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w-Invariants and the Fintushel–Stern Invariants for Plumbed Homology 3-Spheres

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CONTENTS

In this paper, we present numerical computations of the w-invariants and the Fintushel–Stern invariants for plumbed homology 3-spheres and use the results to test a conjecture of Witten suggesting that the invariants carry equivalent information. While the two invariants give nearly the same information for some homology 3-spheres, we present numerous examples in which the information carried by the two invariants is quite different.

1. INTRODUCTION

In this paper, we investigate a conjecture of Witten on the equivalence in gauge theory between Donaldson theory and Seiberg–Witten theory by comparing computations of two invariants of homology spheres arising from these two theories. The first is the Fintushel–Stern invariant, which is defined for Seifert-fibered homology 3-spheres Σ as the virtual dimension of the moduli space of self-dual V-connections on the closed V-manifold constructed as the disk V-bundle associated to the Seifert fibration. The second is the w-invariant, which is an integral lift of the Rohlin invariant and is defined in [Fukumoto and Furuta 00] as the Seiberg–Witten analogue of the Fintushel–Stern invariant.

This invariant turns out to be a homology cobordism invariant for a certain class of homology 3-spheres using the 10/8-inequality. Under the assumption that Seifert homology 3-spheres bound positive definite 4-manifolds, the w-invariant essentially gives the information of the virtual dimension of the moduli space of the Seiberg–Witten monopoles on a closed V-manifold. A numerical computation indicates that these two invariants give almost the same information. However, there certainly exist some homology spheres such that the w-invariants give obstructions to bounding positive definite (or negative definite) manifolds, while the Fintushel–Stern invariant does not, and vice versa. Our goal in this paper is to explore the difference between the Fintushel–Stern

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invariants and the w-invariants in the context of plumbed homology 3-spheres, and use our results to shed light on Witten's conjecture.

In [Fukumoto and Furuta 00], the authors applied a V-manifold version of the 10/8-inequality to define a homology cobordism invariant for certain classes of homology 3-spheres, and we briefly recall the construction here. For a triple (Σ, X, c) consisting of a homology 3-sphere Σ, a 4-V-manifold X with boundary Σ, and a V-spin^c structure c on X, we define a $\mathbb{Z}$-valued invariant $w(\Sigma, X, c)$,

$$w(\Sigma, X, c) := \operatorname{ind}_V \; D(X \cup_\Sigma W) + \frac{1}{8}\operatorname{sign}(W),$$

where W is a spin 4-manifold with $\partial W \cong -\Sigma$. When the V-spin^c structure c on X comes from a V-spin structure, this invariant $w(\Sigma, X, c)$ is an integral lift of the Rochlin invariant $\mu(\Sigma)$, since the Dirac operator corresponding to a V-spin structure is quaternionic linear. If $\Sigma = \Sigma(\Gamma)$ is a plumbed homology 3-sphere associated to a plumbing tree graph Γ, then $\Sigma(\Gamma)$ bounds the plumbed 4-manifold $P(\Gamma)$, and W. Neumann and L. Siebenmann defined an integral lift $\bar{\mu}(\Sigma)$ of the Rohlin invariant using the data of Γ, where $\bar{\mu}(\Sigma)$ does not depend on the choice of Γ. N. Saveliev [Saveliev 02] (and partially the author in a joint work with M. Furuta and M. Ue) proved that the $\bar{\mu}$-invariant $\bar{\mu}(\Sigma)$ is equal to the w-invariant $w(\Sigma(\Gamma), P(\hat{\Gamma}), c)$, where $P(\hat{\Gamma})$ is the plumbed V-manifold associated to a decorated plumbing graph $\hat{\Gamma}$ in the sense of Saveliev. Let $\mathcal{S}(k^+, k^-)$ be the set of homology 3-spheres Σ such that there exists a spin 4-V-manifold X satisfying $b_2^{\pm}(X) \leq k^{\pm}$. If we assume $k^+ + k^- \leq 2$, then we see that $w(\Sigma, X, c)$ does not depend on the pair (X, c) of a spin 4-V-manifold X with boundary Σ satisfying $b_2^{\pm}(X) \leq k^{\pm}$ and a V-spin structure c, and furthermore, we see that the map

$$\mathcal{S}(k^+, k^-) \ni \Sigma \longmapsto w(\Sigma, X, c) \in \mathbb{Z}$$

gives a homology cobordism invariant. This means that the $\bar{\mu}$-invariant is in fact a homology cobordism invariant for plumbed homology spheres in the class $\mathcal{S}(k^+, k^-)$ with $k^+ + k^- \leq 2$.

Let $\Gamma = (V, E, \omega, \varepsilon)$ be a graph with (unnormalized) Seifert invariants

$$\omega(k) = \{(a_{k1}, b_{k1}), \ldots, (a_{kn_k}, b_{kn_k})\}$$

of the rational Euler number $\sum_{i=1}^{n_k} b_{ki}/a_{ki}$ on each vertex $k \in V$ and signs $\varepsilon(e)$ on each edge $e \in E$. Let $P(\Gamma)$ be the 4-V-manifold obtained by plumbing according to Γ. Then the boundary $\Sigma(\Gamma)$ of $P(\Gamma)$ is a homology 3-sphere if and only if Γ satisfies Condition HS, which is given in Section 3.

Let A_Γ be the intersection matrix over $\mathbb{Q}$ with respect to the standard basis of $H_2(P(\Gamma); \mathbb{Q})$. Let $k^+(\Gamma)$ (respectively $k^-(\Gamma)$) be the number of positive (negative) eigenvalues of the matrix A_Γ. Take $\vec{m} = (m_1, \ldots, m_{\#V}) \in \mathbb{Z}^{\#V}$ to be any $\#V$-tuple of integers that parameterizes V-spin^c structures $c(\vec{m})$ on $P(\Gamma)$. The following definition is in fact the explicit formula of the invariant of plumbed homology 3-spheres.

Definition 1.1. Suppose Γ satisfies Condition HS. Then we define

$$\begin{aligned} w(\Gamma, \vec{m}) := \frac{1}{8}\Bigg[& {}^t\vec{s} A_\Gamma \vec{s} - \left(k^+(\Gamma) - k^-(\Gamma)\right) \\ & - \sum_{k \in V} \sum_{i=1}^{n_k} \frac{1}{a_{ki}} \sum_{\ell=1}^{a_{ki}-1} \Bigg\{ \cot\left(\frac{\pi \ell}{a_{ki}}\right) \cot\left(\frac{\pi b_{ki} \ell}{a_{ki}}\right) \\ & + 2\cos\left(\frac{\pi\left(1 + (2m_k + 1) b_{ki}\right)\ell}{a_{ki}}\right) \csc\left(\frac{\pi \ell}{a_{ki}}\right) \\ & \times \csc\left(\frac{\pi b_{ki} \ell}{a_{ki}}\right) \Bigg\} \Bigg], \end{aligned}$$

where $\vec{s} = A_\Gamma^{-1}(\vec{\chi} + \vec{e}) + 2\vec{m} \in \mathbb{Z}^{\#V}$ for $\vec{m} = (m_k)_{k \in V}$ and $\vec{\chi}, \vec{e} \in \mathbb{Q}^{\#V}$ are defined by

$$\chi_k = 2 - \sum_{i=1}^{n_k}\left(1 - \frac{1}{a_{ki}}\right), \quad e_k = \sum_{i=1}^{n_k} \frac{b_{ki}}{a_{ki}}.$$

Remark 1.2. The canonical line V-bundle K can be written as

$$K = \bigotimes_{k \in V} \tilde{L}_k^{-\left(A_\Gamma^{-1}(\vec{\chi} + \vec{e})\right)_k},$$

where $\tilde{L}_k$ is the line V-bundle over $P(\Gamma)$ obtained in Section 3 by trivial extension of the pullback of the line V-bundle $L_k \to Z_k$ corresponding to the vertex $k \in V$ by its projection map $L_k \to Z_k$.

Combining this formula with several properties of the invariant, we obtain the following theorem.

Theorem 1.3. *Suppose Γ satisfies Condition HS, and $k^+(\Gamma) = 0$. If $w(\Gamma, \vec{m}) > 0$ for some $\vec{m} \in \mathbb{Z}^{\#V}$, then the connected sum of any number of copies of the plumbed homology 3-spheres $\Sigma(\Gamma)$ cannot be the boundary of a positive definite 4-manifold.*

Remark 1.4. If $k^-(\Gamma) \leq 2$ and the associated V-plumbing $P(\Gamma)$ admits a V-spin structure $\vec{m} = \vec{m}_{\mathrm{spin}}$, then by a V-manifold version of the 10/8-inequality, if $w(\Gamma, \vec{m}) \neq 0$, then the plumbed homology 3-sphere

$\Sigma(\Gamma)$ cannot be the boundary of an *acyclic* 4-manifold [Fukumoto and Furuta 00, Fukumoto 00].

On the other hand, we calculate the Fintushel–Stern invariant for the plumbed homology 3-spheres $\Sigma(\Gamma)$ as follows. Set $a_k = \mathrm{lcm}(a_{ki})$, $b_k = a_k e_k$. Let $\bar{\Gamma}$ be the plumbing graph with $P(\bar{\Gamma}) = -P(\Gamma)$, and define $\bar{b}_{ki} = -b_{ki}$, $\bar{b}_k = -b_k$, $\bar{e}_k = -e_k$, $\bar{m}_k^i = m_k \bar{b}_{ki}$, and $\bar{m}_k = m_k \bar{b}_k$.

Definition 1.5. Suppose Γ satisfies Condition HS, and $k^+(\Gamma) = 0$. Then we define

$$\begin{aligned} R(\Gamma; \vec{m}) &:= 2\,{}^t\vec{m} A_{\bar{\Gamma}} \vec{m} - 3 \\ &+ \sum_{k\in V} \sum_{i=1}^{n_k} \frac{2}{a_{ki}} \sum_{\ell=1}^{a_{ki}-1} \sin^2 \frac{\pi \bar{m}_k^i \ell}{a_{ki}} \cot \frac{\pi \ell}{a_{ki}} \cot \frac{\pi \bar{b}_{ki} \ell}{a_{ki}} \\ &+ \#\left\{(k,i) \,\middle|\, \bar{m}_k^i \not\equiv 0 \bmod a_{ki},\ k \in V\right\}. \end{aligned}$$

We extend the argument of [Lawson 88] for Seifert-fibered homology 3-spheres to the case of plumbed homology 3-spheres $\Sigma(\Gamma)$. Let $\vec{m} = (m_k)_{k\in V}$ be an element in the lattice $\mathbb{Z}^{\#V}$. Then we call $\vec{m}$ odd if $\bar{m}_k$ is odd for some $k \in V$. Let $\mu(\Gamma, \vec{m})$ be the number defined by

$$\begin{aligned} \mu(\Gamma, \vec{m}) = \#\Big\{ \vec{m}' \in \mathbb{Z}^{\#V} \,\Big|\, &{}^t\vec{m} A_{\bar{\Gamma}} \vec{m} = {}^t\vec{m}' A_{\bar{\Gamma}} \vec{m}', \\ &\bar{m}_k \equiv \bar{m}'_k \bmod 2 \text{ for any } k \in V, \\ &\bar{m}_k^i \equiv \bar{m}_k'^i \bmod a_{ki} \text{ for any } k \in V, \\ &i \in \{1, \ldots, n_{ki}\} \Big\}. \end{aligned}$$

Then we have the following result (cf. [Lawson 88]).

Theorem 1.6. *Let Γ be a graph with Seifert invariants satisfying Condition HS and $k^+(\Gamma) = 0$. Suppose there exists an odd element $\vec{m} \in \mathbb{Z}^{\#V}$ such that*

1. ${}^t\vec{m} A_{\bar{\Gamma}} \vec{m} < 4$ *(< 2 if some a_{ki} is even),*
2. $R(\Gamma; \vec{m}) > 0$,
3. $\mu(\Gamma, \vec{m})$ *is odd, and*
4. *for any odd $\vec{m}'$ such that $\bar{m}_k \equiv \bar{m}'_k \bmod 2$ for any $k \in V$, ${}^t\vec{m}' A_{\bar{\Gamma}} \vec{m}' < {}^t\vec{m} A_{\bar{\Gamma}} \vec{m}$, and $\bar{m}_k^i \equiv \bar{m}_k'^i \bmod 2$ for a_{ki} even, one of the following does not hold:*
   - (P) ${}^t\vec{m} A_{\bar{\Gamma}} \vec{m} - {}^t\vec{m}' A_{\bar{\Gamma}} \vec{m}' = \sum_{k\in V} \sum_i p_1(\bar{m}_k'^i, \bar{m}_k^i)$,
   - (I) $I_{\mathrm{eq}}(\bar{m}_k'^i, \bar{m}_k^i) \geq 0$ *for* $p_1(\bar{m}_k'^i, \bar{m}_k^i) < 4$ *(< 2 if a_{ki} even),*
   - (E) *If* $p_1(\bar{m}_k'^i, \bar{m}_k^i) = 4/a_{ki}$, *then* $1 + \bar{b}_{ki} \equiv \pm \bar{m}_k^i \bmod a_{ki}$ *and* $-1 + \bar{b}_{ki} \equiv \pm \bar{m}_k'^i \bmod a_{ki}$,

where $p_1(\bar{m}_k'^i, \bar{m}_k^i)$ and $I_{\mathrm{eq}}(\bar{m}_k'^i, \bar{m}_k^i)$ are defined as follows:

$$\begin{aligned} p_1(\bar{m}_k'^i, \bar{m}_k^i) &= \frac{4}{a_{ki}} \min \left\{ b \in \mathbb{Z}^+ \,\middle|\, b \equiv \bar{b}_{ki}^* \left(\frac{\left(\bar{m}_k^i\right)^2 - \left(\bar{m}_k'^i\right)^2}{4} \right) \right. \\ &\qquad \left. \bmod \bar{a}_{ki} \right\}, \end{aligned}$$

where $\bar{a}_{ki} = a_{ki}$ if a_{ki} is odd and $\bar{a}_{ki} = a_{ki}/2$ if a_{ki} is even, $\bar{m}_k^i \equiv \bar{m}_k'^i \bmod 2$ if a_{ki} is even, b^* denotes the inverse of $b \bmod a_{ki}$ if it exists, and if a_{ki} is odd the division by 4 is to be interpreted as multiplication by 4^*, and

$$\begin{aligned} I_{\mathrm{eq}}(\bar{m}_k'^i, \bar{m}_k^i) = 2p_1&\left(\bar{m}_k'^i, \bar{m}_k^i\right) - 3 \\ &+ \delta\left(a_{ki}, 1, \bar{b}_{ki}, \bar{m}_k^i\right) - \delta\left(a_{ki}, 1, \bar{b}_{ki}, \bar{m}_k'^i\right) \\ &+ \#\{p \in \{\bar{m}_k^i, \bar{m}_k'^i\} \mid p \not\equiv 0 \bmod a_{ki}\}, \end{aligned}$$

where

$$\delta(a, r, s, p) = \frac{2}{a} \sum_{k=1}^{a-1} \cot\left(\frac{\pi r k}{a}\right) \cot\left(\frac{\pi s k}{a}\right) \sin^2\left(\frac{\pi p k}{a}\right).$$

Then the plumbed homology 3-sphere $\Sigma(\Gamma)$ cannot be the boundary of a positive definite 4-manifold with no 2-torsion in the first homology.

Remark 1.7. Theorem 1.6 concerns only the bubbling phenomena on cone points [Lawson 88]. As pointed out by T. Lawson, we can also consider the bubbling on ordinary points and may obtain stronger conditions in principle.

This paper is organized as follows. In Section 2, we review several facts concerning the invariant studied in [Fukumoto and Furuta 00]. In Section 3, we recall a generalization of the notion of the plumbing to the 4-V-manifold category to apply this invariant to homology 3-spheres of plumbing type, and we describe the set of all V-spin^c structures on the plumbed 4-V-manifold in terms of the plumbing data. In Section 4, we consider the problem of negative definite cobordisms of plumbed homology 3-spheres and calculate the virtual dimension of the Seiberg–Witten moduli space over V-manifolds. In Section 5, we recall the definition of the Fintushel–Stern invariant in the case of plumbed V-manifolds and then apply the argument of [Lawson 88]. In Section 6, we give an explicit computation for plumbed homology 3-spheres to compare the Fintushel–Stern invariant and w-invariants.

The numerical values of the invariants and graphs and scatter plots of Brieskorn homology 3-spheres were obtained using the software Mathematica.

2. *w*-INVARIANT OF HOMOLOGY 3-SPHERES

Let X be an oriented 4-V-manifold with boundary Σ and with only isolated singular points in its interior. We assume that a neighborhood of each singular point is of the form of a cone on the quotient of the 3-sphere by a finite group action. Suppose X admits a spin^c structure c, has first Betti number $b_1(X)=0$, and has boundary Σ, a homology 3-sphere. Let W be a spin 4-manifold with boundary $-\Sigma$. Then we can patch them together to get the closed 4-V-manifold $X\cup_\Sigma W$. Since the normal neighborhood of the boundary $\Sigma\times[0,1]$ admits a unique spin^c structure, we can glue the spin^c structures on X and W along the boundary Σ to get a spin^c structure on $X\cup_\Sigma W$. Since $H^1(\Sigma;\mathbb{Z})=0$, there is a unique homotopy class of automorphisms of the spin^c structure on $\Sigma\times[1,0]$, and so we can patch them in a unique way up to homotopy.

In [Fukumoto and Furuta 00], we studied the following invariant for (Σ, X, c):

$$w(\Sigma,X,c) := \mathrm{ind}_V\ D(X\cup_\Sigma W)+\frac{1}{8}\,\mathrm{sign}\,(W).$$

Here $D(X\cup_\Sigma W)$ is the Dirac operator on the closed V-manifold $X\cup_\Sigma W$ associated to the spin^c structure c on $X\cup_\Sigma W$, and $b_2^+(W)$ (respectively $b_2^-(W)$) is the maximal dimension of positive (negative) definite subspace of $H^2(W,\partial W;\mathbb{R})\cong H^2(W;\mathbb{R})$ with respect to the quadratic form defined by the cup product. Note that each term of the right-hand side is an integer. The invariant $w(\Sigma,X,c)$ satisfies the following property [Fukumoto and Furuta 00]:

$$\begin{aligned} &w(\Sigma_0\#\Sigma_1, X_0\natural X_1, c_0\natural c_1)\\ &\quad = w(\Sigma_0,X_0,c_0)+w(\Sigma_1,X_1,c_1), \end{aligned}$$

where $\Sigma_0\#\Sigma_1$ is the connected sum of Σ_0 and Σ_1, and $X_0\natural X_1$ is the boundary connected sum of X_0 and X_1.

3. PLUMBED HOMOLOGY 3-SPHERES AND *V*-PLUMBING

In this section, we recall the generalization of the notion of plumbing [Neumann 81] to the 4-V-manifold category and consider the case that the boundary is a homology 3-sphere. In this paper, we consider plumbing only among smooth points. It is possible to consider plumbing among V-singular points, but it requires a more complicated treatment.

First we define a graph with Seifert data $\Gamma=(V,E,\omega,\varepsilon)$ as follows: (1) (V,E) is a one-dimensional simplicial complex consisting of a set of vertices $k\in V$ and a set of edges $e\in E$. (2) Each vertex $k\in V$ is assigned an unnormalized Seifert invariant:

$$\omega(k)=\{g_k;(a_{k1},b_{k1}),\ldots,(a_{kn_k},b_{kn_k})\},\quad k\in V,$$

where b_k, g_k are integers, and (a_{ki},b_{ki}) are coprime integers. (3) Each edge $e\in E$ has a sign $\varepsilon(e)=\pm$. The Euler number for the line V-bundle associated to the Seifert fibration is

$$e_k=\sum_{i=1}^{n_k}\frac{b_{ki}}{a_{ki}}.$$

A plumbed 4-V-manifold $P(\Gamma)$ is constructed from a graph with Seifert data Γ as follows. For each vertex $k\in V$, let L_k be the line V-bundle over a Riemannian V-surface Z_k of genus g_k whose associated S^1-V-bundle $S(L_k)$ is the Seifert-fibered space with Seifert invariant $\omega(k)$. Let $D(L_k)$ be the associated disk V-bundle. If two vertices k, k' are connected by an edge $e=(k,k')\in E$ with sign $\varepsilon(e)=\pm$, then we choose a local trivialization of each disk V-bundle $D(L_k)|_{D_{kk'}}\cong D_{kk'}\times D^2$ and glue them by the orientation-preserving map

$$\begin{aligned} \phi_{kk'}^{\varepsilon(e)}: D(L_k)|_{D_{kk'}} &\cong D_{kk'}\times D^2\ni(z,w)\\ &\longmapsto \begin{cases}(w,z)\ (\varepsilon(e)=+)\\ (\bar{w},\bar{z})\ (\varepsilon(e)=-)\end{cases}\in D_{k'k}\times D^2\\ &\cong D(L_{k'})|_{D_{k'k}}. \end{aligned}$$

The plumbed 4-V-manifold $P(\Gamma)$ has singularities in the form of a cone on the lens space. In the following discussion, we assume that the graph (V,E) is a tree and that all genera g_k are zero. If we denote by A_Γ the intersection matrix of $P(\Gamma)$, then the (k,k')-entry of A_Γ is

$$(A_\Gamma)_{k,k'}=\begin{cases}e_k & k=k',\\ 1, & (k,k')\in E,\\ 0, & \text{otherwise}.\end{cases}$$

Let $\bar{\Gamma}=\left(\bar{V},\bar{E},\bar{\omega},\bar{\varepsilon}\right)$ be the graph with Seifert invariants such that

$$\begin{aligned} &\bar{V}=V,\ \bar{E}=E,\\ &\bar{\omega}(k)=\{g_k;\left(a_{k1},\bar{b}_{k1}\right),\ldots,\left(a_{kn_k},\bar{b}_{kn_k}\right)\},\\ &\bar{\varepsilon}(e)=-\varepsilon(e), \end{aligned}$$

where $\bar{b}_{ki} = -b_{ki}$, so that

$$\bar{e}_k = \sum_{i=1}^{n_{ki}} \frac{\bar{b}_{ki}}{a_{ki}} = -\sum_{i=1}^{n_k} \frac{b_{ki}}{a_{ki}} = -e_k.$$

Then $P(\bar{\Gamma}) = -P(\Gamma)$ and $A_{\bar{\Gamma}} = -A_\Gamma$. In fact, the orientation of $D(L_k)$ in $P(\Gamma)$ is induced by that of the base space Z_k and the fiber $D^2 \subset \mathbb{C}$, and if we define an orientation-reversing diffeomorphism $\iota_k : D(L_k) \to D(\bar{L}_k)$ as the complex conjugates of the fibers, then the orientation-preserving gluing map $\phi^+_{kk'} : D_{kk'} \times D^2 \to D_{k'k} \times D^2$ given by $\phi^+_{kk'}(z,w) = (w,z)$ induces the orientation-preserving gluing map $\phi^-_{k'k} = (\iota_{k'}|_{D_{k'k}\times D^2}) \circ \phi^+_{k'k} \circ (\iota_k|_{D_{kk'}\times D^2})^{-1}$ given by $\phi^-_{kk'}(z,w) = (\iota_{k'}|_{D_{k'k}\times D^2}) \circ \phi^+_{k'k}(z,\bar{w}) = \iota_{k'}(\bar{w},z) = (\bar{w},\bar{z})$. Let us denote the boundary of the plumbing $P(\Gamma)$ by $\Sigma(\Gamma)$. Note also that $\Sigma(\bar{\Gamma}) = -\Sigma(\Gamma)$. Now we see that $\Sigma(\Gamma)$ is a homology 3-sphere if and only if the following conditions are satisfied:

Condition HS

1. Γ is a tree graph.
2. $g_k = 0$ for $k \in V$.
3. $\det A_\Gamma = \pm \frac{1}{\prod_{k\in V} a_k}$, $a_k := \prod_i a_{ki}$.

From now on we assume that Γ satisfies Condition HS, and we omit g_k from the notation of the Seifert invariants $\omega(k)$. Then $P(\Gamma)$ is V-spin if and only if one of the following conditions holds for each vertex $k \in V$. Note that if $P(\Gamma)$ has a V-spin structure, then it is unique:

1. One of the a_{ki} is even.
2. All a_{ki} are odd and $a_k e_k$ is even.

Let $\tilde{L}_k$ be the line V-bundle over $P(\Gamma)$ defined by extending trivially the pullback $p_k^* L_k$ of the line V-bundle $p_k : L_k \to Z_k$. Then the tautological section of $p_k^* L_k$ on $D(L_k) \subset P(\Gamma)$ can be extended trivially to $\tilde{L}_k$ on $P(\Gamma)$, and its zero set intersects Z_k with intersection number e_k. We fix a V-complex structure and a V-Hermitian metric on $P(\Gamma)$ that comes from the base Riemannian V-surfaces Z_k and the fibers $\mathbb{C}$ corresponding to each vertex in Γ.

Then we have the canonical V-spin^c structure on $P(\Gamma)$ whose associated line V-bundle is the canonical line V-bundle K. Let $S^\pm_{\mathrm{can}}$ be the spinor V-bundle associated with the canonical V-spin^c structure. Since the set of all V-spin^c structures on $P(\Gamma)$ is the affine space over $\mathrm{Pic}^{\mathrm{t}}_V(P(\Gamma))$, we take the canonical V-spin^c structure as a reference point, and we have the following theorem.

Theorem 3.1. *Suppose $\Gamma = (V, E, \omega)$ satisfies Condition HS. Then there is a one-to-one correspondence between the set of all V-spin^c structures on $P(\Gamma)$ and the lattice $\mathbb{Z}^{\#V}$, and $\vec{m} = (m_1, \ldots, m_{\#V}) \in \mathbb{Z}^{\#V}$ corresponds to the V-spin^c structure whose associated spinor V-bundle is*

$$S^\pm_{\mathrm{can}} \otimes \bigotimes_{k\in V} \tilde{L}_k^{m_k}.$$

Remark 3.2. The line V-bundle associated to the V-spin^c structure on $P(\Gamma)$ that corresponds to $\vec{m} \in \mathbb{Z}^{\#V}$ is the determinant line V-bundle of $S^\pm_{\mathrm{can}} \otimes \bigotimes_{k\in V} \tilde{L}_k^{m_k}$:

$$K^{-1} \otimes \bigotimes_{k\in V} \tilde{L}_k^{2m_k},$$

where K is the canonical line V-bundle of $P(\Gamma)$.

4. THE VIRTUAL DIMENSION OF THE SEIBERG–WITTEN MODULI SPACE

Let $\Gamma = (V, E, \omega)$ be a graph with Seifert data and let $P(\Gamma)$ be the 4-V-manifold obtained by plumbing according to Γ. We assume that Γ satisfies Condition HS. Then the boundary $\Sigma(\Gamma) = \partial P(\Gamma)$ is a homology 3-sphere. We take a V-spin^c structure $c(\vec{m})$ ($\vec{m} \in \mathbb{Z}^{\#V}$) on $P(\Gamma)$ whose associated spinor V-bundle over $P(\Gamma)$ is $S^\pm_{c(\vec{m})} = S^\pm_{\mathrm{can}} \otimes \bigotimes_{k\in V} \tilde{L}_k^{m_k}$ and the associated spin^c line V-bundle over $P(\Gamma)$ is $\tilde{L}_{c(\vec{m})} = K^{-1} \otimes \bigotimes_{k\in V} \tilde{L}_k^{-2m_k}$. Let $k^+(\Gamma)$ (respectively $k^-(\Gamma)$) be the number of positive (negative) eigenvalues of the intersection matrix A_Γ.

Theorem 4.1. *Let Γ be a graph with Seifert invariants satisfying Condition HS and such that $k^+(\Gamma) = 0$. Suppose the homology 3-sphere $\Sigma(\Gamma)$ bounds a positive definite 4-manifold W. Then the virtual dimension of the moduli space $\mathcal{M}^{\mathrm{SW}}_{c(\vec{m})}(Z)$ of Seiberg–Witten monopoles on the negative definite closed 4-V-manifold $Z = P(\Gamma) \cup_{\Sigma(\Gamma)} (-W)$ with spin^c structure $c(\vec{m})$ for some $\vec{m} \in \mathbb{Z}^{\#V}$ is given by*

$$\begin{aligned}
&\mathrm{vir\,dim}\,\mathcal{M}^{\mathrm{SW}}_{c(\vec{m})}(Z) \\
&\quad = 2w(\Sigma(\Gamma), P(\Gamma), c(\vec{m})) + \frac{b_2^+(W)}{4} - 1 \\
&\quad = \frac{1}{4}\Bigg[{}^t\vec{s}A_\Gamma\vec{s} - k^+(\Gamma) + b_2^+(W) \\
&\qquad - \sum_{k\in V}\sum_{i=1}^{n_k}\frac{1}{a_{ki}}\sum_{\ell=1}^{a_{ki}-1}\Bigg\{\cot\left(\frac{\pi\ell}{a_{ki}}\right)\cot\left(\frac{\pi b_{ki}\ell}{a_{ki}}\right) \\
&\qquad + 2\cos\left(\frac{\pi(1+(2m_k+1)b_{ki})\ell}{a_{ki}}\right)\csc\left(\frac{\pi\ell}{a_{ki}}\right) \\
&\qquad \times \csc\left(\frac{\pi b_{ki}\ell}{a_{ki}}\right)\Bigg\}\Bigg] - 1.
\end{aligned}$$

Proof: The result follows from the fact that

$$\begin{aligned}&\mathrm{vir}\dim_{\mathbb{R}}\mathcal{M}^{\mathrm{SW}}_{c(\vec{m})}(Z)\\&=\mathrm{ind}^{\mathbb{R}}_V\left[\left(d^*\oplus d^+\right)(Z)\oplus D_A(Z):\Omega^1(i\mathbb{R})\oplus\Gamma\left(S^+\right)\right.\\&\qquad\left.\to\Omega^0(i\mathbb{R})\oplus\Omega^2_+(i\mathbb{R})\oplus\Gamma\left(S^-\right)\right],\end{aligned}$$

where the V-indices of the operators $(d^*\oplus d^+)(Z)$ and $D_A(Z)$ over the negative definite closed 4-V-manifold Z are given by

$$\begin{aligned}\mathrm{ind}^{\mathbb{R}}_V\left(d^*\oplus d^+\right)(Z)&=b_1(Z)-\left(b_0(Z)+b_2^+(Z)\right)\\&=0-(1+0)=-1\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\mathrm{ind}^{\mathbb{R}}_V D_A(Z)&=\frac{1}{2}\mathrm{ind}^{\mathbb{R}}_V D_A(P(\Gamma)\cup_{\Sigma(\Gamma)}(-W))\\&=w(\Sigma(\Gamma),P(\Gamma),c(\vec{m}))-\frac{\mathrm{sign}(-W)}{8}\\&=w(\Sigma(\Gamma),P(\Gamma),c(\vec{m}))+\frac{\mathrm{sign}(W)}{8}.\end{aligned}$$

The explicit formula is calculated in [Fukumoto 00] using the V-index formula of [Kawasaki 81]:

$$\begin{aligned}&\mathrm{ind}_V\left[D_A:\Gamma\left(S^+\otimes L\right)\to\Gamma\left(S^-\otimes L\right)\right]\\&=\int_Z\hat{A}(TZ)\,\mathrm{ch}(K^{-1/2})\mathrm{ch}(L)\\&\quad+\sum_{k\in V}\sum_{i=1}^{n_k}\frac{1}{a_{ki}}\sum_{\substack{\gamma\in\mathbb{Z}/a_{ki}\\ \gamma\neq1}}\frac{\mathrm{ch}_\gamma\left(S^+-S^-\right)\mathrm{ch}_\gamma(L)}{\mathrm{ch}_\gamma\wedge_{-1}\left(N^\gamma\otimes\mathbb{C}\right)}[p_{ki}]\\&=\int_Z\left(-\frac{p_1}{24}+\frac{1}{8}c_1\left(K^{-1}\otimes L^2\right)^2\right)\\&\quad+\sum_{k\in V}\sum_{i=1}^{n_k}\frac{1}{a_{ki}}\\&\quad\times\sum_{\substack{\zeta_{ki}\in\mathbb{Z}/a_{ki}\\ \zeta_{ki}\neq1}}\frac{\zeta_{ki}^{1/2}\zeta_{ki}^{b_{ki}/2}}{\left(\zeta_{ki}^{1/2}-\zeta_{ki}^{-1/2}\right)\left(\zeta_{ki}^{b_{ki}/2}-\zeta_{ki}^{-b_{ki}/2}\right)}\cdot\zeta_{ki}^m.\end{aligned}$$

By the V-spin theorem,

$$\begin{aligned}&\mathrm{sign}(Z)\\&=\int_Z\frac{p_1}{3}+\sum_{k\in V}\sum_{i=1}^{n_k}\frac{1}{a_{ki}}\\&\quad\times\sum_{\substack{\zeta_{ki}\in\mathbb{Z}/a_{ki}\\ \zeta_{ki}\neq1}}\frac{\zeta_{ki}^{1/2}+\zeta_{ki}^{-1/2}}{\zeta_{ki}^{1/2}-\zeta_{ki}^{-1/2}}\frac{\zeta_{ki}^{b_{ki}/2}+\zeta_{ki}^{-b_{ki}/2}}{\zeta_{ki}^{b_{ki}/2}-\zeta_{ki}^{-b_{ki}/2}},\end{aligned}$$

we have

$$\begin{aligned}&\mathrm{ind}_V\left[D_A:\Gamma\left(S^+\otimes L\right)\to\Gamma\left(S^-\otimes L\right)\right]\\&=\frac{1}{8}\int_Z c_1\left(K^{-1}\otimes L^2\right)^2\\&\quad-\frac{1}{8}\mathrm{sign}(Z)\\&\quad+\frac{1}{8}\sum_{k\in V}\sum_{i=1}^{n_k}\frac{1}{a_{ki}}\\&\quad\times\sum_{\substack{\zeta_{ki}\in\mathbb{Z}/a_{ki}\\ \zeta_{ki}\neq1}}\frac{\zeta_{ki}^{1/2}+\zeta_{ki}^{-1/2}}{\zeta_{ki}^{1/2}-\zeta_{ki}^{-1/2}}\frac{\zeta_{ki}^{b_{ki}/2}+\zeta_{ki}^{-b_{ki}/2}}{\zeta_{ki}^{b_{ki}/2}-\zeta_{ki}^{-b_{ki}/2}}\\&\quad+\sum_{k\in V}\sum_{i=1}^{n_k}\frac{1}{a_{ki}}\\&\quad\times\sum_{\substack{\zeta_{ki}\in\mathbb{Z}/a_{ki}\\ \zeta_{ki}\neq1}}\frac{\zeta_{ki}^{(1+b_{ki}+2m)/2}}{\left(\zeta_{ki}^{1/2}-\zeta_{ki}^{-1/2}\right)\left(\zeta_{ki}^{b_{ki}/2}-\zeta_{ki}^{-b_{ki}/2}\right)}.\end{aligned}$$

□

Theorem 4.2. *Let X be a closed 4-V-manifold with $b_1(X)=0$, $b_2^+(X)=0$. Let c be a V-spin^c structure on X. Then the virtual dimension of the moduli space $\mathcal{M}^{\mathrm{SW}}_c(X)$ of the Seiberg–Witten monopoles on X with V-spin^c structure c is nonpositive:* $\mathrm{vir}\dim\mathcal{M}^{\mathrm{SW}}_c(X)\le0$.

Proof: Note that the Dirac operator D_A associated to the V-connection A on the determinant line-V-bundle $L=\det S^\pm\to X$ is $\mathbb{C}$-linear, and hence $\mathrm{ind}_V D_A$ is even. Hence if $b_1(X)=0$, $b_2^+(X)=0$, then the virtual dimension is odd and there is only one reducible solution modulo gauge equivalence by the Hodge theory on V-manifolds. Now suppose that the virtual dimension is positive: $2d+1$, $d\ge0$. By perturbing the equation if necessary, we make the moduli space $\mathcal{M}^{\mathrm{SW}}_c(X)$ a smooth compact orientable $(2d+1)$-dimensional manifold except for the unique singularity whose neighborhood $\mathcal{N}$ has the form of a cone over the complex projective space $\mathbb{CP}^d$. The base-point fibration $\mathcal{L}$ gives the S^1-bundle on $\mathcal{M}_0:=\mathcal{M}^{\mathrm{SW}}_c(X)-\mathcal{N}$ whose restriction to the link $\partial\mathcal{M}_0\cong\mathbb{CP}^d$ is isomorphic to the Hopf bundle $H\to\mathbb{CP}^d$. Since $\mathcal{M}_0$ is a compact oriented smooth manifold with boundary, it follows that

$$0=c_1(\mathcal{L})^d\left[\partial\mathcal{M}_0\right]=c_1(H)^d\left[\mathbb{CP}^d\right]=1,$$

which is a contradiction. □

Remark 4.3. This argument is applicable even for certain noncompact smooth 4-manifolds as long as the monopole moduli space is compact; see the argument in [Froyshov 96].

Then we have the following theorem.

Theorem 4.4. [Fukumoto and Furuta 00, Theorem 5] *Suppose Γ satisfies Condition HS and $k^+(\Gamma) = 0$. If the plumbed homology 3-sphere $\Sigma(\Gamma)$ bounds a positive definite 4-manifold, then we have*

$$w(\Sigma(\Gamma), P(\Gamma), c(\vec{m})) \leq 0$$

for any $\vec{m} \in \mathbb{Z}^{\# V(\Gamma)}$.

Proof: Suppose that $\Sigma(\Gamma)$ bounds a positive definite 4-manifold W. Using surgery to cut out the free part of $H^1(W;\mathbb{Z})$ if necessary, we may assume that $b_1(W) = 0$. Note that the plumbed 4-V-manifold $P(\Gamma)$ is negative definite, $b_2^+(P(\Gamma)) = 0$. Then the closed V-manifold $Z = P(\Gamma) \cup_{\Sigma(\Gamma)} (-W)$ satisfies $b_1(X) = 0$ and $b_2^+(X) = 0$. Then the virtual dimension of the moduli space $\mathcal{M}^{\mathrm{SW}}_{c(\vec{m})}$ is nonpositive,

$$\begin{aligned}\operatorname{vir}\dim \mathcal{M}^{\mathrm{SW}}_{c(\vec{m})}(Z) \\ = 2w(\Sigma(\Gamma), P(\Gamma), c(\vec{m})) + \frac{b_2^+(W)}{4} - 1 \leq 0,\end{aligned}$$

which means that

$$w(\Sigma(\Gamma), P(\Gamma), c(\vec{m})) \leq \frac{1}{2} - \frac{b_2^+(W)}{8} \leq \frac{1}{2}.$$

□

By the additivity under connected sums of the w-invariant, we have the following theorem.

Theorem 4.5. *Suppose $\Gamma_1, \ldots, \Gamma_s$ all satisfy Condition HS and $k^+(\Gamma_i) = 0$ $(i = 1, \ldots, s)$. If the connected sum $\Sigma(\Gamma_1)\#\cdots\#\Sigma(\Gamma_s)$ bounds a positive definite 4-manifold, then we have*

$$\sum_{i=1}^{s} w(\Sigma(\Gamma_i), P(\Gamma_i), c(\vec{m}_i)) \leq 0$$

for any $\vec{m}_i \in \mathbb{Z}^{\# V(\Gamma_i)}$.

In particular, we have the following corollary.

Corollary 4.6. *Suppose Γ satisfies Condition HS and $k^-(\Gamma) = 0$. If the connected sum of any number of copies of $\Sigma(\Gamma)$ bounds a positive definite 4-manifold, then $w(\Sigma(\Gamma), P(\Gamma), c(\vec{m})) \leq 0$ for any $\vec{m} \in \mathbb{Z}^{\# V(\Gamma)}$.*

5. FINTUSHEL–STERN INVARIANTS

The argument of R. Fintushel and R. Stern for Seifert homology 3-spheres can be extended to the case of plumbed homology 3-spheres. First we compute the Fintushel–Stern invariant of plumbed homology 3-spheres.

Let Γ be a graph with Seifert invariants satisfying Condition HS and $k^+(\Gamma) = 0$. Let $P(\Gamma)$ be the negative definite 4-V-manifold with boundary $\Sigma(\Gamma)$ obtained by plumbing according to Γ. Suppose that $\Sigma(\Gamma)$ bounds a positive definite 4-manifold W. Let $\mathcal{M}^{\mathrm{D}}_{\vec{m}}$ be the moduli space of self-dual V-connections on the positive definite closed 4-V-manifold $Z = W \cup_{\Sigma(\Gamma)} (-P(\Gamma))$ with $\mathrm{SO}(3)$-V-bundle $E = \tilde{L}(\vec{m}) \oplus \varepsilon$, where $\tilde{L}(\vec{m}) := \bigotimes_{k\in V} \tilde{L}_k^{m_k}$ and ε is the trivial $\mathbb{R}$-line V-bundle over Z. Then we have the following theorem.

Theorem 5.1.

$$\operatorname{vir}\dim \mathcal{M}^{\mathrm{D}}_{\vec{m}}(Z) = R(\Gamma;\vec{m}).$$

Proof: Note that the Pontryagin number $p_1(E)$ of the $\mathrm{SO}(3)$-V-bundle $E = \tilde{L}(\vec{m}) \oplus \varepsilon$ over $W \cup_{\Sigma(\Gamma)} (-P(\Gamma))$ is calculated to be

$$c_1\left(\tilde{L}(\vec{m})\right)^2 \left[W \cup_{\Sigma(\Gamma)} (-P(\Gamma))\right] = {}^t\vec{m} A_{\bar{\Gamma}} \vec{m},$$

where $A_{\bar{\Gamma}}$ is the intersection matrix of $P(\bar{\Gamma})$. Then this formula is obtained by the V-index formula of [Kawasaki 81] in the same way as the calculation in [Fintushel and Stern 85].

Let Z be the closed V-manifold $Z = W \cup_{\Sigma(\Gamma)} (-P(\Gamma))$. The virtual dimension of the moduli space $\mathcal{M}^{\mathrm{D}}_{\vec{m}}(Z)$ of the self-dual V-connection over Z associated to the $\mathrm{SO}(3)$-V-bundle $E = L \oplus \varepsilon$, $L = \tilde{L}(\vec{m})$ is given by the V-index of the elliptic operator $d_A^* \oplus d_A^-$,

$$\begin{aligned}\operatorname{vir}\dim \mathcal{M}^{\mathrm{D}}_{\vec{m}}(Z) \\ = \operatorname{ind}_V \left[d_A^* \oplus d_A^- : \Omega^1(\mathbf{g}_E) \to \Omega^0(\mathbf{g}_E) \oplus \Omega^2_-(\mathbf{g}_E)\right],\end{aligned}$$

where $\mathbf{g}_E = P_E \times_{\mathrm{Ad}} \mathrm{so}(3)$ is the adjoint bundle associated to the $\mathrm{SO}(3)$-bundle E. The V-index is equal to that of the twisted Dirac operator $D_{S^+\otimes \mathbf{g}_E} : \Gamma(S^+ \otimes S^- \otimes \mathbf{g}_E) \to \Gamma(S^- \otimes S^- \otimes \mathbf{g}_E)$ and is calculated using Kawasaki's V-index formula:

$$\begin{aligned}&\operatorname{ind}_V\left[d_A^*\oplus d_A^- : \Omega^1(\mathbf{g}_E)\to\Omega^0(\mathbf{g}_E)\oplus\Omega^2_-(\mathbf{g}_E)\right]\\&\quad=\operatorname{ind}_V\left[D_{S^-\otimes\mathbf{g}_E}:\Gamma\left(S^+\otimes S^-\otimes\mathbf{g}_E\right)\right.\\&\qquad\left.\to\Gamma\left(S^-\otimes S^-\otimes\mathbf{g}_E\right)\right]\\&\quad=\int_Z\hat{A}(Z)\operatorname{ch}\left(S^-\right)\operatorname{ch}\left(\mathbf{g}_E\otimes\mathbb{C}\right)\\&\qquad+\sum_{k\in V}\sum_{i=1}^{n_k}\frac{1}{a_{ki}}\\&\qquad\times\sum_{\substack{\gamma\in\mathbb{Z}/a_{ki}\\ \gamma\neq 1}}\frac{\operatorname{ch}_\gamma\left(S^+-S^-\right)\operatorname{ch}_\gamma\left(S^-\right)}{\operatorname{ch}_\gamma\wedge_{-1}\left(N^\gamma\otimes\mathbb{C}\right)}\operatorname{ch}_\gamma\left(\mathbf{g}_E\otimes\mathbb{C}\right)[p_{ki}],\end{aligned}$$

where p_{ki} is the ith cone point on $P(\Gamma)$ corresponding to the vertex $k\in V$. The first term is calculated as follows:

$$\begin{aligned}&\int_Z\hat{A}(Z)\operatorname{ch}\left(S^-\right)\operatorname{ch}\left(\mathbf{g}_E\otimes\mathbb{C}\right)\\&\quad=2\int_Z p_1(\mathbf{g}_E)+3\int_Z\hat{A}(Z)\operatorname{ch}\left(S^-\right)\\&\quad=2\int_Z c_1(L)^2+3\int_Z\hat{A}(Z)\operatorname{ch}\left(S^-\right),\end{aligned}$$

where for the last equality the adjoint bundle $\mathbf{g}_E = P_E\times_{\mathrm{Ad}}\mathrm{so}(3)$ is isomorphic to $L\oplus\varepsilon$, and hence $p_1(\mathbf{g}_E) = p_1(L\oplus\varepsilon)=p_1(L)=c_1(L)^2$. On the other hand, the V-index of the operator $d^*\oplus d^- : \Omega^1\to\Omega^0\oplus\Omega^2_-$ on the de Rham cohomology is calculated to be

$$\begin{aligned}\operatorname{ind}_V\left(d^*\oplus d^-\right)&=\operatorname{ind}_V\left(D_{S^-}\right)\\&=\int_Z\hat{A}(Z)\operatorname{ch}\left(S^-\right)\\&\quad+\sum_{k\in V}\sum_{i=1}^{n_k}\frac{1}{a_{ki}}\sum_{\substack{\gamma\in\mathbb{Z}/a_{ki}\\ \gamma\neq 1}}\chi(p_{ki},\gamma)[p_{ki}]\\&=b_1(Z)-\left(b_0(Z)+b_2^-(Z)\right)=-\frac{\chi-\sigma}{2},\end{aligned}$$

where we define

$$\chi(p_{ki},\gamma):=\frac{\operatorname{ch}_\gamma\left(S^+-S^-\right)\operatorname{ch}_\gamma\left(S^-\right)}{\operatorname{ch}_\gamma\wedge_{-1}\left(N^\gamma\otimes\mathbb{C}\right)}[p_{ki}].$$

If μ is the weight of the action $\gamma\in\mathbb{Z}/a_{ki}$ on the fiber, then the residual Chern character of $\mathbf{g}_E\otimes\mathbb{C}$ is given by

$$\begin{aligned}\operatorname{ch}_\gamma\left(\mathbf{g}_E\otimes\mathbb{C}\right)&=\operatorname{ch}_\gamma\left(L\oplus\bar{L}\oplus\underline{\mathbb{C}}\right)=\operatorname{ch}_\gamma(L)+\operatorname{ch}_\gamma\left(\bar{L}\right)+1\\&=\mu+\mu^{-1}+1.\end{aligned}$$

Therefore

$$\begin{aligned}&\operatorname{ind}_V\left[d_A^*\oplus d_A^- : \Omega^1(\mathbf{g}_E)\to\Omega^0(\mathbf{g}_E)\oplus\Omega^2_-(\mathbf{g}_E)\right]\\&\quad=2\int_Z c_1(L)^2+3\int_Z\hat{A}(Z)\operatorname{ch}\left(S^-\right)\\&\qquad+\sum_{k\in V}\sum_{i=1}^{n_k}\frac{1}{a_{ki}}\sum_{\substack{\gamma\in\mathbb{Z}/a_{ki}\\ \gamma\neq 1}}\chi(p_{ki},\gamma)\left(\mu+\mu^{-1}+1\right)\\&\quad=2\int_Z c_1(L)^2-3\left(\frac{\chi-\sigma}{2}\right)\\&\qquad+\sum_{k\in V}\sum_{i=1}^{n_k}\frac{1}{a_{ki}}\sum_{\substack{\gamma\in\mathbb{Z}/a_{ki}\\ \gamma\neq 1}}\chi(p_{ki},\gamma)\left(\mu+\mu^{-1}-2\right).\end{aligned}$$

The last term is calculated [Fintushel and Stern 85] to be

$$\begin{aligned}&\sum_{k\in V}\sum_{i=1}^{n_k}\frac{1}{a_{ki}}\sum_{\substack{\gamma\in\mathbb{Z}/a_{ki}\\ \gamma\neq 1}}\chi(p_{ki},\gamma)\left(\mu+\mu^{-1}-2\right)\\&\quad=\sum_{k\in V}\sum_{i=1}^{n_k}\frac{2}{a_{ki}}\sum_{\ell=1}^{a_{ki}-1}\cot\frac{\pi\ell}{a_{ki}}\cot\frac{\pi\bar{b}_{ki}\ell}{a_{ki}}\sin^2\frac{\pi\bar{m}_k^i\ell}{a_{ki}}\\&\qquad+\#\{(k,i)\mid\bar{m}_k^i\not\equiv 0 \bmod a_{ki},\ k\in V\}.\end{aligned}$$

Hence the assertion follows. □

We extend the argument of [Lawson 88] for Seifert fibered homology 3-spheres to the case of plumbed homology 3-spheres $\Sigma(\Gamma)$. Let $\mathcal{M}^{\mathrm{eq}}(\bar{m}_k'^i,\bar{m}_k^i)$ be the virtual dimension of the moduli space of equivariant instantons of the SO(3)-bundle over S^4 determined by representations of $\mathbb{Z}/a_{ki}$ in SO(3) corresponding to $\bar{m}_k^i$, $\bar{m}_k'^i$, and let $I_{\mathrm{eq}}(\bar{m}_k'^i,\bar{m}_k^i)$ be its virtual dimension. Let $p_1(\bar{m}_k'^i,\bar{m}_k^i)$ be the first Pontryagin number of the associated SO(3)-bundle. Then we have the following proposition.

Proposition 5.2. *Let L and L' be two line V-bundles over Z. Then $L\oplus\varepsilon$ and $L'\oplus\varepsilon$ are isomorphic as* SO(3)-*V-bundles over Z if and only if*

1. $e^2=e'^2\in\mathbb{Q}$;
2. $e\equiv e' \bmod 2\in H^2(Z_0;\mathbb{Z}/2)$;
3. $i^*(e)=i^*(e')\in H^2(\partial Z_0;\mathbb{Z}),\ i:\partial Z_0\hookrightarrow Z_0$.

Proof: Suppose $L\oplus\varepsilon$ and $L'\oplus\varepsilon$ are isomorphic. Then $p_1(L\oplus\varepsilon)=p_1(L'\oplus\varepsilon)\in H^4(Z;\mathbb{Q})$ and $w_2(L\oplus\varepsilon|_{Z_0}) = w_2(L'\oplus\varepsilon|_{Z_0})\in H^2(Z_0;\mathbb{Z}/2)$. Hence

$$\begin{aligned}e^2&=e(L)^2[Z]=p_1(L\oplus\varepsilon)[Z]=p_1(L'\oplus\varepsilon)[Z]\\&=e(L')^2[Z]=e'^2\end{aligned}$$

and

$$\begin{aligned} e &= e\left(L|_{Z_0}\right) = w_2\left(L \oplus \varepsilon|_{Z_0}\right) = w_2\left(L' \oplus \varepsilon|_{Z_0}\right) \\ &= e\left(L'|_{Z_0}\right) = e'. \end{aligned}$$

Let $V = \cup_i V_i$ be a disjoint union of the neighborhoods V_i of the singularities p_i, which are cones $V_i = (D_i \times \mathbb{C}) / (\mathbb{Z}/a_i)$ over lens spaces. Since $L \oplus \varepsilon|_{V_i}$ are isomorphic to the V-bundle $(D_i \times \mathbb{C}) \times (\mathbb{C} \times \mathbb{R}) / (\mathbb{Z}/a_i) \to V_i$ for some $\mathbb{Z}/a_i$-action on $(D_i \times \mathbb{C}) \times (\mathbb{C} \times \mathbb{R})$, $\zeta_i \cdot (z, w, u, t) = (\zeta_i z, \zeta_i^{b_i} w, \zeta_i^{k_i} u, t)$, and also for $L' \oplus \varepsilon|_{V_i}$, we have some $\mathbb{Z}/a_i$-action on $(D_i \times \mathbb{C}) \times (\mathbb{C} \times \mathbb{R})$, $\zeta_i \cdot (z, w, u, t) = (\zeta_i z, \zeta_i^{b_i} w, \zeta_i^{k_i'} u, t)$. Hence the isomorphism between $L \oplus \varepsilon|_{V_i}$ and $L' \oplus \varepsilon|_{V_i}$ induces one of corresponding $\mathbb{Z}/a_i$-actions on $(D_i \times \mathbb{C}) \times (\mathbb{C} \times \mathbb{R})$, and therefore $k_i \equiv k_i' \bmod a_i$. The weights k_i determine the Euler classes of the line bundles $L|_{\partial V_i} \cong (\partial (D_i \times \mathbb{C}) \times \mathbb{C}) / (\mathbb{Z}/a_i) \to \partial V_i$ and vice versa, and hence $i^*(e) = i^*(e') \in H^2(\partial Z_0; \mathbb{Z})$.

Conversely, assume that items 1–3 hold. Then by the condition $i^*(e) = i^*(e')$, we have an isomorphism of the line V-bundles $L|_V \cong L'|_V$, and hence $L \oplus \varepsilon$ and $L' \oplus \varepsilon$ are isomorphic over V. There are two obstructions to extending this isomorphism $L \oplus \varepsilon|_{\partial V} \cong L' \oplus \varepsilon|_{\partial V}$ as a usual SO(3)-bundle over all of Z_0 that lie in $H^2(Z_0, \partial Z_0; \mathbb{Z}/2)$ and $H^4(Z_0, \partial Z_0; \mathbb{Z})$. The first obstruction vanishes because $e \equiv e' \bmod 2$. For the second obstruction, although $p_1(L \oplus \varepsilon|_{Z_0})$, $p_1(L \oplus \varepsilon|_{Z_0})$ belong to $H^4(Z_0; \mathbb{Z})$, the difference $p_1(L \oplus \varepsilon|_{Z_0}) - p_1(L \oplus \varepsilon|_{Z_0})$ lies in $H^4(Z_0, \partial Z_0; \mathbb{Z})$, since $L \oplus \varepsilon|_V \cong L' \oplus \varepsilon|_V$, and therefore

$$\begin{aligned} &\left(p_1\left(L \oplus \varepsilon|_{Z_0}\right) - p_1\left(L \oplus \varepsilon|_{Z_0}\right)\right)[Z_0, \partial Z_0] \\ &\quad = p_1(L \oplus \varepsilon)[Z] - p_1(L \oplus \varepsilon)[Z] = 0, \end{aligned}$$

and the second obstruction vanishes. □

We have the following result.

Theorem 5.3. *Let Γ be a graph with Seifert invariants satisfying Condition HS and $k^+(\Gamma) = 0$. Suppose there exists an odd element $\vec{m} \in \mathbb{Z}^{\#V}$ such that*

1. ${}^t\vec{m}A_{\bar{\Gamma}}\vec{m} < 4$ *(< 2 if some a_{ki} is even);*
2. $R(\Gamma; \vec{m}) > 0$*;*
3. $\mu(\Gamma, \vec{m})$ *is odd;*
4. *for any odd $\vec{m}'$ such that $\bar{m}_k \equiv \bar{m}_k' \bmod 2$ for any $k \in V$, ${}^t\vec{m}'A_{\bar{\Gamma}}\vec{m}' < {}^t\vec{m}A_{\bar{\Gamma}}\vec{m}$, and $\bar{m}_k^i \equiv \bar{m}_k'^i \bmod 2$ for a_{ki} even, one of the following does not hold:*
   (P) ${}^t\vec{m}A_{\bar{\Gamma}}\vec{m} - {}^t\vec{m}'A_{\bar{\Gamma}}\vec{m}' = \sum_{k \in V} \sum_i p_1(\bar{m}_k'^i, \bar{m}_k^i)$.
   (I) $I_{\mathrm{eq}}(\bar{m}_k'^i, \bar{m}_k^i) \geq 0$ *for* $p_1(\bar{m}_k'^i, \bar{m}_k^i) < 4$ *(< 2 if a_{ki} even).*
   (E) *If* $p_1(\bar{m}_k'^i, \bar{m}_k^i) = 4/a_{ki}$*, then* $\mathcal{M}^{\mathrm{eq}}(\bar{m}_k'^i, \bar{m}_k^i) \neq \varnothing$.

Then the plumbed homology 3-sphere $\Sigma(\Gamma)$ cannot be the boundary of a positive definite 4-manifold with no 2-torsion in the first homology.

Remark 5.4. The above conditions can be written explicitly using the data of Γ. Let E be the reducible SO(3)-V-bundle $E = L \oplus \varepsilon$ over $S^4/(\mathbb{Z}/a)$ determined by the representations (a, r, s, l) (where (r, s) describes the $\mathbb{Z}/a$-action on the disk B^4, and l describes the action on the fiber of L) and $(a, -r, s; m)$. Then:

1. The first Pontryagin number $p_1(E)$ of E is given by
$$p_1(E) = \frac{4}{a} \deg(a, r, s, l, m)$$
and
$$\deg(a, r, s, l, m) \equiv r^* s^* \left(\frac{l^2 - m^2}{4}\right) \bmod \bar{a},$$
where $\bar{a} = a$ if a is odd and $\bar{a} = a/2$ if a is even, $l \equiv m \bmod 2$ if a is even, b^* denotes the inverse of $b \bmod a$ if it exists, and if a is odd, then the division by 4 is to be interpreted as multiplication by 4^* [Fintushel and Lawson 86, Lawson 88].
2. The virtual dimension $I_{\mathrm{eq}}(E)$ of the moduli space $\mathcal{M}^{\mathrm{eq}}(E)$ of self-dual V-connections on the SO(3)-V-bundle E over $S^4/(\mathbb{Z}/a)$ is given by
$$\begin{aligned} I_{\mathrm{eq}}(E) = 2p_1(E) - 3 + \delta(a, r, s, l) - \delta(a, r, s, m) \\ + \#\{p \in \{l, m\} | p \not\equiv 0 \bmod a\}, \end{aligned}$$
where
$$\begin{aligned} &\delta(a, r, s, p) \\ &\quad = \frac{2}{a} \sum_{k=1}^{a-1} \cot\left(\frac{\pi r k}{a}\right) \cot\left(\frac{\pi s k}{a}\right) \sin^2\left(\frac{\pi p k}{a}\right). \end{aligned}$$
3. The moduli space $\mathcal{M}^{\mathrm{eq}}(E)$ is nonempty if and only if $r + s \equiv \pm l \bmod a$ and $-r + s \equiv \pm m \bmod a$, [Lawson 88].

Proof: Suppose that the plumbed homology 3-sphere $\Sigma(\Gamma)$ is the boundary of a positive definite 4-manifold W with no 2-torsion in the first homology. By condition 1, the bubbling phenomena of instantons can occur only at the cone points in $Z = W \cup_{\Sigma(\Gamma)} (-P(\Gamma))$. By Uhlenbeck's removable singularity theorem [Uhlenbeck 82a, Uhlenbeck 82b], for a sequence of gauge equivalence classes $\{[A_i]\}$ with no convergent subsequence on $E = \tilde{L}(\vec{m}) \oplus \varepsilon$, there exist finite points $\{x_1, \dots, x_n\}$ on Z and a subsequence converging to $[A_\infty]$ over $Z - \{x_1, \dots, x_n\}$, and $[A_\infty]$ extends to a self-dual V-connection on a lower SO(3)-V-bundle E' over Z.

The lower V-bundle E' is obtained by regluing the fibration over the neighborhood of the cone points generating equivariant instantons on S^4 with charge conservation. Note that the set of all line V-bundles on $P(\Gamma)$ is generated by $\tilde{L}_k$'s, and Z is positive definite. Hence if the bubbling occurred at a cone point of $P(\Gamma)$ on $E = \tilde{L}(\vec{m}) \oplus \varepsilon$, then the limiting bundle would be of the form $E' = \tilde{L}(\vec{m}') \oplus \varepsilon$ of lower Euler number $c_1\left(\tilde{L}(\vec{m}')\right) = {}^t\vec{m}' A_{\bar{\Gamma}} \vec{m}' < {}^t\vec{m} A_{\bar{\Gamma}} \vec{m} = c_1\left(\tilde{L}(\vec{m})\right)$ for some $\vec{m}' \in \mathbb{Z}^{\#V}$. Note also that the second Stiefel–Whitney class $w_2(E)$ is nontrivial if and only if $\vec{m}$ is odd, i.e., some $\bar{m}_k = m_k \bar{b}_k$ is odd, where $a_k = \mathrm{lcm}(a_{ki})$, $\bar{b}_k = a_k \bar{e}_k$. In fact, $w_2(E)\left[\tilde{\Sigma}_k\right] \equiv 1 \bmod 2$ for the closed smooth oriented surface $\tilde{\Sigma}_k$ with $\tilde{\Sigma}_k^2 = a_k \bar{b}_k$ in $P(\Gamma)$ away from the singularities corresponding to the vertex $k \in V$ if and only if $\bar{m}_k$ is odd. In fact, the first Chern number evaluated on the surface $\tilde{\Sigma}_k$ is

$$\begin{aligned} c_1(L(\vec{m}))[\tilde{\Sigma}_k] &= c_1(\tilde{L}_k^{m_k})[\tilde{\Sigma}_k] = m_k c_1(\tilde{L}_k)[\tilde{\Sigma}_k] \\ &= \mathrm{lcm}(a_{ki}) m_k c_1(\tilde{L}_k)[\Sigma_k] = a_k m_k \frac{\bar{b}_k}{a_k} \\ &= m_k \bar{b}_k = \bar{m}_k, \end{aligned}$$

and hence

$$\begin{aligned} w_2(E)[\tilde{\Sigma}_k] &= w_2(L(\vec{m}) \oplus \varepsilon)[\tilde{\Sigma}_k] \equiv c_1(L(\vec{m}))[\tilde{\Sigma}_k] \\ &= m_k \bar{b}_k = \bar{m}_k \equiv 1 \bmod 2. \end{aligned}$$

Note that $w_2(E) = w_2(E')$ if and only if

$$\begin{aligned} w_2(L(\vec{m}))\left[\tilde{\Sigma}_k\right] &\equiv m_k \bar{b}_k = m'_k \bar{b}_k \\ &\equiv w_2(L(\vec{m}'))\left[\tilde{\Sigma}_k\right] \bmod 2 \end{aligned}$$

for any $k \in V$. Hence we have $\bar{m}_k \equiv \bar{m}'_k \bmod 2$ for any $k \in V$. By the condition that $\mu(\Gamma, \vec{m})$ is odd, the number of reductions of the SO(3)-V-bundle, the number of singularities of the moduli space is odd. Now [Lawson 88, Theorems 2 and 5] shows that by cutting the moduli space $\mathcal{M}^{\mathrm{D}}_{\vec{m}}(Z)$ and using dimension-counting arguments, we obtain a compact 1-dimensional manifold with an odd number of boundary components, which is a contradiction. □

6. COMPUTATIONS

6.1. Behaviors of the Fintushel–Stern invariants and *w*-invariants

Let Γ be the graph with Seifert invariants

$$\begin{aligned} &\omega = \{\{(2,-1),(3,1),(5,1)\},\{(2,59),(3,1),(5,1)\}\}, \\ &V = \{1,2\},\ E = \{(1,2)\}, \\ &\Gamma = (V, E, \omega). \end{aligned}$$

The behavior of the Fintushel–Stern invariant and w-invariant of the plumbed homology 3-sphere $\Sigma(\Gamma)$ under varying instanton numbers or spin^c structure is as follows:

1. Fintushel–Stern invariants $R(\Gamma; m)$ for the instanton numbers ${}^t m A_\Gamma m$. Numerical computation shows that

$$R(\Gamma, (177, -6)) = 5$$

with the no-bubbling condition true; see Figure 1. Hence we see that the plumbed homology 3-sphere $\Sigma(\Gamma)$ cannot be the boundary of any positive definite 4-manifold with no 2-torsion. The shape of the graph is nearly a lower convex paraboloid, so that we have more chance of obstruction by $R(m)$ for large values of m, but at the same

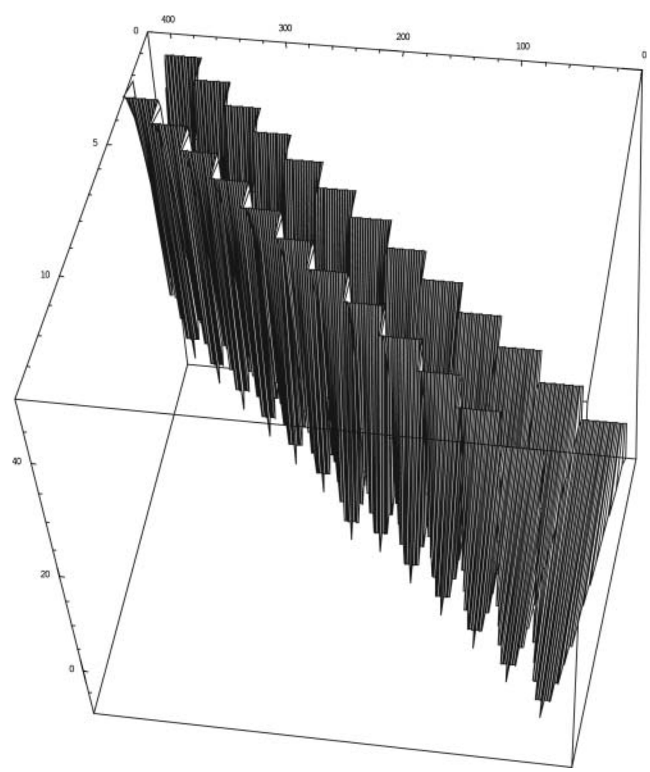

FIGURE 1. The graph of $R(\Gamma; m)$ for $-213 \le m_1 \le 207$, $-7 \le m_2 \le 7$ with scaling 20 times in the direction of m_2.

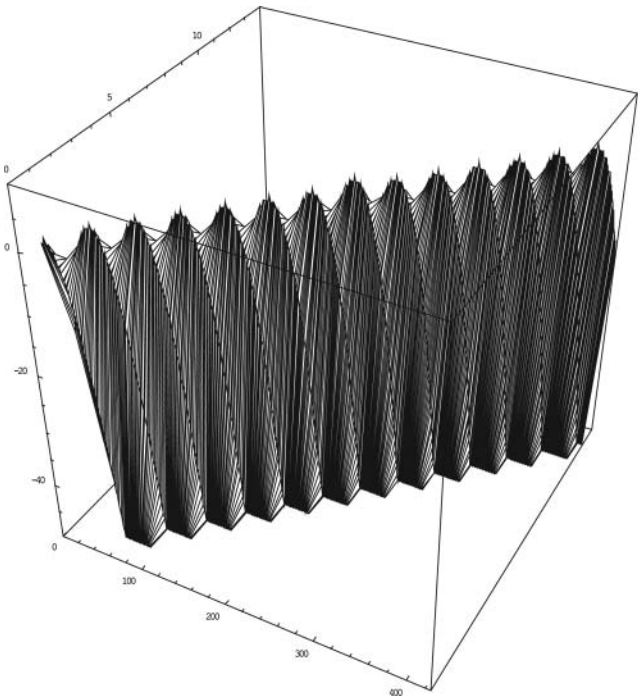

FIGURE 2. The graph of $w(\bar{\Gamma}, m)$ for $13287 \leq m_1 \leq 13713$, $443 \leq m_2 \leq 457$ with scaling 20 times in the direction of m_2.

time, it will be hard for a no-bubbling condition to hold, and the moduli space tends to be noncompact.

2. w-invariants $w(\Gamma, m)$ for spin^c structure m.
   Numerical computation shows that

$$w(\Gamma, (13320, 444)) = 1;$$

see Figure 2. Hence we see that the plumbed homology 3-sphere $\Sigma(\Gamma)$ cannot be the boundary of any positive-definite 4-manifold. Since the moduli space of the Seiberg–Witten monopoles on closed 4-V-manifold is compact for any V-spin^c structures, any integers m are available for obstruction using the w-invariant. However, the shape of the graph is nearly an upper convex paraboloid, so that for large values of m the moduli space tends to be empty.

6.2. Comparison of the Fintushel–Stern invariants and *w*-Invariants

6.2.1. Brieskorn Homology 3-Spheres

Table 1 shows the total number of Brieskorn homology 3-spheres that are obstructions to the boundary of positive definite 4-manifolds using the Fintushel–Stern invariant $R(m)$, w-invariant with varying V-spin^c structures, and w-invariant for V-spin structure. For example, the row corresponding to pattern (5) in the table means that the total number of Brieskorn homology 3-spheres $\Sigma(a_1, a_2, a_3)$, $2 \leq a_1 < a_2 < a_3 \leq 100$, that are obstructed by $R(m)$ and $w(\text{spin})$ but are not obstructed by $w(m)$ is 72. Here $w(\text{spin})$ is the w-invariant $w(m_{\text{spin}})$ corresponding to the V-spin structure m_{spin}; see Remark 1.4.

Pattern	$R(m)$	$w(m)$	$w(\text{spin})$	Total
(1)	1	1	1	16676
(2)	1	1		17373
(3)	1		1	0
(4)	1			0
(5)		1	1	132
(6)		1		65
(7)			1	4627
(8)				5269
Total	34049	34246	21435	44142

TABLE 1. Table of the number of Seifert homology 3-spheres $\Sigma(a_1, a_2, a_3)$, $2 \leq a_1 < a_2 < a_3 \leq 100$, detected using the invariants $R(m)$, $w(m)$, $w(\text{spin})$. (1: detected, blank: not detected).

Note that there are no Brieskorn homology 3-spheres $\Sigma(a_1, a_2, a_3)$, $2 \leq a_1 < a_2 < a_3 \leq 100$, that are obstructed by $R(m)$ and $w(\text{spin})$ but are not obstructed by $w(m)$. Figures 3, 4, and 5 show scatter plots of all of the Brieskorn homology 3-spheres $\Sigma(a_1, a_2, a_3)$, $2 \leq a_1 < a_2 < a_3 \leq 100$, that (i) are obstructed using $R(m)$ and $w(m)$ and so cannot be the boundary of any positive definite 4-manifold, (ii) are not obstructed by $R(m)$, $w(m)$, and $w(\text{spin})$ and so it is undetermined whether they can be the boundary of any positive definite 4-manifold (without 2-torsion), (iii) are obstructed by $w(m)$ and not by $R(m)$ and hence cannot be the boundary of any positive definite 4-manifold.

For the Brieskorn homology 3-sphere $\Sigma(7, 9, 43)$, the value of the Fintushel–Stern invariant is

$$R(\Sigma(7, 9, 43), 45) = 1$$

with the no-bubbling condition false, and in fact, any integer m with $m^2/(7 \cdot 9 \cdot 43) < 4$ such that $R(\Sigma(7, 9, 43), m) > 0$ does not satisfy the no-bubbling condition (Figure 6). So we can say nothing about $\Sigma(7, 9, 43)$ using the Fintushel–Stern invariant.

On the other hand, the value of the w-invariant is

$$w(\Sigma(7, 9, 43), 979) = 1$$

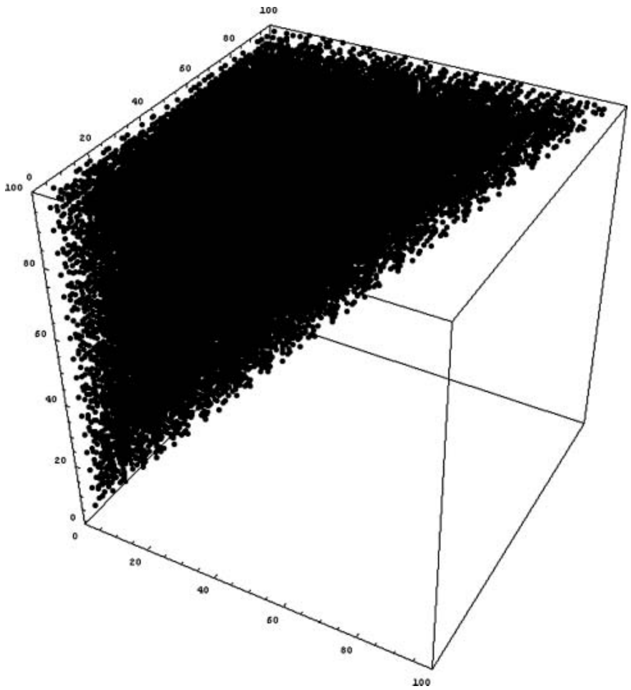

FIGURE 3. $\Sigma(a_1, a_2, a_3)$, $2 \leq a_1 < a_2 < a_3 \leq 100$, that are obstructed by $R\,(m)$ and $w\,(m)$ and so cannot be the boundary of any positive definite 4-manifold (without 2-torsion).

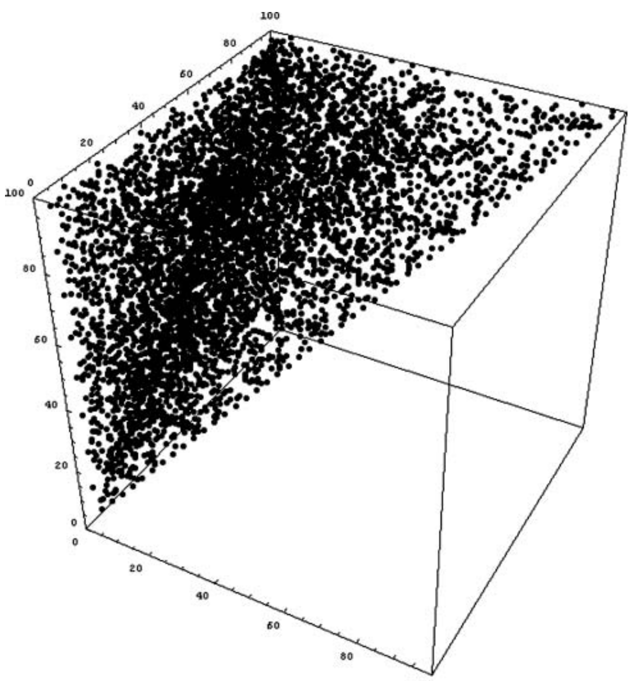

FIGURE 4. $\Sigma(a_1, a_2, a_3)$, $2 \leq a_1 < a_2 < a_3 \leq 100$, that are not obstructed by $R(m)$, $w(m)$, and $w(\text{spin})$, and hence we cannot determine whether they can be the boundary of any positive definite 4-manifold (without 2-torsion).

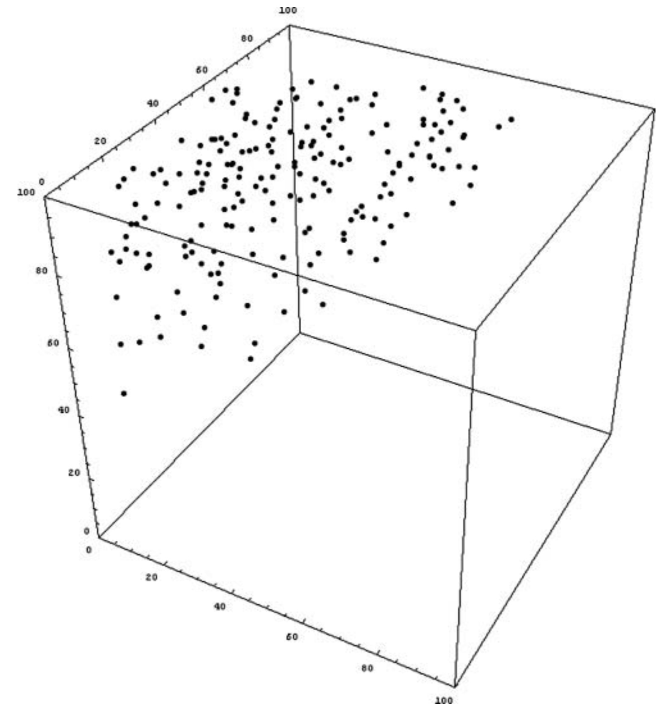

FIGURE 5. $\Sigma(a_1, a_2, a_3)$, $2 \leq a_1 < a_2 < a_3 \leq 100$, that are obstructed by $w\,(m)$ and not by $R\,(m)$.

(see Figure 7), so we see that $\Sigma\,(7, 9, 43)$ cannot be the boundary of any positive definite 4-manifold. Note that R may possibly give an obstruction if we take into consideration the bubbling phenomena on usual points; see Remark 1.7.

For the Brieskorn homology 3-sphere $\Sigma\,(67, 69, 73)$, the value of the Fintushel–Stern invariant is

$$R\,(\Sigma\,(67, 69, 73)\,, 561) = 1$$

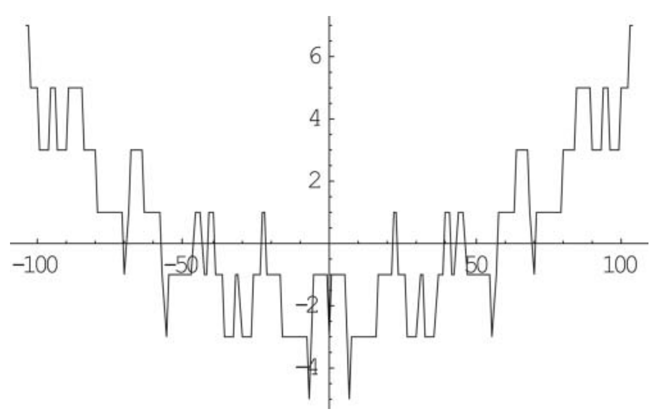

FIGURE 6. The graph of $R(7, 9, 43; m)$.

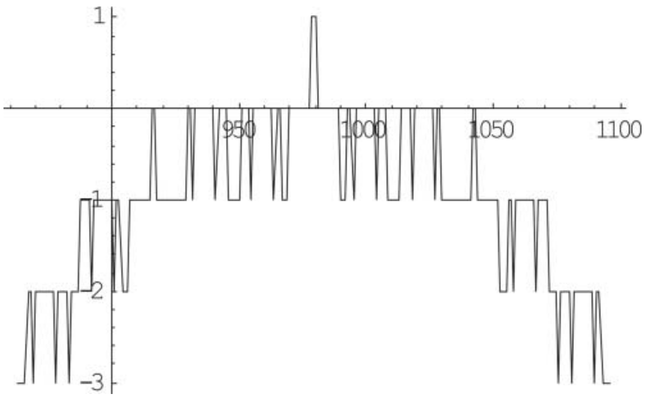

FIGURE 7. The graph of $w(7, 9, 43; m)$.

with no-bubbling condition false, and in fact, any integer m with $m^2/(67 \cdot 69 \cdot 73) < 4$ such that $R(\Sigma(67, 69, 73), m) > 0$ does not satisfy the no-bubbling condition (Figure 8). Hence we cannot say that $\Sigma(67, 69, 73)$ is the boundary of any positive definite 4-manifold without 2-torsion in the first homology.

On the other hand, the value of the w-invariant is

$$\begin{aligned} &w(\Sigma(67, 69, 73), 161464) \\ &\quad = 0 = \max\{w(\Sigma(67, 69, 73), m) \,|\, m \in \mathbb{Z}\} \end{aligned}$$

(Figure 9), so we cannot say that $\Sigma(67, 69, 73)$ cannot be the boundary of any positive definite 4-manifold from the w-invariant.

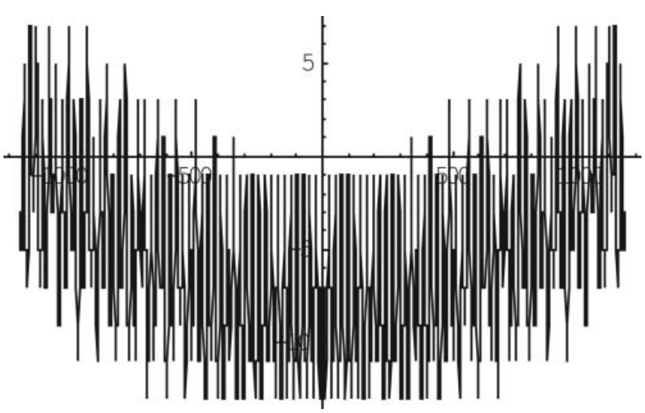

FIGURE 8. The graph of $R(67, 69, 73; m)$.

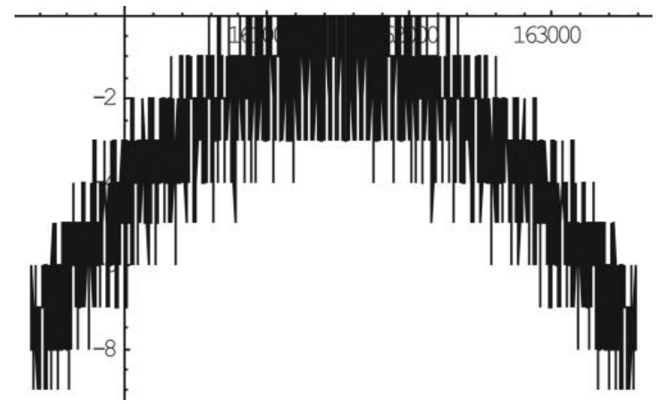

FIGURE 9. The graph of $w(67, 69, 73; m)$.

6.2.2. Plumbed Homology 3-Spheres

For the plumbing graphs Γ,

$$\begin{aligned} \omega &= \{\{(a_{11}, b_{11}), (a_{12}, b_{12}), (a_{13}, b_{13})\}, \\ &\qquad \{(a_{21}, b_{21}), (a_{22}, b_{22}), (a_{23}, b_{23})\}\}, \\ V &= \{1, 2\},\ E = \{(1, 2)\}, \\ \Gamma &= (V, E, \omega), \end{aligned}$$

with $2 \le a_{11} < a_{12} < a_{13} \le 7$ and $2 \le a_{21} < a_{22} < a_{23} \le 7$ and with $(a_{11}, a_{12}, a_{13}) \le (a_{21}, a_{22}, a_{23})$ in alphabetical order. We see that the Fintushel–Stern invariants obstruct all 36 of the corresponding plumbed homology 3-spheres $\Sigma(\Gamma)$ from bounding a positive definite 4-manifold with no 2-torsion in the first homology. On the other hand, the w-invariant obstructs them except for the plumbed homology 3-spheres $\Sigma(\Gamma)$ corresponding to the graph with Seifert invariants as given in Table 2.

For example, let Γ be the graph with Seifert invariants in the first row

$$\begin{aligned} \omega &= \{\{(2, -3), (3, 2), (7, 6)\}, \{(2, 81), (3, 2), (7, 6)\}\}, \\ V &= \{1, 2\},\ E = \{(1, 2)\}, \\ \Gamma &= (V, E, \omega). \end{aligned}$$

The Fintushel–Stern invariant and the w-invariant of the corresponding plumbed homology 3-sphere $\Sigma(\Gamma)$ are calculated as follows:

$$R(\Gamma, (251, -6)) = 1$$

with no-bubbling condition true. So this example means that this plumbed homology 3-sphere $\Sigma(\Gamma)$ cannot be the boundary of any positive definite 4-manifold with no 2-torsion in the first homology. On the other hand,

$$w(\Gamma, (38636, 920)) = 0 = \max\{w(\Gamma, \vec{m}) \,|\, \vec{m} \in \mathbb{Z}^2\},$$

$\{\{(2,-3),(3,2),(7,6)\},\{(2,81),(3,2),(7,6)\}\},$	$\{\{(2,-3),(3,2),(7,6)\},\{(2,81),(5,4),(7,5)\}\},$
$\{\{(2,-3),(3,2),(7,6)\},\{(3,122),(4,3),(5,3)\}\},$	$\{\{(2,-3),(3,2),(7,6)\},\{(4,163),(5,2),(7,6)\}\},$
$\{\{(2,-3),(3,2),(7,6)\},\{(5,203),(6,5),(7,4)\}\},$	$\{\{(2,-3),(5,4),(7,5)\},\{(2,137),(5,4),(7,5)\}\},$
$\{\{(2,-3),(5,4),(7,5)\},\{(3,206),(4,3),(5,3)\}\},$	$\{\{(2,-3),(5,4),(7,5)\},\{(4,275),(5,2),(7,6)\}\},$
$\{\{(2,-3),(5,4),(7,5)\},\{(5,343),(6,5),(7,4)\}\},$	$\{\{(3,-4),(4,3),(5,3)\},\{(3,176),(4,3),(5,3)\}\},$
$\{\{(3,-4),(4,3),(5,3)\},\{(4,235),(5,2),(7,6)\}\},$	$\{\{(3,-4),(4,3),(5,3)\},\{(5,293),(6,5),(7,4)\}\},$
$\{\{(4,-5),(5,2),(7,6)\},\{(4,555),(5,2),(7,6)\}\},$	$\{\{(4,-5),(5,2),(7,6)\},\{(5,693),(6,5),(7,4)\}\},$
$\{\{(5,-7),(6,5),(7,4)\},\{(5,1043),(6,5),(7,4)\}\}$	

TABLE 2. A list of plumbed homology spheres such that the Fintushel-Stern invariant give obstructions to bounding positive definite (or negative definite) manifolds, while the w-invariants does not.

so the w-invariant does not give an obstruction for $\Sigma(\Gamma)$ to be the boundary of positive definite 4-manifolds in this case.

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