Finding Patterns Avoiding Many Monochromatic Constellations

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CONTENTS

Given fixed $0 = q_0 < q_1 < q_2 < \cdots < q_k = 1$, a constellation in $[n]$ is a scaled translated realization of the q_i with all elements in $[n]$, i.e.,

$$p, \ \ p+q_1d, \ \ p+q_2d, \ \ \dots, \ \ p+q_{k-1}d, \ \ p+d.$$

We consider the problem of minimizing the number of monochromatic constellations in a two-coloring of $[n]$. We show how, given a coloring based on a block pattern, to find the number of monochromatic solutions to a lower-order term, and also how experimentally we might find an optimal block pattern. We also show for the case $k = 2$ that there is always a block pattern that beats random coloring.

1. INTRODUCTION

A *constellation pattern* is given by

$$\mathcal{Q} = [q_0 = 0, q_1, q_2, \dots, q_{k-1}, q_k = 1]$$

with q_i rational and $0 < q_1 < q_2 < \cdots < q_{k-1} < 1$. Given $\mathcal{Q}$, a *constellation* in $[n] = \{1, 2, \dots, n\}$ is

$$p, \ \ p+q_1d, \ \ p+q_2d, \ \ \dots, \ \ p+q_{k-1}d, \ \ p+d,$$

where each term is in $[n]$. In other words, a constellation in $[n]$ is a scaled and translated copy of the constellation pattern. We allow for d to be negative (i.e., the pattern to be reflected), so that it does not matter whether we work with $[0, q_1, q_2, \dots, q_{k-1}, 1]$ or $[0, 1-q_{k-1}, 1-q_{k-2}, \dots, 1-q_1, 1]$ (the mirrored version of the pattern).

The most-studied example of constellations is that of k-term arithmetic progressions, which correspond to the case $q_i = i/(k-1)$ for $i = 0, 1, \dots, k$. Another example is solutions to equations of the form $ax + by = (a+b)z$, where $x, y, z \in [n]$, which corresponds to $[0, a/(a+b), 1]$.

For any constellation pattern $\mathcal{Q}$ we will let D be the least common denominator of the q_i. The number of constellations in $[n]$ is $n^2/D + O(n)$ if the pattern is not

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symmetric and $n^2/(2D)+O(n)$ if the pattern is symmetric. One way to see this is to pick two elements $p, q \in [n]$ (which can be done in n^2 ways), at which point p and q are the start and end of a constellation if and only if $D \mid (p-q)$, which happens with probability $1/D$, giving $n^2/D + O(n)$ constellations. When the pattern is symmetric we can interchange p and q, so we divide by 2.

A natural question that arises is the following: Given a constellation $\mathcal{Q}$ and a fixed number r of colors, can we color $[n]$ in such a way as to avoid having a monochromatic constellation (i.e., one in which all the $p + q_i d$ are colored with the same color)? The answer to this is a resounding no, in that not only must we have monochromatic constellations for n large, but a positive fraction of all constellations must be monochromatic.

Fact 1.1. *For any constellation pattern $\mathcal{Q}$ there is a constant $c(\mathcal{Q})$ such that for* every *r-coloring of $[n]$ there are at least $c(\mathcal{Q})n^2$ monochromatic constellations.*

To see this, given $\mathcal{Q}$, we note that the constellation corresponding to the arithmetic progression of length $D+1$ contains all the q_i in $\mathcal{Q}$. In particular, the constellation $\mathcal{Q}$ is contained in an arithmetic progression, so if there are monochromatic progressions, there must also be monochromatic constellations. So it suffices to show that for any two-coloring of $[n]$, there must be at least $d(k)n^2$ monochromatic arithmetic progressions of length k, where $d(k) > 0$ is some constant. This was proved for the case $k = 3$ in [Frankl et al. 88]. The same proof works for arbitrary k; we include it here for completeness.

By a theorem of [van der Waerden 27] there is number $W := W(r, k)$ such that any r-coloring of $[W]$ must have a monochromatic k-term arithmetic progression. We first note that as indicated above, there are $n^2/\big(2(W+1)\big)$ different arithmetic progressions of length W inside $[n]$. Each one of these must contain a monochromatic progression in an r-coloring of $[n]$. To correct for any overcounting, we note that each monochromatic progression in $[n]$ will be counted at most $\binom{W}{2}$ times, since there are at most $\binom{W}{2}$ ways for us to put the progression into $[W]$. Therefore there are at least n^2/W^3 monochromatic progressions.

Since we must have some positive fraction of the constellations be monochromatic, the next natural problem is to determine the smallest number of monochromatic constellations, i.e., the smallest coefficient γ such that there are $\gamma n^2 + o(n^2)$ monochromatic constellations for n arbitrarily large, and how to achieve this lower bound.

One obvious candidate is to consider random colorings. Since a constellation with $k+1$ points will be monochromatic with probability $1/2^k$, then by coloring randomly we get a coefficient of $\gamma = 1/(2^k D)$ if the constellation pattern is not symmetric and $\gamma = 1/(2^{k-1}D)$ if the pattern is symmetric.

In [Parrilo et al. 08], the authors considered this problem for three-term arithmetic progressions (which corresponds to the constellation $\mathcal{Q} = [0, 1/2, 1]$). They showed that by subdividing $[n]$ into 12 appropriately sized blocks, we can have $(117/2192)n^2+O(n)$ monochromatic constellations. Note that $117/2192 \approx 0.05337591\ldots < 1/16 = 0.0625$, so their coloring has significantly fewer progressions than a typical random coloring (roughly 85.4% of what we would expect if we colored randomly).

In this paper we show how one could find this subdivision for three-term arithmetic progressions experimentally. We also generalize the approach for other constellations and find colorings that beat random for four- and five-term arithmetic progressions as well as other constellation patterns. We show that for the constellation $[0, q, 1]$, there is a way of coloring $[n]$ that beats random. We also relate some of our techniques to problems not involving constellations and conclude with some open problems.

2. FINDING A COEFFICIENT OF A BLOCK COLORING

Given a coloring of $[n]$ where there are large runs of a single color, we naturally can group these runs into blocks. A block pattern $\mathcal{B} = \langle b_1, b_2, \ldots, b_m\rangle$ then represents the relative sizes of blocks to one another. Since we care only about the relative sizes of the blocks, we can scale all numbers by any constant. As an example the block pattern found by Parrilo et al. is

$$\langle 28, 6, 28, 37, 59, 116, 116, 59, 37, 28, 6, 28\rangle.$$

Pictorially, this is shown in Figure 1.

Closely related to a block pattern is a subdivision pattern $\mathcal{X} = \langle\!\langle \beta_0, \beta_1, \ldots, \beta_m\rangle\!\rangle$, which gives the subdivision of the interval $[0, 1]$ according to the block pattern. It is easy to go back and forth between these two. Namely, given a block pattern, the subdivision pattern is found

FIGURE 1. A good block coloring for avoiding three-term arithmetic progressions.

by letting

$$\beta_i = \frac{\sum_{j=1}^{i} b_j}{\sum_{j=1}^{m} b_j} \quad \text{for } i = 0, 1, \ldots, m,$$

while given a subdivision pattern, to find the block pattern we let $b_i = \beta_i - \beta_{i-1}$ for $i = 1, 2, \ldots, m$ and then, if desired, scale all the blocks by some constant.

Given a block pattern $\mathcal{B} = \langle b_i \rangle$ with corresponding subdivision pattern $\mathcal{X}$, the $\mathcal{B}$ coloring of $[n]$ is a two-coloring found by coloring with the first color (either red or black in this paper) all m with $\beta_{2i} n \leq m \leq \beta_{2i+1} n$, and with the second color (either blue or white) all m with $\beta_{2i-1} n \leq m \leq \beta_{2i} n$, with any blocks left over colored arbitrarily.

Theorem 2.1. *Given a constellation pattern $\mathcal{Q} = [q_0, q_1, \ldots, q_k]$ and a block pattern $\mathcal{B} = \langle b_1, \ldots, b_m \rangle$, the number of monochromatic constellations of $\mathcal{Q}$ in a $\mathcal{B}$-coloring of $[n]$ is*

$$\begin{cases} \frac{\alpha}{2D} n^2 + O(n) & \text{if } \mathcal{Q} \text{ is symmetric,} \\ \frac{\alpha}{D} n^2 + O(n) & \text{if } \mathcal{Q} \text{ is not symmetric,} \end{cases}$$

where

$$\alpha = \int_0^1 \int_0^1 \left(\prod_{i=0}^{k} \frac{1 + f\big(q_i x + (1-q_i) y\big)}{2} + \prod_{i=0}^{k} \frac{1 - f\big(q_i x + (1-q_i) y\big)}{2} \right) dy\, dx, \tag{2–1}$$

and

$$f(x) = \begin{cases} 1 & \text{for } \beta_{2i} \leq x \leq \beta_{2i+1}, \\ -1 & \text{for } \beta_{2i-1} \leq x \leq \beta_{2i}. \end{cases}$$

The function $f(x)$ is acting as an indicator function for whether we are in a red or a blue block. If we let $g(x, y)$ be the function inside the integral in (2–1), then $g(x, y)$ is also acting as an indicator function, but in this case it takes values 0 and 1, where $g(x, y) = 1$ if and only if x and y are (respectively) the start and end of a monochromatic constellation in $[0, 1]$. The basic idea is that if we know where the monochromatic constellations of the block pattern in $[0, 1]$ are, then we also know where the monochromatic constellations in $[n]$ are.

An important aspect about $g(x, y)$ is that it can change value only when (x, y) crosses a line of the form $q_i x + (1 - q_i) y = \beta_j$. In Figure 2 we have plotted the function for $\mathcal{Q} = [0, 1/2, 1]$ using the block pattern from Figure 1, where red indicates where $g(x, y) = 1$ and $f(x) = 1$, while blue indicates where $g(x, y) = 1$ and $f(x) = -1$ (i.e., location of red and blue progressions respectively), and white indicates where $g(x, y) = 0$ (i.e., a location where there are no progressions). In the figure we have also drawn all the lines of the form $q_i x + (1 - q_i) y = \beta_j$. In particular, note that *every* region where $g(x, y) = 1$ will be a convex polygon.

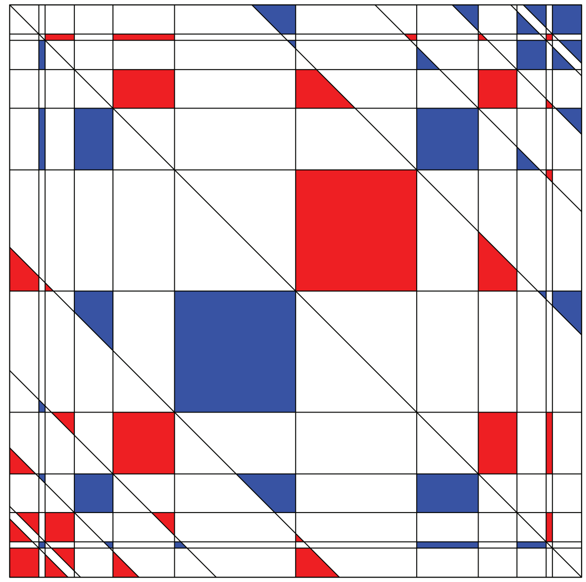


FIGURE 2. Indicator function for $\mathcal{Q} = [0, 1/2, 1]$ using the block pattern from Figure 1.

Proof: Let $C(\mathcal{Q}, \mathcal{B}, n)$ be the number of monochromatic constellations of $\mathcal{Q}$ in a $\mathcal{B}$-coloring of $[n]$. We now approximate the integral for $g(x, y)$ in terms of $C(\mathcal{Q}, \mathcal{B}, n)$.

We make the following claim: p and q are the start and end of a monochromatic constellation in the $\mathcal{B}$-coloring of $[n]$ if $D \mid (p - q)$ and $g(x, y) = 1$ in a neighborhood around $(p/n, q/n)$. Similarly, p and q are not the start and end of a monochromatic constellation in the $\mathcal{B}$-coloring of $[n]$ if $B \nmid (p - q)$ or $g(x, y) = 0$ in a neighborhood around $(p/n, q/n)$.

The divisibility condition follows from what was done in the introduction. The requirement $g(x, y) = 1$ is to ensure that each $q_i p + (1 - q_i) q$ is in the same color class as p and q. The reason we insist that it hold for a neighborhood is to avoid any ambiguity that might occur in the coloring on a border between blocks.

Subdivide $[0, 1] \times [0, 1]$ into squares of the form $[iD/n, (i+1)D/n] \times [jD/n, (j+1)D/n]$ for $0 \leq i, j \leq \lfloor n/D \rfloor$. The function $g(x, y)$ is not constant in a square and a small neighborhood of the square only if one of the lines $q_i x + (1 - q_i) y = \beta_j$ hits the square. Since there are $(m + 1)(k + 1)$ lines and each line can cross at most $2n/D$ squares, it follows that there are at most $2(m+1)(k+1)n/D$ of the $(\lfloor n/D \rfloor)^2$ squares in our subdivision that are not constant in the square and its neighborhood.

Finally, by divisibility considerations each square contains D points that correspond to the start and end of constellations.

We now approximate α. We have that α is at least D^2/n^2 times the number of squares in the subdivision that are identically 1 in the square and a neighborhood. On the other hand, counting monochromatic constellations, we get D constellations for every such square, and this misses at most $2(m+1)(k+1)n$ monochromatic constellations for squares we threw out that intersected a line. In particular, we have that the number of squares is at least $(C(\mathcal{Q}, \mathcal{B}, n) - 2(m+1)(k+1)n)/D$. So we have

$$\alpha \geq \frac{D^2}{n^2} \frac{C(\mathcal{Q}, \mathcal{B}, n) - 2(m+1)(k+1)n}{D},$$

or rearranging,

$$C(\mathcal{Q}, \mathcal{B}, n) \leq \frac{\alpha}{D} n^2 + 2(m+1)(k+1)n.$$

A similar argument in which we overcount monochromatic constellations and overestimate α gives

$$C(\mathcal{Q}, \mathcal{B}, n) \geq \frac{\alpha}{D} n^2 - 2(m+1)(k+1)n.$$

Combining the above two inequalities establishes the result for the asymmetric case. For the symmetric case we divide by a factor of 2 because we restrict to the case $p \leq q$. □

3. PERTURBATION TO FIND GOOD BLOCK PATTERNS

Given a block pattern that colors $[n]$ we now know how to find the number of monochromatic constellations using the block coloring. To make use of this, we first need to find a good candidate block pattern. The goal of this section is to outline an approach for how such a pattern might be found. We will make use of the following observation twice.

Observation 3.1. If an optimal coloring for some fixed constellation pattern is given, then a small perturbation cannot decrease the number of monochromatic constellations.

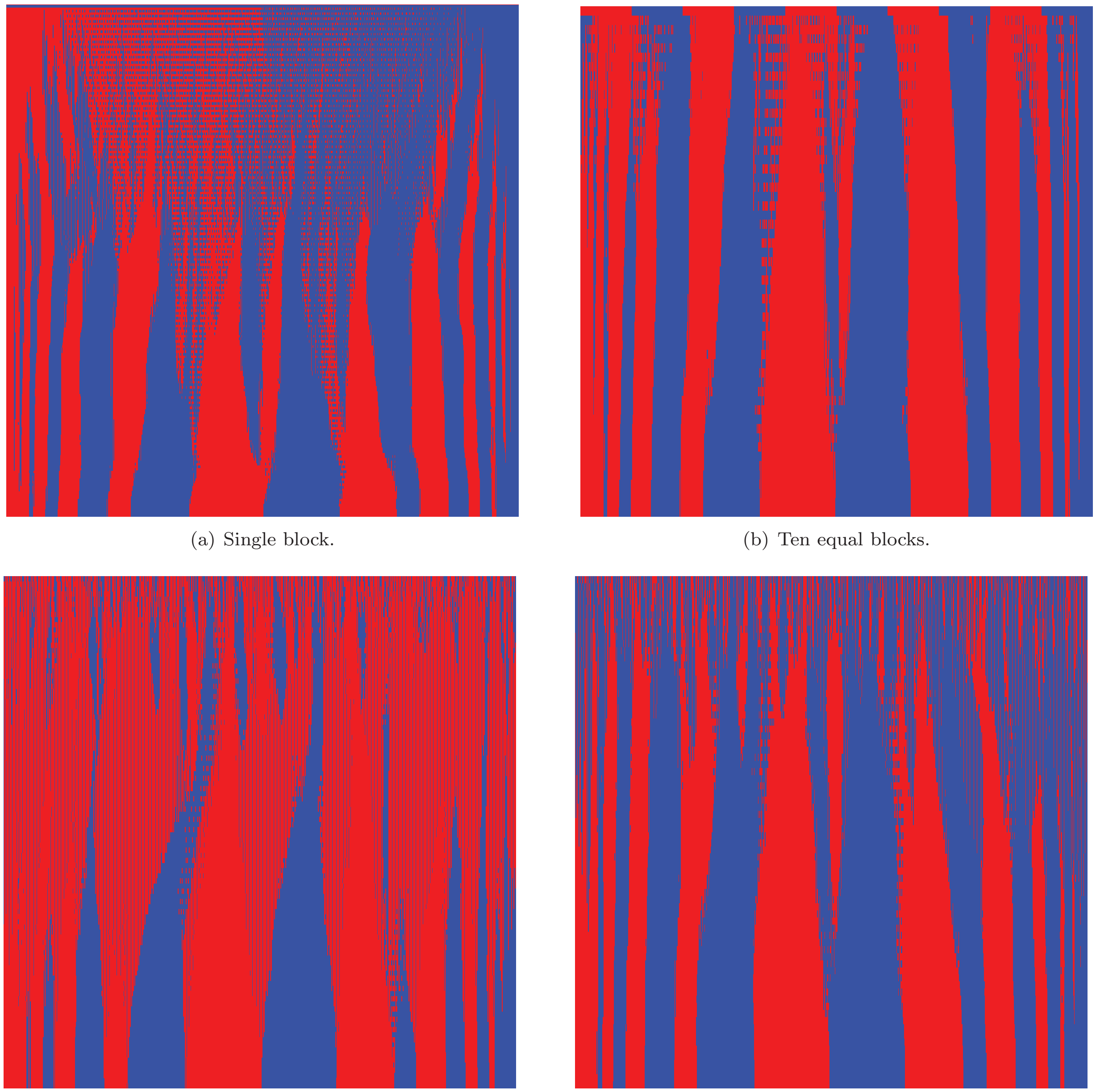
Let us first make use of the observation discretely. We fix n large, say 100,000, and color $[n]$ arbitrarily. Now scan the elements of $[n]$. If we find an element for which switching the color decreases the number of monochromatic constellations, then we switch and continue scanning. This process continues until we get to a coloring such that changing the color on any single term will not decrease the number of monochromatic solutions. We will call such a coloring a *locally optimal coloring* on $[n]$. Note that one single element might change color multiple times in this process. Since the number of monochromatic constellations strictly decreases on each pass, the the process will terminate in finite time.

When implementing this there are two major decisions: how to start the initial coloring and how to scan for the next element to test for switching. In Figure 3 we show the evolution of a red/blue coloring on [1000] using several different starting colorings that converge to a locally optimal coloring for avoiding the constellation $[0, 1/3, 1]$ (this corresponds to avoiding monochromatic solutions to $x + 2y = 3z$). Our rule for scanning is to alternate between going left to right and right to left. We then output the current coloring when we hit the end of a row.

Note that in Figure 3, starting with several different configurations, they all converged to approximately the same block pattern, which consists of 18 blocks. Intuitively, a block pattern that emerges by starting with $[n]$ and running this process should be an approximation to the optimal block pattern (if such a structure exists).

There are two problems. The first is that there is generally not a unique local optimal coloring; in other words, there can be many patterns for which we cannot decrease the number of monochromatic constellations by changing the color of a single element. To deal with this we can run many iterations whereby after each iteration we flip some large fraction of the colors and run the process again. We then make some choice as to which patterns are best, usually based on the ones having the fewest monochromatic constellations. While this does not guarantee that we find the best block structure, it helps to rule out some possibilities.

The second problem is that the block pattern that we find is, at best, an approximate blowup of the optimal block pattern. For example, if the best block structure has a block with small width, say less than $1/n$, then when we blow it up we might not catch the block in our pattern. To deal with this we generally choose n large (depending on the constellation). Another problem is that we do not have the precise relative sizes of the optimal block pattern. To deal with this we now perturb this near-optimal block structure to settle into a locally



(a) Single block.

(b) Ten equal blocks.

(c) Random coloring.

(d) Random coloring.

FIGURE 3. Evolution of a locally minimal coloring for constellation $[0, 1/3, 1]$ with different starting colorings.

optimal block structure, i.e., a block structure for which an ϵ change in any of the β_i in $\mathcal{X}$ will increase the corresponding coefficient of the block coloring.

To do this we use the observation made earlier that if our block structure is optimal, then any small perturbation of the block sizes should increase the coefficient. This implies that if we look along the set of lines $q_i x + (1 - q_i)y = \beta_j$, then a small change in β_j will add as much area as it removes. In terms of the colored indicator function such as shown in Figure 2, we have that a small perturbation should add as much red (blue) as it will remove blue (red). If we are near the optimum, then this allows us to set up a system of linear equations that the β_j in an optimal block structure must satisfy, i.e.,

$$\begin{pmatrix}\text{amount of change in red}\\ \text{under } \epsilon \text{ perturbation of } \beta_j\end{pmatrix} + \begin{pmatrix}\text{amount of change in blue}\\ \text{under } \epsilon \text{ perturbation of } \beta_j\end{pmatrix} = 0,$$

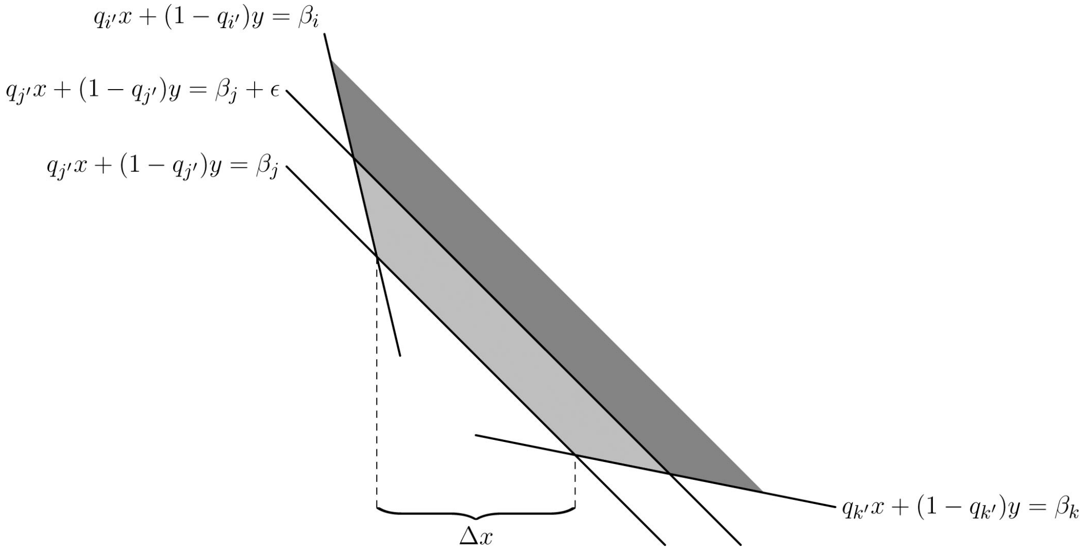


FIGURE 4. Change in area under an ϵ perturbation of β_j.

where by the amount of change in red or blue we mean the change in area of all the polygons under a small ϵ perturbation of one of the β_j. For instance, for the side of the polygon in Figure 4 we have

$$\begin{aligned}\Delta\text{Area} &\approx \frac{\Delta x}{1-q_{j'}}\epsilon \\ &\approx \Bigg(\frac{1}{q_{j'}-q_{i'}}\beta_i + \frac{1}{1-q_{j'}}\left(\frac{1-q_{k'}}{q_{j'}-q_{k'}} + \frac{1-q_{i'}}{q_{i'}-q_{j'}}\right)\beta_j \\ &\qquad + \frac{1}{q_{k'}-q_{j'}}\beta_k\Bigg)\epsilon.\end{aligned}$$

This must hold for $j = 1, 2, \ldots, k-1$. To this we also add $\beta_0 = 0$ and $\beta_k = 1$ to get a system of $k+1$ linear equations in $k+1$ unknowns. It is important that our block structure be near optimum, since the set of linear equations is based on the polygons defining the regions where $g(x, y) = 1$, as used in Theorem 2.1. A different set of polygons in $g(x, y)$ leads to different equations that might produce an even worse coloring or even be undefined. (The important parts of the polygons are the bounding lines of each polygon, so when we are near the optimum, we do have the correct bounding lines and can perturb.)

Remark 3.2. The process of setting up the linear equations can be completely automated, and a Maple worksheet that implements this local perturbation is available at the first author's website.[1]

[1]Currently at http://www.math.ucla.edu/~butler/.

For example, for the constellation pattern $[0, 1/2, 1]$, a local perturbation on $[10000]$ yields an approximate block structure of

$$\langle 508, 109, 511, 674, 1076, 2116, 2117, 1077, 676, 512,\\ 110, 514\rangle.$$

This corresponds to the system of linear equations

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & -12 & 10 & -6 & 6 & -6 & 2 & -2 & 2 & -2 & 2 & 0 & 0 \\ 2 & -10 & 16 & -10 & 6 & -6 & 2 & -2 & 2 & -2 & 4 & -2 & 0 \\ 4 & -6 & 10 & -14 & 10 & -6 & 2 & -2 & 2 & 0 & 2 & -2 & 0 \\ 2 & -6 & 6 & -10 & 16 & -10 & 2 & -2 & 4 & -2 & 2 & -2 & 0 \\ 4 & -6 & 6 & -6 & 10 & -14 & 6 & 0 & 2 & -2 & 2 & -2 & 0 \\ 2 & -2 & 2 & -2 & 2 & -6 & 8 & -6 & 2 & -2 & 2 & -2 & 2 \\ 0 & -2 & 2 & -2 & 2 & 0 & 6 & -14 & 10 & -6 & 6 & -6 & 4 \\ 0 & -2 & 2 & -2 & 4 & -2 & 2 & -10 & 16 & -10 & 6 & -6 & 2 \\ 0 & -2 & 2 & 0 & 2 & -2 & 2 & -6 & 10 & -14 & 10 & -6 & 4 \\ 0 & -2 & 4 & -2 & 2 & -2 & 2 & -6 & 6 & -10 & 16 & -10 & 2 \\ 0 & 0 & 2 & -2 & 2 & -2 & 2 & -6 & 6 & -6 & 10 & -12 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \\ \beta_9 \\ \beta_{10} \\ \beta_{11} \\ \beta_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Solving, we get

$$\langle\!\langle \beta_i \rangle\!\rangle = \left\langle\!\!\left\langle 0, \frac{7}{137}, \frac{17}{274}, \frac{31}{274}, \frac{99}{548}, \frac{79}{274}, \frac{1}{2}, \frac{195}{274}, \frac{449}{548}, \frac{243}{274}, \right.\right.\\ \left.\left. \frac{257}{274}, \frac{130}{137}, 1 \right\rangle\!\!\right\rangle,$$

which translated back into blocks gives us

$$\langle 28, 6, 28, 37, 59, 116, 116, 59, 37, 28, 6, 28\rangle.$$

Now using this block structure, we easily get our coefficient of $117/2192$.

This same technique can be used for any constellation (with the limitations mentioned above). For instance, for

FIGURE 5. A good block coloring for avoiding four-term arithmetic progressions.

FIGURE 6. A good block coloring for avoiding five-term arithmetic progressions.

the constellation $[0, 1/3, 1]$, doing local perturbation on $[25000]$ we get an approximate block structure

$$\langle 1101, 193, 577, 583, 989, 1434, 1115, 2833, 3680, 3681, 2830, 1113, 1434, 988, 582, 575, 194, 1098\rangle.$$

Solving the system of linear equations, we get a locally optimal block structure of

$$\langle 1552213, 272415, 813251, 822338, 1394548, 2025068, 1572841, 3995910, 5196075, 5196075, 3995910, 1572841, 2025068, 1394548, 822338, 813251, 272415, 1552213\rangle,$$

giving a coefficient of

$$\frac{16040191}{211735908} \approx 0.075755\ldots < \frac{1}{12} \approx 0.083333\ldots,$$

showing that again in this case we can beat random.

For four-term arithmetic progressions, several different runs of $[100000]$ gave us an approximate block structure with 36 blocks. Perturbing, we found a pattern that has coefficient

$$\frac{17939629302218100912470205240133659380304674379 75}{104177418768222598213753754515890676996254443021344} \approx 0.0172202\ldots < \frac{1}{48} \approx 0.020833\ldots,$$

again showing that we can beat random (we give the corresponding block structure for this coefficient in the appendix, Section 8). The corresponding coloring is shown in Figure 5.

For five-term arithmetic progressions, we found a block structure with 117 blocks that gives a coefficient of $0.005719619\ldots < 1/128 = 0.0078125$, showing yet again that we can beat random (the corresponding block structure is available at the first author's website). The corresponding coloring is shown in Figure 6.

While we do not know whether any of these block structures are optimal, we still note that the number of blocks seems to rise dramatically. We use 12, 36, and 117 blocks respectively for the colorings avoiding three-, four-, and five-term arithmetic progressions. In general, we note that for a constellation with k points we need more than 2^{k-1} blocks in any block structure that beats random, so that the number of blocks needed grows exponentially with the number of points. To see this, we note that the integral in Theorem 2.1 is at least as large as the area in the squares along the main diagonal. So if we have m blocks and we beat random, then

$$\frac{1}{2^{k-1}} \geq \int_0^1 \int_0^1 g(x,y)\,dx\,dy \geq \sum_{i=1}^{m} \left(\beta_i - \beta_{i-1}\right)^2 \geq \frac{1}{m}.$$

In particular, this shows that the experimental approach runs into severe limitations as the number of points in the constellation gets large.

4. SOLID BLOCKS MIGHT NOT ALWAYS BE BEST

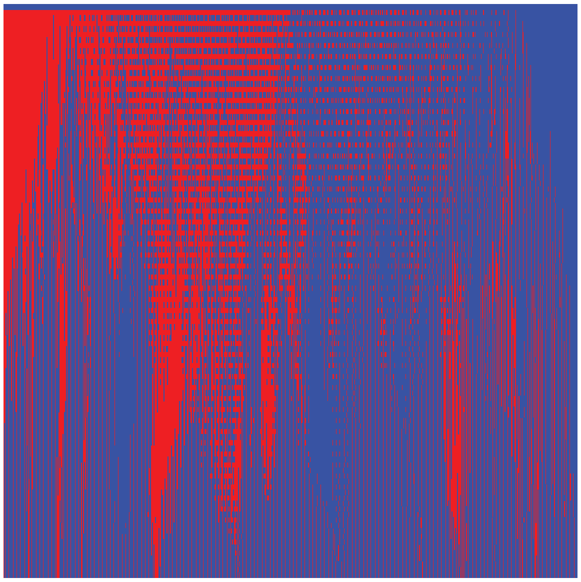
So far, we have assumed that the optimal coloring of $[n]$ is done by blowing up large monochromatic blocks. But this might not always be the case. For example, consider the constellation $[0, 2/5, 1]$ (which corresponds to avoiding monochromatic solutions to $2x + 3y = 5z$). In Figure 7 we show the evolution of a coloring on $[1000]$ to a locally optimal red/blue coloring for two starts (one monochromatic and one random).

The pattern that emerges in both of these runs (and many additional runs done for various block sizes, starts, and scanning rules) does not appear to be solid blocks but rather "alternating blocks," i.e., blocks that alternate red and blue in every entry, and between blocks there is an extra entry, i.e.,

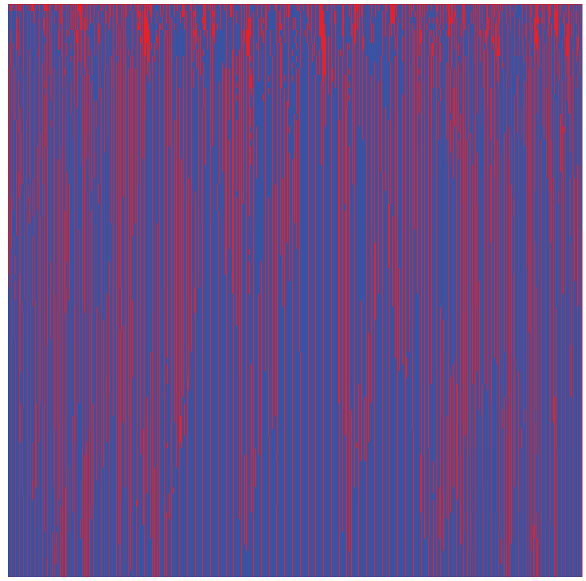
$$\cdots RBRBRBRBR\overset{\downarrow}{R}BRBRBRBRBR\cdots.$$

This extra entry has the property of shifting the modulus of the location of red and blue between two consecutive alternating blocks.

We can do the same process as before whereby we blow up a block pattern but only make each block alternating and switch the modulus between blocks. Also, as before, we can compute the coefficient to which this corresponds. The trick in doing this is to observe that a monochromatic constellation corresponds to a solution of $2x + 3y = 5z$, and if we look at the equations modulo 2, then we have $y \equiv z \pmod 2$. We can break our count into two situations, one in which $x \equiv y \pmod 2$ and one in which $x \not\equiv y \pmod 2$. The first case is counted as before, while the second case is counted by switching the



(a) Single block.



(b) Random coloring.

FIGURE 7. Evolution of a locally minimal coloring for constellation $[0, 2/5, 1]$ with different starting colorings.

color of the x term. This gives us a coefficient κ, where

$$\begin{aligned}\kappa &= \frac{1}{10}\int_0^1\int_0^1\left(\frac{(1+f(x))(1+f(y))(1+f\left(\frac{2x+3y}{5}\right))}{8}\right.\\&\qquad\left.+\frac{(1-f(x))(1-f(y))(1-f\left(\frac{2x+3y}{5}\right))}{8}\right)dx\,dy\\&\quad+\frac{1}{10}\int_0^1\int_0^1\left(\frac{(1-f(x))(1+f(y))(1+f\left(\frac{2x+3y}{5}\right))}{8}\right.\\&\qquad\left.+\frac{(1+f(x))(1-f(y))(1-f\left(\frac{2x+3y}{5}\right))}{8}\right)dx\,dy\\&=\frac{1}{20}\int_0^1\int_0^1\left(1+f(y)f\left(\frac{2x+3y}{5}\right)\right)dx\,dy\\&=\frac{1}{20}+\frac{1}{8}\int_0^1\int_{3y/5}^{(2+3y)/5} f(x)f(y)\,dx\,dy,\end{aligned}$$

where f is as in Theorem 2.1. The $1/20$ is fixed (and corresponds to the coefficient expected in a random coloring), so that our goal becomes minimizing the integral term, which is an integral over a parallelogram. Since $f(x)f(y) = \pm 1$, then when we plot the function $f(x)f(y)$ we can mark where it is 1 by coloring it white and -1 by coloring it black; see Figure 8. Minimizing the integral then becomes equivalent to finding a pattern that maximizes the amount of black inside of the parallelogram.

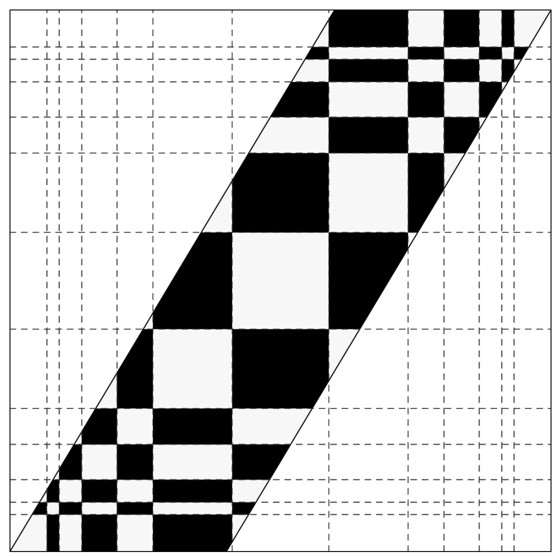


FIGURE 8. Maximizing the amount of black inside the parallelogram.

Experimentally, we find that an approximate (alternating) block pattern is

$$\langle 348, 113, 208, 325, 331, 731, 894, 731, 331, 325, 208, 113, 348\rangle.$$

As before, we can locally optimize, which in this case means that for each β_j we must have the amount of black immediately to the right of the line $x = \beta_j$ and above the line $y = \beta_j$ equal to the amount of white there (if this is not the case, we can increase the black by slightly increasing or decreasing β_j). As before, this sets up a system of linear equations that can be solved to give a local optimum. In doing so, we get the following block pattern:

$$\langle 9098298, 3018600, 5562432, 8660160, 8833560, 19511900, 23766825, 19511900, 8833560, 8660160, 5562432, 3018600, 9098298\rangle,$$

which gives a coefficient of $18447862/399410175 \approx 0.046187 \leq 1/20 = 0.05$, and again we have a coloring that beats random.

5. BEATING THE RANDOM COLORING FOR CONSTELLATIONS $[0, q, 1]$

In the previous sections we have seen colorings that beat random for the constellations $[0, 1/2, 1]$, $[0, 1/3, 1]$, and $[0, 2/5, 1]$. In this section we show that we can always beat random for any constellation of the form $[0, q, 1]$. We have already established the result for $[0, 1/2, 1]$, and by symmetry we need to do only the case $[0, q, 1]$ with $q < 1/2$, which is handled by the following fact.

Fact 5.1. *Let a, b be relatively prime natural numbers with $2a < b$ and*

$$0 < \epsilon < 1 + \frac{a}{b} - \frac{a}{b}\left\lceil \frac{b}{a} \right\rceil.$$

Then for the constellation pattern $[0, a/b, 1]$ and the block pattern

$$\langle 1-\epsilon, 1+\epsilon, \underbrace{1, 1, \ldots, 1}_{2b-2 \text{ terms}}\rangle,$$

there are $\gamma n^2 + O(n)$ monochromatic constellations, where

$$\gamma = \begin{cases} \frac{1}{4b} + \frac{(2a - a\lceil b/a\rceil)}{8ab^2(b-a)}\epsilon + O(\epsilon^2) & \text{if } \lceil b/a\rceil \text{ is odd,} \\ \frac{1}{4b} + \frac{(a - 2b + a\lceil b/a\rceil)}{8ab^2(b-a)}\epsilon + O(\epsilon^2) & \text{if } \lceil b/a\rceil \text{ is even.} \end{cases}$$

Since randomly we expect $(1/4b)n^2$ monochromatic constellations and in both cases above, the coefficient for ϵ is negative, then for a small enough choice of ϵ the above subdivision pattern beats random. The key to this argument is that the block pattern with $\langle 1, 1, 1, \ldots, 1\rangle$ with $2b$ blocks gives the coefficient $1/4b$, and so we need only find a slight perturbation that will cause the coefficient to drop.

To see this, note that taking (2–1), expanding, and substituting yields

$$\begin{aligned} \frac{\alpha}{b} &= \frac{1}{4b} + \frac{1}{4b}\int_0^1\int_0^1 f(x)f(y)\,dx\,dy \\ &\quad + \frac{1}{4b}\int_0^1\int_0^1 f(x)f\left(\frac{ax+(b-a)y}{b}\right)dx\,dy \\ &\quad + \frac{1}{4b}\int_0^1\int_0^1 f(y)f\left(\frac{ax+(b-a)y}{b}\right)dx\,dy \\ &= \frac{1}{4b} + \frac{1}{4b}\int_0^1\int_0^1 f(u)f(v)\,du\,dv \\ &\quad + \frac{1}{4(b-a)}\int_0^1\int_{av/b}^{(b+a(v-1))/b} f(u)f(v)\,du\,dv \\ &\quad + \frac{1}{4a}\int_0^1\int_{(b-a)v/b}^{(bv-a(v-1))/b} f(u)f(v)\,du\,dv. \end{aligned}$$

Note the similarity to what we did in the previous section. In particular, calculating the coefficient reduces to calculating the difference between black and white in the whole square and inside two parallelograms.

For the block pattern $\langle 1, 1, 1, \ldots, 1\rangle$ with $2b$ blocks, if we look at the inside integral of each term, we see that the first one will look over the entire interval $[0, 1]$, the second one will look over an interval of $[0, 1]$ with width $(b-a)/b$, and the third one will look over an interval of $[0, 1]$ with width a/b. Since the function f will change sign at regular steps of $1/2b$, it is easy to see that each of these inside integrals is 0. In particular, for the block pattern all the integrals vanish, and we are left with the constant term $1/4b$, which corresponds to random.

Now we simply perturb the pattern in the location of the first sign change of f. Estimating the change of this perturbation to the integral reduces to estimating the difference between black and white along the first line in the parallelograms, giving us the desired result.

The coloring we have produced in the above argument is almost certainly far from the best possible. To get a sense of how much better than random we can do, we looked for optimal block colorings for $[0, q, 1]$ for some simple q and plotted the ratio of the coefficient of this optimal coloring and the coefficient of the random coloring in Figure 9. The symmetry of the figure follows,

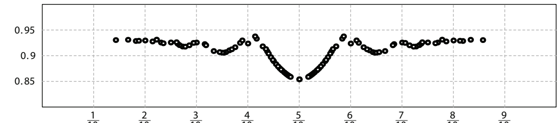
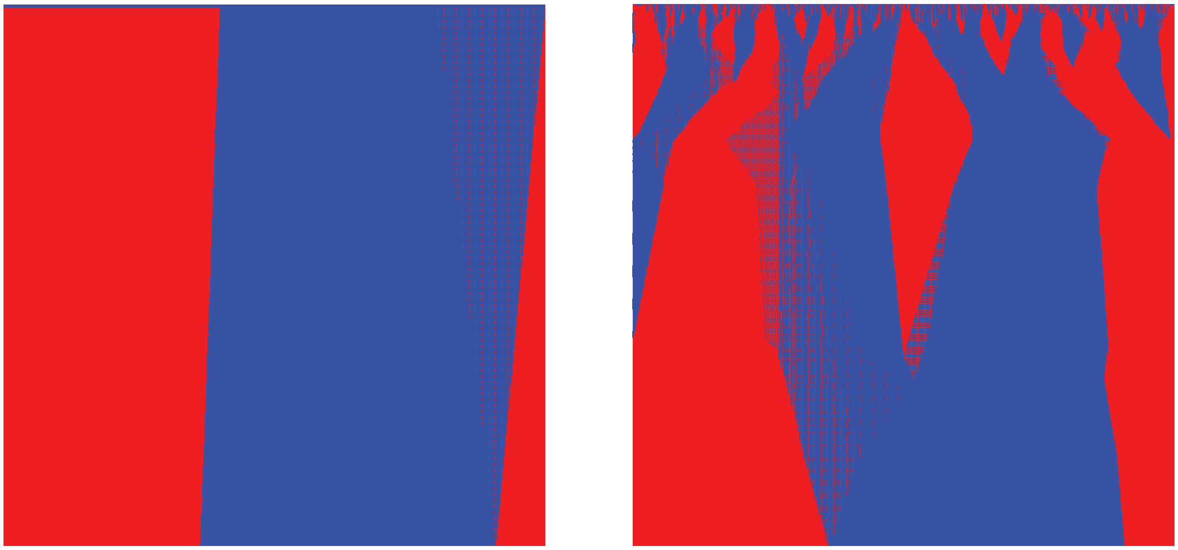


FIGURE 9. Good block colorings versus random for some $[0, q, 1]$.



(a) Single block.

(b) Random coloring.

FIGURE 10. Evolution of a locally minimal coloring for Schur triples with different starting colorings.

since we avoid $[0, q, 1]$ if and only if we avoid $[0, 1-q, 1]$. We note that the lowest point we have found is at the three-term arithmetic progressions, which corresponds to the ratio of $0.854\ldots$. Also, there seems to be a transition in behavior around $q = 2/5, 3/5$, which corresponds to the problem of avoiding monochromatic solutions of $2x + 3y = 5z$.

6. NONCONSTELLATION PATTERNS

There are related questions of minimizing monochromatic solutions to equations whose solutions are not constellations. The best known example is that of Schur triples, which are solutions to $x + y = z$ (since solutions to this equation are not invariant under translation, they are not constellations). This is currently the only known situation in which the minimal number of monochromatic solutions in a coloring is known, namely there are at least $(1/22)n^2 + O(n)$ monochromatic solutions (see [Datskovsky 01, Robertson and Zeilberger 98, Schoen 99]). The lower bound achieving this is the block pattern $\langle 4, 6, 1\rangle$.

While Theorem 2.1 no longer applies in this situation (and so we cannot do local minimization of block structures), we can still experimentally find what should happen. For instance, in Figure 10 we show the evolution of red/blue colorings on $[1000]$ that avoid monochromatic solutions to $x + y = z$. In both of these runs (and many more) we see that the minimum has the form (medium block)–(large block)–(small block). Given this pattern, it is not too hard to set up some variables for the three block sizes and to find the pattern that achieves the minimum number of monochromatic solutions.

We can also carry out the same process for other equations. For instance, for $x + ky = z$, $k \geq 2$, experimentally we see that we get three blocks, again in the form medium–large–small. Suppose that we use the subdivision pattern $\langle\langle 0, \alpha, \beta, 1\rangle\rangle$. It is easy to show that the number of solutions in a monochromatic block $[p, q]$ (contained in $[n]$ with $(k+1)p < q$) is

$$\frac{\big((q-kp)-p\big)^2}{2k} + O(n).$$

This gives $(\alpha^2/2k)n^2 + O(n)$ monochromatic solutions from the interval $[1, \alpha n]$, and $\big((\beta-(k+1)\alpha)^2/2k\big)n^2 + O(n)$ monochromatic solutions from the interval $[\alpha n, \beta n]$. The remaining solutions come from the situation in which x, z are in the third block and y in the first block; there are $\big((1-\beta)^2/2k\big)n^2 + O(n)$ such solutions. So altogether there are

$$\left(\frac{\alpha^2 + \big(\beta-(k+1)\alpha\big)^2 + (1-\beta)^2}{2k}\right)n^2 + O(n)$$

monochromatic solutions. Optimizing our choice of α and β gives us a block pattern

$$\left\langle \frac{k+1}{k^2+k+3}, \frac{k^2+k+1}{k^2+2k+3}, \frac{1}{k^2+2k+3}\right\rangle,$$

which gives

$$\frac{1}{2k(k^2+2k+3)}n^2 + O(n)$$

monochromatic solutions. (This same pattern was found independently in [Thanatipanonda 09].)

For $ax + by = az$ with relatively prime a, b with $a > b \geq 2$, experimentally the optimal pattern appears to be obtained by coloring $m \equiv 0 \pmod a$ red and the remaining terms blue, which gives a coefficient of $\big((2a-b)/(2a^4)\big)n^2 + O(n)$ monochromatic solutions. For $ax + by = az$ with $b > a \geq 2$, then, experimentally the optimal pattern appears to be obtained by coloring $m \equiv 0 \pmod a$ red for m small and the remainder blue. By optimizing as we have done above, we conclude that we should color red for $m \equiv 0 \pmod a$ and

$$m < \frac{ab^2(a-1)}{b\big(b^2(a-1)+a\big)}n.$$

Doing so gives us

$$\frac{a-1}{2b\big(b^2(a-1)+a\big)}n^2 + O(n)$$

monochromatic solutions.

7. CONCLUDING REMARKS

What we have done in the preceding sections is to give a systematic way to look for colorings having fewer than the random number of monochromatic constellations and other patterns. This gives a way to give upper bounds that we expect to be nearly optimal for the minimum number of monochromatic constellations in such a coloring. However, there still remains the question of determining corresponding lower bounds. The only pattern for which the best known coloring matches (up to lower-order terms) the best known lower bound is that of the Schur triples. (The lower bound given in the introduction is fairly weak and makes a poor candidate.)

For three-term arithmetic progressions there is a lower bound of $(1675/32768)n^2 + o(n^2)$ given in [Parrilo et al. 08], which differs by about five percent from the previously mentioned upper bound. We believe that the correct value for three-term progressions is the one given by the known locally optimal coloring, i.e., $(117/2192)n^2+O(n)$. Through several hundred runs with various sizes, starts, and scanning rules the same block pattern emerged repeatedly.

An interesting problem related to three-term arithmetic progressions is the following: For a partition $0 = a_0 < a_1 < a_2 < \cdots < a_\ell = 1/2$ create a *checkerboard* pattern by taking a square of side length $1/2$ and coloring the rectangle $[a_i, a_{i+1}] \times [a_j, a_{j+1}]$ white if $i + j$ is even and black if $i + j$ is odd. What is the maximum amount of black that can be enclosed in the triangle with vertices at $(0, 0)$, $(0, 1/2)$, and $(1/2, 1/4)$, and what partition (if any) produces this maximum? The best known pattern is found by scaling the block pattern $\langle 28, 6, 28, 37, 59, 116\rangle$; it is shown in Figure 11. (The connection is seen by looking at the integral for the case $q = 1/2$ in Section 5 and assuming that the pattern is antisymmetric, i.e., $f(1-x) = -f(x)$.)

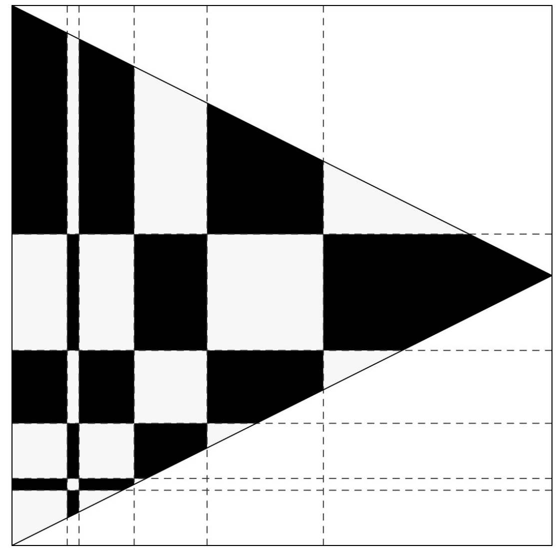


FIGURE 11. Best known pattern maximizing black inside of the triangle.

Any improvement on this pattern would automatically produce a lower constant for the number of three-term arithmetic progressions. By an exhaustive computer search we have determined that there is no block pattern with 11 or fewer blocks that beats this pattern. This is additional evidence to support the supposition that the current pattern for three-term arithmetic progressions is optimal.

One striking thing to notice is that for $[0, 1/2, 1]$, $[0, 1/3, 1]$, and $[0, 2/5, 1]$, all of the patterns we found are antisymmetric; that is, the colors of i and $n+1-i$ are opposite. This same behavior occurs frequently for many (but not all) of the locally optimal block colorings that we found for constellations $[0, q, 1]$. It would be interesting to know whether there is a reason for such a prevalence of antisymmetry.

In a related question, it would be interesting to know why $[0, 2/5, 1]$ goes into large alternating blocks. More generally, we might have a coloring in which a block pattern emerges only when we look at what is happening modulo some appropriate p.

Is there a way to predict beforehand, given a constellation pattern, whether the optimal coloring consists of solid blocks or some sort of alternating structure? Perhaps even more basic, is there a reason that we should expect block structures?

We have also seen that for the case $[0, q, 1]$, we can always beat a random coloring. We conjecture that this holds in general.

Conjecture 7.1. *For any constellation pattern $\mathcal{Q}$ there is a coloring pattern of $[n]$ that has $\gamma n^2 + o(n^2)$ monochromatic constellations, where γ is smaller than the coefficient for a random coloring.*

This conjecture is related to an idea in Ramsey theory, where for some time it was thought that the best way to avoid monochromatic K_t's in a two-coloring of K_n was to color randomly. It was shown in [Thomason 89] that this is not the case; the author produced colorings that beat random.

The key to proving the special case $[0, q, 1]$ is that we had a simple coloring that had the same number of monochromatic constellations as a random coloring, which we could then perturb. A first step in trying to prove the conjecture might be to try to find some "simple" block pattern that *matched* random, and then try to perturb it. This is not trivial, since even for four-term arithmetic progressions no simple pattern is known.

Another idea might be to try to bootstrap our way up. For instance, one would expect that since every four-term arithmetic progression has a three-term arithmetic progression inside, using the pattern for avoiding three-term arithmetic progressions we would also avoid many four-term arithmetic progressions. However, this is not the case. Indeed, using the pattern for three-term arithmetic progressions we do worse than random coloring for avoiding four-term arithmetic progressions, and also conversely.

One might also consider this problem for two-colorings of $\mathbb{Z}_p$, for prime p. It is shown in [Cameron et al. 07] that for the constellations of the form $[0, q, 1]$, the number of monochromatic constellations depends only on the amount of each color used, and not the distribution of the coloring in $\mathbb{Z}_p$. The authors also give some lower bounds for the number of four-term arithmetic progressions in colorings of $\mathbb{Z}_p$, which have been improved in [Wolf 10].

Finally, using roots of unity, it is not hard to adapt Theorem 2.1 to the case of $r > 2$ colors. However, we have found enough beauty and mystery in the $r = 2$ case to keep us occupied for some time. We hope to see some of the problems addressed above in future work.

8. APPENDIX

The locally optimal block pattern for four-term arithmetic progressions that gives the coefficient of

$$\frac{1793962930221810091247020524013365938030467437975}{10417741876822259821375375451589067699625444302134 4}$$

is given using the following 36 blocks:

$$\begin{aligned}
\langle &566124189415440472939626834822903743300467940483,\\
&115903533761943477398551818347715476722877927241,\\
&568011813340950665677009286694526323061781532322,\\
&472083073090028493914605548954507028673457587863,\\
&174690683867336844297305424871758992360029965453,\\
&98464537567500111285074159909918993309405119848,\\
&737681146409933099806596775369238915383890216793,\\
&881071132072892536672404740128385947619685842609,\\
&387204684955306822603896642766296540832568256888,\\
&340852889156784985628080980878507258595675472221,\\
&1398355239284691808801098670395696996980804292522,\\
&2015438904391090234472652819593929714355629836078,\\
&354924006068259988552316716495705798216298952575,\\
&917029329994691011286378833655488533756529343857,\\
&1246774229265384930907724953794401373314144038191,\\
&543203071437439856124749368271693956037186323582,\\
&2179716742907087903057122171392866104311765026441,\\
&2172387005301046067153961748343296914044366107569,\\
&546203621232713973260465876924982631234637232779,\\
&1296607046453245562932414262768745367411919249702,\\
&848633230480614872785768439513578746631939174778,\\
&332362434790023921274974476865878572983006589230,\\
&2079963873190082657423397539748746717308015584742,\\
&1352139932444260494597496699603210730948918467199,\\
&339606780510267312616862401149984633870549046619,\\
&373718051493152648659948917556716014372715915533,\\
&786614601718483336599780069288734659594156639775,\\
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&51505717888223458966003645016452574647172214751,\\
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&660659764939424259451601477074423264167731648445,\\
&458220356230536594713018106829384942175966684332,\\
&25106402444485772567927008361180285102508619312,\\
&396852300398188165681981456085190506968094545183,\\
&59492524857712314099066167971046557078820629622,\\
&398049321798723913182570904641775780071858799086\rangle
\end{aligned}$$

The corresponding diagram for the function $g(x, y)$ from Theorem 2.1 is shown in Figure 12.

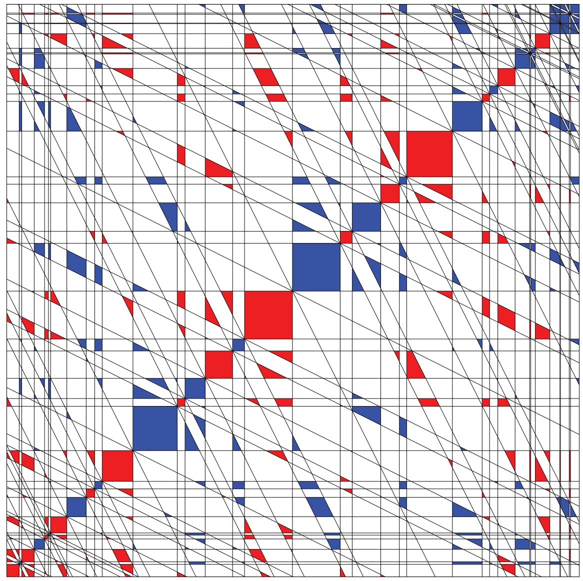

FIGURE 12. Indicator function for $\mathcal{Q} = [0, 1/3, 2/3, 1]$ using block pattern given in the appendix.

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