More Torsion in the Homology of the Matching Complex

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CONTENTS

A matching on a set X is a collection of pairwise disjoint subsets of X of size two. Using computers, we analyze the integral homology of the matching complex M_n, which is the simplicial complex of matchings on the set $\{1, \dots, n\}$. The main result is the detection of elements of order p in the homology for $p \in \{5, 7, 11, 13\}$. Specifically, we show that there are elements of order 5 in the homology of M_n for $n \geq 18$ and for $n \in \{14, 16\}$. The only previously known value was $n = 14$, and in this particular case we have a new computer-free proof. Moreover, we show that there are elements of order 7 in the homology of M_n for all odd n between 23 and 41 and for $n = 30$. In addition, there are elements of order 11 in the homology of M_{47} and elements of order 13 in the homology of M_{62}. Finally, we compute the ranks of the Sylow 3- and 5-subgroups of the torsion part of $\tilde{H}_d(\mathsf{M}_n; \mathbb{Z})$ for $13 \leq n \leq 16$; a complete description of the homology already exists for $n \leq 12$. To prove the results, we use a representation-theoretic approach, examining subcomplexes of the chain complex of M_n obtained by letting certain groups act on the chain complex.

1. INTRODUCTION

Recall that a *matching* on a set X is a collection of pairwise disjoint subsets of X of size two. Using terminology from graph theory, we refer to a subset of size two as an *edge on* X. The *matching complex* M_n is the family of matchings on the set $\{1, \dots, n\}$. Since M_n is closed under deletion of edges, M_n is an abstract simplicial complex.

Despite its simple definition, the topology of M_n remains a mystery. Its rational simplicial homology is well known and has been computed by Bouc and others [Bouc 92, Karaguezian 04, Reiner and Roberts 00, Dong and Wachs 02], but the integral homology is known only in special cases. Specifically, the bottom nonvanishing homology group of M_n is known to be an elementary 3-group for almost all n [Bouc 92, Shareshian and Wachs 07]. In fact, there is a nearly complete characterization of all (n, d) such that the homology group

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$\tilde{H}_d(\mathsf{M}_n;\mathbb{Z})$ contains elements of order 3 [Jonsson 08]; see Proposition 1.8(4), (5) for a summary. As for the existence of elements of order p for primes different from 3, nothing is known besides the recent discovery [Jonsson 09] that $\tilde{H}_4(\mathsf{M}_{14};\mathbb{Z})$ contains elements of order 5.

Computers have not been able to tell us much more about the homology of M_n. A complete description is known only for $n \leq 12$; already for $n = 13$, there are too many cells in M_n to allow for a direct homology computation, at least with the existing software.

The goal of this paper is to cut M_n into smaller pieces and then use a computer to search for torsion in the homology of those pieces. Via this technique, we come fairly close to a complete description of the homology of M_n for $13 \leq n \leq 16$. Moreover, and maybe more importantly, for each of the primes 5, 7, 11, and 13, we find new values of n such that the homology of M_n contains elements of order p.

More precisely, we do the following: First, we split the chain complex $\mathcal{C}(\mathsf{M}_n;\mathbb{Z})$ of M_n into smaller pieces with the property that the homology of M_n is isomorphic to the direct sum of the homology of the smaller complexes, except that the Sylow 2-subgroups of the torsion part may differ. Using this splitting technique, we determine the 3- and 5-ranks of the homology of M_n for $13 \leq n \leq 16$. Here we define the p-rank of an abelian group to be the rank of the Sylow p-subgroup of the torsion part of the group.

Second, we use a similar technique to produce even smaller pieces. While it does not seem to be possible to compute the entire homology of M_n from these pieces, we may still deduce useful information about the existence of elements of order p for various primes p. Specifically, our computations show that there are elements of order 5 in $\tilde{H}_{4+u}(\mathsf{M}_{14+2u};\mathbb{Z})$ for $0 \leq u \leq 8$, in $\tilde{H}_{6+u}(\mathsf{M}_{19+2u};\mathbb{Z})$ for $0 \leq u \leq 4$, and in $\tilde{H}_8(\mathsf{M}_{24};\mathbb{Z})$.

This turns out to imply that there are elements of order 5 in the homology of M_n whenever $n = 14$, $n = 16$, or $n \geq 18$. Moreover, there are elements of order 7 in $\tilde{H}_{8+u}(\mathsf{M}_{23+2u};\mathbb{Z})$ for $0 \leq u \leq 9$ and in $\tilde{H}_{11}(\mathsf{M}_{30};\mathbb{Z})$, elements of order 11 in $\tilde{H}_{13}(\mathsf{M}_{47};\mathbb{Z})$, and elements of order 13 in $\tilde{H}_{19}(\mathsf{M}_{62};\mathbb{Z})$.

Almost all results are computer-based, but in Section 7 we present a computer-free proof that $\tilde{H}_4(\mathsf{M}_{14};\mathbb{Z})$ contains elements of order 5.

Based on our computations, we derive conjectures about the existence of elements of order p in the homology of M_n for arbitrary odd primes p. For example, the above data suggest that there are elements of order p in the homology of $\mathsf{M}_{(p^2+6p+1)/4}$ for any odd prime p.

Let us explain the underlying method in some detail. For a given group G acting on the simplicial chain complex $\mathcal{C}(\mathsf{M}_n;\mathbb{Z})$ of M_n, let $\tilde{C}_d(\mathsf{M}_n;\mathbb{Z})/G$ be the subgroup of the chain group $\tilde{C}_d(\mathsf{M}_n;\mathbb{Z})$ consisting of all sums $\sum_{g\in G} g(c)$ such that $c \in \tilde{C}_d(\mathsf{M}_n;\mathbb{Z})$. As long as the order of G is not a multiple of a given prime p, the homology of the subcomplex does not contain elements of order p unless the homology of M_n contains elements of order p; see Section 2.

In particular, if we indeed detect elements of order p in the subcomplex, then we can deduce that the homology of M_n also contains elements of order p.

To give an example, we first introduce some notation. For a sequence $\lambda = (\lambda_1, \ldots, \lambda_r)$ of nonnegative integers summing to n, let $\mathfrak{S}_\lambda$ be the Young group $\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_r}$. Given a set partition $(U_1, \ldots, U_r)$ of $\{1, \ldots, n\}$ such that $|U_a| = \lambda_a$ for $1 \leq a \leq r$, we obtain a natural action on $\mathcal{C}(\mathsf{M}_n;\mathbb{Z})$ by letting $\mathfrak{S}_{\lambda_a}$ act on U_a in the natural manner for each a; for a group element $\pi \in \mathfrak{S}_\lambda$, the action is given by replacing the edge $xy = \{x, y\}$ with the edge $\pi(x)\pi(y)$ for each choice of distinct elements $x, y \in \{1, \ldots, n\}$.

In Section 4, we demonstrate that $\mathcal{C}(\mathsf{M}_n;\mathbb{Z})/\mathfrak{S}_\lambda$ is isomorphic to the chain complex of a certain simplicial complex BD_r^λ. The vertices of this complex are all edges and loops on $\{1, \ldots, r\}$; a loop is a multiset of the form $xx = \{x, x\}$). A collection of edges and loops is a face of BD_r^λ if and only if the total number of occurrences of the element a in the edges and loops is at most λ_a for $1 \leq a \leq r$. By a computer-based result from [Andersen 92], $\tilde{H}_4(\mathsf{BD}_7^2;\mathbb{Z})$ is a cyclic group of order five. As a consequence, the following is true.

Proposition 1.1. [Jonsson 09] *The homology group $\tilde{H}_4(\mathsf{M}_{14};\mathbb{Z})$ contains elements of order* 5*.*

The proof in [Jonsson 09] of Proposition 1.1 uses a definition of $\mathcal{C}(\mathsf{M}_n;\mathbb{Z})/\mathfrak{S}_\lambda$ that differs slightly from the one given above. As alluded to earlier, we present a computer-free proof of the proposition in Section 7.

Using similar but more-refined methods, we have managed to deduce more information about the presence of elements of order 5 in the homology of M_{14} and other matching complexes.

Theorem 1.2. *By computer calculations, the following properties hold:*

(i) *The* 5*-rank of $\tilde{H}_4(\mathsf{M}_{14};\mathbb{Z})$ is* 233*.*

(ii) *The* 5*-rank of $\tilde{H}_5(\mathsf{M}_{16};\mathbb{Z})$ is* 8163*.*

(iii) $\tilde{H}_{4+u}(\mathsf{M}_{14+2u};\mathbb{Z})$ *contains elements of order* 5 *for* $0 \le u \le 8$, *as do* $\tilde{H}_{6+u}(\mathsf{M}_{19+2u};\mathbb{Z})$ *for* $0 \le u \le 4$ *and* $\tilde{H}_8(\mathsf{M}_{24};\mathbb{Z})$.

Note that 233 is a Fibonacci number. As we will see in Section 9.1, this does not appear to be purely a coincidence.

To prove the theorem, we consider various actions of the Young group $\mathfrak{S}_\lambda$ on the chain complex of M_n for certain choices of $\lambda = (\lambda_1, \dots, \lambda_r)$. Specifically, for each sequence $\mathsf{s} = (s_1, \dots, s_r)$ of signs, we obtain an action given by

$$(\pi, c) \mapsto \prod_{1 \le a \le r : s_a = -1} \operatorname{sgn}(\pi_a) \cdot \pi(c), \qquad (1\text{–}1)$$

where $\pi = \pi_1 \cdots \pi_r$ and $\pi_a \in \mathfrak{S}_{U_a}$. Here $\mathfrak{S}_{U_a}$ denotes the symmetric group $\mathfrak{S}_{\lambda_a}$, viewed as the group of permutations on the set U_a. Whenever we refer to an action on M_n as being induced by the pair (λ, s), we mean this action. For clarity, we will often write the pair in matrix form as

$$\begin{pmatrix} \lambda \\ \mathsf{s} \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ s_1 & s_2 & \cdots & s_r \end{pmatrix}. \qquad (1\text{–}2)$$

If all s_a are equal to -1, then the action is simply $(\pi, c) \mapsto \operatorname{sgn}(\pi) \cdot \pi(c)$. We refer to this action as the *natural signed action* and the action $(\pi, c) \mapsto \pi(c)$ as the *natural unsigned action*.

To prove (i) and (ii) in Theorem 1.2, we split the chain complexes into many small pieces using the actions defined in (1–1) for the group $(\mathfrak{S}_2)^r$ for $2r = 14$ and $2r = 16$, respectively. For the proof of (iii), we consider the natural signed action of certain Young groups $\mathfrak{S}_\lambda$ on the chain complex of M_n. In some cases, we need to pick an even larger group whose elements permute not only the elements within each U_a but also the sets U_a themselves. We describe this construction in greater detail in Section 6.

It is important to stress that the choice of signs has a significant impact on the homology of the resulting piece. For many of the choices of λ used for proving Theorem 1.2, the piece resulting from the natural *signed* action of $\mathfrak{S}_\lambda$ contains plenty of elements of order 5, whereas the piece resulting from the *unsigned* action of $\mathfrak{S}_\lambda$ does not contain any such elements at all.

By the following proposition, the result that $\tilde{H}_6(\mathsf{M}_{19};\mathbb{Z})$ and $\tilde{H}_8(\mathsf{M}_{24};\mathbb{Z})$ contain elements of order 5 turns out to be of particular importance.

Proposition 1.3. [Jonsson 08] *For* $q \ge 3$, *if* $\tilde{H}_{2q}(\mathsf{M}_{5q+4};\mathbb{Z})$ *contains elements of order* 5, *then so does* $\tilde{H}_{2q+u}(\mathsf{M}_{5q+4+2u};\mathbb{Z})$ *for each* $u \ge 0$.

Theorem 1.2 and Proposition 1.3 imply the following result.

Theorem 1.4. *The homology groups* $\tilde{H}_{6+u}(\mathsf{M}_{19+2u};\mathbb{Z})$ *and* $\tilde{H}_{8+u}(\mathsf{M}_{24+2u};\mathbb{Z})$ *contain elements of order* 5 *for each* $u \ge 0$. *Thus the homology of* M_n *contains elements of order* 5 *for all* $n \ge 18$ *and for* $n = 14$ *and* $n = 16$.

We do not know whether Proposition 1.3 can be extended to include $q = 2$. In particular, we cannot tell whether $\tilde{H}_{4+u}(\mathsf{M}_{14+2u};\mathbb{Z})$ contains elements of order 5 for $u \ge 9$, though we do conjecture that this is the case.

Next, we consider higher torsion, in which case we have the following results.

Theorem 1.5. *By computer calculations, the following properties hold:*

(i) *The homology group* $\tilde{H}_{8+u}(\mathsf{M}_{23+2u};\mathbb{Z})$ *contains elements of order* 7 *for* $0 \le u \le 9$. *The same is true for the group* $\tilde{H}_{11}(\mathsf{M}_{30};\mathbb{Z})$.

(ii) *The homology group* $\tilde{H}_{19}(\mathsf{M}_{47};\mathbb{Z})$ *contains elements of order* 11.

(iii) *The homology group* $\tilde{H}_{26}(\mathsf{M}_{62};\mathbb{Z})$ *contains elements of order* 13.

To prove Theorem 1.5(i) for $u = 0$, we consider the action induced by $\left(\begin{smallmatrix} \lambda \\ \mathsf{s} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 3 & 4 & 4 & 4 & 4 & 4 \\ + & - & - & - & - & - \end{smallmatrix}\right)$ as defined in (1–1). A computer calculation yields that the homology in degree eight of the resulting chain complex is a group of order 7.

The action induced by $\left(\begin{smallmatrix} 3 & 4 & 4 & 4 & 4 & 4 \\ + & - & - & - & - & - \end{smallmatrix}\right)$ fits nicely into a pattern starting with $\left(\begin{smallmatrix} 1 & 2 & 2 & 2 \\ + & - & - & - \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 2 & 3 & 3 & 3 & 3 \\ + & - & - & - & - \end{smallmatrix}\right)$. The action induced by $\left(\begin{smallmatrix} 1 & 2 & 2 & 2 \\ + & - & - & - \end{smallmatrix}\right)$ on M_7 yields a chain complex with a homology group of order 3 in degree one, whereas the corresponding action induced by $\left(\begin{smallmatrix} 2 & 3 & 3 & 3 & 3 \\ + & - & - & - & - \end{smallmatrix}\right)$ on M_{14} induces a chain complex with a homology group of order 5 in degree four.

Parts (ii) and (iii) in Theorem 1.5 fit the same pattern; we pick the group actions induced by $\left(\begin{smallmatrix} 5 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ + & - & - & - & - & - & - & - \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 6 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\ + & - & - & - & - & - & - & - & - \end{smallmatrix}\right)$. The resulting chain complexes are much too large for a computer to handle, but since almost all λ_a are the same, we may extend the group to include elements that permute the sets U_a; see the discussion above after Theorem 1.2.

	$d=0$	1	2	3	4	5	6
$n=3$	$\mathbb{Z}^2$	–	–	–	–	–	–
4	$\mathbb{Z}^2$	–	–	–	–	–	–
5	–	$\mathbb{Z}^6$	–	–	–	–	–
6	–	$\mathbb{Z}^{16}$	–	–	–	–	–
7	–	$\mathbb{Z}_3$	$\mathbb{Z}^{20}$	–	–	–	–
8	–	–	$\mathbb{Z}^{132}$	–	–	–	–
9	–	–	$\mathbb{Z}^{42} \oplus (\mathbb{Z}_3)^8$	$\mathbb{Z}^{70}$	–	–	–
10	–	–	$\mathbb{Z}_3$	$\mathbb{Z}^{1216}$	–	–	–
11	–	–	–	$\mathbb{Z}^{1188} \oplus (\mathbb{Z}_3)^{45}$	$\mathbb{Z}^{252}$	–	–
12	–	–	–	$(\mathbb{Z}_3)^{56}$	$\mathbb{Z}^{12440}$	–	–
13	–	–	–	$\mathbb{Z}_3$	$\mathbb{Z}^{24596} \oplus (\mathbb{Z}_3)^{220}$	$\mathbb{Z}^{924}$	–
14	–	–	–	–	$(\mathbb{Z}_5)^{233} \oplus (\mathbb{Z}_3)^{2157}$	$\mathbb{Z}^{138048}$	–
15	–	–	–	–	$(\mathbb{Z}_3)^{92}$	$\mathbb{Z}^{472888} \oplus (\mathbb{Z}_3)^{1001}$	$\mathbb{Z}^{3432}$
16	–	–	–	–	$\mathbb{Z}_3$	$\mathbb{Z}^{24024} \oplus (\mathbb{Z}_5)^{8163} \oplus (\mathbb{Z}_3)^{60851}$	$\mathbb{Z}^{1625288}$

TABLE 1. The homology $\tilde{H}_d(\mathsf{M}_n;\mathbb{Z})$ for $n \le 16$. For $n \ge 13$, the second-highest nonvanishing homology group is a guess based on computations over some small fields $\mathbb{Z}_p$ of odd characteristic; the torsion part might be a strictly larger group.

For completeness, let us mention that the corresponding group action induced by $\left(\begin{smallmatrix}4 & 5 & 5 & 5 & 5 & 5 & 5\\ + & - & - & - & - & - & -\end{smallmatrix}\right)$ yields a cyclic homology group of order 9 in degree 13. Since the order of the acting group is a multiple of 9, we cannot conclude anything about the existence of elements of order 9 in the homology of M_{34}.

In addition, we establish precise results about the 3-rank of the homology of M_n for $13 \le n \le 16$. The methods used are exactly the same as those used for proving Theorem 1.2(i) and (ii).

Theorem 1.6. *By computer calculations, the following properties hold:*

(i) *The* 3*-rank of* $\tilde{H}_4(\mathsf{M}_{13};\mathbb{Z})$ *is* 220.

(ii) *The* 3*-rank of* $\tilde{H}_4(\mathsf{M}_{14};\mathbb{Z})$ *is* 2157.

(iii) *The* 3*-ranks of* $\tilde{H}_4(\mathsf{M}_{15};\mathbb{Z})$ *and* $\tilde{H}_5(\mathsf{M}_{15};\mathbb{Z})$ *are* 92 *and* 1001*, respectively.*

(iv) *The* 3*-rank of* $\tilde{H}_5(\mathsf{M}_{16};\mathbb{Z})$ *is* 60851.

It is well known [Shareshian and Wachs 07] that $\tilde{H}_4(\mathsf{M}_{15};\mathbb{Z})$ is an elementary 3-group. We do not know whether the other 3-groups are elementary. See Table 1 for a summary of the situation for $n \le 16$.

It is known that $\tilde{H}_1(\mathsf{M}_7;\mathbb{Z}) \cong \mathbb{Z}_3$, $\tilde{H}_2(\mathsf{M}_9;\mathbb{Z}) \cong \mathbb{Z}^{42} \oplus (\mathbb{Z}_3)^8$, and $\tilde{H}_3(\mathsf{M}_{11};\mathbb{Z}) \cong \mathbb{Z}^{1188} \oplus (\mathbb{Z}_3)^{45}$. This suggests the following conjecture.

Conjecture 1.7. *For $k \ge 0$, we have that $\tilde{H}_{k+1}(\mathsf{M}_{2k+7};\mathbb{Z})$ is the direct sum of a free group and an elementary* 3*-group of rank $\binom{2k+6}{k}$.*

See Table 3 for a schematic overview of the situation for $17 \le n \le 28$.

The proofs of Theorems 1.2, 1.5, and 1.6 involve chain complexes that are not simplicial. To handle such chain complexes, we use Pilarczyk's excellent computer program HOMCHAIN (version 2.08), which is part of the advanced version of the CHomP package [Pilarczyk 04].

1.1 The Big Picture

Note that the first occurrence of elements of order 3 in $H_d(\mathsf{M}_n;\mathbb{Z})$ is for $(n,d) = (7,1)$, and the first occurrence of elements of order 5 is for $(n,d) = (14,4)$. Let us provide a heuristic argument explaining why it seemed reasonable to look for elements of orders 7, 11, and 13 in $H_d(\mathsf{M}_n;\mathbb{Z})$ for $(n,d) = (23,8)$, $(47,19)$, and $(62,26)$, respectively. The argument is best understood when expressed in terms of a pair (k,r) of parameters, introduced in a previous paper [Jonsson 08], defined as

$$\begin{cases} k = 3d - n + 4, \\ r = n - 2d - 3, \end{cases} \iff \begin{cases} n = 2k + 1 + 3r, \\ d = k - 1 + r. \end{cases} \tag{1–3}$$

The following proposition and corollary provide a rationale for this parameter choice.

	$k=0$	1	2	3	4	5	6	7	8	9	10
$r=0$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
1	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
2	3	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
3			5	∞	∞	∞	∞	∞	∞	∞	∞
4						7	∞	∞	∞	∞	∞
5										$(*)$	∞

TABLE 2. Boxes marked with "∞" correspond to pairs (k, r) such that $\tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r}; \mathbb{Z})$ is infinite. We have marked boxes corresponding to the first known occurrences of elements of order p for $p = 3, 5, 7$. The next box to form a corner of the region of finite homology is marked with $(*)$.

	$d=5$	6	7	8	9	10	11	12
$n=17$	3	$\infty, (3)$	∞	–	–	–	–	–
18	3	$\infty, 3, 5$	∞	–	–	–	–	–
19	3	$3, 5$	$\infty, (3)$	∞	–	–	–	–
20	–	3	$\infty, 3, 5$	∞	–	–	–	–
21	–	3	$3, 5$	$\infty, (3)$	∞	–	–	–
22	–	3	3	$\infty, 3, 5$	∞	–	–	–
23	–	–	3	$3, 5, 7$	$\infty, (3)$	∞	–	–
24	–	–	3	$3, 5$	$\infty, 3, 5$	∞	–	–
25	–	–	3	3	$\infty, 3, 5, 7$	$\infty, (3)$	∞	–
26	–	–	–	3	$3, 5$	$\infty, 3, 5$	∞	–
27	–	–	–	3	3	$\infty, 3, 5, 7$	$\infty, (3)$	∞
28	–	–	–	3	3	$3, 5$	$\infty, 3, 5$	∞

TABLE 3. List of known infinite and prime orders of elements in the group $\tilde{H}_d(\mathsf{M}_n; \mathbb{Z})$ for $17 \leq n \leq 28$. Legend: ∞ = group is infinite; p = group contains elements of order p; (p) = group is conjectured to contain elements of order p.

Proposition 1.8. *Let $n \geq 1$. Then the following hold for the homology group $H_d(\mathsf{M}_n; \mathbb{Z}) = \tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r}; \mathbb{Z})$:*

(1) [Bouc 92] *The group is infinite if and only if $k \geq r(r-1)/2$ and $r \geq 0$.*

(2) [Björner et al. 94] *The group is zero unless $k \geq 0$ and $r \geq 0$.*

(3) [Bouc 92, Shareshian and Wachs 07] *The group is a nonvanishing elementary 3-group whenever $k \in \{0, 1, 2\}$ and $r \geq k+2$.*

(4) [Jonsson 08] *The group is a nonvanishing 3-group whenever $0 \leq k \leq r-2$.*

(5) [Jonsson 08] *The group contains elements of order 3 whenever $k \geq 0$ and $r \geq 3$.*

Corollary 1.9. *For $n \geq 1$, the group $\tilde{H}_d(\mathsf{M}_n; \mathbb{Z}) = \tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r}; \mathbb{Z})$ is nonzero if and only if $k \geq 0$ and $r \geq 0$.*

Regarding Proposition 1.8(4), (5), we conjecture that the following stronger properties hold.

Conjecture 1.10. *The group $\tilde{H}_d(\mathsf{M}_n; \mathbb{Z}) = \tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r}; \mathbb{Z})$ contains elements of order 3 if and only if $k \geq 0$ and $r \geq 2$. The homology group is an elementary nonvanishing 3-group if and only if $0 \leq k \leq r-2$.*

In Conjecture 1.7 above, we conjectured that the 3-rank of $\tilde{H}_{k+1}(\mathsf{M}_{2k+7}; \mathbb{Z})$ is $\binom{2k+6}{k}$ for $k \geq 0$. This would imply the first part of Conjecture 1.10; it is well known and easy to prove that $\tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r}; \mathbb{Z})$ is free if $r \leq 1$ (equivalently, $d \geq \frac{n-4}{2}$). Conjecture 1.12 below implies the "only if" part of the second statement.

Note that $(n, d) = (7, 1)$ corresponds to $(k, r) = (0, 2)$ and that $(n, d) = (14, 4)$ corresponds to $(k, r) = (2, 3)$. Both these pairs satisfy the equation

$$k = \frac{r(r-1)}{2} - 1. \tag{1–4}$$

In particular, the pairs just barely fail to meet the requirement in Proposition 1.8(1). An alternative way to put it is to say that the pairs (k, r) satisfying (1–4) are the corners of the region of finite homology as illustrated in Table 2. The next corner after $(0, 2)$ and $(2, 3)$ is $(5, 4)$, which yields $(n, d) = (23, 8)$, exactly the pair for which we detected elements of order 7. The subsequent corners are

	$k=0$	1	2	3	4	5	6	7	8
$r=0$	∞	∞	∞	∞	∞	∞	∞	∞	∞
1	∞	∞	∞	∞	∞	∞	∞	∞	∞
2	3	$\infty, 3$	$\infty, 3$	$\infty, 3$	$\infty, 3$	$\infty, (3)$	$\infty, (3)$	$\infty, (3)$	$\infty, (3)$
3	3	3	$3, 5$	$\infty, 3, 5$	$\infty, 3, 5$	$\infty, 3, 5$	$\infty, 3, 5$	$\infty, 3, 5$	$\infty, 3, 5$
4	3	3	3	$3, 5$	$3, 5$	$3, 5, 7$	$\infty, 3, 5, 7$	$\infty, 3, 5, 7$	$\infty, 3, 5, 7$
5	3	3	3	3	$3, 5$	$3, 5$	$3, 5$	$3, 5, 7$	$3, 5, (7)$
6	3	3	3	3	3	$3, (5)$	$3, (5)$	$3, (5)$	$3, (5)$
7	3	3	3	3	3	3	$3, (5)$	$3, (5)$	$3, (5)$

TABLE 4. List of known infinite and prime orders of elements in the group $\tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r};\mathbb{Z})$ for $k \le 8$ and $r \le 7$; notation is as in Table 3.

$(9,5)$, $(14,6)$, and $(20,7)$, which yield $(n,d) = (34,13)$, $(47,19)$, and $(62,26)$, respectively. The latter two pairs are the places where we detected elements of order 11 and 13, respectively.

One may ask whether there are elements of order $2r-1$ in the group corresponding to the pair

$$(k,r) = \left(\frac{r(r-1)}{2} - 1, r\right)$$

for all $r \ge 2$ or at least all r such that $2r-1$ is a prime. As already alluded to, we do not know whether there are elements of order 9 in the group corresponding to $(k,r) = (9,5)$.

Expressing Theorems 1.2 and 1.4 in terms of the parameters k and r, we obtain the following characterization of groups known to contain elements of order 5.

Corollary 1.11. *The homology group* $\tilde{H}_d(\mathsf{M}_n;\mathbb{Z}) = \tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r};\mathbb{Z})$ *contains elements of order* 5 *whenever either of the following holds:*

- $r = 3$ *and* $2 = r-1 \le k \le 10$;
- $r \in \{4,5\}$ *and* $k \ge r-1$.

This suggests the following conjecture.

Conjecture 1.12. *The group* $\tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r};\mathbb{Z})$ *contains elements of order* 5 *whenever* $r \ge 3$ *and* $k \ge r-1$. *Equivalently,* $\tilde{H}_d(\mathsf{M}_n;\mathbb{Z})$ *contains elements of order* 5 *whenever*

$$\frac{2n-8}{5} \le d \le \frac{n-6}{2}.$$

Proceeding to the next prime, Theorem 1.5 yields that there are elements of order 7 in $\tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r};\mathbb{Z})$ for $r = 4$ and $5 \le k \le 14$ and also for $(k,r) = (7,5)$. While this is very little evidence for a conjecture, we do hope that the following is true.

Conjecture 1.13. *The group* $\tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r};\mathbb{Z})$ *contains elements of order* 7 *whenever* $r \ge 4$ *and* $k \ge 2r-3$. *Equivalently,* $\tilde{H}_d(\mathsf{M}_n;\mathbb{Z})$ *contains elements of order* 7 *whenever*

$$\frac{3n-13}{7} \le d \le \frac{n-7}{2}.$$

Table 4 gives a schematic overview similar to the one in Table 3 but with rows and columns indexed by r and k rather than n and d.

We have even less evidence for the following conjecture about the existence of elements of order p for general p in the homology of M_n.

Conjecture 1.14. *Let* $p = 2q-1$ *be an odd prime. Then* $\tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r};\mathbb{Z})$ *contains elements of order* p *whenever* $r \ge q$ *and* $k \ge (q-2)r - \binom{q-1}{2}$. *Equivalently,* $\tilde{H}_d(\mathsf{M}_n;\mathbb{Z})$ *contains elements of order* p *whenever*

$$\frac{(q-1)(n-q/2)}{2q-1} - 1 \le d \le \frac{n-q-3}{2}.$$

2. GENERAL CONSTRUCTION

Let

$$\mathcal{C} : \cdots \xrightarrow{\partial} C_{d+1} \xrightarrow{\partial} C_d \xrightarrow{\partial} C_{d-1} \xrightarrow{\partial} \cdots$$

be a chain complex of abelian groups. Let G be a finite group acting on $\mathcal{C}$; thus we have a map $\alpha : G \times \bigoplus_d C_d \to \bigoplus_d C_d$ satisfying

$$\alpha_g(\partial(c)) = \partial(\alpha_g(c)),$$
$$\alpha_g \circ \alpha_h(c) = \alpha_{gh}(c),$$

for all $g,h \in G$ and $c \in \bigoplus_d C_d$.

Let $C_d/(G,\alpha)$ be the subgroup of C_d consisting of all elements

$$[c] = \sum_{g \in G} \alpha_g(c),$$

$c \in C_d$. Writing $C_d/G = C_d/(G, \alpha)$, this yields a chain subcomplex

$$\begin{aligned}\mathcal{C}/G : \cdots &\xrightarrow{\partial} C_{d+1}/G \xrightarrow{\partial} C_d/G \\ &\xrightarrow{\partial} C_{d-1}/G \xrightarrow{\partial} \cdots .\end{aligned}$$

Proposition 2.1. *Let q be a positive integer. If $H_d(\mathcal{C}/G)$ contains elements of order q, then $H_d(\mathcal{C})$ contains elements of order $q/\gcd(q, |G|)$. In particular, if q and $|G|$ are coprime, then $H_d(\mathcal{C})$ contains elements of order q.*

Proof. Defining $\iota : C_d/G \mapsto C_d$ to be the natural inclusion map, we obtain the following diagram of maps between homology groups:

$$H_d(\mathcal{C}/G) \xrightarrow{\iota^*} H_d(\mathcal{C}) \xrightarrow{[\cdot]^*} H_d(\mathcal{C}/G).$$

Now suppose that z is an element of order q in $H_d(\mathcal{C}/G)$. Let w be an element from C_d/G in the homology class z. By construction, $w = [w']$ for some w'. As a consequence,

$$\begin{aligned}[\iota(w)] &= \sum_{g\in G}\sum_{h\in G} \alpha_g(\alpha_h(w')) = \sum_{g\in G}\sum_{h\in G} \alpha_{gh}(w') = \sum_{g\in G} w \\ &= |G| \cdot w.\end{aligned}$$

This implies that $[\iota^*(z)]^* = |G| \cdot z$, which is an element of order $q/\gcd(q, |G|)$. It follows that the order of $\iota^*(z)$ in $H_d(\mathcal{C})$ is divisible by $q/\gcd(q, |G|)$. □

Note that $\mathcal{C}/G$ is a subcomplex of the complex of G-invariant elements. In our applications with Young groups acting on matching complexes, the two complexes are not identical in general. The reason for examining $\mathcal{C}/G$ rather than the G-invariant complex is that the former complex turns out to be more attractive in our situation. For example, the results presented in Sections 4 and 5 are not valid in general for the G-invariant complex unless $|G|$ is a unit in the underlying coefficient ring.

3. A CHAIN COMPLEX STRUCTURE ON A FAMILY OF MULTISETS

We want to understand the chain complex obtained from that of M_n by acting on M_n as in (1–1). In this section, we look at a more general situation, thereby postponing the special case of importance to us until the next section.

A *loop* on a set X is a multiset of the form $xx = \{x, x\}$, where $x \in X$. Let V be a finite totally ordered set, and let E be a subset of the set of edges and loops on X. We say that two elements a and b in V *commute* if ab belongs to E; a commutes with itself if the loop aa belongs to E. Otherwise, a and b *anticommute*. Given a sequence $(v_0, \ldots, v_d)$ of elements from V, we say that the pair (i, j) forms an *inversion* if $i < j$ and $v_i > v_j$. The inversion is *commuting* if v_i and v_j commute, and *anticommuting* otherwise.

In what follows, we will consider multisets. Given a multiset σ and an element $x \in \sigma$, we define $\sigma \setminus \{x\}$ to be the multiset obtained by decreasing the multiplicity of x in σ by one. We extend this to larger submultisets in the obvious manner: $\sigma \setminus \{x, y\} = (\sigma \setminus \{x\}) \setminus \{y\}$ and so on.

Let Δ be a family of multisets of elements from V satisfying the following properties:

(A) If $\sigma, \tau \in \Delta$ and $\sigma \subseteq \rho \subseteq \tau$, then $\rho \in \Delta$.

(B) For each multiset σ in Δ, if x and y commute and $\{x, y\} \subseteq \sigma$, then the multiset $\sigma \setminus \{x, y\}$ does not belong to Δ.

(C) For each $\sigma \in \Delta$, any element $a \in V$ appearing more than once in σ commutes with itself.

Our goal is to define a chain complex associated to Δ and E. As we will see, if E is empty, meaning that no pairs of elements commute, and Δ is closed under deletion of elements, then the resulting chain complex coincides with the simplicial chain complex on Δ.

For a coefficient ring R, define a chain complex $\mathcal{C}((\Delta, E); R)$ in the following manner. The chain group $C_d((\Delta, E); R)$ is the free R-module with one generator $x_0 \otimes \cdots \otimes x_d$ for each multiset $\sigma = \{x_0, \ldots, x_d\}$ in Δ of size $d + 1$ ($x_0 \leq \cdots \leq x_d$). By convention, we set $x_0 \otimes \cdots \otimes x_d$ equal to zero whenever $\{x_0, \ldots, x_d\} \notin \Delta$.

Sometimes we will need to consider generators in which the elements are not arranged according to the total order on V. For a permutation $\pi \in \mathfrak{S}_{\{0,\ldots,d\}}$, we define

$$x_{\pi(0)} \otimes \cdots \otimes x_{\pi(d)} = (-1)^\eta \cdot x_0 \otimes \cdots \otimes x_d, \qquad (3\text{–}1)$$

where η is the number of anticommuting inversions of $(x_{\pi(0)}, \ldots, x_{\pi(d)})$. Equivalently, for any sequence $(x_0, \ldots, x_d)$, ordered or not, and any integer i, we have that the element $\gamma' = x_0 \otimes \cdots \otimes x_{i-2} \otimes x_i \otimes x_{i-1} \otimes x_{i+1} \otimes \cdots \otimes x_d$ obtained by swapping x_{i-1} and x_i in $\gamma = x_0 \otimes \cdots \otimes x_d$ equals γ if x_{i-1} and x_i commute and $-\gamma$ if x_{i-1} and x_i anticommute (hence the choice of terminology). Here note that property (C) yields that x_{i-1} and x_i always commute when they are equal.

Let ∂ be the boundary operator defined on a given generator $\gamma = x_0 \otimes \cdots \otimes x_d$ as

$$\partial(\gamma) = \sum_{i=0}^{d} (-1)^{d-\eta_i} \cdot \gamma_i. \qquad (3\text{–}2)$$

Here γ_i denotes the element $x_0 \otimes \cdots \otimes x_{i-1} \otimes \hat{x}_i \otimes x_{i+1} \otimes \cdots \otimes x_d$ obtained by removing the element x_i, and $\eta_i = \eta_i(\gamma)$ is the number of indices $j \in \{i+1, \ldots, d\}$ such that x_i and x_j anticommute.

We need to show that (3–1) and (3–2) are consistent for any $(x_0, \ldots, x_d)$. Now, the coefficient of γ_j in the boundary of γ equals $(-1)^{d-\eta_j}$. Let γ' be the element obtained from γ by swapping x_{i-1} and x_i. For $j \notin \{i-1, i\}$, let γ'_j be the element obtained from γ_j by swapping x_{i-1} and x_i. The coefficient of γ'_j in the boundary of γ' remains equal to $(-1)^{d-\eta_j}$, aligning with the fact that either $(\gamma', \gamma'_j) = (\gamma, \gamma_j)$ or $(\gamma', \gamma'_j) = (-\gamma, -\gamma_j)$. For $j = i$, the coefficient of γ_i in the boundary of γ' equals $(-1)^{d-\eta_i}$ if and only if x_{i-1} and x_i commute, aligning with the fact that $\gamma' = \gamma$ if and only if x_{i-1} and x_i commute. The case $j = i-1$ follows by symmetry.

Proposition 3.1. *We have that $\mathcal{C}((\Delta, E); R) = (C_*((\Delta, E); R), \partial)$ defines a chain complex. Equivalently, $\partial \circ \partial = 0$.*

Proof. Let γ and γ_i be defined as above. We want to prove that $\partial \circ \partial(\gamma) = 0$.

It suffices to show that either the coefficient of $\gamma_{i,j} = x_0 \otimes \cdots \otimes \hat{x}_i \otimes \cdots \otimes \hat{x}_j \otimes \cdots \otimes x_d$ in $\partial \circ \partial(\gamma)$ is zero or $\gamma_{i,j}$ itself is zero whenever $0 \le i < j \le d$. We obtain $\gamma_{i,j}$ either by first removing x_i to get γ_i and then removing x_j or by first removing x_j to get γ_j and then removing x_i. By properties (A) and (B), $\gamma_{i,j}$ is zero unless each of γ_i and γ_j is nonzero and x_i and x_j anticommute; hence assume that these properties are satisfied.

If we remove first x_i and then x_j, then the sign of $\gamma_{i,j}$ equals

$$(-1)^{d-\eta_i(\gamma)} \cdot (-1)^{d-1-\eta_{j-1}(\gamma_i)} = (-1)^{2d-1-\eta_i(\gamma)-\eta_{j-1}(\gamma_i)}.$$

If we proceed the other way around, then the sign becomes

$$(-1)^{d-\eta_j(\gamma)} \cdot (-1)^{d-1-\eta_i(\gamma_j)} = (-1)^{2d-1-\eta_j(\gamma)-\eta_i(\gamma_j)}.$$

In the first case, we take $2d-1$ and subtract from it the number of indices $k > i$ such that x_i and x_k anticommute and then the number of indices $k > j$ such that x_j and x_k anticommute. In the second case, we again take $2d-1$ and subtract from it the number of indices $k > j$ such that x_j and x_k anticommute and then the number of indices $k > i$, excluding $k = j$, such that x_i and x_k anticommute. Thus since x_i and x_j anticommute, the two signs cancel out. This concludes the proof. $\square$

4. PROPERTIES OF THE CHAIN COMPLEX INDUCED BY (λ, s)

Let $n \ge 1$, let $\lambda_1, \ldots, \lambda_r$ be positive integers summing to n, and let $\mathsf{s} = (s_1, \ldots, s_r)$ be a sequence of signs. Let $(U_1, \ldots, U_r)$ be a set partition of $\{1, \ldots, n\}$ such that $|U_a| = \lambda_a$ for $1 \le a \le r$. In this section, we examine the chain complex obtained from that of M_n by acting on M_n as in (1–1).

In this chain complex, it turns out that we may choose generators that admit an interpretation as multisets of edges and loops on the set $\{1, \ldots, r\}$. Specifically, a given matching $\{x_0y_0, \ldots, x_dy_d\} \in \mathsf{M}_n$ corresponds to the multiset $\{a_0b_0, \ldots, a_db_d\}$, where a_i and b_i are such that $x_i \in U_{a_i}$ and $y_i \in U_{b_i}$ for $0 \le i \le d$.

Let R be a commutative ring. We consider the action in (1–1). To be precise, this is the action by the Young group $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_r}$ on $\mathcal{C}(\mathsf{M}_n; R)$ given by

$$(\pi, c) \mapsto \prod_{\substack{1 \le a \le r \\ s_a = -1}} \operatorname{sgn}(\pi_a) \cdot \pi(c),$$

where $\pi = \pi_1 \cdots \pi_r$, $\pi_i \in \mathfrak{S}_{U_a} \cong \mathfrak{S}_{\lambda_a}$ for $1 \le a \le r$, and

$$\pi(x_0y_0 \wedge \cdots \wedge x_dy_d) = \pi(x_0)\pi(y_0) \wedge \cdots \wedge \pi(x_d)\pi(y_d).$$

Write

$$\pi^{(\mathsf{s})}(c) = \prod_{\substack{1 \le a \le r \\ s_a = -1}} \operatorname{sgn}(\pi_a) \cdot \pi(c).$$

For a chain group element c, write

$$[c] = \sum_{\pi = \pi_1 \cdots \pi_r \in \mathfrak{S}_\lambda} \pi^{(\mathsf{s})}(c).$$

We want to describe the chain subcomplex generated by elements of the form $[c]$. By some abuse of notation, we write this chain complex as $\mathcal{C}(\mathsf{M}_n; R)/(\lambda, \mathsf{s})$. For any $\pi \in \mathfrak{S}_\lambda$ and chain group element c, we note for future reference that

$$[\pi^{(\mathsf{s})}(c)] = [c]. \tag{4–1}$$

It turns out to be helpful to describe generators of $\mathcal{C}(\mathsf{M}_n; R)/(\lambda, \mathsf{s})$ in terms of multisets of edges and loops on the set $\{1, \ldots, r\}$. We refer to an element $a \in \{1, \ldots, r\}$ as *positively charged* if $s_a = +1$ and *negatively charged* if $s_a = -1$. Two edges ab and cd *commute* if the intersection $\{a, b\} \cap \{c, d\}$ contains exactly one negatively charged element. Otherwise, the two edges *anticommute*. Every loop anticommutes with all edges and loops. As we will see, this aligns with the terminology used in Section 3.

For a multiset σ of edges and loops on $\{1, \ldots, r\}$ and an element $a \in \{1, \ldots, r\}$, define $\deg_\sigma(a)$ to be the *degree* of a in σ; this is the number of times a occurs as a member of an edge or a loop in σ. Note that a occurs twice in the loop aa, which means that this loop contributes two to the degree of a. Let $\Delta_{\lambda,\mathsf{s}}$ be the family of all multisets σ of edges and loops on the set $\{1,\dots,r\}$ satisfying the following properties:

(i) If the element a is positively charged, then $0 \le \deg_\sigma(a) \le \lambda_a$.

(ii) If the element a is negatively charged, then $\lambda_a - 1 \le \deg_\sigma(a) \le \lambda_a$, and the loop aa does not appear in σ.

(iii) An edge ab does not appear more than once in σ unless a and b have opposite charges. In particular, no loop appears more than once in σ.

Let $E_{\lambda,\mathsf{s}}$ be the set consisting of those edges and loops $\{ab, cd\}$ on the set $\{ab : 1 \le a \le b \le r\}$ with the property that ab and cd commute. In Section 3, we needed a total order on the underlying set V. In the case of $E_{\lambda,\mathsf{s}}$, the set V is the set of edges and loops on the set $\{1,\dots,r\}$. We order the elements of V lexicographically: $ab \le cd$ if either $a < c$ or $a = c$ and $b \le d$, where we assume that $a \le b$ and $c \le d$ (recall that ab and ba denote the same edge).

Choosing $\Delta = \Delta_{\lambda,\mathsf{s}}$ and $E = E_{\lambda,\mathsf{s}}$, note that properties (i)–(iii) imply properties (A)–(C) in Section 3. Namely, each of the families defined by (i), (ii), and (iii) satisfies (A); hence $\Delta_{\lambda,\mathsf{s}}$ satisfies (A), being the intersection of these families. Moreover, suppose that $\sigma \in \Delta_{\lambda,\mathsf{s}}$ contains two commuting edges e and e'; let a be the unique negatively charged element in the intersection of e and e'. Since the degree of a in σ is at most λ_a, the degree of a in $\sigma \setminus \{e, e'\}$ is at most $\lambda_a - 2$, which implies by (ii) that $\sigma \setminus \{e, e'\}$ does not belong to $\Delta_{\lambda,\mathsf{s}}$; thus (ii) implies (B).

Finally, by construction, an edge commutes with itself if and only if it contains exactly one negatively charged element; hence (iii) implies (C). In particular, the construction in Section 3 yields a well-defined chain complex $\mathcal{C}((\Delta_{\lambda,\mathsf{s}}, E_{\lambda,\mathsf{s}}); R)$.

Theorem 4.1. *Suppose that* $|\mathfrak{S}_\lambda| = \prod_{a=1}^r \lambda_a!$ *is not a zero divisor in* R. *Then* $\mathcal{C}(\mathsf{M}_n; R)/(\lambda, \mathsf{s})$ *and* $\mathcal{C}((\Delta_{\lambda,\mathsf{s}}, E_{\lambda,\mathsf{s}}); R)$ *are isomorphic as chain complexes.*

Proof. Let $\sigma = \{a_0b_0, \dots, a_db_d\}$ be a multiset on $\{1,\dots,r\}$ such that $0 \le \deg_\sigma(a) \le \lambda_a$ for all $a \in \{1,\dots,r\}$ and such that $a_0b_0 \le \cdots \le a_db_d$; we assume that $a_k \le b_k$ for $0 \le k \le d$. Since $|U_a| \ge \deg_\sigma(a)$ for all a, there exist matchings $\{x_0y_0, \dots, x_dy_d\}$ satisfying $x_k \in U_{a_k}$ and $y_k \in U_{b_k}$ for $0 \le k \le d$. Among all such matchings, let $\{x_0y_0, \dots, x_dy_d\}$ be the lexicographically smallest matching when viewed as a sequence $(x_0, y_0, x_1, y_1, \cdots, x_d, y_d)$. Write

$$\hat{\gamma} = x_0y_0 \wedge \cdots \wedge x_dy_d$$

and $\gamma = [\hat{\gamma}]$, and define

$$\varphi(a_0b_0 \otimes \cdots \otimes a_db_d) = \gamma.$$

We want to show that φ defines an isomorphism from $\mathcal{C}((\Delta_{\lambda,\mathsf{s}}, E_{\lambda,\mathsf{s}}); R)$ to $\mathcal{C}(\mathsf{M}_n; R)/(\lambda, \mathsf{s})$.

First, note that φ would be surjective if we extended its domain to include all elements $a_0b_0 \otimes \cdots \otimes a_db_d$ such that $0 \le \deg_\sigma(a) \le \lambda_a$ for all $a \in \{1,\dots,r\}$; every generator of $\mathcal{C}(\mathsf{M}_n; R)$ corresponds to such a generator. In particular, to prove that φ is surjective, it suffices to prove that $\varphi(a_0b_0 \otimes \cdots \otimes a_db_d)$ is zero whenever $\sigma = \{a_0b_0, \dots, a_db_d\}$ violates (i), (ii), or (iii). Since (i) is true by assumption, we may assume that either (ii) or (iii) is violated.

Suppose first that condition (ii) is violated for some negatively charged element a; hence the loop aa appears in σ or $\deg_\sigma(a) \le \lambda_a - 2$.

- If the loop aa appears in σ, then $\hat{\gamma}$ contains an edge x_iy_i such that $x_i, y_i \in U_a$. Let $x = x_i$ and $y = y_i$ in this case.
- If $\deg_\sigma(a) \le \lambda_a - 2$, then there are elements $x, y \in U_a$ such that $x, y \notin \{x_i, y_i\}$ for all i.

In both cases, the action of the transposition (x, y) on $\hat{\gamma}$ yields $-\hat{\gamma}$, implying that $2\gamma = 0$ by (4–1). Since $|U_a| \ge 2$, we obtain that $|\mathfrak{S}_\lambda|$ is divisible by 2. In particular, 2 is not a zero divisor in R, which yields that $\gamma = 0$.

Now suppose that condition (iii) is violated. This means that there are two identical edges or loops $a_ib_i = a_jb_j$ such that a_i and b_i have the same charge. Swapping x_iy_i and x_jy_j, we obtain an element ϵ that is equal to $-\hat{\gamma}$. However, ϵ is also the element obtained by acting on $\hat{\gamma}$ with the group element $(x_i, x_j)(y_i, y_j)$, because a_i and b_i have the same charge. We deduce that

$$-\gamma = [\epsilon] = [\hat{\gamma}] = \gamma;$$

the second equality is (4–1). Again, we obtain that $\gamma = 0$.

To prove that φ defines an isomorphism of modules, it remains to show that $\varphi(\kappa)$ is nonzero for every nonzero generator κ of $\mathcal{C}((\Delta_{\lambda,\mathsf{s}}, E_{\lambda,\mathsf{s}}); R)$.

Saying that $\gamma = x_0y_0 \wedge \cdots \wedge x_dy_d$ is zero in $\mathcal{C}(\mathsf{M}_n; R)/(\lambda, \mathsf{s})$ is equivalent to saying that there is a group element $\pi \in \mathfrak{S}_\lambda$ such that $\pi^{(\mathsf{s})}(\hat{\gamma}) = -\hat{\gamma}$. Namely, suppose that such a group element π exists. Then

$$\gamma = [\hat{\gamma}] = [\pi^{(\mathsf{s})}(\hat{\gamma})] = [-\hat{\gamma}] = -\gamma,$$

which implies that $\gamma = 0$ by the assumption on R. If such an element π does not exist, then the coefficient of $\hat{\gamma}$ in γ is a divisor of $|\mathfrak{S}_\lambda|$, which is not a zero divisor in R by assumption.

Let π be such that $\pi^{(\mathsf{s})}(\hat{\gamma}) = -\hat{\gamma}$. For each $a \leq b \in \{1, \dots, r\}$, let $\pi_{a,b}$ be the restriction of π to the union of the sets $\{x_i, y_i\}$ satisfying $x_i \in U_a$ and $y_i \in U_b$. For each $a \in \{1, \dots, r\}$, let π_a be the restriction of π to the set of elements $x \in U_a$ such that $x \notin \{x_i, y_i : i \in \{0, \dots, d\}\}$. We may decompose π as the product of all $\pi_{a,b}$ and π_a, extending each factor to $\{1, \dots, n\}$ by defining it to be the identity outside its domain. In particular, either $\pi_a^{(\mathsf{s})}(\hat{\gamma}) = -\hat{\gamma}$ for some a or $\pi_{a,b}^{(\mathsf{s})}(\hat{\gamma}) = -\hat{\gamma}$ for some $a \leq b$.

In the former case, a is negatively charged, and there are at least two elements x and y in U_a outside the set $\{x_i, y_i : i \in \{1, \dots, r\}\}$. This violates (ii), since the degree of a is then at most $\lambda_a - 2$. In the latter case, ab anticommutes with itself, meaning that a and b have the same charge. This violates (iii).

It remains to show that the boundary operators coincide. Consider the ith term $\epsilon_i = (-1)^i \cdot x_0y_0 \wedge \cdots \wedge \widehat{x_iy_i} \wedge x_dy_d$ in the boundary of $\hat{\gamma}$ in $\mathcal{C}(\mathsf{M}_N; R)$. Write $\delta_i = [\epsilon_i]$. If $a_i \neq b_i$, then let $(z_1 = x_i, z_2, \dots, z_q)$ be the sequence of elements appearing in $U_{a_i} \cap \{x_j, y_j : i \leq j \leq d\}$ and let $(z'_1 = y_i, z'_2, \dots, z'_{q'})$ be the sequence of elements appearing in $U_{b_i} \cap \{x_j, y_j : i \leq j \leq d\}$, arranged in increasing order. To obtain the lexicographically smallest element $\hat{\delta}_i$ from ϵ_i, we need to act on ϵ_i with the permutation $(z_q, \dots, z_2, z_1)(z'_{q'}, \dots, z'_2, z'_1)$. There are three cases:

Case 1: a_i and b_i are both negatively charged. By construction, a_ib_i appears only once in γ, meaning that there are $q + q' - 2$ edges a_jb_j commuting with a_ib_i such that $j > i$; hence $\eta_i = d - i - (q + q' - 2)$, where η_i is the number of indices $j > i$ such that a_ib_i and a_jb_j anticommute. We conclude that the sign of $\hat{\delta}_i$ equals $(-1)^{i+q+q'-2} = (-1)^{d-\eta_i}$, which aligns with (3–2).

Case 2: a_i and b_i are both positively charged. This means that no edges commute with a_ib_i; hence $\eta_i = d - i$ and the sign of $\hat{\delta}_i$ equals $(-1)^i = (-1)^{d-\eta_i}$.

Case 3: a_i and b_i have opposite charges. By symmetry, we may assume that a_i is negatively charged and b_i positively charged. In that case, there are $q - 1$ edges a_jb_j commuting with a_ib_i such that $j > i$; hence $\eta_i = d - i - (q - 1)$. We conclude that the sign of $\hat{\delta}_i$ equals $(-1)^{i+q-1} = (-1)^{d-\eta_i}$.

If $a_i = b_i$, then a_i is positively charged by construction, and there are no edges or loops commuting with a_ia_i; hence $\eta_i = d - i$. We conclude that the sign of $\hat{\delta}_i$ is $(-1)^i = (-1)^{d-\eta_i}$. □

When all signs s_i are positive, we may identify $\mathcal{C}(\mathsf{M}_n; R)/(\lambda, \mathsf{s})$ with the chain complex of a certain simplicial complex. Specifically, define BD_r^λ to be the family of sets σ of edges and loops on the set $\{1, \dots, r\}$ such that $\deg_\sigma(a) \leq \lambda_a$ for each $a \in \{1, \dots, r\}$. It is clear that BD_r^λ is a simplicial complex.

Proposition 4.2. *If $s_a = +1$ for all $a \in \{1, \dots, r\}$, then $\mathcal{C}((\Delta_{\lambda,\mathsf{s}}, E_{\lambda,\mathsf{s}}); R)$ and $\mathcal{C}(\mathsf{BD}_r^\lambda; R)$ are isomorphic. In particular, if $|\mathfrak{S}_\lambda| = \prod_{a=1}^r \lambda_a!$ is not a zero divisor in R, then $\mathcal{C}(\mathsf{M}_n; R)/(\lambda, \mathsf{s})$ and $\mathcal{C}(\mathsf{BD}_r^\lambda; R)$ are isomorphic.*

Proof. By assumption, all elements are positively charged, which implies by (i)–(iii) that the individual chain groups are isomorphic for the two chain complexes. Since all pairs of edges anticommute, the boundary operator ∂ on $\mathcal{C}((\Delta_{\lambda,\mathsf{s}}, E_{\lambda,\mathsf{s}}); R)$ defined in (3–2) has the property that

$$\partial(\gamma) = \sum_{i=0}^{d} (-1)^i \cdot \gamma_i,$$

which is the usual simplicial boundary operator. For the last statement of the proposition, apply Theorem 4.1. □

It turns out that we can transform $\mathcal{C}(\mathsf{M}_n; R)/(\lambda, \mathsf{s})$ into a smaller chain complex with the same homology such that the description of the generators is symmetric with respect to charge.

Theorem 4.3. *The homology of $\mathcal{C}((\Delta_{\lambda,\mathsf{s}}, E_{\lambda,\mathsf{s}}); R)$ is isomorphic to the homology of the quotient complex $\mathcal{C}((\lambda, \mathsf{s}); R)$ obtained by restricting to multisets σ satisfying the following two conditions:*

(I) *For each element a, we have that $\lambda_a - 1 \leq \deg_\sigma(a) \leq \lambda_a$, and the loop aa does not appear in σ.*

(II) *An edge ab does not appear more than once in σ unless a and b have opposite charges.*

In particular, if $|\mathfrak{S}_\lambda| = \prod_{a=1}^r \lambda_a!$ is not a zero divisor in R, then the homology of $\mathcal{C}(\mathsf{M}_n; R)/(\lambda, \mathsf{s})$ is isomorphic to that of $\mathcal{C}((\lambda, \mathsf{s}); R)$.

Proof. Let $a_1, \dots, a_p$ be the positively charged elements. For $0 \leq q \leq p$, let $\mathcal{C}^{(q)}$ be the quotient complex obtained by removing all generators in which a_i does not satisfy (I) for some $i \leq q$. Note that $\mathcal{C}^{(0)}$ coincides with $\mathcal{C}((\Delta_{\lambda,\mathsf{s}}, E_{\lambda,\mathsf{s}}); R)$ and that $\mathcal{C}^{(p)}$ coincides with $\mathcal{C}((\lambda, \mathsf{s}); R)$. To prove the theorem, it suffices to prove that the homology of $\mathcal{C}^{(q-1)}$ is isomorphic to that of $\mathcal{C}^{(q)}$ for $1 \leq q \leq p$ and that an isomorphism is induced by the natural quotient map. Namely, this implies that the natural quotient map from $\mathcal{C}((\Delta_{\lambda,\mathsf{s}}, E_{\lambda,\mathsf{s}}); R)$ to $\mathcal{C}((\lambda, \mathsf{s}); R)$ induces an isomorphism on homology.

Note that we may view $\mathcal{C}^{(q)}$ as the quotient complex of $\mathcal{C}^{(q-1)}$ by the subcomplex $\mathcal{W}^{(q)}$ consisting of those generators in which a_q does not satisfy (I). Given a chain group element c in $\mathcal{W}^{(q)}$, we may decompose c as $c = a_q a_q \otimes c' + c''$, where no generators appearing in c' and c'' contain the loop $a_q a_q$. This means that the degree of a_q is at most $\lambda_{a_q} - 2$ in each generator appearing in c' or c''. Let ∂ denote the boundary operator in $\mathcal{W}^{(q)}$ induced by that in $\mathcal{C}((\Delta_{\lambda,\mathsf{s}}, E_{\lambda,\mathsf{s}}); R)$. Since

$$\partial(c) = c' - a_q a_q \otimes \partial(c') + \partial(c''),$$

c is a cycle if and only if $\partial(c') = 0$ and $c' = -\partial(c'')$. However, in this case we have that c equals $\partial(a_q a_q \otimes c'')$. This is indeed a boundary in $\mathcal{W}^{(i)}$, because the degree of a_q is at most $\lambda_{a_q} - 2$ in each generator appearing in c''. As a consequence, the homology of $\mathcal{W}^{(i)}$ is zero. By the long exact sequence for the pair $(\mathcal{C}^{(i-1)}, \mathcal{W}^{(i)})$, it follows that $\mathcal{C}^{(i-1)}$ and $\mathcal{C}^{(i)}$ have the same homology. This concludes the proof. $\square$

5. CONNECTION TO THE FREE TWO-STEP NILPOTENT LIE ALGEBRA

Before proceeding, we discuss a closely related Koszul complex [Józefiak and Weyman 88, Sigg 96]. Let R be a commutative ring. Let $\lambda = (\lambda_1, \dots, \lambda_r)$ and

$$\lambda^- = \begin{pmatrix} \lambda_1 & \cdots & \lambda_r \\ - & \cdots & - \end{pmatrix}.$$

Let X^λ_{d+1} be the free R-module generated by elements of the form $e_0 \wedge \cdots \wedge e_d \otimes v_1 \wedge \cdots \wedge v_t$, where $v_1, \dots, v_t \in \{1, \dots, r\}$ and $e_0, \dots, e_d$ are edges on $\{1, \dots, r\}$ (loops are not allowed) such that the total number of occurrences of a in $(e_0, \dots, e_d, v_1, \dots, v_t)$ is λ_a for $1 \leq a \leq r$. If $e_i = e_j$ or $v_i = v_j$ for some $i \neq j$, then we define $e_0 \wedge \cdots \wedge e_d \otimes v_1 \wedge \cdots \wedge v_t$ to be zero. For any permutations ρ of $\{0, \dots, d\}$ and τ of $\{1, \dots, t\}$, we define

$$\begin{aligned} &e_{\rho(0)} \wedge \cdots \wedge e_{\rho(d)} \otimes v_{\tau(1)} \wedge \cdots \wedge v_{\tau(t)} \\ &\quad = \operatorname{sgn}(\rho) \operatorname{sgn}(\tau) \cdot e_0 \wedge \cdots \wedge e_d \otimes v_1 \wedge \cdots \wedge v_t. \end{aligned}$$

Note that $2d+2+t = |\lambda|$. For example, for $\lambda = (2, 2, 2, 1)$, the element $12 \wedge 23 \otimes 1 \wedge 3 \wedge 4$ appears in X^λ_2.

We define a boundary operator by

$$\begin{aligned} &\delta(x_0 y_0 \wedge \cdots \wedge x_d y_d \otimes v_1 \wedge \cdots \wedge v_t) \\ &\quad = \sum_{i=0}^{d} (-1)^i \cdot x_0 y_0 \wedge \cdots \wedge \widehat{x_i y_i} \wedge \cdots \wedge x_d y_d \\ &\qquad\qquad \otimes x_i \wedge y_i \wedge v_1 \wedge \cdots \wedge v_t; \end{aligned}$$

we assume that $x_j < y_j$ for $0 \leq j \leq d$. This indeed yields a chain complex $\mathcal{X}^\lambda = (X^\lambda_*, \delta)$. What we just defined is the chain complex associated to the free two-step nilpotent Lie algebra analyzed in [Józefiak and Weyman 88] and [Sigg 96].

For a sequence $w = (w_1, \dots, w_k)$ of elements from $\{1, \dots, r\}$, let $\operatorname{inv}(w)$ be the number of inversions of w, i.e., the number of pairs (i, j) such that $i < j$ and $w_i > w_j$. For a generator $\gamma = x_0 y_0 \wedge \cdots \wedge x_d y_d \otimes v_1 \wedge \cdots \wedge v_t$, let

$$\langle \gamma \rangle = (x_0, y_0, \dots, x_d, y_d, v_1, \dots, v_t).$$

Define $\operatorname{sgn}(\gamma)$ to be $(-1)^{\operatorname{inv}(\langle \gamma \rangle)}$. For example,

$$\operatorname{sgn}(12 \wedge 23 \otimes 1 \wedge 3 \wedge 4) = -1,$$

because $(1, 2, 2, 3, 1, 3, 4)$ contains three inversions: $(2, 5)$, $(3, 5)$, $(4, 5)$.

Define a map φ from X^λ_{d+1} to $\tilde{C}_d(\lambda^-; R)$ by

$$\varphi(\gamma) = \varphi(e_0 \wedge \cdots \wedge e_d \otimes v_1 \wedge \cdots \wedge v_t) = \operatorname{sgn}(\gamma) \cdot e_0 \otimes \cdots \otimes e_d.$$

We need to show that this map is well defined. For this, it suffices to show that $\varphi(\gamma') = -\varphi(\gamma)$ for any γ' obtained from γ by swapping either e_{i-1} and e_i or v_{i-1} and v_i for some i.

Suppose we obtain γ' by swapping e_{i-1} and e_i. If the two edges are disjoint, then the number of inversions changes by an even number, because the inversion status changes for exactly four pairs of indices. In particular, $\operatorname{sgn}(\gamma') = \operatorname{sgn}(\gamma)$. Since e_{i-1} and e_i anticommute, we obtain that $\varphi(\gamma') = -\varphi(\gamma)$.

If the two edges have one element in common, then the number of inversions changes by an odd number, because there are three pairs of indices for which the inversion status changes. This means that $\operatorname{sgn}(\gamma') = -\operatorname{sgn}(\gamma)$. Since e_{i-1} and e_i commute, we again deduce that $\varphi(\gamma') = -\varphi(\gamma)$.

Now suppose we obtain γ' by swapping v_{i-1} and v_i. Since the number of inversions either increases or decreases by one, we obtain that $\operatorname{sgn}(\gamma') = -\operatorname{sgn}(\gamma)$, which implies that $\varphi(\gamma') = -\varphi(\gamma)$.

Complex	Group Action (G,α)	d	β_d	$\rho_d(2)$	$\rho_d(3)$	$\rho_d(5)$	$\rho_d(7)$
M_{14}	$\begin{pmatrix} 2\,2\,2\,2\,3\,3 \\ -\,-\,-\,-\,-\,- \end{pmatrix}$	4 5	– 112	– –	19 –	2 –	– –
M_{16}	$\begin{pmatrix} 2\,2\,3\,3\,3\,3 \\ -\,-\,-\,-\,-\,- \end{pmatrix}$	5 6	6 76	– –	43 –	8 –	– –
M_{18}	$\begin{pmatrix} 2\,2\,3\,3\,4\,4 \\ -\,-\,-\,-\,-\,- \end{pmatrix}$	6 7	20 20	– –	18 –	2 –	– –
	$\begin{pmatrix} 2\,3\,3\,3\,3\,4 \\ -\,-\,-\,-\,-\,- \end{pmatrix}$	6 7	28 28	– –	44 –	8 –	– –
	$\begin{pmatrix} 3\,3\,3\,3\,3\,3 \\ -\,-\,-\,-\,-\,- \end{pmatrix}$	6 7	40 40	– –	90 –	20 –	– –
M_{20}	$\begin{pmatrix} 3\,3\,3\,3\,4\,4 \\ -\,-\,-\,-\,-\,- \end{pmatrix}$	7 8	76 6	– –	43 –	8 –	– –
M_{22}	$\begin{pmatrix} 3\,3\,4\,4\,4\,4 \\ -\,-\,-\,-\,-\,- \end{pmatrix}$	8 9	112 –	– –	19 –	2 –	– –

TABLE 5. Examples yielding elements of order 5 in $\tilde{H}_{4+u}(\mathsf{M}_{14+2u};\mathbb{Z})$ for $0 \le u \le 4$; $\rho_d(p) = \dim \mathrm{Tor}(\tilde{H}_d((G,\alpha);\mathbb{Z});\mathbb{Z}_p)$ is the p-rank of $\tilde{H}_d((G,\alpha);\mathbb{Z})$ and β_d is the free rank of the same group.

Complex	Group Action (G,α)	d	β_d	$\rho_d(2)$	$\rho_d(3)$	$\rho_d(5)$	$\rho_d(7)$
M_{22}	$\left\{4 \times \begin{matrix}3\\-\end{matrix}\right\} * \begin{pmatrix} 3\,3\,4 \\ -\,-\,- \end{pmatrix}$	8 9	58 –	36 –	53 –	14 –	– –
M_{24}	$\left\{4 \times \begin{matrix}3\\-\end{matrix}\right\} * \begin{pmatrix} 4\,4\,4 \\ -\,-\,- \end{pmatrix}$	9 10	93 –	32 1	29 –	5 –	– –
M_{26}	$\left\{4 \times \begin{matrix}4\\-\end{matrix}\right\} * \begin{pmatrix} 3\,3\,4 \\ -\,-\,- \end{pmatrix}$	10 11	141 –	25 –	22 –	2 –	– –
M_{28}	$\left\{4 \times \begin{matrix}4\\-\end{matrix}\right\} * \begin{pmatrix} 4\,4\,4 \\ -\,-\,- \end{pmatrix}$	11 12	167 –	35 –	18 –	3 –	– –
M_{30}	$\left\{4 \times \begin{matrix}4\\-\end{matrix}\right\} * \left\{3 \times \begin{matrix}4\\-\end{matrix}\right\} * \begin{pmatrix} 2 \\ - \end{pmatrix}$	11 12	– 550	– 53	8 29	– 2	– –

TABLE 6. Examples yielding elements of order 5 in $\tilde{H}_{4+u}(\mathsf{M}_{14+2u};\mathbb{Z})$ for $4 \le u \le 8$ (notation as in Table 5).

Theorem 5.1. *The map φ defines an isomorphism between $\mathcal{X}^\lambda$ and $\mathcal{C}(\lambda^-;R)$.*

Proof. It is straightforward to check that we have an isomorphism between the individual chain groups. Consider a generator

$$\gamma = e_0 \wedge \cdots \wedge e_d \otimes v_1 \wedge \cdots \wedge v_t,$$

and define

$$\gamma_i = e_0 \wedge \cdots \wedge \hat{e_i} \wedge \cdots \wedge e_d \otimes x_i \wedge y_i \wedge v_1 \wedge \cdots \wedge v_t,$$

where $e_i = x_i y_i$. The coefficient of γ_i in the boundary of γ in $\mathcal{X}^\lambda$ equals $(-1)^i$, which implies that the coefficient of $e_0 \otimes \cdots \otimes \hat{e_i} \otimes \cdots \otimes e_d$ in $\varphi \circ \delta(\gamma)$ equals $\mathrm{sgn}(\gamma_i) \cdot (-1)^i$. Moreover, the coefficient of the same generator in $\partial \circ \varphi(\gamma)$ equals $(-1)^{d-\eta_i} \cdot \mathrm{sgn}(\gamma)$, where η_i is the number of edges among $e_{i+1}, \ldots, e_d$ that are disjoint from e_i. For these two coefficients to be the same, we need that

$$\mathrm{sgn}(\gamma_i) = (-1)^{d-\eta_i} \cdot (-1)^i \cdot \mathrm{sgn}(\gamma). \tag{5–1}$$

Now, we may get from $\langle \gamma \rangle$ to $\langle \gamma_i \rangle$ by applying $4(d-i)$ transpositions of adjacent elements; we first move y_i to its new position via $2(d-i)$ transpositions and then move x_i via the same number of transpositions.

The number of inversions increases or decreases by one whenever we transpose two distinct elements, and stays the same otherwise. Since all edges are distinct, the total number of times the number of inversions changes is equal to $4(d-i)$ minus the number of edges among $e_{i+1}, \ldots, e_d$

Complex	Group Action (G,α)	d	β_d	$\rho_d(2)$	$\rho_d(3)$	$\rho_d(5)$	$\rho_d(7)$
M_{19}	$\left\{4 \times \genfrac{}{}{0pt}{}{3}{-}\right\} * \begin{pmatrix} 2 & 2 & 3 \\ - & - & - \end{pmatrix}$	6	–	–	4	1	–
		7	96	5	1	–	–
M_{21}	$\left\{4 \times \genfrac{}{}{0pt}{}{3}{-}\right\} * \begin{pmatrix} 3 & 3 & 3 \\ - & - & - \end{pmatrix}$	7	–	1	20	3	–
		8	167	7	–	–	–
M_{23}	$\left\{4 \times \genfrac{}{}{0pt}{}{3}{-}\right\} * \begin{pmatrix} 3 & 4 & 4 \\ - & - & - \end{pmatrix}$	8	–	–	22	2	–
		9	141	–	2	–	–
M_{25}	$\left\{4 \times \genfrac{}{}{0pt}{}{4}{-}\right\} * \begin{pmatrix} 3 & 3 & 3 \\ - & - & - \end{pmatrix}$	9	–	2	27	5	–
		10	93	1	1	–	–
M_{27}	$\left\{4 \times \genfrac{}{}{0pt}{}{4}{-}\right\} * \begin{pmatrix} 3 & 4 & 4 \\ - & - & - \end{pmatrix}$	10	–	12	51	14	–
		11	58	2	2	–	–

TABLE 7. Examples yielding elements of order 5 in $\tilde{H}_{6+u}(\mathsf{M}_{19+2u};\mathbb{Z})$ for $0 \le u \le 4$ (notation as in Table 5).

Complex	Group Action (G,α)	d	β_d	$\rho_d(3)$	$\rho_d(5)$	$\rho_d(7)$
M_{23}	$\left\{3 \times \genfrac{}{}{0pt}{}{2}{-}\right\} * \left\{3 \times \genfrac{}{}{0pt}{}{3}{-}\right\} * \left\{2 \times \genfrac{}{}{0pt}{}{4}{-}\right\}$	8	–	29	3	1
		9	332	–	–	–
M_{25}	$\left\{2 \times \genfrac{}{}{0pt}{}{2}{-}\right\} * \left\{3 \times \genfrac{}{}{0pt}{}{3}{-}\right\} * \left\{3 \times \genfrac{}{}{0pt}{}{4}{-}\right\}$	9	–	73	11	2
		10	407	1	–	–
M_{27}	$\left\{5 \times \genfrac{}{}{0pt}{}{4}{-}\right\} * \begin{pmatrix} 2 & 2 & 3 \\ - & - & - \end{pmatrix}$	10	–	77	27	9
		11	150	1	–	–
M_{29}	$\left\{5 \times \genfrac{}{}{0pt}{}{4}{-}\right\} * \begin{pmatrix} 2 & 2 & 5 \\ - & - & - \end{pmatrix}$	11	4	84	33	7
		12	66	–	–	–
M_{31}	$\left\{5 \times \genfrac{}{}{0pt}{}{4}{-}\right\} * \begin{pmatrix} 2 & 3 & 6 \\ - & - & - \end{pmatrix}$	12	9	98	36	4
		13	32	1	–	–
M_{33}	$\left\{5 \times \genfrac{}{}{0pt}{}{4}{-}\right\} * \begin{pmatrix} 2 & 5 & 6 \\ - & - & - \end{pmatrix}$	13	32	98	36	4
		14	9	1	–	–
M_{35}	$\left\{5 \times \genfrac{}{}{0pt}{}{4}{-}\right\} * \begin{pmatrix} 3 & 6 & 6 \\ - & - & - \end{pmatrix}$	14	66	94	33	7
		15	4	–	–	–
M_{37}	$\left\{5 \times \genfrac{}{}{0pt}{}{4}{-}\right\} * \begin{pmatrix} 5 & 6 & 6 \\ - & - & - \end{pmatrix}$	15	150	86	27	9
		16	–	–	–	–
M_{39}	$\left\{3 \times \genfrac{}{}{0pt}{}{4}{-}\right\} * \left\{3 \times \genfrac{}{}{0pt}{}{5}{-}\right\} * \left\{2 \times \genfrac{}{}{0pt}{}{6}{-}\right\}$	16	407	75	11	2
		17	–	1	–	–
M_{41}	$\left\{2 \times \genfrac{}{}{0pt}{}{4}{-}\right\} * \left\{3 \times \genfrac{}{}{0pt}{}{5}{-}\right\} * \left\{3 \times \genfrac{}{}{0pt}{}{6}{-}\right\}$	17	332	30	3	1
		18	–	–	–	–

TABLE 8. Examples yielding elements of order 7 in $\tilde{H}_{8+u}(\mathsf{M}_{23+2u};\mathbb{Z})$ for $0 \le u \le 9$ (notation as in Table 5).

Complex	Group Action (G,α)	d	β_d	$\rho_d(3)$	$\rho_d(5)$	$\rho_d(7)$
M_{24}	$\left\{4 \times \genfrac{}{}{0pt}{}{3}{-}\right\} * \left\{4 \times \genfrac{}{}{0pt}{}{3}{-}\right\}$	8	–	–	1	–
		9	67	28	6	–
		10	–	2	–	–
M_{30}	$\left\{3 \times \genfrac{}{}{0pt}{}{2}{-}\right\} * \left\{6 \times \genfrac{}{}{0pt}{}{4}{-}\right\}$	11	–	6	–	1
		12	174	8	–	–
		13	1	2	–	–

TABLE 9. One example yielding elements of order 5 in $\tilde{H}_8(\mathsf{M}_{24};\mathbb{Z})$ and another yielding elements of order 7 in $\tilde{H}_{11}(\mathsf{M}_{30};\mathbb{Z})$ (notation as in Table 5).

that are not disjoint from e_i. This equals

$$4(d-i)-((d-i)-\eta_i) = 3d-3i+\eta_i \equiv i+d-\eta_i \pmod 2,$$

which implies (5–1). □

For $R = \mathbb{Q}$, it is shown in [Dong and Wachs 02] that the homology of $\mathcal{X}^\lambda$ is isomorphic to that of $\mathcal{C}(\lambda^+;\mathbb{Q})$, where λ^+ is obtained from λ^- by replacing all minus signs with plus signs. As it turns out, this property does not remain true if we replace $\mathbb{Q}$ with $\mathbb{Z}$. Namely, in general, the homology of $\mathcal{C}(\lambda^+;\mathbb{Z})$ is not isomorphic to the homology of $\mathcal{C}(\lambda^-;\mathbb{Z})$. The smallest example is given by $\lambda = (1,2,2,2)$, in which case we obtain elements of order 3 for the natural signed action but not for the natural unsigned action. Computational evidence suggests that the torsion subgroup of the homology of $\mathcal{C}(\lambda^+;\mathbb{Z})$ tends to be smaller than the torsion subgroup of the homology of $\mathcal{C}(\lambda^-;\mathbb{Z})$, but we have not been able to make this observation more precise, let alone prove anything in this direction.

All results listed in Tables 5 through 9 can be interpreted as results about the homology of $\mathcal{X}^\lambda$ for various λ; all signs s_i are -1. For example, the homology of each of $\mathcal{X}^{(3,3,3,3,3,3,3)}$, $\mathcal{X}^{(3,3,3,3,3,3,3,3)}$, $\mathcal{X}^{(3,3,3,3,3,3,3,3,3)}$, and $\mathcal{X}^{(4,4,4,4,4,4,4)}$ contains elements of order 5. Keep in mind that all degrees should be shifted one step up.

6. ACTIONS INDUCED BY DIRECT PRODUCTS OF WREATH PRODUCTS

In some situations in which the complex $\mathcal{C}((\lambda,\mathsf{s});R)$ remains too large to admit a direct analysis, we will consider a group action on $\mathcal{C}((\lambda,\mathsf{s});R)$. Specifically, let $T = (T_1,\dots,T_q)$ be a partition of $\{1,\dots,r\}$ such that $(\lambda_a,s_a) = (\lambda_b,s_b)$ for all a and b in the same set T_k. The Young group $\mathfrak{S}_T = \mathfrak{S}_{T_1}\times\cdots\times\mathfrak{S}_{T_q}$ acts on $\mathcal{C}((\lambda,\mathsf{s});R)$ by permuting the elements in each T_k. In this paper, we will consider only the natural unsigned action.

Assume that $|\mathfrak{S}_\lambda|$ and $|\mathfrak{S}_T|$ are not zero divisors in R. Note that we obtain the chain complex $\mathcal{C}((\lambda,\mathsf{s});R)/\mathfrak{S}_T$ from $\mathcal{C}(\mathsf{M}_n;R)$ in three steps:

1. Let $\mathfrak{S}_\lambda$ act on $\mathcal{C}(\mathsf{M}_n;R)$ to form $\mathcal{C}(\mathsf{M}_n;R)/(\lambda,\mathsf{s})$.
2. Transform $\mathcal{C}(\mathsf{M}_n;R)/(\lambda,\mathsf{s}) \cong \mathcal{C}((\Delta_{\lambda,\mathsf{s}},E_{\lambda,\mathsf{s}});R)$ into $\mathcal{C}((\lambda,\mathsf{s});R)$ via the natural quotient map.
3. Let $\mathfrak{S}_T$ act on $\mathcal{C}((\lambda,\mathsf{s});R)$ to form $\mathcal{C}((\lambda,\mathsf{s});R)/\mathfrak{S}_T$.

One may interchange steps 2 and 3, thus first letting the group $\mathfrak{S}_T$ act on $\mathcal{C}(\mathsf{M}_n;R)/(\lambda,\mathsf{s})$ and then forming a quotient complex satisfying properties I and II in Theorem 4.3. In this manner, we may view $(\mathcal{C}((\lambda,\mathsf{s});R)/\mathfrak{S}_T$ as a quotient complex of $\tilde{C}_d(\mathsf{M}_n;\mathbb{Z})/G(\lambda,\mathsf{s},T)$, where $G(\lambda,\mathsf{s},T)$ is a direct product of certain wreath products defined in terms of $\mathfrak{S}_\lambda$ and $\mathfrak{S}_T$.

Specifically, define $\mu_k = |T_k|$. For simplicity, assume that $k \in T_k$ for $1 \le k \le q$ and that $\mu_k = 1$ if and only if k is greater than a given value p. Then $G(\lambda,\mathsf{s},T)$ is isomorphic to

$$(\mathfrak{S}_{\lambda_1}\wr\mathfrak{S}_{\mu_1})\times\cdots\times(\mathfrak{S}_{\lambda_p}\wr\mathfrak{S}_{\mu_p})\times\mathfrak{S}_{\lambda_{p+1}}\times\cdots\times\mathfrak{S}_{\lambda_q},$$

where $\mathfrak{S}_{\lambda_k}\wr\mathfrak{S}_{\mu_k}$ denotes the wreath product of $\mathfrak{S}_{\lambda_k}$ by $\mathfrak{S}_{\mu_k}$. We represent the action as

$$\left\{\mu_1\times\begin{matrix}\lambda_1\\ s_1\end{matrix}\right\} * \left\{\mu_2\times\begin{matrix}\lambda_2\\ s_2\end{matrix}\right\} *\cdots* \left\{\mu_p\times\begin{matrix}\lambda_p\\ s_p\end{matrix}\right\} * \begin{pmatrix}\lambda_{p+1} & \lambda_{p+2} & \cdots & \lambda_q\\ s_{p+1} & s_{p+2} & \cdots & s_q\end{pmatrix}. \tag{6–1}$$

Let $\mathcal{W}$ be the subcomplex of $\mathcal{C}(\mathsf{M}_n;R)/(\lambda,\mathsf{s})$ with the property that the complex $\mathcal{C}((\lambda,\mathsf{s});R)$ is the quotient of $\mathcal{C}(\mathsf{M}_n;R)/(\lambda,\mathsf{s})$ by $\mathcal{W}$. Since $\mathcal{W}$ is acyclic by the proof of Theorem 4.3, the homology of $\mathcal{W}/\mathfrak{S}_T$ has exponent dividing $|\mathfrak{S}_T|$ and is hence finite; apply Proposition 2.1. By the exact sequence

$$\begin{aligned}\tilde{H}_i(\mathcal{W}/\mathfrak{S}_T) &\longrightarrow \tilde{H}_i((\mathcal{C}(\mathsf{M}_n;R)/(\lambda,\mathsf{s}))/\mathfrak{S}_T)\\ &\longrightarrow \tilde{H}_i(\mathcal{C}((\lambda,\mathsf{s});R)/\mathfrak{S}_T)\\ &\longrightarrow \tilde{H}_{i-1}(\mathcal{W}/\mathfrak{S}_T),\end{aligned}$$

we may hence deduce that $\tilde{H}_i((\mathcal{C}(\mathsf{M}_n;R)/(\lambda,\mathsf{s}))/\mathfrak{S}_T)$ and $\tilde{H}_i(\mathcal{C}((\lambda,\mathsf{s});R)/\mathfrak{S}_T)$ nearly coincide, the exception being that the Sylow p-subgroups of their torsion parts may differ when p divides $|\mathfrak{S}_T|$.

If some block T_j of size at least two consists of positively charged elements, then the homology of $\mathcal{W}/\mathfrak{S}_T$ is indeed not necessarily zero. For example, if $\lambda = \left(\begin{smallmatrix}3&3&3\\+&+&+\end{smallmatrix}\right)$ and $\mathfrak{S}_T = \mathfrak{S}_3$, then the homology in degree two is a group of order two.

However, if each block T_j of size at least two consists of negatively charged elements, then the homology groups do coincide; it is not hard to adapt the proof of Theorem 4.3 to prove that $\mathcal{W}/\mathfrak{S}_T$ has vanishing homology. All examples considered in this paper are of this type.

The problem of finding a combinatorial description of the generators of $\mathcal{C}((\lambda,\mathsf{s});R)/\mathfrak{S}_T$ appears to be immensely hard in general; we are not aware of any simple characterization similar to the one in Theorem 4.3.

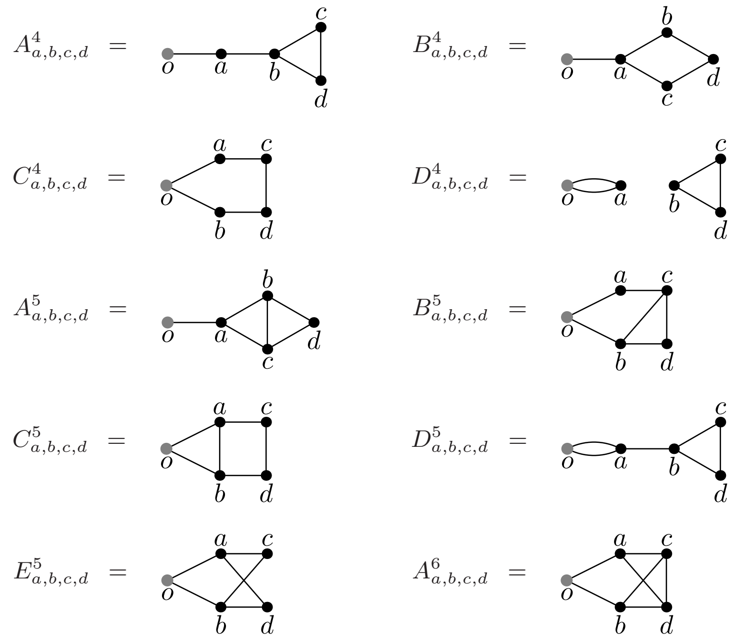

FIGURE 1. Multigraphs for which we have one generator, where $\{a, b, c, d\} = \{1, 2, 3, 4\}$.

7. DETECTING ELEMENTS OF ORDER 5 IN THE HOMOLOGY OF M_{14}

We present a computer-free proof that $\tilde{H}_4(\mathsf{M}_{14}; \mathbb{Z})$ contains elements of order 5. The proof consists of two steps. In the first step we consider the action on M_{14} induced by $\left(\begin{smallmatrix} 2 & 3 & 3 & 3 & 3 \\ + & - & - & - & - \end{smallmatrix}\right)$. In the second step we proceed with the unsigned action induced by the natural action by $\mathfrak{S}_4$ on the four parts of size three, thus analyzing the action induced by $\left(\begin{smallmatrix} 2 \\ + \end{smallmatrix}\right) * \left\{4 \times \begin{smallmatrix} 3 \\ - \end{smallmatrix}\right\}$. The resulting chain complex consists of two free groups of rank four, one in degree four and one in degree five. An explicit calculation of the boundary map yields a matrix with determinant ± 5.

Unfortunately, the proof does not shed much light on *why* we end up with the value 5. In particular, we do not know how to generalize the proof to deduce the existence of elements of order $(2r-1)$ in the homology of $\mathsf{M}_{(r+1)^2-2}$ for general r. We expect that this can be done by analyzing the action induced by $\left(\begin{smallmatrix} r-1 \\ + \end{smallmatrix}\right) * \left\{(r+1) \times \begin{smallmatrix} r \\ - \end{smallmatrix}\right\}$; recall the discussion after Theorem 1.5 in Section 1 and in Section 1.1.

Thus pick the action on M_{14} induced by $\left(\begin{smallmatrix} \lambda \\ \mathsf{s} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & 3 & 3 & 3 & 3 \\ + & - & - & - & - \end{smallmatrix}\right)$. By Theorem 4.3, the homology of the resulting chain complex $\mathcal{C}(\mathsf{M}_{14}; \mathbb{Z})/(\lambda, \mathsf{s})$ is isomorphic to the homology of the chain complex $\mathcal{C}((\lambda, \mathsf{s}); \mathbb{Z})$. In this complex, we have one generator for each loopless multigraph on the vertex set $\{o, 1, 2, 3, 4\}$ such that the degree of o is 1 or 2, the degrees of the other vertices are 2 or 3, and there are no multiple edges between vertices in $\{1, 2, 3, 4\}$; we identify each multigraph with its multiset of edges. A careful examination yields that we have one generator for each of the multigraphs in Figure 1, where $\{a, b, c, d\} = \{1, 2, 3, 4\}$.

Arrange the set of edges on $\{o, 1, 2, 3, 4\}$ lexicographically according to the order $o < 1 < 2 < 3 < 4$:

$$o1 < o2 < o3 < o4 < 12 < 13 < 14 < 23 < 24 < 34.$$

Consider the unsigned action on $\mathcal{C}((\lambda, \mathsf{s}); \mathbb{Z})$ by $\mathfrak{S}_4$ induced by the natural action on the set $\{1, 2, 3, 4\}$. In the resulting chain complex $\mathcal{C}'$, we have one generator for each of the above isomorphism classes of multigraphs.

Yet note that

$$E^5_{1,2,3,4} = E^5_{2,1,3,4} = \{o1, o2, 13, 14, 23, 24\}.$$

Writing $\gamma = o1 \otimes o2 \otimes 13 \otimes 14 \otimes 23 \otimes 24$, we have that

$$\begin{aligned}(1,2) \circ \gamma &= o2 \otimes o1 \otimes 23 \otimes 24 \otimes 13 \otimes 14 \\ &= -o1 \otimes o2 \otimes 13 \otimes 14 \otimes 23 \otimes 24 = -\gamma;\end{aligned}$$

this is because the sequence $(o2, o1, 23, 24, 13, 14)$ contains the three anticommuting inversions $(1, 2)$, $(3, 6)$, and $(4, 5)$. In particular, the generator corresponding to the isomorphism class of $E^5_{1,2,3,4}$ is zero. For similar reasons, the generator corresponding to the isomorphism class of $A^6_{1,2,3,4}$ is also zero.

The remaining eight generators are nonzero, though. For example, the identity and $(1, 2)(3, 4)$ are the two group elements in $\mathfrak{S}_4$ that leave $C^4_{1,2,3,4}$ fixed. Since

$$(1,2)(3,4)\circ(o1\otimes o2\otimes 13\otimes 24\otimes 34) = o2\otimes o1\otimes 24\otimes 13\otimes 34,$$

and since the sequence $(o2, o1, 24, 13, 34)$ contains two anticommuting inversions $(1,2)$ and $(3,4)$, we obtain the desired claim. The seven other generators are treated similarly.

For any multigraph G on the vertex set $\{o,1,2,3,4\}$, fix the orientation given by the lexicographic order defined earlier. For example, we identify $B^4_{3,1,4,2} = \{o3, 13, 34, 12, 24\}$ with the oriented simplex $o3\otimes 12\otimes 13\otimes 24\otimes 34$. Identify each generator G with its isomorphism class. Moreover, write $A^4 = A^4_{1,2,3,4}$ and so on. We obtain the following:

$$\begin{aligned}
\partial(A^5) &= A^4_{1,2,3,4} + A^4_{1,3,2,4} - B^4_{1,2,3,4} = 2A^4 - B^4;\\
\partial(B^5) &= -A^4_{1,3,2,4} - C^4_{1,2,3,4} = -A^4 - C^4;\\
\partial(C^5) &= -B^4_{1,2,3,4} + B^4_{2,1,4,3} + C^4_{1,2,3,4} = -2B^4 + C^4;\\
\partial(D^5) &= A^4_{1,2,3,4} + A^4_{1,2,3,4} + D^4_{1,2,3,4} = 2A^4 + D^4.
\end{aligned}$$

In matrix form, we get

$$\partial\begin{pmatrix} A^5\\ B^5\\ C^5\\ D^5\end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0\\ -1 & 0 & -1 & 0\\ 0 & -2 & 1 & 0\\ 2 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix} A^4\\ B^4\\ C^4\\ D^4\end{pmatrix}.$$

Since the determinant is -5, we conclude that $\tilde{H}_4(\mathcal{C}') \cong \mathbb{Z}_5$. As a consequence, since the order of $\mathfrak{S}_4$ is not divisible by five, $\tilde{H}_4((\lambda, \mathsf{s}); \mathbb{Z})$ contains elements of order five, as does $\tilde{H}_4(\mathsf{M}_{14}; \mathbb{Z})$ for similar reasons.

8. THE CASE OF AN ABELIAN 2-GROUP

Let us discuss the special case that all values in the sequence λ are at most two. Before considering the matching complex, we look at a more general situation. Let R be a commutative ring such that 2 is a unit. Let

$$\begin{aligned}\mathcal{C} : \cdots &\xrightarrow{\partial} C_{d+1} \xrightarrow{\partial} C_d\\ &\xrightarrow{\partial} C_{d-1} \xrightarrow{\partial} \cdots\end{aligned}$$

be a chain complex of R-modules. Write $C = \bigoplus_d C_d$. Suppose that τ is an involution on $\mathcal{C}$; τ generates a group of size 2 acting on $\mathcal{C}$. Consider the subgroups C^+_d and C^-_d of C_d induced by the unsigned and signed actions on $\mathcal{C}$, respectively:

$$\begin{aligned} C^+_d &= \{c + \tau(c) : c \in C_d\},\\ C^-_d &= \{c - \tau(c) : c \in C_d\}.\end{aligned}$$

We obtain two chain complexes $\mathcal{C}^+$ and $\mathcal{C}^-$. Write $C^+ = \bigoplus C^+_d$ and $C^- = \bigoplus C^-_d$.

Proposition 8.1. *With notation as above, we have that*

$$\mathcal{C} = \mathcal{C}^+ \oplus \mathcal{C}^-.$$

In particular,

$$H_d(\mathcal{C}) \cong H_d(\mathcal{C}^+) \oplus H_d(\mathcal{C}^-).$$

Proof. Note that we may decompose the identity as a sum of two orthogonal idempotent endomorphisms; we have that

$$\mathrm{Id} = \frac{1}{2}(\mathrm{Id} + \tau) + \frac{1}{2}(\mathrm{Id} - \tau).$$

Since $\frac{1}{2}(\mathrm{Id} + \tau)(\mathcal{C}) = \mathcal{C}^+$ and $\frac{1}{2}(\mathrm{Id} - \tau)(\mathcal{C}) = \mathcal{C}^-$, we are done. □

More generally, assume that we have r pairwise commuting involutions $\tau_1, \dots, \tau_r$ and hence an elementary abelian 2-group of order 2^r. For any given sequence of signs $\mathsf{s} = (s_1, \dots, s_r) \in \{+1, -1\}^r$, define

$$C^s_d = C^{s_1,\dots,s_r}_d = \Big\{\prod_{i=1}^r (\mathrm{id} + s_i\tau_i)\circ c : c \in C_d\Big\}.$$

Similarly to the proof of Proposition 8.1, we may write the identity as a sum

$$\sum_{\mathsf{s}\in\{+1,-1\}^r} \frac{1}{2^r}\prod_{i=1}^r(\mathrm{id} + s_i\tau_i)$$

of endomorphisms. Since the endomorphisms are mutually orthogonal and idempotent, we obtain the following generalization of Proposition 8.1.

Proposition 8.2. *With notation as above, we have that*

$$\mathcal{C} = \bigoplus_{s\in\{+1,-1\}^r} \mathcal{C}^s.$$

In particular,

$$H_d(\mathcal{C}) \cong \bigoplus_{s\in\{+1,-1\}^r} H_d(\mathcal{C}^s).$$

For the matching complex M_n, a natural choice of involutions is $\tau_1 = (1,2), \tau_2 = (3,4), \dots, \tau_r = (2r-1,$

s	$d=1$	Factor
$++$	$\tilde{\mathbb{Z}}$	1
$+-$	$\tilde{\mathbb{Z}}^2$	2
$--$	$\tilde{\mathbb{Z}}$	1
$\tilde{H}_d(\mathsf{M}_5;\tilde{\mathbb{Z}})$	$\tilde{\mathbb{Z}}^6$	

TABLE 10. The groups $\tilde{H}_d(((2,2,1),\mathsf{s});\tilde{\mathbb{Z}})$ yielding $\tilde{H}_d(\mathsf{M}_5;\tilde{\mathbb{Z}})$.

s	$d=1$	Factor
$+++$	$\tilde{\mathbb{Z}}^2$	1
$++-$	$\tilde{\mathbb{Z}}^2$	3
$+--$	$\tilde{\mathbb{Z}}^2$	3
$---$	$\tilde{\mathbb{Z}}^2$	1
$\tilde{H}_d(\mathsf{M}_6;\tilde{\mathbb{Z}})$	$\tilde{\mathbb{Z}}^{16}$	

TABLE 11. The groups $\tilde{H}_d(((2,2,2),\mathsf{s});\tilde{\mathbb{Z}})$ yielding $\tilde{H}_d(\mathsf{M}_6;\tilde{\mathbb{Z}})$.

$2r$), where $2r \leq n$. Defining $\lambda = (2^r, 1^{n-2r}) = (2,\dots,2,1,\dots,1)$, we obtain that

$$\tilde{C}_d(\mathsf{M}_n;R)/(\lambda,\mathsf{s}) = \Big\{\prod_{i=1}^{r}(\mathrm{id}+s_i\tau_i)\circ c : c \in C_d(\mathsf{M}_n;R)\Big\}.$$

By Proposition 8.2 and Theorem 4.3, we have that

$$\tilde{H}_d(\mathsf{M}_n;R) \cong \bigoplus_{s\in\{+1,-1\}^r} \tilde{H}_d(\mathsf{M}_n;R)/(\lambda,\mathsf{s}) \cong \bigoplus_{s\in\{+1,-1\}^r} \tilde{H}_d((\lambda,\mathsf{s});R).$$

Define $(+)^a(-)^b$ to be the sequence consisting of a plus signs followed by b minus signs.

Proposition 8.3. *With notation as above, we have that*

$$\tilde{H}_d(\mathsf{M}_n;R) \cong \bigoplus_{i=0}^{r}\bigoplus_{\binom{r}{i}} \tilde{H}_d\big((\lambda,(+)^{r-i}(-)^i);R\big).$$

Proof. This is immediate from the fact that $\mathcal{C}((\lambda,\mathsf{s});R)$ and $\mathcal{C}((\lambda,\mathsf{s}');R)$ are isomorphic whenever s and s' consist of the same number of plus signs. □

We use this decomposition to analyze the homology of M_n for $5 \leq n \leq 16$; see Section 9.1.

s	$d=1$	$d=2$	Factor
$+++$	$-$	$\tilde{\mathbb{Z}}$	1
$++-$	$-$	$\tilde{\mathbb{Z}}^3$	3
$+--$	$-$	$\tilde{\mathbb{Z}}^3$	3
$---$	$\mathbb{Z}_3$	$\tilde{\mathbb{Z}}$	1
$\tilde{H}_d(\mathsf{M}_7;\tilde{\mathbb{Z}})$	$\mathbb{Z}_3$	$\tilde{\mathbb{Z}}^{20}$	

TABLE 12. The groups $\tilde{H}_d(((2,2,2,1),\mathsf{s});\tilde{\mathbb{Z}})$ yielding $\tilde{H}_d(\mathsf{M}_7;\tilde{\mathbb{Z}})$.

s	$d=2$	Factor
$++++$	$\tilde{\mathbb{Z}}^6$	1
$+++-$	$\tilde{\mathbb{Z}}^9$	4
$++--$	$\tilde{\mathbb{Z}}^8$	6
$+---$	$\tilde{\mathbb{Z}}^9$	4
$----$	$\tilde{\mathbb{Z}}^6$	1
$\tilde{H}_d(\mathsf{M}_8;\tilde{\mathbb{Z}})$	$\tilde{\mathbb{Z}}^{132}$	

TABLE 13. The groups $\tilde{H}_d(((2,2,2,2),\mathsf{s});\tilde{\mathbb{Z}})$ yielding $\tilde{H}_d(\mathsf{M}_8;\tilde{\mathbb{Z}})$.

9. OVERVIEW OF COMPUTATIONS

The purpose of this section is to present an overview of the computations leading to the results presented in Section 1.

9.1 The Homology of M_n for $n \leq 16$

Applying the decomposition described in Section 8, we analyze the homology of M_n for $5 \leq n \leq 16$ using the computer program HOMCHAIN [Pilarczyk 04]; see Tables 10–19 for a summary. Since the underlying group has order a power of 2, our computations do not give us any immediate information about the existence of elements of order 2 in the homology of M_n. For this reason, we express our results in terms of the coefficient ring

$$\tilde{\mathbb{Z}} = \{a \cdot 2^{-b} : a, b \in \mathbb{Z}, b \geq 0\},$$

thus ignoring the Sylow 2-subgroup of the torsion part of the homology.

In some cases, we managed to compute the homology only over finite fields $\mathbb{Z}_p$ and not over $\tilde{\mathbb{Z}}$. Using the universal coefficient theorem, one may still obtain the p-rank of the homology; see [Hatcher 02, Corollary 3A.6]. We refer to Theorems 1.2 and 1.6 and Table 1 for a summary of our results.

s	$d=2$	$d=3$	Factor
$++++$	$\tilde{\mathbb{Z}}^3$	$\tilde{\mathbb{Z}}$	1
$+++-$	$\tilde{\mathbb{Z}}^3$	$\tilde{\mathbb{Z}}^4$	4
$++--$	$\tilde{\mathbb{Z}}^2$	$\tilde{\mathbb{Z}}^6$	6
$+---$	$\tilde{\mathbb{Z}}^3 \oplus \mathbb{Z}_3$	$\tilde{\mathbb{Z}}^4$	4
$----$	$\tilde{\mathbb{Z}}^3 \oplus (\mathbb{Z}_3)^4$	$\tilde{\mathbb{Z}}$	1
$\tilde{H}_d(\mathsf{M}_9;\tilde{\mathbb{Z}})$	$\tilde{\mathbb{Z}}^{42} + (\mathbb{Z}_3)^8$	$\tilde{\mathbb{Z}}^{70}$	

TABLE 14. The groups $\tilde{H}_d(((2,2,2,2,1),\mathsf{s});\tilde{\mathbb{Z}})$ yielding $\tilde{H}_d(\mathsf{M}_9;\tilde{\mathbb{Z}})$.

s	$d=2$	$d=3$	Factor
$+++++$	$-$	$\tilde{\mathbb{Z}}^{28}$	1
$++++-$	$-$	$\tilde{\mathbb{Z}}^{36}$	5
$+++--$	$-$	$\tilde{\mathbb{Z}}^{40}$	10
$++---$	$-$	$\tilde{\mathbb{Z}}^{40}$	10
$+----$	$-$	$\tilde{\mathbb{Z}}^{36}$	5
$-----$	$\mathbb{Z}_3$	$\tilde{\mathbb{Z}}^{28}$	1
$\tilde{H}_d(\mathsf{M}_{10};\tilde{\mathbb{Z}})$	$\mathbb{Z}_3$	$\tilde{\mathbb{Z}}^{1216}$	

TABLE 15. The groups $\tilde{H}_d(((2,2,2,2,2),\mathsf{s});\tilde{\mathbb{Z}})$ yielding $\tilde{H}_d(\mathsf{M}_{10};\tilde{\mathbb{Z}})$.

s	$d=3$	$d=4$	Factor
$+++++$	$\tilde{\mathbb{Z}}^{39}$	$\tilde{\mathbb{Z}}$	1
$++++-$	$\tilde{\mathbb{Z}}^{39}$	$\tilde{\mathbb{Z}}^5$	5
$+++--$	$\tilde{\mathbb{Z}}^{36}$	$\tilde{\mathbb{Z}}^{10}$	10
$++---$	$\tilde{\mathbb{Z}}^{36} + \mathbb{Z}_3$	$\tilde{\mathbb{Z}}^{10}$	10
$+----$	$\tilde{\mathbb{Z}}^{39} + (\mathbb{Z}_3)^5$	$\tilde{\mathbb{Z}}^5$	5
$-----$	$\tilde{\mathbb{Z}}^{39} + (\mathbb{Z}_3)^{10}$	$\tilde{\mathbb{Z}}$	1
$\tilde{H}_d(\mathsf{M}_{11};\tilde{\mathbb{Z}})$	$\tilde{\mathbb{Z}}^{1188} + (\mathbb{Z}_3)^{45}$	$\tilde{\mathbb{Z}}^{252}$	

TABLE 16. The groups $\tilde{H}_d(((2,2,2,2,2,1),\mathsf{s});\tilde{\mathbb{Z}})$ yielding $\tilde{H}_d(\mathsf{M}_{11};\tilde{\mathbb{Z}})$.

s	$d=3$	$d=4$	Factor
$++++++$	$-$	$\tilde{\mathbb{Z}}^{140}$	1
$+++++-$	$-$	$\tilde{\mathbb{Z}}^{170}$	6
$++++--$	$-$	$\tilde{\mathbb{Z}}^{200}$	15
$+++---$	$-$	$\tilde{\mathbb{Z}}^{206}$	20
$++----$	$\mathbb{Z}_3$	$\tilde{\mathbb{Z}}^{200}$	15
$+-----$	$(\mathbb{Z}_3)^5$	$\tilde{\mathbb{Z}}^{170}$	6
$------$	$(\mathbb{Z}_3)^{11}$	$\tilde{\mathbb{Z}}^{140}$	1
$\tilde{H}_d(\mathsf{M}_{12};\tilde{\mathbb{Z}})$	$(\mathbb{Z}_3)^{56}$	$\tilde{\mathbb{Z}}^{12440}$	

TABLE 17. The groups $\tilde{H}_d(((2,2,2,2,2,2),\mathsf{s});\tilde{\mathbb{Z}})$ yielding $\tilde{H}_d(\mathsf{M}_{12};\tilde{\mathbb{Z}})$.

The next value, $n = 17$, is certainly not out of reach of the existing software, but it does not seem worth the considerable effort to compute the homology of M_{17}. We already know the homology in the top dimension 7 [Bouc 92] and also have a qualified guess about the homology in degree 6 (see Conjecture 1.7). This leaves us with the rank of the elementary 3-group in degree 5 (the smallest degree with nonvanishing homology), likely a random-looking number that will not tell us much useful unless we also get to know the rank of the bottom nonvanishing homology of M_n for more values of n.

Define $f_0 = 1$, $f_1 = 0$, and $f_i = f_{i-1} + f_{i-2}$ for $i \geq 2$; these are the Fibonacci numbers. By Table 21, the 5-rank of $\tilde{H}_4(((2,2,2,2,2,2,2),(+)^{7-i}(-)^i);\tilde{\mathbb{Z}})$ is equal to f_i. It is not hard to show that this implies the following; see Section 4 for the definition of BD_r^λ.

Corollary 9.1. *For $0 \leq i \leq 7$, the 5-rank of $\tilde{H}_4(\mathsf{BD}_{7+i}^{2^{7-i}1^{2i}};\tilde{\mathbb{Z}})$ is f_{2i}.*

It is known [Andersen 92, Jonsson 09] that the Sylow 5-subgroup is elementary for $i = 0$.

9.2 Detecting Elements of Order 5

To detect elements of order 5 in $\tilde{H}_{4+u}(\mathsf{M}_{14+2u};\mathbb{Z})$ for $0 \leq u \leq 4$, we use the action induced by

$$\begin{pmatrix} \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6 \\ ------ \end{pmatrix}$$

for choices of partitions $(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6)$ such that $2 \leq \lambda_a \leq 4$ for all a. This does not bring us beyond $u = 4$, but a similar action based on a partition with seven instead of six parts helps us find elements of order 5 for $5 \leq u \leq 7$. The resulting chain complexes are too large for our computer to handle, so we reduce them further using group actions of the form

$$\left\{ 4 \times \begin{matrix} \lambda_1 \\ - \end{matrix} \right\} * \begin{pmatrix} \lambda_2\lambda_3\lambda_4 \\ --- \end{pmatrix}.$$

Yet another construction yields elements of order 5 in $\tilde{H}_{12}(\mathsf{M}_{30};\mathbb{Z})$. See Tables 5 and 6 for a summary of our computations for even n.

The same kind of action turns out to yield elements of order 5 in the group $\tilde{H}_{6+u}(\mathsf{M}_{19+2u};\mathbb{Z})$ for $0 \leq u \leq 4$; see

s	$d=4$	$d=5$	$d=6$	Factor
$+++++++$	$-$	$\tilde{\mathbb{Z}}^{3309}$	$\tilde{\mathbb{Z}}$	1
$++++++-$	$-$	$\tilde{\mathbb{Z}}^{3543}$	$\tilde{\mathbb{Z}}^{7}$	7
$+++++--$	$-$	$\tilde{\mathbb{Z}}^{3689}$	$\tilde{\mathbb{Z}}^{21}$	21
$++++---$	$-$	$\tilde{\mathbb{Z}}^{3739}+\mathbb{Z}_3$	$\tilde{\mathbb{Z}}^{35}$	35
$+++----$	$-$	$\tilde{\mathbb{Z}}^{3739}+(\mathbb{Z}_3)^7$	$\tilde{\mathbb{Z}}^{35}$	35
$++-----$	$\mathbb{Z}_3$	$\tilde{\mathbb{Z}}^{3689}+(\mathbb{Z}_3)^{21}$	$\tilde{\mathbb{Z}}^{21}$	21
$+------$	$(\mathbb{Z}_3)^7$	$\tilde{\mathbb{Z}}^{3543}+(\mathbb{Z}_3)^{35}$	$\tilde{\mathbb{Z}}^{7}$	7
$-------$	$(\mathbb{Z}_3)^{22}$	$\tilde{\mathbb{Z}}^{3309}+(\mathbb{Z}_3)^{35}$	$\tilde{\mathbb{Z}}$	1
$\tilde{H}_d(\mathsf{M}_{15};\tilde{\mathbb{Z}})$	$(\mathbb{Z}_3)^{92}$	$\tilde{\mathbb{Z}}^{472888}+(\mathbb{Z}_3)^{1001}$	$\tilde{\mathbb{Z}}^{3432}$	

TABLE 18. The groups $\tilde{H}_d(((2,2,2,2,2,2,2,1),\mathsf{s});\tilde{\mathbb{Z}})$ yielding $\tilde{H}_d(\mathsf{M}_{15};\tilde{\mathbb{Z}})$. The torsion parts for $d=5$ are guesses based on computations over $\mathbb{Z}_3$ and $\mathbb{Z}_5$.

s	$d=4$	$d=5$	$d=6$	Factor
$++++++++$	$-$	$\tilde{\mathbb{Z}}^{126}+(\mathbb{Z}_5)^7$	$\tilde{\mathbb{Z}}^{4060}$	1
$+++++++-$	$-$	$\tilde{\mathbb{Z}}^{105}+(\mathbb{Z}_5)^7$	$\tilde{\mathbb{Z}}^{5019}$	8
$++++++--$	$-$	$\tilde{\mathbb{Z}}^{100}+(\mathbb{Z}_5)^{12}+(\mathbb{Z}_3)^{10}$	$\tilde{\mathbb{Z}}^{5894}$	28
$+++++---$	$-$	$\tilde{\mathbb{Z}}^{91}+(\mathbb{Z}_5)^{18}+(\mathbb{Z}_3)^{55}$	$\tilde{\mathbb{Z}}^{6545}$	56
$++++----$	$-$	$\tilde{\mathbb{Z}}^{90}+(\mathbb{Z}_5)^{27}+(\mathbb{Z}_3)^{159}$	$\tilde{\mathbb{Z}}^{6768}$	70
$+++-----$	$-$	$\tilde{\mathbb{Z}}^{91}+(\mathbb{Z}_5)^{41}+(\mathbb{Z}_3)^{350}$	$\tilde{\mathbb{Z}}^{6545}$	56
$++------$	$-$	$\tilde{\mathbb{Z}}^{100}+(\mathbb{Z}_5)^{61}+(\mathbb{Z}_3)^{635}$	$\tilde{\mathbb{Z}}^{5894}$	28
$+-------$	$-$	$\tilde{\mathbb{Z}}^{105}+(\mathbb{Z}_5)^{91}+(\mathbb{Z}_3)^{966}$	$\tilde{\mathbb{Z}}^{5019}$	8
$--------$	$\mathbb{Z}_3$	$\tilde{\mathbb{Z}}^{126}+(\mathbb{Z}_5)^{134}+(\mathbb{Z}_3)^{1253}$	$\tilde{\mathbb{Z}}^{4060}$	1
$\tilde{H}_d(\mathsf{M}_{16};\tilde{\mathbb{Z}})$	$\mathbb{Z}_3$	$\tilde{\mathbb{Z}}^{24024}+(\mathbb{Z}_5)^{8163}+(\mathbb{Z}_3)^{60851}$	$\tilde{\mathbb{Z}}^{1625288}$	

TABLE 19. The groups $\tilde{H}_d(((2,2,2,2,2,2,2,2),\mathsf{s});\tilde{\mathbb{Z}})$ yielding $\tilde{H}_d(\mathsf{M}_{16};\tilde{\mathbb{Z}})$. The torsion parts for $d=5$ are guesses based on computations over $\mathbb{Z}_3$, $\mathbb{Z}_5$, and $\mathbb{Z}_7$.

Table 7. Moreover, the specific action represented as

$$\left\{4\times\frac{3}{-}\right\}*\left\{4\times\frac{3}{-}\right\}$$

yields elements of order 5 in $\tilde{H}_8(\mathsf{M}_{24};\mathbb{Z})$; see Table 9.

9.3 Detecting Elements of Order 7

To detect elements of order 7 in $\tilde{H}_{8+u}(\mathsf{M}_{23+2u};\mathbb{Z})$ for $0\le u\le 9$, we use partitions with eight parts as summarized in Table 8. The actions are of the forms

$$\left\{5\times\frac{\lambda_1}{-}\right\}*\binom{\lambda_2\lambda_3\lambda_4}{---}$$

and

$$\left\{2\times\frac{\lambda_1}{-}\right\}*\left\{3\times\frac{\lambda_2}{-}\right\}*\left\{3\times\frac{\lambda_3}{-}\right\}.$$

We detect elements of order 7 in $\tilde{H}_{11}(\mathsf{M}_{30};\mathbb{Z})$ by analyzing the complex induced by the action represented as

$$\left\{3\times\frac{2}{-}\right\}*\left\{6\times\frac{4}{-}\right\};$$

see Table 9.

9.4 Detecting Elements of Orders 11 and 13

As alluded to in the introduction, one may disclose elements of order $(2r-1)$ in the group $\tilde{H}_{\binom{r+1}{2}-2}(\mathsf{M}_{(r+1)^2-2};\mathbb{Z})$ for $2r-1\in\{5,7,11,13\}$ by analyzing the complex induced by the action

$$\left\{(r+1)\times\frac{r}{-}\right\}*\binom{r-1}{+};$$

see Table 22. We have not been able to detect any further elements of order 11 or 13, let alone larger primes p.

s	$d=3$	$d=4$	$d=5$	Factor
$++++++$	$-$	$\tilde{\mathbb{Z}}^{369}$	$\tilde{\mathbb{Z}}$	1
$+++++-$	$-$	$\tilde{\mathbb{Z}}^{384}$	$\tilde{\mathbb{Z}}^{6}$	6
$++++--$	$-$	$\tilde{\mathbb{Z}}^{387}$	$\tilde{\mathbb{Z}}^{15}$	15
$+++---$	$-$	$\tilde{\mathbb{Z}}^{382}+\mathbb{Z}_3$	$\tilde{\mathbb{Z}}^{20}$	20
$++----$	$-$	$\tilde{\mathbb{Z}}^{387}+(\mathbb{Z}_3)^6$	$\tilde{\mathbb{Z}}^{15}$	15
$+-----$	$-$	$\tilde{\mathbb{Z}}^{384}+(\mathbb{Z}_3)^{15}$ $(*)$	$\tilde{\mathbb{Z}}^{6}$	6
$------$	$\mathbb{Z}_3$	$\tilde{\mathbb{Z}}^{369}+(\mathbb{Z}_3)^{20}$	$\tilde{\mathbb{Z}}$	1
$\tilde{H}_d(\mathsf{M}_{13};\tilde{\mathbb{Z}})$	$\mathbb{Z}_3$	$\tilde{\mathbb{Z}}^{24596}+(\mathbb{Z}_3)^{220}$ $(*)$	$\tilde{\mathbb{Z}}^{924}$	

TABLE 20. The groups $\tilde{H}_d(((2,2,2,2,2,2,1),\mathsf{s});\tilde{\mathbb{Z}})$ yielding $\tilde{H}_d(\mathsf{M}_{13};\tilde{\mathbb{Z}})$. In boxes marked with $(*)$, the torsion part is a guess based on computations over $\mathbb{Z}_3$ and some additional small fields $\mathbb{Z}_p$.

s	$d=4$	$d=5$	Factor
$+++++++$	$\mathbb{Z}_5$	$\tilde{\mathbb{Z}}^{732}$	1
$++++++-$	$-$	$\tilde{\mathbb{Z}}^{900}$	7
$+++++--$	$\mathbb{Z}_5$	$\tilde{\mathbb{Z}}^{1052}$	21
$++++---$	$\mathbb{Z}_5+(\mathbb{Z}_3)^3$	$\tilde{\mathbb{Z}}^{1140}$	35
$+++----$	$(\mathbb{Z}_5)^2+(\mathbb{Z}_3)^{15}$	$\tilde{\mathbb{Z}}^{1140}$	35
$++-----$	$(\mathbb{Z}_5)^3+(\mathbb{Z}_3)^{40}$	$\tilde{\mathbb{Z}}^{1052}$	21
$+------$	$(\mathbb{Z}_5)^5+(\mathbb{Z}_3)^{81}$	$\tilde{\mathbb{Z}}^{900}$	7
$-------$	$(\mathbb{Z}_5)^8+(\mathbb{Z}_3)^{120}$	$\tilde{\mathbb{Z}}^{732}$	1
$\tilde{H}_d(\mathsf{M}_{14};\tilde{\mathbb{Z}})$	$(\mathbb{Z}_5)^{233}+(\mathbb{Z}_3)^{2157}$	$\tilde{\mathbb{Z}}^{138048}$	

TABLE 21. The groups $\tilde{H}_d(((2,2,2,2,2,2,2),\mathsf{s});\tilde{\mathbb{Z}})$ yielding $\tilde{H}_d(\mathsf{M}_{14};\tilde{\mathbb{Z}})$. The torsion parts are guesses based on computations over $\mathbb{Z}_3$, $\mathbb{Z}_5$, and $\mathbb{Z}_7$.

Complex	Group Action (G,α)	d	$\tilde{H}_d((G,\alpha);\mathbb{Z})$
M_7	$\begin{pmatrix}1\\+\end{pmatrix}*\left\{3\times\begin{matrix}2\\-\end{matrix}\right\}$	1 2	$\mathbb{Z}_3$ $-$
M_{14}	$\begin{pmatrix}2\\+\end{pmatrix}*\left\{4\times\begin{matrix}3\\-\end{matrix}\right\}$	4 5	$\mathbb{Z}_5$ $-$
M_{23}	$\begin{pmatrix}3\\+\end{pmatrix}*\left\{5\times\begin{matrix}4\\-\end{matrix}\right\}$	8 9	$\mathbb{Z}_7$ $\mathbb{Z}$
M_{34}	$\begin{pmatrix}4\\+\end{pmatrix}*\left\{6\times\begin{matrix}5\\-\end{matrix}\right\}$	13 14	$\mathbb{Z}_9$ $\mathbb{Z}^6\oplus\mathbb{Z}_3$
M_{47}	$\begin{pmatrix}5\\+\end{pmatrix}*\left\{7\times\begin{matrix}6\\-\end{matrix}\right\}$	19 20 21	$\mathbb{Z}_{11}$ $\mathbb{Z}^{18}\oplus(\mathbb{Z}_3)^2$ $\mathbb{Z}_3\oplus\mathbb{Z}_2$
M_{62}	$\begin{pmatrix}6\\+\end{pmatrix}*\left\{8\times\begin{matrix}7\\-\end{matrix}\right\}$	26 27 28	$\mathbb{Z}_{13}$ $\mathbb{Z}^{55}\oplus(\mathbb{Z}_3)^6$ $(\mathbb{Z}_3)^3\oplus(\mathbb{Z}_2)^4$

TABLE 22. The integral homology $\tilde{H}_d((G,\alpha);\mathbb{Z})$ for certain choices of parameters.

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