 3.3,  $S(\alpha_1(g + 1)) = \alpha_1(g)$ , and  $\alpha_1(g + 1)$  is the smallest number with this property. Since  $\alpha_1(2) = E_{2r} + 1$ ,  $S(E_n) = E_{n-1}$ , and  $S(E_n + 1) > E_n$ , we can use induction together with the discussion following Theorem 3.3 to deduce that  $\alpha_1(g) = E_{2r+g-2} + 1$ .

We conclude our consideration of Conway sequence variants with the sequences defined by (1–2) for  $k > 1$ , and with initial conditions  $A(1) = A(2) = 1$  (see [Grytczuk 04]). For  $k = 2$ , Grytczuk proved that the resulting sequence is slow-growing. He also showed that the last occurrence of the Fibonacci number  $F_n$  in the sequence occurs at position  $F_{n+1}$ . This prompts Grytczuk to observe that this sequence has a clearly identifiable block structure, in which the Fibonacci numbers play a prominent role. He states, “This suggests to divide  $A(n)$  into segments of the form  $[A(F_n + 1), A(F_n + 2), \dots, A(F_{n+1})]$ ” [Grytczuk 04, p. 149].

Once again, just as we found for the Conway and Newman–Conway sequences, the maternal generation structure for Grytczuk’s sequence matches the natural block structure that he identified. In fact, even more is true: an analogous result holds for (1–2) with any positive  $k > 1$  and the same initial conditions

$$A(1) = A(2) = 1.$$

We now outline the argument for this, which follows the same lines as that given above.<sup>5</sup> For any fixed  $k > 1$  the sequence  $A(n)$  defined by (1–2) is slow-growing. For

<sup>5</sup>The interested reader may contact us for additional details.

$n > k$ , define  $E_n$  by  $E_n = E_{n-1} + E_{n-k}$ , with initial conditions  $E_i = 1$  for  $1 \leq i \leq k$ . Then for  $n > k$ , we can show that  $A(E_{n+1}) = E_n$ ,  $E_{n+1}$  marks the last occurrence of the value  $E_n$ , and  $A^k(E_n - 1) = E_{n-k}$ . It follows that  $A(n)$  has a natural block structure whose division points are at the points  $E_n$ .

Using these properties, we will show that the maternal generations of  $A(n)$  based on the spot  $n - A^k(n - 1)$  have an interval structure, where for  $g > 1$  the  $g$ th generation begins at  $\alpha_1(g) = E_{k+g-1} + 1$ . As such, the maternal generation structure coincides with the block structure for  $A(n)$  that we just described. To see this, note first that  $\alpha_1(2) = 3 = E_{k+1} + 1$ . For  $g > 2$ , using the aforementioned properties of  $A(n)$ , we get

$$\begin{aligned} S_1(E_{k+g-1} + 1) &= E_{k+g-1} + 1 - A^k(E_{k+g-1}) \\ &= E_{k+g-1} - E_{g-1} + 1 \\ &= E_{k+g-2} + 1. \end{aligned}$$

Also, we have that

$$\begin{aligned} S_1(E_{k+g-1}) &= E_{k+g-1} - A^k(E_{k+g-1} - 1) \\ &= E_{k+g-1} - E_{g-1} \\ &= E_{k+g-2}. \end{aligned}$$

By the remarks immediately following Theorem 3.3, these properties uniquely determine the start points for the maternal generations. Thus, for  $g > 2$ , we have that  $\alpha_1(g) = E_{k+g-1}$ .

In our final example we show, as asserted in Section 2, that the maternal generation sequence of the Conolly sequence  $C(n)$  discussed in Section 2 is slow-growing. Further, for  $g > 1$ ,  $G_1(g) = [2^{g-1} + 1, 2^g]$ , that is, the  $g$ th maternal generation is a shift of 1 from the  $g$ th block  $[2^{g-1}, 2^g - 1]$  identified in [Conolly 89].

We want to show that for  $g > 1$  we have  $\alpha_1(g) = 2^{g-1} + 1$ . Similar to our earlier arguments, since

$$C(n) = C(n - C(n - 1)) + C(n - 1 - C(n - 2)),$$

it suffices to verify that

$$S_1(2^{g-1} + 1) = 2^{g-2} + 1 \quad \text{and} \quad S_1(2^{g-1}) = 2^{g-2}.$$

Since  $S_1(n) = n - C(n - 1)$ , these are equivalent to  $C(2^g) = C(2^g - 1) = 2^{g-1}$ , which is a well-known property of the Conolly sequence (see, for example, [Tanny 92]). This completes the proof.

#### 4. GENERATIONAL STRUCTURE FOR SELECTED NONSLOW SEQUENCES

Based on our initial experimental evidence, we believe that an analysis of generation structures can provide use-

	$n$									
	1	2	3	4	5	6	7	8	9	10
$\mu(n + 0)$	1	1	1	2	2	2	3	3	4	4
$\mu(n + 10)$	4	5	5	6	7	7	7	8	8	8
$\mu(n + 20)$	9	9	10	11	11	11	13	12	14	13
$\mu(n + 30)$	14	15	15	15	16	16	16	17	17	18
$\mu(n + 40)$	19	19	19	21	20	22	21	22	24	24

TABLE 2. First 50 terms of  $\mu(n)$ .

ful insights for sequences with more erratic behavior than that of the slow-growing sequences addressed in the previous section. For example, a sequence of interest is  $\mu(n)$  generated by the recurrence  $(1, 2, 2, 1)$  with three initial conditions all equal to 1:

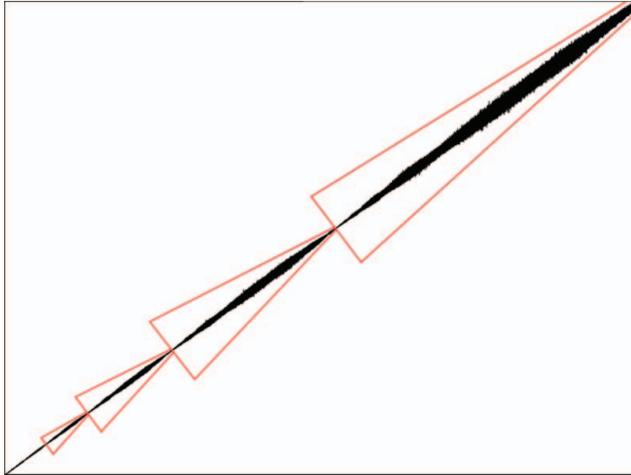
$$\begin{aligned} \mu(n) &= \mu(n - 1 - \mu(n - 2)) + \mu(n - 2 - \mu(n - 1)), \\ \mu(1) &= \mu(2) = \mu(3) = 1. \end{aligned} \tag{4-1}$$

Table 2 contains the first 50 values of  $\mu(n)$ . The sequence  $\mu(n)$  is neither slow-growing nor monotonic. It is not even known whether  $\mu(n)$  is defined for all positive integers  $n$ , that is, whether  $n - 2 - \mu(n - 1) > 0$  for all integers  $n \geq 4$ .

At the same time, an examination of the first  $10^6$  terms of  $\mu(n)$  indicates that the sequence appears to have many regularities. For example,  $\mu(n)$  hits every power of 2 exactly three times, and for each power of 2, the three occurrences are for consecutive arguments. Each of these runs of  $2^k$  is also preceded by at least two consecutive occurrences of  $2^k - 1$  and succeeded by precisely two occurrences of  $2^k + 1$  (see Table 2 for examples of this behavior). Additional regularities are evident in the graph of  $\mu(n)$ : the graph alternately widens and narrows, and the general appearances on each interval, defined by successive narrowing of the sequence, are similar (see Figure 2). The narrowing of  $\mu(n)$  occurs where the sequence takes on the values of a power of 2. The lengths of these successive intervals double, as does the amplitude of the variation about the trend line of the graph.

It is fascinating that the maternal generation structure for the first  $10^6$  terms of  $\mu(n)$  corresponds precisely to the intervals identified from the graph. Based on our experimental evidence we conjecture the following:

**Conjecture 4.1.** *The sequence  $\mu(n)$  is defined for all positive integers  $n$ . The maternal generation sequence of  $\mu(n)$  is slow-growing. For each  $g \geq 3$ , the  $g$ th maternal*

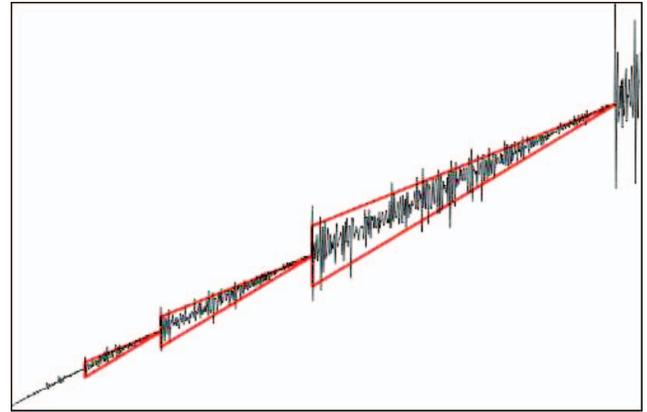


**FIGURE 2.** Graph of  $\mu(n)$  for  $1 \leq n \leq 10^6$ . (Figure is available in color online).

generation begins at index  $2^{g-1} + g$ , which is the first occurrence of  $2^{g-2} + 1$  in  $\mu(n)$ , and ends at index  $2^g + g$ , which is the last occurrence of  $2^{g-1}$  in  $\mu(n)$ .

We conclude by describing some intriguing experimental findings concerning the maternal generation structure for Hofstadter’s famous  $Q$ -sequence  $(0, 1, 0, 2)$  [Hofstadter 79]. First we set the stage. It is well known (for example, see [Pinn 99]) that  $Q(n)$  exhibits the following repetitive behavior about its underlying trend line  $y = n/2$ : a period of relatively large oscillations, sometimes initiated by a large “spike,” followed by gradually decreasing oscillation that tapers to a portion of relative quiet with very small differences, and then the process repeats (see Figure 3). In [Pinn 99], the first 20 “transition points” where the large oscillations in  $Q(n) - n/2$  recur following a period of relative calm are identified (see the right-hand side of Table 3). Pinn identifies these points with the start points of the intervals that partition the domain into an underlying block structure for the  $Q$ -sequence (he terms these blocks “generations”). Pinn finds the first eleven of these transition points “by eye” from the graph of  $Q(n)$ .<sup>6</sup> For subsequent generations, Pinn observes that “the onset of the new generations is a little less well defined” [Pinn 99, p. 8]. By applying certain statistical approaches, Pinn concludes

<sup>6</sup>The transition points double for generations 2 through 10, as does the value of  $Q(n)$  at each of these nine points, these values being the first occurrences of  $2, \dots, 2^9$  respectively. This pattern fails for the start point of the 11th Pinn generation.



**FIGURE 3.** Graph of  $Q(n)$  for  $1 \leq n \leq 800$ . (Figure is available in color online).

that a good estimate for the start point for generation  $g > 11$  is  $\lfloor 2^{g-1/2} \rfloor$ .<sup>7</sup>

As we now show, our spot-based maternal generation structure for  $Q(n)$  does a much better job than Pinn’s statistical estimation techniques at identifying the locations of the transitions in  $Q(n)$ , at least for all the first 18 generations that we have checked. Let  $\alpha(g, \pi)$  denote the start point of the  $g$ th Pinn generation, while  $\alpha_1(g)$  will denote the start point of the  $g$ th maternal generation of  $Q(n)$  based on spot  $n - Q(n - 1)$ . In Table 3 we compare the start points for the first 20 maternal generations of  $Q(n)$  with those for the first 20 Pinn generations.

From Table 3 we see that the start points match for the first eleven maternal and Pinn generations, respectively. This is very surprising, since the two approaches for identifying these points are entirely unrelated. For the remaining generations there are substantial differences in the start points, with the maternal generation start point bigger in every case.

In Table 4 we calculate the absolute percent deviation of  $Q(\alpha_1(g))$  from its previous value  $Q(\alpha_1(g) - 1)$ , that is, the value

$$\frac{|Q(\alpha_1(g)) - Q(\alpha_1(g) - 1)|}{Q(\alpha_1(g) - 1)} \times 100\%,$$

for maternal generations 12 to 18. We then compare that with the corresponding absolute deviation for  $Q(\alpha(g, \pi))$ . Note how these deviations are almost always higher for the maternal generation start point, suggesting that these points are more closely located to the upcoming transition point. In the last column of Table 4 we locate the

<sup>7</sup>Observe the typographical error in [Pinn 99, p. 8], where he writes  $\lfloor 2^{g+1/2} \rfloor$ .

	Maternal Generation $g$					Pinn Generation $g$			
	1	2	3	4		1	2	3	4
$\alpha_1(g+0)$	1	3	6	12	$\alpha(g+0;\pi)$	1	3	6	12
$\alpha_1(g+4)$	24	48	96	192	$\alpha(g+4;\pi)$	23	48	96	192
$\alpha_1(g+8)$	384	768	1522	3031	$\alpha(g+8;\pi)$	384	768	1522	2896
$\alpha_1(g+12)$	6043	12056	24086	48043	$\alpha(g+12;\pi)$	5792	11585	23170	46340
$\alpha_1(g+16)$	95286	189268	376996	750285	$\alpha(g+16;\pi)$	92681	185363	370727	741455

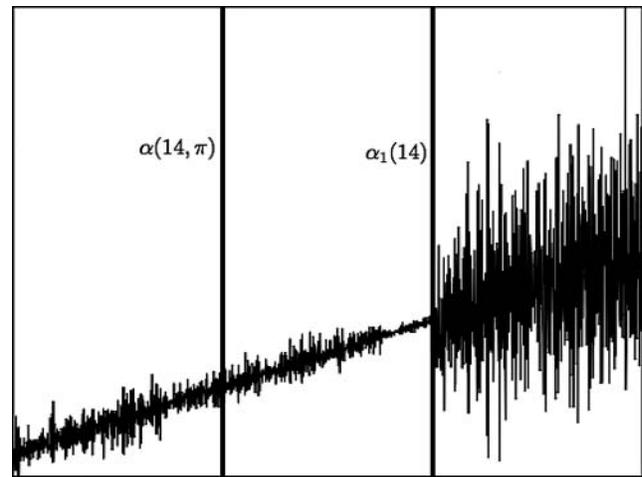
**TABLE 3.** Comparison of start points for maternal generations with Pinn’s generations.

$g$	$\alpha_1(g)$	Absolute Deviation (%)	$\alpha(g, \pi)$	Absolute Deviation (%)	Transition Points
12	3031	9.48	2896	0.68	3032
13	6043	1.52	5792	0.48	6042
14	12056	1.00	11585	0.22	12069
15	24086	5.72	23170	0.46	24064
16	48043	0.42	46340	1.00	48013
17	95286	2.60	92681	0.36	95182
18	189268	1.73	185363	0.28	189266

**TABLE 4.** Deviations in  $Q(n)$  at start points of maternal and Pinn generations. The last column shows the transition points for  $Q(n) - n/2$ .

actual transition points of  $Q(n)$ .<sup>8</sup> We observe that for each generation  $g$  from 12 through 18,  $\alpha_1(g)$  is much closer to the transition point for generation  $g$  than  $\alpha(g, \pi)$ . For example, for  $g = 14$ ,  $\alpha_1(14) = 12,056$ ,  $\alpha(14, \pi) = 11,585$ , and we estimate the transition point to be 12,069 (see Figure 4; notice that Pinn’s start point for generation 14 is located in the midst of a relatively quiet portion of the values of  $Q(n) - n/2$ , while the start point for maternal generation 14 is much closer to the upcoming spike in  $Q(n)$  situated at the end of this quiet region).

Further investigation is required to determine for how many generations beyond 18 this apparent connection between the start point of the maternal generations and the transition points for  $Q(n)$  persists, and to understand its significance. But this tantalizing property of the maternal generation function for  $Q(n)$  provides a further suggestion that a close examination of spot-based generation structures may provide potentially important insights into the intrinsic structure of many meta-Fibonacci sequences, including those with highly complex behavior such as  $Q(n)$ .



**FIGURE 4.** Comparison of  $\alpha_1(14) = 12,056$  and  $\alpha(14; \pi) = 11,585$  on the graph of  $Q(n)$ .

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<sup>8</sup>This is done via a careful examination of the behavior of the sequence  $Q(n) - n/2$ . The interested reader may contact us for details.

## REFERENCES

- [Balamohan et al. 07] B. Balamohan, A. Kuznetsov, and S. Tanny. “On the Behavior of a Variant of Hofstadter’s  $Q$ -Sequence.” *J. Integer Sequences* 10 (2007), Article 07.7.1.
- [Callaghan et al. 05] J. Callaghan, J. J. Chew III, and S. Tanny. “On the Behaviour of a Family of Meta-Fibonacci Sequences.” *SIAM J. Discrete Math.* 18 (2005), 794–824.
- [Conolly 89] B. W. Conolly. “Fibonacci and Meta-Fibonacci Sequences.” In *Fibonacci and Lucas Numbers and the Golden Section: Theory and Applications*, edited by S. Vajda, pp. 127–139. New York: Halsted Press, 1989.
- [Grytczuk 04] J. Grytczuk. “Another Variation on Conway’s Recursive Sequence.” *Discrete Mathematics* 282 (2004), 149–161.
- [Guy 04] R. K. Guy. *Unsolved Problems in Number Theory*, third edition, Problem Books in Mathematics. New York: Springer, 2004.
- [Hofstadter 79] D. R. Hofstadter. *Gödel, Escher, Bach: An Eternal Golden Braid*, New York: Random House, 1979.
- [Isgur et al. 09] A. Isgur, D. Reiss, and S. Tanny. “Trees and Meta-Fibonacci Sequences.” *Electron. J. of Combin.* 16 (2009), R129.
- [Kleitman 91] D. Kleitman. “Solution to AMM Problem 3274” (Another Fibonacci Generalization, proposed by David Newman). *Amer. Math. Monthly* 98 (1991), 958–959.
- [Kubo and Vakil 96] T. Kubo and R. Vakil. “On Conway’s Recursive Sequence.” *Discrete Mathematics* 152 (1996), 225–252.
- [Mallows 91] C. Mallows. “Conway’s Challenging Sequence.” *Amer. Math. Monthly* 98 (1991), 5–20.
- [Newman 88] D. Newman. “Problem E3274.” *Amer. Math. Monthly* 95 (1988), 555.
- [Pinn 99] K. Pinn. “Order and Chaos in Hofstadter’s  $Q(n)$  Sequence.” *Complexity* 4:3 (1999), 41–46.
- [Pinn 00] K. Pinn. “A Chaotic Cousin of Conway’s Recursive Sequence.” *Experimental Mathematics* 9:1 (2000), 55–66.
- [Ruskey and Deugau 09] F. Ruskey and C. Deugau. “The Combinatorics of Certain  $k$ -ary Meta-Fibonacci Sequences.” *J. Integer Sequences* 12 (2009), Article 09.4.3.
- [Tanny 92] S. M. Tanny. “A Well-Behaved Cousin of the Hofstadter Sequence.” *Discrete Math.* 105 (1992), 227–239.

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