Calculation of Hilbert Borcherds Products

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CONTENTS

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In [Brunier and Bundschuh 03], the authors use Borcherds lifts to obtain Hilbert modular forms. Another approach is to calculate Hilbert modular forms using the Jacquet–Langlands correspondence, which was implemented by Lassina Dembele in MAGMA. In [Mayer 09] we use [Brunier and Bundschuh 03] to determine the rings of Hilbert modular forms for $\mathbb{Q}(\sqrt{13})$ and $\mathbb{Q}(\sqrt{17})$. In the present note we give the major calculational details and present some results for $\mathcal{K} = \mathbb{Q}(\sqrt{5})$, $\mathcal{K} = \mathbb{Q}(\sqrt{13})$, and $\mathcal{K} = \mathbb{Q}(\sqrt{17})$. For calculations in the ring $\mathfrak{o}$ of integers of $\mathcal{K}$ we order $\mathfrak{o}$ by the norm of its elements and get for fixed norm, modulo multiplication by $\pm\varepsilon_0^{2\mathbb{Z}}$, a finite set. We use this decomposition to describe Weyl chambers and their boundaries, to determine the Weyl vector of Borcherds products, and hence to calculate Borcherds products. As a further example we calculate Fourier expansions of Eisenstein series.

1. INTRODUCTION

In [Brunier and Bundschuh 03] the authors outline a path to calculate Hilbert Borcherds products for totally real number fields $\mathcal{K}$ and calculate some Borcherds products in the case $\mathcal{K} = \mathbb{Q}(\sqrt{5})$. In [Mayer 09] we use this to determine the rings of Hilbert modular forms for $\mathbb{Q}(\sqrt{13})$ and $\mathbb{Q}(\sqrt{17})$. In this note we present the major calculational details and give examples in the cases of $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{13})$, and $\mathbb{Q}(\sqrt{17})$. We start with a treatment of calculations in the ring $\mathfrak{o}$ of integers of $\mathcal{K}$. Especially, we write $\mathfrak{o}$ in Lemma 2.2 as $\pm\varepsilon_0^{2\mathbb{Z}} \bigcup_{m\in\mathbb{Z}} \mathcal{J}(m)$ with, for each $m \in \mathbb{Z}$, a finite set $\mathcal{J} = \mathcal{J}(m)$ of elements of norm m and the fundamental unit $\varepsilon_0 > 1$.

The corresponding decomposition of $\mathfrak{o}$ is then used to describe Weyl chambers and their boundaries, to calculate Eisenstein series, and to determine the Weyl vectors of the Borcherds products. We give an algorithm to calculate the Borcherds lift of nearly holomorphic modular forms in $A_k^+(p, \chi_p)$. For the calculation of a basis of $A_k^+(p, \chi_p)$ we refer to [Brunier and Bundschuh 03] (in the case of $\mathbb{Q}(\sqrt{5})$) and to [Mayer 09] or [Mayer 07].

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2. INTEGERS IN $\mathbb{Q}(\sqrt{p})$

Let p be a prime number and $\mathcal{K} = \mathbb{Q}(\sqrt{p})$ the corresponding quadratic number field with ring of integers $\mathfrak{o}$. Denote the nontrivial field automorphism by $\bar{\cdot}$ and for $\lambda \in \mathcal{K}$ the norm by $\mathrm{N}(\lambda) = \lambda\overline{\lambda}$ and the trace by $\mathrm{S}(\lambda) = \lambda + \overline{\lambda}$. In order to calculate Borcherds products, we have to investigate some number-theoretic properties of $\mathfrak{o}$. In particular, we give in Lemma 2.2, for fixed norm, a finite set of representatives of $\mathfrak{o}$ modulo multiplication by $\pm\varepsilon_0^2$ and use this to investigate the sets $S(m)$, which bound the union of all Weyl chambers. This is important in the calculation of Borcherds products and in addition, proves that Weyl chambers are open sets.

Furthermore, Lemma 2.2 is quite useful in calculating Fourier expansions, especially in the case of Eisenstein series.

Lemma 2.1. (Fundamental unit.) *We write $\varepsilon_0 := x_0 + y_0\sqrt{p}$ for the fundamental unit of $\mathfrak{o}$ with $x_0, y_0 \in \mathbb{Q}$. Then $x_0 > 0$ and $y_0 > 0$.*

Proof. We have $\mathrm{N}(\varepsilon_0) = \varepsilon_0\overline{\varepsilon_0} = \pm 1$ and $\varepsilon_0 = x_0 + y_0\sqrt{p} > 1$. So we obtain $\varepsilon_0 > 1 > |\overline{\varepsilon_0}| > 0$ and conclude that $y_0 = (\varepsilon_0 - \overline{\varepsilon_0})/(2\sqrt{p}) > 0$ and $x_0 = (\varepsilon_0 + \overline{\varepsilon_0})/2 > 0$, independent of the sign of $\overline{\varepsilon_0}$. $\square$

This result is important in the proof of the following lemma.

Lemma 2.2. (Numbers of fixed norm.) *Let p be a prime number, $\mathcal{K} = \mathbb{Q}(\sqrt{p})$, and let $\mathfrak{o}$ be the ring of integers in $\mathcal{K}$. For every m in $\mathbb{Z} \setminus \{0\}$ there is a finite set $\mathcal{J}$ such that*

$$\mathcal{I} := \left\{\lambda \in \frac{\mathfrak{o}}{\sqrt{p}} : \mathrm{N}(\lambda) = -\frac{m}{p}\right\} = \left\{\pm\lambda\varepsilon_0^{2k} : k \in \mathbb{Z}, \lambda \in \mathcal{J}\right\}.$$

More precisely, if we write $\breve{\lambda}_1 + \breve{\lambda}_2\sqrt{p}/p := \varepsilon_0^{-2}\lambda$ for all $\lambda \in \mathfrak{o}/\sqrt{p}$, we obtain that

$$\mathcal{J} := \left\{\lambda = \lambda_1 + \lambda_2\sqrt{p}/p \in \frac{\mathfrak{o}}{\sqrt{p}} : \mathrm{N}(\lambda) = -\frac{m}{p},\ \lambda_1 > 0,\ \lambda_2 > 0,\ \breve{\lambda}_1\breve{\lambda}_2 \leq 0\right\}$$

is a set of representatives of $\mathcal{I}/\sim$ with respect to the equivalence relation $\sim$ induced by multiplication by ε_0^2 and -1.

For $\lambda = \lambda_1 + \lambda_2\sqrt{p}/p$ in $\mathcal{J}$ we have, depending on $m = -p\,\mathrm{N}(\lambda)$ and p, the results given in Table 1.

Proof. A detailed proof can be found in [Mayer 07, Lemma 3.2.2]. Clearly, along with $\lambda \in \mathcal{I}$, we also have $-\lambda \in \mathcal{I}$ as well as $\varepsilon_0^{2k}\lambda \in \mathcal{I}$ for all $k \in \mathbb{Z}$.

For $m \in \mathbb{Z} \setminus \{0\}$ and $\lambda = \lambda_1 + \lambda_2\frac{\sqrt{p}}{p} \in \mathcal{I}$ we obtain

$$\varepsilon_0^{\pm 2}\lambda = \lambda_1\left(x_0^2 + py_0^2\right) \pm \lambda_2\left(2x_0y_0\right) + \sqrt{p}\left(\pm\lambda_1(2x_0y_0) + \lambda_2\frac{x_0^2 + py_0^2}{p}\right) \tag{2–1}$$

and $N(\lambda) = -m/p < 0$; hence $p\lambda_1^2 + m = \lambda_2^2$.

Given $m > 0$, we let $\lambda_2 \geq 0$ without loss of generality. From $p\lambda_1^2 + m = \lambda_2^2$ it follows that $\lambda_2 > \sqrt{p}|\lambda_1| \geq 0$. Calculations prove that for $\varepsilon_0^2\lambda = \tilde{\lambda} = \tilde{\lambda}_1 + \tilde{\lambda}_2\frac{\sqrt{p}}{p}$, we have $\tilde{\lambda}_2 > 0$ and with (2–1), that $\tilde{\lambda}_1 > \lambda_1\varepsilon_0^{-2}$.

Since $\varepsilon_0^{-2} < 1$ and for $\lambda \in \mathfrak{o}$ the coefficients λ_1 and λ_2 take values in a discrete set, there is $k \in \mathbb{N}$ such that $\varepsilon_0^{2k}\lambda = \acute{\lambda}_1 + \acute{\lambda}_2\frac{\sqrt{p}}{p}$ satisfies $\acute{\lambda}_1 > 0$ and $\acute{\lambda}_2 > 0$. For $\acute{\lambda}\varepsilon_0^{-2} = \breve{\lambda}_1 + \breve{\lambda}_2\frac{\sqrt{p}}{p}$ we get $\breve{\lambda}_2 > 0$ and $\breve{\lambda}_1 < \acute{\lambda}_1$ from (2–1), and the given restrictions in the lemma for $\breve{\lambda}_2 \leq 0$ can be calculated.

The case $m < 0$ can be treated analogously. Without loss of generality we then have $\lambda_1 \geq 0$ and achieve $\acute{\lambda}_2 > 0$ by multiplication by some appropriate power of ε_0^2. Then we multiply by ε_0^{-2} to obtain $\breve{\lambda} = \varepsilon_0^{-2}\acute{\lambda}$ and read the stated shape of $\mathcal{J}$ from the equation $\breve{\lambda}_2 = 0$. $\square$

Definition 2.3. We denote the complex upper half-plane by $\mathbb{H}$, and the Hilbert modular group by Γ and its subgroup fixing $\infty = (\infty, \infty)$ by Γ_∞:

$$T(m) := \cup_{a,b,\lambda M(a,b,\lambda)},$$

with

$$M(a,b,\lambda) = \{(\tau_1,\tau_2) \in \mathbb{H}^2;\quad a\tau_1\tau_2 + \lambda\tau_1 + \overline{\lambda}\tau_2 + b = 0\},$$

where the summation is over

$$\left\{a, b, \lambda \in \mathfrak{L}' = \mathbb{Z}^2 \times \frac{1}{\sqrt{p}}\mathfrak{o} : ab - \mathrm{N}(\lambda) = \frac{m}{p}\right\},$$

and

$$S(m) := \bigcup_{\substack{\lambda \in \mathfrak{o}/\sqrt{p} \\ -\mathrm{N}(\lambda) = m/p}} M(\lambda),$$

where

$$M(\lambda) := \left\{(\tau_1,\tau_2) \in \mathbb{H} \times \mathbb{H} : \lambda\,\mathrm{Im}\,(\tau_1) + \overline{\lambda}\,\mathrm{Im}\,(\tau_2) = 0\right\}$$

and $T(m)$ is called *Hirzebruch–Zagier divisor of discriminant m*, where one assigns the multiplicity 1 to every irreducible component of $T(m)$.

	$m>0$		$m<0$	
p	$\lambda_1 \le \sqrt{\frac{m\alpha_p}{1-p\alpha_p}}$	$\lambda_2 \le \sqrt{\frac{m}{1-p\alpha_p}}$	$\lambda_1 \le \sqrt{\frac{-m}{p(1-p\alpha_p)}}$	$\lambda_2 \le \sqrt{\frac{-mp\alpha_p}{1-p\alpha_p}}$
$p=5$	$\lambda_1 \le \frac{1}{2}\sqrt{m}$	$\lambda_2 \le \frac{3}{2}\sqrt{m}$	$\lambda_1 \le \frac{3\sqrt{5}}{10}\sqrt{-m}$	$\lambda_2 \le \frac{\sqrt{5}}{2}\sqrt{-m}$
$p=13$	$\lambda_1 \le \frac{3}{2}\sqrt{m}$	$\lambda_2 \le \frac{11}{2}\sqrt{m}$	$\lambda_1 \le \frac{11}{2\sqrt{13}}\sqrt{-m}$	$\lambda_2 \le \frac{3\sqrt{13}}{2}\sqrt{-m}$
$p=17$	$\lambda_1 \le 8\sqrt{m}$	$\lambda_2 \le 33\sqrt{m}$	$\lambda_1 \le \frac{33\sqrt{17}}{17}\sqrt{-m}$	$\lambda_2 \le 8\sqrt{17}\sqrt{-m}$

TABLE 1. Estimates for $\lambda_1+\lambda_2\sqrt{p}/p$ in the set $\mathcal{J}$ of representatives of $\mathcal{I}/_\sim$, where $\alpha_p=(2x_0y_0)^2/(x_0^2+py_0^2)^2$ with the fundamental unit $\varepsilon_0=x_0+y_0\sqrt{p}$.

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\chi_5(m)$	1	−1	−1	1		1	−1	−1	1		1	−1	−1	1		1
$\chi_{13}(m)$	1	−1	1	1	−1	−1	−1	−1	1	1	−1	1		1	−1	1
$\chi_{17}(m)$	1	1	−1	1	−1	−1	−1	1	1	−1	−1	−1	1	−1	1	1

TABLE 2. $\chi_p(m)$ for $m\le 16$ and $p=5,13,17$.

All divisors of Borcherds products are Hirzebruch–Zagier divisors and conversely (cf. Remark 4.3). In [van der Geer 88], the author describes for discriminant D the shape of some special sets of quadratic equations. A special case of this gives us the number of generating equations of $T(m)$ for given p and m.

Definition 2.4. (Nearly holomorphic modular forms.) We denote by $A_k(p,\chi_p)$ the space of *nearly holomorphic modular forms* of weight k for the group $\Gamma_0(p)$ with the Dirichlet character χ_p given by the Legendre symbol (cf. Table 2). Here "nearly holomorphic" means that the modular forms are holomorphic on $\mathbb{H}$ and meromorphic at the cusps. Then the plus space $A_k^+(p,\chi_p)$ is the subspace of $A_k(p,\chi_p)$ where for $f\in A_k^+(p,\chi_p)$ with $f(z)=\sum_{n\in\mathbb{Z}} a(n)q^n$, $q=e^{2\pi iz}$, we have $a(n)=0$ for all $n\in\mathbb{Z}$ with $\chi_p(n)=-1$.

Definition 2.5. (Weyl chamber.) For $f=\sum_{n\in\mathbb{Z}} a(n)q^n \in A_0^+(p,\chi_p)$ we call $W\subset\mathbb{H}\times\mathbb{H}$ a *Weyl chamber* attached to f if W is a connected component of

$$\mathbb{H}\times\mathbb{H}\setminus \bigcup_{\substack{n<0\\ a(n)\neq 0}} S(-n).$$

Definition 2.6. ($(W,\lambda)>0$.) For $W\subset\mathbb{H}\times\mathbb{H}$, especially if W is a Weyl chamber, and $\lambda\in\mathfrak{o}/\sqrt{p}$ we write $(W,\lambda)=(\lambda,W)>0$ if $\lambda\,\mathrm{Im}\,(\tau_1)+\overline{\lambda}\,\mathrm{Im}\,(\tau_2)>0$ holds for all (τ_1,τ_2) in W.

Lemma 2.7. (Shape of $S(m)$.) *For every prime number p and every $m>0$ the set $S(m)$ is the intersection of $\mathbb{H}^2$ with an empty or an infinite union of hyperplanes of the real vector space $\mathbb{C}^2$. We have*

$$S(m) = \bigcup_{\lambda\in\mathcal{I}} \left\{(z_1,z_2)\in\mathbb{H}\times\mathbb{H} : \lambda\,\mathrm{Im}\,(z_1)+\overline{\lambda}\,\mathrm{Im}\,(z_2)=0\right\},$$

where $\mathcal{I}$ is as in Lemma 2.2. In particular, $S(m)$ is invariant under the stabilizer Γ_∞ of infinity.

Proof. The set $S(m)$ has the given shape by Definition 2.3. Let $\mathcal{I}$ be the set of λ in $\mathfrak{o}/\sqrt{p}$ with $-\mathrm{N}(\lambda)=m/p$ and let $\mathcal{I}$ be nonempty, e.g., let $\lambda\in\mathcal{I}$ be an element. Clearly, the set

$$M(\lambda)=\left\{(\tau_1,\tau_2)\in\mathbb{H}\times\mathbb{H} : \lambda\,\mathrm{Im}\,(\tau_1)+\overline{\lambda}\,\mathrm{Im}\,(\tau_2)=0\right\}$$

is mapped onto itself by real transformations $\mathbb{H}^2\to\mathbb{H}^2, \tau\mapsto\tau+r,\ r\in\mathbb{R}^2$. Let τ in $M(\lambda)$ and $k\in\mathbb{Z}$. Then defining $\tau^{(k)}:=\varepsilon_0^{2k}\tau=\left(\varepsilon_0^{2k}\tau_1,\overline{\varepsilon_0}^{2k}\tau_2\right)$, we have

$$\varepsilon_0^{-2k}\lambda\,\mathrm{Im}\left(\tau_1^{(k)}\right)+\overline{\varepsilon_0}^{-2k}\overline{\lambda}\,\mathrm{Im}\left(\tau_2^{(k)}\right)=0,$$

so $\tau^{(k)}$ is an element of $M(\varepsilon_0^{-2k}\lambda)$.

The group Γ_∞ is generated by real transformations and multiplication by ε_0^{2k} ($k\in\mathbb{Z}$), so we have shown the invariance under Γ_∞. We rewrite $M(\lambda)$ as

$$M(\lambda)=\left\{(z_1,z_2)\in\mathbb{H}\times\mathbb{H} : \mathrm{Im}\,(z_2)=\frac{-\lambda}{\overline{\lambda}}\,\mathrm{Im}\,(z_1)\right\}.$$

Since for all $k\in\mathbb{Z}\setminus\{0\}$ we have $\varepsilon_0^{-2k}/\overline{\varepsilon_0}^{-2k}\neq 1$, the sets $M(\lambda)$ and $M(\varepsilon_0^{-2k}\lambda)$ do not coincide, so $\mathcal{I}$ is either empty or has an infinite number of elements. □

Remark 2.8. (Calculation of $S(m)$.) Let $m>0$. If we use both Lemma 2.7 and Lemma 2.2, we obtain a

<table>
<tr><td></td><td>weight</td><td>μ</td><td>diagonal</td></tr>
<tr><td colspan="4">Fourier expansion</td></tr>
<tr><td colspan="4">Fourier expansion on the diagonal</td></tr>
<tr><td>E_2^H</td><td>2</td><td>1</td><td>$E_2^H(\tau,\tau) = E_4(\tau)$</td></tr>
<tr><td colspan="4">$1 + 120g\left(h + \frac{1}{h}\right) + g^2\left(720 + 600\left(h^2 + \frac{1}{h^2}\right) + 120\left(h^4 + \frac{1}{h^4}\right)\right) + O\left(g^3\right)$</td></tr>
<tr><td colspan="4">$1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + 30240q^5 + O\left(q^6\right)$</td></tr>
<tr><td>E_4^H</td><td>4</td><td>1</td><td>$E_4^H(\tau,\tau) = (E_4(\tau))^2 = E_8(\tau)$</td></tr>
<tr><td colspan="4">$1 + 240g\left(h + \frac{1}{h}\right) + g^2\left(30240 + 15600\left(h^2 + \frac{1}{h^2}\right) + 240\left(h^4 + \frac{1}{h^4}\right)\right) + O\left(g^3\right)$</td></tr>
<tr><td colspan="4">$1 + 480q + 61920q^2 + 1050240q^3 + 7926240q^4 + 37500480q^5 + O\left(q^6\right)$</td></tr>
<tr><td>E_6^H</td><td>6</td><td>1</td><td>$E_6^H(\tau,\tau) = \frac{42}{67}\left(E_4(\tau)\right)^3 + \frac{25}{67}\left(E_6(\tau)\right)^2$</td></tr>
<tr><td colspan="4">$1 + \frac{2520}{67}g\left(h + \frac{1}{h}\right) + g^2\left(\frac{7877520}{67} + \frac{2583000}{67}\left(h^2 + \frac{1}{h^2}\right) + \frac{2520}{67}\left(h^4 + \frac{1}{h^4}\right)\right) + O\left(g^3\right)$</td></tr>
<tr><td colspan="4">$1 + \frac{5040}{67}q + \frac{13048560}{67}q^2 + \frac{1125069120}{67}q^3 + \frac{26660859120}{67}q^4 + \frac{310192878240}{67}q^6 + O\left(q^5\right)$</td></tr>
</table>

TABLE 3. Eisenstein series in case $p = 5$ ($g = \exp(\pi i(\tau_1 + \tau_2))$ and $h = \exp(\pi i(\tau_1 - \tau_2)/\sqrt{p})$).

program for the calculation of $S(m)$. We take all positive λ_2 in $\frac{1}{2}\mathbb{Z}$ smaller than $\sqrt{\frac{m}{1-p\alpha_p}}$. Then $\lambda_1 > 0$ is uniquely determined by the formula $p\lambda_1^2 + m = \lambda_2^2$. We have only to check whether $\lambda_1 \in \mathbb{Z}/2$. Then we have calculated $S(m)$ modulo multiplication by ε_0^2.

Corollary 2.9. *Weyl chambers are open sets.*

Proof. Each Weyl chamber is given as a component of $\mathbb{H}^2 \setminus \bigcup_{n<0, a(n)\neq 0} S(-n)$, where the $a(n)$ are the coefficients of the principal part of a nearly holomorphic modular form of weight 0 with character χ_p for the group $\Gamma_0(p)$, so this is a finite union. By Lemma 2.7 together with Lemma 2.2, the set $S(m)$ is a locally finite union of hyperplanes, where we consider $S(m) \subset \mathbb{H}^2$, so $S(m)$ is a closed subset of $\mathbb{H}^2$. Then each of component of the complement, i.e., each Weyl chamber, is open (cf. Figure 1). $\square$

Remark 2.10. Lemma 2.2 together with Lemma 2.7 shows that $S(j)$ is a countable (or empty) union of hyperplanes

$$E_\lambda = \left\{\tau \in \mathbb{H}^2 : \mathrm{Im}\,(\tau_1) = \frac{\overline{\lambda}}{\lambda}\,\mathrm{Im}\,(\tau_2)\right\},$$

which is, modulo multiplication by ε_0^2, a finite union of hyperplanes. The sketch shows the case $S(j) = \bigcup_{m\in\mathbb{Z}}\left\{E_{\varepsilon_0^{2m}a} \bigcup E_{\varepsilon_0^{2m}b}\right\}$ in a projection of $S(j)$ and its hyperplanes and the Weyl chambers onto the imaginary parts. Each Weyl chamber is the product of its projection on the imaginary part and $\mathbb{R}^2$, if we write $\mathbb{H}^2 = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^2$.

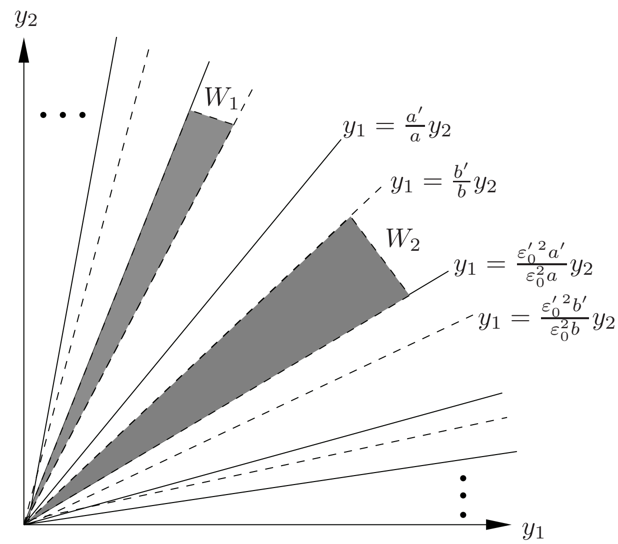

FIGURE 1. Imaginary parts of Weyl chambers.

Remark 2.11. If W is a Weyl chamber attached to $f = \sum_{n\in\mathbb{Z}} a(n)q^n \in A_0^+(p,\chi_p)$ and $\lambda \in \mathfrak{o}\,/\sqrt{p}$, then for every $a(-p\,\mathrm{N}(\lambda)) \neq 0$, the condition $(\lambda, W) > 0$ is equivalent to the existence of a point $(\tau_1,\tau_2) \in W$ with $\lambda\,\mathrm{Im}\,(\tau_1) + \overline{\lambda}\,\mathrm{Im}\,(\tau_2) > 0$.

Proof. This follows from Corollary 2.9, since all zeros of $\lambda\,\mathrm{Im}\,(\tau_1) + \overline{\lambda}\,\mathrm{Im}\,(\tau_2)$ are contained in $S(-p\,\mathrm{N}(\lambda))$. $\square$

3. CALCULATION OF EISENSTEIN SERIES

We denote the Hilbert modular group by $\Gamma = \mathrm{SL}(2,\mathfrak{o})$ and the subgroup fixing $\infty = (\infty,\infty)$ by Γ_∞. For $c, d \in \mathfrak{o}$

	weight	μ	diagonal
Fourier expansion			
Fourier expansion on the diagonal			
E_2^H	2	1	$E_2^H(\tau,\tau) = E_4(\tau)$
$1 + g\left(96\left(h+\frac{1}{h}\right) + 24\left(h^3+\frac{1}{h^3}\right)\right) + O\left(g^2\right)$			
$1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + 30240q^5 + 60480q^6 + 82560q^7 + O\left(q^8\right)$			
$4\Psi_1^4 \cdot E_2^H = \Psi_4^2 + 4\Psi_1^2\Psi_3$			
E_4^H	4	1	$E_4^H(\tau,\tau) = \left(E_4(\tau)\right)^2$
$1 + g\left(\frac{6720}{29}\left(h+\frac{1}{h}\right) + \frac{240}{29}\left(h^3+\frac{1}{h^3}\right)\right) + O\left(g^2\right)$			
$1 + 480q + 61920q^2 + 1050240q^3 + 7926240q^4 + 37500480q^5 + 135480960q^6 + O\left(q^7\right)$			
E_6^H	6	1	$E_6^H(\tau,\tau) = \frac{21378}{33463}E_4^3(\tau) + \frac{12085}{33463}E_6^2(\tau)$
$1 + g\left(\frac{1598688}{33463}\left(h+\frac{1}{h}\right) + \frac{6552}{33463}g\left(h^3+\frac{1}{h^3}\right)\right)$			
$1 + \frac{3210480}{33463}q + \frac{6500435760}{33463}q^2 + \frac{562087955520}{33463}q^3 + \frac{13314685915440}{33463}q^4 + \frac{154928487036960}{33463}q^5 + O\left(q^6\right)$			

TABLE 4. Eisenstein series in case $p = 13$ ($g = \exp(\pi i(\tau_1+\tau_2))$ and $h = \exp(\pi i(\tau_1-\tau_2)/\sqrt{p})$).

	weight	μ	diagonal
Fourier expansion			
Fourier expansion on the diagonal			
E_2^H	2	1	$E_2^H(\tau,\tau) = E_4(\tau)$
$1 + g\left(84\left(h+\frac{1}{h}\right) + 36\left(h^3+\frac{1}{h^3}\right)\right) + O\left(g^2\right)$			
$1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + 30240q^5 + 66312q^6 + 82560q^7 + O\left(q^8\right)$			
$3\Psi_1^3 \cdot E_2^H = 3\Psi_1 \cdot \Psi_2^2 - \Psi_9$			
E_4^H	4	1	$E_4^H(\tau,\tau) = \left(E_4(\tau)\right)^2 = E_8(\tau)$
$1 + g\left(\frac{8760}{41}\left(h+\frac{1}{h}\right) + \frac{1080}{41}\left(h^3+\frac{1}{h^3}\right)\right) + O\left(g^2\right)$			
$1 + 480q + 61920q^2 + 1050240q^3 + 7926240q^4 + O\left(q^5\right)$			
E_6^H	6	1	$E_6^H(\tau,\tau) = \frac{3696}{5791}E_4^3(\tau) + \frac{2095}{5791}E_6^2(\tau)$
$1 + g\left(\frac{266364}{5791}\left(h+\frac{1}{h}\right) + \frac{8316}{5791}\left(h^3+\frac{1}{h^3}\right)\right)$			
$1 + \frac{549360}{5791}q + \frac{1125094320}{5791}q^2 + \frac{97271576640}{5791}q^3 + \frac{2304206236080}{5791}q^4 + O\left(q^5\right)$			
$\frac{5791}{2095}\left(E_6^H - \frac{3696}{5791}\left(E_2^H\right)^3\right)$	6	1	$\frac{5791}{2095}\left(E_6^H - \frac{3696}{5791}\left(E_2^H\right)^3\right)(\tau,\tau) = E_6^2(\tau)$
$1 - g\left(\frac{665028}{2095}\left(h+\frac{1}{h}\right) + \frac{390852}{2095}\left(h^3+\frac{1}{h^3}\right)\right) + O\left(g^2\right)$			
$1 - 1008q + 220752q^2 + 16519104q^3 + 399517776q^4 + 4624512480q^5 + O\left(q^6\right)$			
$\frac{41}{2^4 \cdot 3^2}\left(E_4^H - \left(E_2^H\right)^2\right)$	4	1	0
$13g\left(h+\frac{1}{h}-h^3-\frac{1}{h^3}\right) + g^2\left(-784 + 349\left(h^2+\frac{1}{h^2}\right) + 14\left(h^4+\frac{1}{h^4}\right) + O\left(h^6+\frac{1}{h^6}\right)\right)$			

TABLE 5. Eisenstein series in case $p = 17$ ($g = \exp(\pi i(\tau_1+\tau_2))$ and $h = \exp(\pi i(\tau_1-\tau_2)/\sqrt{p})$).

p	$B(1)$	$B(2)$	$B(3)$	$B(4)$	$B(5)$	$B(6)$	$B(7)$	$B(8)$	$B(9)$	$B(10)$	$B(11)$	$B(12)$	$B(13)$
5	-10			-30	-30	-20			-70	-20	-120		
13	-2		-8	-6					-26	-8		-24	-14
17	-1	-3		-7				-15	-7				-14

TABLE 6. Fourier coefficients of $E_2^+ = 1 + \sum_{n\in\mathbb{N}} B(n)q^n$.

p	Ψ_1	Ψ_2	Ψ_3	Ψ_4	Ψ_5	Ψ_6	Ψ_7	Ψ_8	Ψ_9	Ψ_{10}	Ψ_{11}	Ψ_{12}	Ψ_{13}
5	5			15	15	10			35	10	60		
13	1		4	3					13	4		12	7
17	1/2	3/2		7/2				15/2	7/2				7
p	Ψ_{14}	Ψ_{15}	Ψ_{16}	Ψ_{17}	Ψ_{18}	Ψ_{19}	Ψ_{20}	Ψ_{21}	Ψ_{22}	Ψ_{23}	Ψ_{24}	Ψ_{25}	Ψ_{26}
5	30	20	55			100	45	60			50	65	60
13	6		11	18					10	24		21	6
17		4	31/2	9/2	21/2	10		6				21/2	21
p	Ψ_{27}	Ψ_{28}	Ψ_{29}	Ψ_{30}	Ψ_{31}	Ψ_{32}	Ψ_{33}	Ψ_{34}	Ψ_{35}	Ψ_{36}	Ψ_{37}	Ψ_{38}	Ψ_{39}
5			150	30	160			80	60	105			120
13	40		30	16					24	39		18	28
17				12		63/2	10	27/2	12	49/2		30	

TABLE 7. The weights of Borcherds products and of some of their holomorphic quotients.

p	5	13	17
T_1	$\Gamma M(0,0,\frac{1}{5}\sqrt{5})$	$\Gamma M(0,0,\frac{1}{13}\sqrt{13})$	$\Gamma M(0,0,\frac{1}{17}\sqrt{17})$
T_2	0	0	$\Gamma M(0,0,\frac{1}{2}+\frac{5}{34}\sqrt{17})$
T_3	0	$\Gamma M(0,0,\frac{-1}{2}+\frac{5}{26}\sqrt{13})$	0
T_4	$T_1+\Gamma M(0,-1,\frac{2}{5}\sqrt{5})$	$T_1+\Gamma M(0,-1,\frac{2}{13}\sqrt{13})$	$T_1+\Gamma M(0,0,\frac{-3}{2}+\frac{13}{34}\sqrt{17})$
T_5	$\Gamma M(0,0,\frac{1}{2}+\frac{1}{2}\sqrt{5})$	0	0
T_6	$\Gamma M(1,-1,\frac{-1}{2}+\frac{7}{10}\sqrt{5})$	0	0
T_7	0	0	0
T_8	0	0	$T_2+\Gamma M(0,0,\frac{-1}{2}+\frac{7}{34}\sqrt{17})$
T_9	$T_1+\Gamma M(0,1,\frac{3}{5}\sqrt{5})$	$T_1+\Gamma M(0,0,\frac{-1}{2}+\frac{7}{26}\sqrt{13})$	$T_1+\Gamma M(0,1,\frac{3}{17}\sqrt{17})$
T_{10}	$\Gamma M(1,1,\frac{1}{2}+\frac{1}{2}\sqrt{5})$	$\Gamma M(-1,-1,\frac{1}{2}+\frac{1}{26}\sqrt{13})$	0
T_{11}	$\Gamma M(0,0,\frac{-1}{2}+\frac{7}{10}\sqrt{5})$	0	0
T_{12}	0	$T_3+\Gamma M(0,-1,-1+\frac{5}{13}\sqrt{13})$	0
T_{13}	0	$\Gamma M(0,0,\frac{3}{2}+\frac{1}{2}\sqrt{13})$	$\Gamma M(0,0,-2+\frac{9}{17}\sqrt{17})$
T_{14}	$\Gamma M(1,-1,\frac{1}{2}+\frac{9}{10}\sqrt{5})$	$\Gamma M(1,-1,\frac{-1}{2}+\frac{11}{26}\sqrt{13})$	0
T_{15}	$\Gamma M(1,-1,1+1\sqrt{5})$	0	$\Gamma M(1,-1,1+\frac{7}{17}\sqrt{17})$
T_{16}	$T_4+\Gamma M(0,-1,\frac{4}{5}\sqrt{5})$	$T_4+\Gamma M(0,-1,\frac{4}{13}\sqrt{13})$	$T_4+\Gamma M(0,0,\frac{-1}{2}+\frac{9}{34}\sqrt{17})$
T_{17}	0	$\Gamma M(0,0,\frac{-1}{2}+\frac{9}{26}\sqrt{13})$	$\Gamma M(0,0,4+\sqrt{17})$
T_{18}	0	0	$T_2+\Gamma M(0,1,\frac{-3}{2}+\frac{15}{34}\sqrt{17})$
T_{19}	$\Gamma M(0,0,\frac{1}{2}+\frac{9}{10}\sqrt{5})$	0	$\Gamma M(0,0,-1+\frac{6}{17}\sqrt{17})$
T_{20}	$T_5+\Gamma M(0,-1,1+1\sqrt{5})$	0	0
T_{21}	$\Gamma M(1,1,\frac{4}{5}\sqrt{5})$	0	$\Gamma M(1,-1,\frac{-1}{2}+\frac{13}{34}\sqrt{17})$
T_{22}	0	$\Gamma M(1,1,\frac{1}{2}+\frac{7}{26}\sqrt{13})$	0
T_{23}	0	$\Gamma M(0,0,-1+\frac{6}{13}\sqrt{13})$	0
T_{24}	$T_6+\Gamma M(1,-1,\frac{-1}{2}+\frac{11}{10}\sqrt{5})$	0	0

TABLE 8. Divisors of the Borcherds products ($\Gamma=\mathrm{SL}(2,\mathfrak{o})$).

and $\tau\in\mathbb{H}^2$ we extend the notion of the norm to

$$N(c\tau+d)=(c\tau_1+d)(\overline{c}\tau_2+\overline{d}).$$

The matrix $M=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$ operates on $\mathbb{H}^2$ by

$$M\tau:=\left(\frac{a\tau_1+b}{c\tau_1+d},\frac{\overline{a}\tau_2+\overline{b}}{\overline{c}\tau_2+\overline{d}}\right).$$

Definition 3.1. A *Hilbert modular form f of (parallel) weight $k\in\mathbb{Q}$ with multiplier system $\mu:\Gamma\to\mathbb{C}\setminus\{0\}$ for the quadratic number field $\mathcal{K}$* is a holomorphic map $f:\mathbb{H}^2\to\mathbb{C}$ with the property that

$$f\mid_k^\mu M:=\mu(M)^{-1}\,\mathrm{N}(c\tau+d)^{-r}f(M\tau)=f.$$

Note that the Goetzky–Koecher principle grants that f is regular at the cusps.

For every $k \in \mathbb{N}$ the Eisenstein series of weight $2k$ is the Hilbert modular form

$$E_{2k}^H : \mathbb{H}^2 \longrightarrow \mathbb{C},$$
$$\tau \longmapsto \sum_{M \in \Gamma_\infty \backslash \Gamma} 1|_{2k} M = \sum_{M \in \Gamma_\infty \backslash \Gamma} \mathrm{N}(c\tau + d)^{-2k},$$

where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. The Eisenstein series E_{2k}^H converges absolutely for $k \geq 1$ and represents a Hilbert modular form of weight $2k$ with trivial multiplier system, which has the value 1 at the cusp ∞.

Siegel describes a way of calculating the Fourier coefficients of Hilbert Eisenstein series [Siegel 69]. He considers a more general definition for Hilbert Eisenstein series (he calls them Hecke Eisenstein series) than we do. We obtain

$$\begin{aligned}\zeta(2k) E_{2k}^H(\tau) &= F_{2k}(\mathfrak{o}, \tau) \\ &= \zeta(2k) + \left(\frac{(2\pi i)^{2k}}{(2k-1)!}\right)^2 \sqrt{p}^{1-4k} \\ &\quad \times \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} \sigma_{2k-1}(\nu) e^{2\pi i S(\nu\tau)},\end{aligned}$$

where

$$\zeta(2k) = \sum_{\text{ideals } (\mu)} \mathrm{N}(\mu^{-2k}) = \sum_{\text{ideals } (\mu)} \mathrm{N}(\mu)^{-2k}$$

and

$$\sigma_{2k-1}(\nu) = \sum_{\substack{(t) | (\sqrt{p}\nu) \\ t \in \sqrt{p}\,\mathfrak{o}}} \mathrm{N}((t)^{2k-1}) = \sum_{\substack{(t) | (\sqrt{p}\nu) \\ t \in \sqrt{p}\,\mathfrak{o}}} \mathrm{N}(t)^{2k-1}.$$

The norm $\mathrm{N}(\mu)$ of a prime ideal (μ) is given by the norm of a generating element $\mu \in \mathfrak{o}$.

Note that we have written

$$\zeta(2k) E_{2k}^H(\tau) = \zeta(2k) + \sum_{\nu} a_\nu e^{2\pi i S(\nu\tau)}$$

with some known a_ν, since we can calculate the finite sum $\sigma_{2k-1}(\nu)$ if we can calculate a set of representatives of $\mathfrak{o} \,/\, \mathfrak{o}^*$ ordered by the corresponding absolute value of the norm. We also need a way to order $\mathfrak{d}^{-1} = \frac{1}{\sqrt{p}}\,\mathfrak{o}$. Both can be achieved by Lemma 2.2.

In order to calculate $\zeta(2k)$, Siegel advises that one restrict the Fourier expansion of E_{2k}^H to the diagonal $\mathrm{Diag} = \left\{\tau \in \mathbb{H}^2 : \tau_1 = \tau_2\right\}$, which yields an elliptic modular form of weight $4k$. A basis of all elliptic modular forms of weight $4k$ is known, and $E_{2k}^H(\tau)$ tends to 1 as the imaginary part of τ goes to infinity. Hence we can determine $\zeta(2k)$ by linear algebra.

Remark 3.2. Some of the (truncated) Fourier expansions of Eisenstein series can be found in Tables 3, 4, and 5.

4. THE THEOREM OF BORCHERDS, BRUINIER, AND BUNDSCHUH

Hilbert Borcherds products are Hilbert modular forms that vanish on some Hirzebruch–Zagier divisors $T(m)$, have absolutely convergent product expansion on Weyl chambers, and are lifts of modular forms in $A_0^+(p, \chi_p)$. We give the theorem of Borcherds, which is [Borcherds 98, Theorem 13.3], about Borcherds products in the version of Bruinier and Bundschuh; compare [Brunier and Bundschuh 03, Theorem 9] and [Brunier and Bundschuh 03, Theorem 3.1].

Theorem 4.1. (Borcherds, Bruinier, Bundschuh.) *Let $f = \sum_{n \in \mathbb{Z}} a(n) q^n \in A_0^+(p, \chi_p)$ and assume that $s(n)a(n) \in \mathbb{Z}$ for all $n < 0$, where $s(n) = 2$ if $p \mid n$ and $s(n) = 1$ otherwise. Then there is a meromorphic function Ψ on $\mathbb{H} \times \mathbb{H}$ with the following properties:*

(i) *Ψ is a meromorphic modular form for Γ with some multiplier system of finite order. The weight of Ψ is equal to the constant coefficient $a(0)$ of f.*

(ii) *The divisor of Ψ is determined by the principal part of f. It equals*

$$\sum_{n<0} s(n) a(n) T(-n).$$

(iii) *Let $W \subset \mathbb{H} \times \mathbb{H}$ be a Weyl chamber attached to f and put $N = \min\{n; a(n) \neq 0\}$. The function Ψ has the Borcherds product expansion*

$$\begin{aligned}\Psi(\tau_1, \tau_2) &= \mathbf{e}(\rho_W \tau_1 + \overline{\rho_W} \tau_2) \\ &\quad \times \prod_{\substack{\nu \in \mathfrak{o}/\sqrt{p} \\ (\nu, W) > 0}} \left(1 - \mathbf{e}(\nu\tau_1 + \overline{\nu}\tau_2)\right)^{s(p\nu\overline{\nu}) a(p\nu\overline{\nu})},\end{aligned}$$

where $\mathbf{e}(\tau) = e^{2\pi i \tau}$. Here ρ_W and $\overline{\rho_W}$ are algebraic numbers in K that can be computed explicitly. The product converges normally for all $\tau \in W$ with $\mathrm{Im}\,(\tau_1)\,\mathrm{Im}\,(\tau_2) > |N|/p$ outside the set of poles.

(iv) *There is a positive integer c such that Ψ^c has integral rational Fourier coefficients with greatest common divisor 1.*

Definition 4.2. If W is a Weyl chamber and n an integer, we define the set

$$R(n) := \{\lambda \in \mathfrak{o}/\sqrt{p} : \lambda > 0,\ \mathrm{N}(\lambda) = n/p\}$$

n	$p=5$	$p=13$	$p=17$
1	$\{\frac{1}{5}\sqrt{5}\}$	$\{\frac{1}{13}\sqrt{13}\}$	$\{\frac{1}{17}\sqrt{17}\}$
2	$\{\}$	$\{\}$	$\{-\frac{1}{2}+\frac{5}{34}\sqrt{17},\ \frac{1}{2}+\frac{5}{34}\sqrt{17}\}$
3	$\{\}$	$\{-\frac{1}{2}+\frac{5}{26}\sqrt{13},\ \frac{1}{2}+\frac{5}{26}\sqrt{13}\}$	$\{\}$
4	$\{\frac{2}{5}\sqrt{5}\}$	$\{\frac{2}{13}\sqrt{13}\}$	$\{-\frac{3}{2}+\frac{13}{34}\sqrt{17},\ \frac{3}{2}+\frac{13}{34}\sqrt{17},\ \frac{2}{17}\sqrt{17}\}$
5	$\{\frac{1}{2}\sqrt{5}-\frac{1}{2}\}$	$\{\}$	$\{\}$
8	$\{\}$	$\{\}$	$\left\{\begin{matrix}\frac{1}{2}+\frac{7}{34}\sqrt{17},\ 1+\frac{5}{17}\sqrt{17},\\ -1+\frac{5}{17}\sqrt{17},\ -\frac{1}{2}+\frac{7}{34}\sqrt{17}\end{matrix}\right\}$
9	$\{\frac{3}{5}\sqrt{5}\}$	$\left\{\begin{matrix}\frac{1}{2}+\frac{7}{26}\sqrt{13},\ \frac{3}{13}\sqrt{13},\\ -\frac{1}{2}+\frac{7}{26}\sqrt{13}\end{matrix}\right\}$	$\{\frac{3}{17}\sqrt{17}\}$
10	$\{\}$	$\{\}$	$\{\}$
11	$\left\{\begin{matrix}\frac{1}{2}+\frac{7}{10}\sqrt{5},\\ -\frac{1}{2}+\frac{7}{10}\sqrt{5}\end{matrix}\right\}$	$\{\}$	$\{\}$
12	$\{\}$	$\{1+\frac{5}{13}\sqrt{13},\ -1+\frac{5}{13}\sqrt{13}\}$	$\{\}$
13	$\{\}$	$\{-\frac{3}{2}+\frac{1}{2}\sqrt{13}\}$	$\{-2+\frac{9}{17}\sqrt{17},\ 2+\frac{9}{17}\sqrt{17}\}$
16	$\{\frac{4}{5}\sqrt{5}\}$	$\{\frac{4}{13}\sqrt{13}\}$	$\left\{\begin{matrix}\frac{4}{17}\sqrt{17},\ 3+\frac{13}{17}\sqrt{17},\\ \frac{1}{2}+\frac{9}{34}\sqrt{17},\ -\frac{1}{2}+\frac{9}{34}\sqrt{17},\\ -3+\frac{13}{17}\sqrt{17}\end{matrix}\right\}$
17	$\{\}$	$\{\frac{1}{2}+\frac{9}{26}\sqrt{13},\ -\frac{1}{2}+\frac{9}{26}\sqrt{13}\}$	$\{\sqrt{17}-4\}$
18	$\{\}$	$\{\}$	$\{\frac{3}{2}+\frac{15}{34}\sqrt{17},\ -\frac{3}{2}+\frac{15}{34}\sqrt{17}\}$
19	$\left\{\begin{matrix}\frac{1}{2}+\frac{9}{10}\sqrt{5},\\ -\frac{1}{2}+\frac{9}{10}\sqrt{5}\end{matrix}\right\}$	$\{\}$	$\{-1+\frac{6}{17}\sqrt{17},\ 1+\frac{6}{17}\sqrt{17}\}$
20	$\{-1+\sqrt{5}\}$	$\{\}$	$\{\}$
23	$\{\}$	$\{1+\frac{6}{13}\sqrt{13},\ -1+\frac{6}{13}\sqrt{13}\}$	$\{\}$
24	$\{\}$	$\{\}$	$\{\}$
25	$\{\sqrt{5}\}$	$\{\frac{5}{13}\sqrt{13}\}$	$\{\frac{5}{17}\sqrt{17}\}$

TABLE 9. $R(W,-n)$: For $p \in \{5,13,17\}$ and $n \in \{6,7,14,15,21,22\}$ the set $R(W,-n)$ is empty.

and write $R(W,n)$ for the finite set

$$R(W,n) = \big\{\lambda \in R(n) : \lambda \operatorname{Im}(\tau_1) + \overline{\lambda} \operatorname{Im}(\tau_2) < 0, \quad \varepsilon_0^2 \lambda \operatorname{Im}(\tau_1) + \overline{\varepsilon_0}^2 \overline{\lambda} \operatorname{Im}(\tau_2) > 0, \forall \tau \in W\big\}. \tag{4–1}$$

From [Brunier and Bundschuh 03] we take the following remark to Theorem 4.1.

Remark 4.3. Additionally, we have

1. For all $\tau \in W$ and $y_1 = \operatorname{Im}(\tau_1)$ and $y_2 = \operatorname{Im}(\tau_2)$ the Weyl vector $(\rho_W, \overline{\rho_W})$ is given by

$$\rho_W y_1 + \overline{\rho_W} y_2 = \frac{1}{\varepsilon_0 + \overline{\varepsilon_0}} \sum_{n<0} s(n)a(n) \sum_{\lambda \in R(W,n)} \big(\varepsilon_0 \lambda y_1 + \overline{\varepsilon_0}\overline{\lambda} y_2\big). \tag{4–2}$$

2. Every modular form for Γ whose divisor is a linear combination of Hirzebruch–Zagier divisors $T(m)$ is given as a Borcherds product as in Theorem 4.1.

For concrete calculations we reformulate this as the following lemma.

Lemma 4.4. (ρ_W and $\overline{\rho_W}$.) *Let $f = \sum_{n\in\mathbb{Z}} a(n)q^n \in A_0^+(p,\chi_p)$. If W is a Weyl chamber attached to f, we have*

$$\rho_W = \frac{1}{\varepsilon_0 + \overline{\varepsilon_0}} \sum_{n<0} s(n)a(n) \sum_{\lambda \in R(W,n)} \lambda\varepsilon_0$$

and

$$\overline{\rho_W} = \frac{1}{\varepsilon_0 + \overline{\varepsilon_0}} \sum_{n<0} s(n)a(n) \sum_{\lambda \in R(W,n)} \overline{\lambda}\overline{\varepsilon_0}.$$

Proof. Since Weyl chambers are open in $\mathbb{H}^2$ by Corollary 2.9, equation (4–2) holds for $\tau+(\delta_1,\delta_2)$ if $\delta_1 \geq 0$ and $\delta_2 \geq 0$ are sufficiently small. We subtract both equations and get

$$\rho_W\delta_1 + \overline{\rho_W}\delta_2 = \frac{1}{\varepsilon_0+\overline{\varepsilon_0}}\sum_{n<0} s(n)a(n) \sum_{\lambda\in R(W,n)} \left(\varepsilon_0\lambda\delta_1 + \overline{\varepsilon_0}\overline{\lambda}\delta_2\right).$$

Differentiation by δ_1 and δ_2 gives the stated result. □

Lemma 4.5. (Choice of Weyl chamber.) *Let $m=-n$ be a natural number and let $\tau\in\mathbb{H}^2$ be a point. Then $W(\tau)$ defines the Weyl chamber attached to $\tilde{\tau} := \tau + (i\overline{\delta}, i\delta)$ for sufficiently small $\delta \in \mathcal{K}$, $\delta \geq 0$, in the following sense: If τ is contained in a Weyl chamber, then we define $W(\tau)$ to be this Weyl chamber ($\delta = 0$). Otherwise, if $\mathrm{Im}\,(\tau_1) \neq \mathrm{Im}\,(\tau_2)$, then there are a Weyl chamber, which we denote by $W(\tau)$, and some $\delta_0 > 0$ such that for $\delta = (i\delta_1, i\delta_1)$ we have $\tau+\delta \in W(\tau)$ for all $0<\delta_1<\delta_0$, $\delta_1\in\mathbb{Q}$.*

In the case that τ is not contained in a Weyl chamber and $\mathrm{Im}\,(\tau_1) = \mathrm{Im}\,(\tau_2)$, there are a unique Weyl chamber, which we denote by $W(\tau)$, and some $\delta_0 > 0$ such that $\tau + (-i\delta_2\sqrt{p}, i\delta_2\sqrt{p})$ is contained in $W(\tau)$ for all $0 < \delta_2 < \delta_0$ with $\delta_2 \in \mathbb{Q}$.

Our standard choice for τ will be $\tau = (-i\overline{\varepsilon_0} + i\varepsilon_0)$ and $\tilde{\tau} := (-i\overline{\varepsilon_0} + i\overline{\delta}, i\varepsilon_0 + i\delta)$.

Proof. If $\tau \in \mathbb{H}$ is not contained in a Weyl chamber, then $\tau \in S(m)$. By Lemma 2.7 we know that $S(m)$ is a finite union of hyperplanes $M(\lambda)$ modulo multiplication by ε_0^2. The projection of these hyperplanes onto the imaginary parts are straight lines through 0 intersected with $\mathbb{H}^2$. Hence for $\tau \in S(m)$, the point $(\mathrm{Im}\,(\tau_1), \mathrm{Im}\,(\tau_2))$ lies on the straight line through 0 with direction $(\mathrm{Im}\,(\tau_1), \mathrm{Im}\,(\tau_2))$, and the choice of $W(\tau)$ described in the lemma is unique and well defined. □

With respect to Lemma 2.11 it suffices to choose one point in W in order to calculate Borcherds products on W (without knowing about the concrete shape of W). Now we can easily calculate $R(W,n)$.

Lemma 4.6. (Calculation of $R(W,n)$.) *Let $m=-n$ be a natural number and $\tau\in W$ for some Weyl chamber W. Then $R(W,n)$ can be calculated by the following algorithm: For every element λ in a set of representatives of $R(-m)$ modulo multiplication by ε_0^2 (such as $\mathcal{J}\bigcup -\mathcal{J}$ in Lemma 2.2) do*

(i) *Multiply λ by ε_0^2 (and denote the result again by λ) until $\lambda y_1 + \overline{\lambda} y_2 > 0$ for the imaginary part y of τ.*

(ii) *Multiply λ by ε_0^{-2} until $\lambda y_1 + \overline{\lambda} y_2 < 0$.*

The resulting $M(\lambda)$ is an element of $R(W,n)$ and this procedure gives all of its elements when applied to all λ in $R(-m)/\varepsilon_0^2$.

Proof. We have $\varepsilon_0 > 1$ and $\mathrm{N}(\varepsilon_0) = \pm 1$, so $0 < \overline{\varepsilon_0^2} = \varepsilon_0^{-2} < 1$. Let $\tau\in\mathbb{H}^2$ and $\lambda \in \mathfrak{o}/\sqrt{p}$ with $\lambda > 0$. Write $y_1 = \mathrm{Im}\,(\tau_1)$ and $y_2 = \mathrm{Im}\,(\tau_2)$. Then $\overline{\lambda} = \mathrm{N}(\lambda)/\lambda = -\frac{m}{p}\lambda < 0$ and we get

$$\varepsilon_0^{2k}\underbrace{\lambda y_1}_{>0} + \overline{\varepsilon_0^{2k}}\overline{\lambda} y_2 \overset{k\to\infty}{\longrightarrow} +\infty,$$

$$\varepsilon_0^{-2k}\lambda y_1 + \overline{\varepsilon_0^{-2k}}\underbrace{\overline{\lambda} y_2}_{<0} \overset{k\to\infty}{\longrightarrow} -\infty,$$

and

$$\underbrace{\varepsilon_0^2}_{>1}\underbrace{\lambda y_1}_{>0} + \underbrace{\overline{\varepsilon_0^2}}_{<1}\underbrace{\overline{\lambda} y_2}_{<0} > \lambda y_1 + \overline{\lambda} y_2.$$

So the algorithm described in the lemma gives some $\tilde{\lambda} = \varepsilon_0^{2k}\lambda$ with $\tilde{\lambda}\in R(W,-n)$, $k\in\mathbb{Z}$, and clearly it suffices to apply this algorithm to a set of representatives of $\mathcal{I}$ modulo multiplication by ε_0^2. □

Remark 4.7. (Interpretation of $R(W,n)$.) From (4–1), we get that if W is a Weyl chamber attached to f_n, then the boundary of W in $\mathbb{H}^2$ is a subset of

$$\bigcup_{\lambda\in R(W,n)} \left(M(\lambda)\bigcup M(\varepsilon_0^2\lambda)\right).$$

In particular, the boundary is the union of two $M(\mu)$.

5. CALCULATION OF BORCHERDS PRODUCTS

We investigate the remaining tasks for the concrete calculation of Borcherds products as described in Theorem 4.1. A basis of $A_0^+(p,\chi_p)$ is calculated via Eisenstein series in some space $M_k(p,1)$, and rational functions in η and $\eta^{(p)}$ in the cases $p\in\{5,13,17\}$ (cf. [Brunier and Bundschuh 03] in case $p=5$, [Mayer 09] in case $p=13$, and $p=17$, and [Mayer 07] in the cases $p=5$, $p=13$, and $p=17$).

We will determine the multiplier system of a Borcherds product from the Weyl vector, as was suggested by Bruinier, and give a method to calculate the Fourier expansion of a Borcherds product up to arbitrary finite degree. For calculations in $A_k(p,\chi_p)$ note the following remark:

Remark 5.1. (Precision invariant under multiplication and division.) If f is a meromorphic modular form with $f = q^k \sum_{n=0}^{M} a(n)q^n + O(q^{k+M+1})$ and $a(0) \neq 0$, then we say that f is given with precision $M+1$. Then the product and the quotient of two modular forms given with precision N is given with precision N again. The same holds for the inverse of a modular form. So, in order to determine the first N coefficients of a product or quotient of Fourier expansions, the first N coefficients have to be determined for each of the factors.

Remark 5.2. We can show by calculation of Fourier exponents that the Eisenstein series E_2^+ considered in [Brunier and Bundschuh 03] is a theta nullwert in case $p \in \{5, 13, 17\}$. In particular, we set

$$M_5 := \begin{pmatrix} 2 & 1 & & \\ 1 & 2 & 1 & \\ & 1 & 4 & 5 \\ & & 5 & 10 \end{pmatrix}, \quad M_{13} := \begin{pmatrix} 2 & 1 & & \\ 1 & 4 & 3 & \\ & 3 & 10 & 13 \\ & & 13 & 26 \end{pmatrix},$$

and

$$M_{17} := \begin{pmatrix} 2 & 1 & & \\ 1 & 4 & 1 & \\ & 1 & 10 & 17 \\ & & 17 & 34 \end{pmatrix}.$$

The inverse matrices M_p^{-1} are each contained in $(\mathbb{Z}/p)^{4\times 4}$, so the functions

$$z \mapsto \sum_{g \in \mathbb{Z}^4} e^{\pi i g^t M_p g z}$$

are modular forms for $\Gamma_0(p)$. Then we can compare Fourier coefficients and obtain

$$E_2^+(z) = \sum_{g \in \mathbb{Z}^4} e^{\pi i g^t M_p g z} \quad \text{for all } p \in \{5, 13, 17\}.$$

With [Brunier and Bundschuh 03, Theorems 6 and 9] we can use E_2^+ to calculate the weights of the Borcherds products (cf. Tables 6 and 7 as well as Tables 8 and 9).

Next, we investigate the weights and multiplier systems possible for Hilbert modular forms following the work of [Gundlach 88]. We investigate the multiplier system of Borcherds products in dependence of the Weyl vector. In the cases $p \in \{5, 13, 17\}$, this suffices to determine the multiplier system of Borcherds products.

Definition 5.3. For all $w \in \mathfrak{o}$ we define

$$T_w = \left(\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \overline{w} \\ 0 & 1 \end{pmatrix} \right)$$

and write

$$J = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)$$

and $T := T_1$.

Remark 5.4. (Symmetric multiplier systems.) Let μ be a multiplier system. We define $\overline{\mu}(M) = \mu(\overline{M})$ for all matrices $M \in \mathrm{SL}(2, \mathfrak{o})$, where $\overline{M}$ is the matrix derived from M by componentwise conjugation. Then $\overline{\mu}$ is again a multiplier system and is given by $\overline{\mu}(J) = \mu(J)$, $\overline{\mu}(T) = \mu(T)$, and $\overline{\mu}(T_w) = \mu(T \cdot T_w^{-1}) = \frac{\mu(T)}{\mu(T_w)}$. In case $\mu = \overline{\mu}$, we call μ symmetric. In [Mayer 09] we prove the following theorem.

Theorem 5.5. (Multiplier systems of Borcherds products.) *If Ψ is a Borcherds product with multiplier system μ, then the values $\mu(T_\lambda)$ of all translations T_λ, $\lambda \in \mathfrak{o}$, can be determined by $\mu(T_\lambda) = \mathbf{e}((\mathrm{S}(\rho_W \lambda))$. Especially in the case $\mathbb{Q}(\sqrt{13})$ and $\mathbb{Q}(\sqrt{17})$, the multiplier system can be read from the Weyl vector.*

Remark 5.6. For $\mathbb{Q}(\sqrt{5})$, only the trivial multiplier system exists.

Proof. Let Ψ be a Borcherds product with multiplier system μ. Then Ψ has the Fourier expansion

$$\Psi(\tau_1, \tau_2) = \mathbf{e}(\rho_W \tau_1 + \overline{\rho_W} \tau_2) \times \prod_{\substack{\nu \in \mathfrak{o}/\sqrt{p} \\ (\nu, W) > 0}} (1 - \mathbf{e}(\nu \tau_1 + \overline{\nu} \tau_2))^{s(p\nu\overline{\nu}) a(p\nu\overline{\nu})},$$

and the product is invariant with respect to the operation of T and T_w, since $\mathbf{e}(\nu \tau_1 + \overline{\nu} \tau_2)$ itself is invariant.

So we have $\mu(T) = \mathbf{e}(\mathrm{S}(\rho_W))$ and $\mu(T_w) = \mathbf{e}(\mathrm{S}(\rho_W w))$. By [Mayer 09, Corollary 23 and 24], in case $p = 13$ and $p = 17$ the multiplier system μ is uniquely determined by $\mu(T)$ and $\mu(T_w)$. Hence in the cases $p = 13$ and $p = 17$, the multiplier system can be read from the Weyl vector. $\square$

Definition 5.7. If f is a Hilbert modular form with Fourier expansion

$$f(\tau) = \sum_{\substack{a, b \in \mathcal{Z} \\ a \geq 0 \\ |b| \leq \sqrt{p} a}} c(a, b) g^a h^b,$$

where $g = \mathbf{e}((\tau_1 + \tau_2)/2)$ and $h = \mathbf{e}((\tau_1 - \tau_2)/2\sqrt{p})$, and there is $N \in \mathbb{N}$ such that $c(a, b)$ is known for all $a \leq N$, $|b| \leq \sqrt{p} a$, then f and

$$\sum_{\substack{a, b \in \mathcal{Z} \\ 0 \leq a \leq N \\ |b| \leq \sqrt{p} a}} c(a, b) g^a h^b$$

Ψ_k	weight	Fourier expansion	divisor
μ	other	Fourier expansion on the diagonal (if not 0)	diagonal
Ψ_1	5	$g(h-\frac{1}{h})-10g^2(h^2-\frac{1}{h^2})-g^2(h^4-\frac{1}{h^4})+O(g^3)$	T_1
1	$\overline{\Psi_1}=-\Psi_1$		
$\frac{\Psi_4}{\Psi_1}$	10	$g\left(h+\frac{1}{h}\right)+g^2\left(454+228\left(h^2+\frac{1}{h^2}\right)+\left(h^4+\frac{1}{h^4}\right)\right)$	T_4-T_1
1	$\overline{\left(\frac{\Psi_4}{\Psi_1}\right)}=\frac{\Psi_4}{\Psi_1}$	$2g+912g^2+101304g^3-632704g^4+O(g^5)$	$2E_4^2\cdot\Delta$
Ψ_4	15	$g^2(h^2-\frac{1}{h^2})+216g^3(h+h^3-\frac{1}{h}-\frac{1}{h^3})+O(g^4)$	T_4
1	$\overline{\Psi_4}=-\Psi_4$		
Ψ_5	15	$g^2-275g^3(h+\frac{1}{h})-g^3(h^5+\frac{1}{h^5})+O(g^4)$	T_5
1	$\overline{\Psi_5}=\Psi_5$	$g^2-552g^3+8640g^4+116000g^5+O(g^6)$	$E_6\cdot\Delta^2$
Ψ_6	10	$1-264g(h+\frac{1}{h})+O(g^2)$	T_6
1	$\overline{\Psi_6}=\Psi_6$	$1-528g-201168g^2+61114944g^3+O(g^4)$	$E_4^2E_6^2$
Ψ_9	35	$g^3(h^3-\frac{1}{h^3})+3555g^4(h^2+h^4-\frac{1}{h^2}-\frac{1}{h^4})+O(g^5)$	T_9
1	$\overline{\Psi_9}=-\Psi_9$		
Ψ_{10}	10	$1-3400g(h+\frac{1}{h})+O(g^2)$	T_{10}
1	$\overline{\Psi_{10}}=\Psi_{10}$	$1-6800g-3061200g^2-256574400g^3+O(g^4)$	$\frac{5^2}{3^3}E_4^2E_6^2-\frac{2\cdot7^2}{3^3}E_4^5$
Ψ_{11}	60	$-g^6+3256g^7(h+\frac{1}{h})+g^7(h^7+\frac{1}{h^7})+O(g^8)$	T_{11}
1	$\overline{\Psi_{11}}=\Psi_{11}$	$-g^6+6514g^7+O(g^8)$	
Ψ_{14}	30	$1+25704g(h+\frac{1}{h})+O(g^2)$	T_{14}
1	$\overline{\Psi_{14}}=\Psi_{14}$	$1+51408g+146187664g^2+O(g^3)$	
Ψ_{15}	20	$1-22425f(h+\frac{1}{h})+O(g^2)$	T_{15}
1	$\overline{\Psi_{15}}=\Psi_{15}$	$1-44850g-428741775g^2+O(g^3)$	

TABLE 10. Borcherds products in case $p=5$ for the Weyl chamber $W(-i\overline{\varepsilon_0}, i\varepsilon_0)$ (($g=\exp(\pi i(\tau_1+\tau_2))$ and $h=\exp(\pi i(\tau_1-\tau_2)/\sqrt{p})$)).

are said to be given with precision g^N. Here $\mathcal{Z}$ is a rational ideal in $\mathbb{Q}$ depending on the multiplier system of f. In case of the trivial multiplier system, it is $\mathbb{Z}$.

Hence we have the following results.

Lemma 5.8. *If $f_{(1)}$ and $f_{(2)}$ are Hilbert modular forms given with precision g^N, then their product $f_{(1)}f_{(2)}$ is given with precision g^N.*

Lemma 5.9. (Calculation of Borcherds products with given precision.) *Let $p\equiv 1 \pmod 4$ be a prime, and for $m\in\mathbb{N}$ with $\chi_p(m)\geq 0$, denote by f_m the unique basis element of $A_0^+(p,\chi_p)$ with Fourier expansion $s(-m)^{-1}q^{-m}+\sum_{k\geq 0}a(k)q^k$. Let W be a Weyl chamber attached to f_m and $\tau\in W$ with $y_1=\mathrm{Im}\,(\tau_1)$ and $y_2=\mathrm{Im}\,(\tau_2)$. Define $a(-m)=s(-m)^{-1}$ and $a(-k)=0$ for all $k\in\mathbb{N}_0\setminus\{m\}$. Then for every $N\in\mathbb{N}$ the Borcherds product Ψ_m, given by*

$$\Psi_m(\tau_1,\tau_2)=\mathbf{e}(\rho_W\tau_1+\overline{\rho_W}\tau_2)\times\prod_{\substack{\nu\in\mathfrak{o}/\sqrt{p}\\(\nu,W)>0}}(1-\mathbf{e}(\nu\tau_1+\overline{\nu}\tau_2))^{s(p\nu\overline{\nu})a(p\nu\overline{\nu})}$$

for all $\tau\in W$ can be calculated with precision g^N by the following algorithm:

1: *Calculate $R(W,n)$ with Lemma 4.6.*

2: *Calculate ρ_W with Lemma 4.4.*

3: *Calculate the leading coefficient $a(h,-k)g^{-k}$ of*

$$\prod_\nu (1-\mathbf{e}(\nu\tau_1+\overline{\nu}\tau_2))^{s(p\nu\overline{\nu})a(p\nu\overline{\nu})},$$

where the product is over

$$\nu=\nu_1+\nu_2\sqrt{p}\in\mathfrak{o}/\sqrt{p},\quad \nu_2^2\leq\nu_1^2/p+m/p^2,$$
$$-\sqrt{(y_1-y_2)^2m/(4py_1y_2)}<\nu_1<0.$$

4: *If $\mathrm{S}(\rho_W)$ is negative, then rewrite $k:=k-\mathrm{S}(\rho_W)$.*

5: *Expand*

$$R=g^{\mathrm{S}(\rho_W)}h^{(\rho_W-\overline{\rho_W})\sqrt{p}}\prod_\nu\left(1-g^{2\nu_1}h^{2p\nu_2}\right)^{s(p\nu\overline{\nu})a(p\nu\overline{\nu})},$$

where the product over ν is as in step 3, and we expand each factor $\left(1-g^{2\nu_1}h^{2p\nu_2}\right)^{s(p\nu\overline{\nu})a(p\nu\overline{\nu})}$ with precision

Ψ_k	weight	divisor	ρ_W	μ	other	diagonal
Fourier expansion						
Fourier expansion on the diagonal (if not 0)						
Ψ_1	1	T_1	$\frac{1}{6}+\frac{\sqrt{13}}{26}$	$\mu_{1,2}$	$\overline{\Psi_1}=-\Psi_1$	$\Psi_1(\tau,\tau)\equiv 0$
$g^{1/3}\left(h-\frac{1}{h}\right)-g^{4/3}\left(2\left(h^2-\frac{1}{h^2}\right)+\left(h^4-\frac{1}{h^4}\right)\right)+O\left(g^{7/3}\right)$						
Ψ_3	4	T_3	$\frac{5}{6}+5\frac{\sqrt{13}}{26}$	$\mu_{2,1}$		$\Psi(\tau,\tau)=(\eta(\tau))^{16}$
$-g^{2/3}+g^{5/3}\left(-2\left(h+\frac{1}{h}\right)+9\left(h^3+\frac{1}{h^3}\right)+\left(h^5+\frac{1}{h^5}\right)\right)+g^{8/3}\left(16+O(h^2+\frac{1}{h^2})\right)+O\left(g^{11/3}\right)$						
$-q^{2/3}+16q^{5/3}-104q^{8/3}+O(q^{11/3})$						
Ψ_4	3	T_4	$\frac{1}{3}+\frac{\sqrt{13}}{13}$	$\mu_{2,1}$	$\overline{\Psi_4}=-\Psi_4$	$\Psi_4(\tau,\tau)\equiv 0$
$g^{2/3}\left(h^2-\frac{1}{h^2}\right)+g^{5/3}\left(-24\left(h-\frac{1}{h}\right)-16\left(h^3-\frac{1}{h^3}\right)+8\left(h^5-\frac{1}{h^5}\right)\right)+O\left(g^{8/3}\right)$						
$\frac{\Psi_4}{2\Psi_1}$	2	T_4-T_1	$\frac{1}{6}+\frac{\sqrt{13}}{26}$	$\mu_{1,2}$	symmetric	$\frac{\Psi_4}{2\Psi_1}(\tau,\tau)=\eta^8(\tau)$
$\frac{1}{2}g^{1/3}\left(h+\frac{1}{h}\right)+g^{4/3}\left(-26-4\left(h^2+\frac{1}{h^2}\right)+9\left(h^4+\frac{1}{h^4}\right)\right)+O\left(g^{7/3}\right)$						
$-q^{1/3}\left(-1+8q-20q^2+70q^4\right)+O\left(q^{16/3}\right)$						
Ψ_{10}	4	T_{10}	0	1	$\overline{\Psi_{10}}=\Psi_{10}$	$\Psi_{10}(\tau,\tau)=(E_4(\tau))^2$
$1+g\left(200\left(h+\frac{1}{h}\right)+40\left(h^3+\frac{1}{h^3}\right)\right)+O\left(g^2\right)$						
$1+480q+61920q^2+1050240q^3+7926240q^4+O\left(q^5\right)$						
Ψ_{13}	7	T_{13}	$\frac{1}{3}$	$\mu_{2,1}$	symmetric	$\Psi_{13}(\tau,\tau)=\eta^{16}(\tau)\cdot E_6(\tau)$
$g^{2/3}+g^{5/3}\left(-221\left(h+\frac{1}{h}\right)-39\left(h^3+\frac{1}{h^3}\right)\right)$						
$q^{2/3}-520q^{5/3}-8464q^{8/3}+O\left(q^{11/3}\right)$						
Ψ_{14}	6	T_{14}	0	1	$\overline{\Psi_{14}}=\Psi_{14}$	$\Psi_{14}(\tau,\tau)=E_6^2(\tau)$
$1-504g\left(h+\frac{1}{h}\right)+O\left(g^2\right)$						
$1-1008q+220752q^2+16519104q^3+399517776q^4+O\left(q^5\right)$						
The restriction of Ψ_{14} to the diagonal has trivial character.						
Ψ_{26}	6	T_{26}	0	1	$\overline{\Psi_{26}}=\Psi_{26}$	$\Psi_{26}(\tau,\tau)=\frac{125}{27}\left(E_6(\tau)\right)^2-\frac{98}{27}\left(E_4(\tau)\right)^3$
$1-g\left(3432\left(h+\frac{1}{h}\right)+208\left(h^3+\frac{1}{h^3}\right)\right)+O\left(g^2\right)$						
$1-7280q+371280q^2+14938560q^3+408750160q^4+O\left(q^5\right)$						

TABLE 11. Borcherds products in case $p = 13$ for the Weyl chamber $W(-i\overline{\varepsilon_0}, i\varepsilon_0)$ ($g = \exp(\pi i(\tau_1+\tau_2))$ and $h = \exp(\pi i(\tau_1-\tau_2)/\sqrt{p})$).

g^{k+N} *and neglect higher-order terms. For negative exponents use the geometric series*

$$(1-x)^{-1} = \sum_{n=0}^{\infty} x^n \quad \textit{for } |x| < 1.$$

Then Ψ_m *is given by* R *with precision* g^N*.*

Proof. Let $\mu = \mu_1 + \mu_2\sqrt{p} \in \mathfrak{o}/\sqrt{p}$. Then $\mathbf{e}(\mathrm{S}(\nu\tau)) = g^{2\nu_1}h^{2p\nu_2}$ and the factor $\left(1-g^{2\nu_1}h^{2p\nu_2}\right)$ has a negative power of g if and only if $\nu_1 < 0$. In this case we get from $(W,\nu) > 0$ the following:

$$\begin{aligned}
&\nu_1(y_1+y_2) + \nu_2(y_1-y_2)\sqrt{p} > 0 \\
&\iff \nu_2(y_1-y_2)\sqrt{p} > \underbrace{-\nu_1}_{>0}\underbrace{(y_1+y_2)}_{>0} \\
&\iff |\nu_2||y_1-y_2|\sqrt{p} > -\nu_1(y_1+y_2) \\
&\iff |\nu_2| > |\nu_1|\frac{y_1+y_2}{|y_1-y_2|\sqrt{p}}.
\end{aligned}$$

<table>
<tr><th>Ψ_k</th><th>weight</th><th>divisor</th><th>ρ_W</th><th>μ</th><th>other</th><th>diagonal</th></tr>
<tr><th colspan="7">Fourier expansion</th></tr>
<tr><th colspan="7">Fourier expansion on the diagonal (if not 0)</th></tr>
<tr><td>Ψ_1</td><td>$\frac{1}{2}$</td><td>T_1</td><td>$\frac{1}{8}+\frac{\sqrt{17}}{34}$</td><td>$\mu_{3,4}$</td><td>$\overline{\Psi_1}=-\Psi_1$</td><td>0</td></tr>
<tr><td colspan="7">$g^{1/4}\left(h-h^{-1}\right)-g^{5/4}\left(\left(h^2-\frac{1}{h^2}\right)+\left(h^4-\frac{1}{h^4}\right)\right)+g^{9/4}\left(h^9-\frac{1}{h^9}\right)+O\left(g^{13/4}\right)$</td></tr>
<tr><td>Ψ_2</td><td>$\frac{3}{2}$</td><td>T_2</td><td>$\frac{5}{8}+\frac{5\sqrt{17}}{34}$</td><td>$\mu_{3,3}=\mu_{3,4}^5$</td><td>$\overline{\Psi_2}=\Psi_2$</td><td>$\Psi_2(\tau,\tau)=-\left(\eta(\tau)\right)^6$</td></tr>
<tr><td colspan="7">$-g^{1/4}+g^{5/4}\left(-\left(h+\frac{1}{h}\right)+3\left(h^3+\frac{1}{h^3}\right)+\left(h^5+\frac{1}{h^5}\right)\right)+O\left(g^{9/4}\right)$</td></tr>
<tr><td colspan="7">$-q^{1/4}+6q^{5/4}-9q^{9/4}-10q^{13/4}+30q^{17/4}+O\left(q^{21/4}\right)$</td></tr>
<tr><td>Ψ_4</td><td>$\frac{7}{2}$</td><td>T_4</td><td>$\frac{15}{8}+\frac{15\sqrt{17}}{34}$</td><td>$\mu_{3,5}=\mu_{3,4}^7$</td><td>$\overline{\Psi_4}=-\Psi_4$</td><td>0</td></tr>
<tr><td colspan="7">$-g^{3/4}\left(h^2-\frac{1}{h^2}\right)+g^{7/4}\left(13\left(h+\frac{1}{h}\right)+11\left(h^3-\frac{1}{h^3}\right)-2\left(h^5-\frac{1}{h^5}\right)\right)+O\left(g^{11/4}\right)$</td></tr>
<tr><td>$\frac{\Psi_4}{\Psi_1}$</td><td>3</td><td>T_4-T_1</td><td>$\frac{7}{4}+\frac{7\sqrt{17}}{17}$</td><td>$\mu_{2,3}=\mu_{3,4}^6$</td><td>$\overline{\frac{\Psi_4}{\Psi_1}}=\frac{\Psi_4}{\Psi_1}$</td><td>$\frac{\Psi_4}{\Psi_1}=2\left(\eta(\tau)\right)^{12}$</td></tr>
<tr><td>Ψ_8</td><td>$\frac{15}{2}$</td><td>T_8</td><td>$\frac{17}{8}+\frac{\sqrt{17}}{2}$</td><td>$\mu_{3,4}$</td><td>$\overline{\Psi_8}=\Psi_8$</td><td>$\Psi_8(\tau,\tau)=\left(\eta(\tau)\right)^{30}=\Delta(\tau)\cdot\left(\eta(\tau)\right)^6$</td></tr>
<tr><td colspan="7">$g^{5/4}+g^{9/4}\left(10\left(h+\frac{1}{h}\right)-24\left(h^3+\frac{1}{h^3}\right)-\left(h^7+\frac{1}{h^7}\right)\right)+O\left(g^{13/4}\right)$</td></tr>
<tr><td>Ψ_9</td><td>$\frac{7}{2}$</td><td>T_9</td><td>$\frac{3}{8}+\frac{3\sqrt{17}}{34}$</td><td>$\mu_{3,6}=\mu_{3,4}^3$</td><td>$\overline{\Psi_9}=-\Psi_9$</td><td>0</td></tr>
<tr><td colspan="7">$g^{3/4}\left(h^3-h^{-3}\right)+g^{7/4}\left(-36\left(h^2-\frac{1}{h^2}\right)-36\left(h^4-\frac{1}{h^4}\right)+27\left(h^6-\frac{1}{h^6}\right)\right)+O\left(g^{7/4}\right)$</td></tr>
<tr><td>$\frac{\Psi_9}{\Psi_1}$</td><td>3</td><td>T_9-T_1</td><td>$\frac{1}{4}+\frac{\sqrt{17}}{17}$</td><td>$\mu_{2,2}=\mu_{3,4}^2$</td><td>$\overline{\frac{\Psi_9}{\Psi_1}}=\frac{\Psi_9}{\Psi_1}$</td><td>$\frac{\Psi_9}{\Psi_1}(\tau,\tau)=3\cdot\left(\eta(\tau)\right)^{12}$</td></tr>
<tr><td colspan="7">$g^{1/2}\left(h^2+1+\frac{1}{h^2}\right)g^{3/2}\left(-40\left(h+\frac{1}{h}\right)-6\left(h^3+\frac{1}{h^3}\right)+28\left(h^5+\frac{1}{h^5}\right)\right)+O\left(g^{5/2}\right)$</td></tr>
<tr><td>Ψ_{13}</td><td>7</td><td>T_{13}</td><td>$\frac{9}{4}+\frac{9\sqrt{17}}{17}$</td><td>$\mu_{2,2}=\mu_{3,4}^2$</td><td>$\overline{\Psi_{13}}=\Psi_{13}$</td><td>$\Psi_{13}(\tau,\tau)=-E_4(\tau)^2\cdot\left(\eta(\tau)\right)^{12}$</td></tr>
<tr><td colspan="7"></td></tr>
<tr><td>Ψ_{15}</td><td>4</td><td>T_{15}</td><td>0</td><td>1</td><td>$\overline{\Psi_{15}}=\Psi_{15}$</td><td>$\Psi_{15}(\tau,\tau)=E_4^2(\tau)=E_8(\tau)$</td></tr>
<tr><td colspan="7">$1+240g\left(h+\frac{1}{h}\right)+O\left(g^2\right)$</td></tr>
<tr><td colspan="7">$1+480q+61920q^2+1050240q^3+7926240q^4+O\left(q^5\right)$</td></tr>
<tr><td>Ψ_{17}</td><td>$\frac{9}{2}$</td><td>T_{17}</td><td>$\frac{1}{8}$</td><td>$\mu_{3,3}=\mu_{3,4}^5$</td><td>$\overline{\Psi_{17}}=\Psi_{17}$</td><td>$\Psi_{17}(\tau,\tau)=\left(\eta(\tau)\right)^6\cdot E_6(\tau)$</td></tr>
<tr><td colspan="7">$g^{1/4}-g^{5/4}\left(204\left(h+\frac{1}{h}\right)+51\left(h^3+\frac{1}{h^3}\right)\right)+O\left(g^{9/4}\right)$</td></tr>
<tr><td colspan="7">$q^{1/4}-510q^{5/4}-13599q^{9/4}-27710q^{13/4}+50370q^{17/4}+360194q^{21/4}-19479432q^{25/4}$</td></tr>
<tr><td>Ψ_{21}</td><td>6</td><td>T_{21}</td><td>0</td><td>1</td><td>$\overline{\Psi_{21}}=\Psi_{21}$</td><td>$\Psi_{21}=E_6^2(\tau)$</td></tr>
<tr><td colspan="7">$1-630g(h+h^{-1})+126g(h^3+h^{-3})+O\left(g^2\right)$</td></tr>
<tr><td colspan="7">$1-1008q+220752q^2+16519104q^3+399517776q^4+O\left(q^5\right)$</td></tr>
</table>

TABLE 12. Borcherds products in case $p=17$ for the Weyl chamber $W(-i\overline{\varepsilon_0},i\varepsilon_0)$ ($g=\exp(\pi i(\tau_1+\tau_2))$ and $h=\exp(\pi i(\tau_1-\tau_2)/\sqrt{p})$).

Moreover, for $\mathrm{N}(\nu)<-m/p$ we have $a(p\,\mathrm{N}(\nu))=0$, so we can skip

$$(1-\mathbf{e}(\nu_1\tau_1+\nu_2\tau_2))^{s(p\,\mathrm{N}(\nu))a(p\,\mathrm{N}(\nu))}=1$$

in the product expansion of Ψ_m whenever $\mathrm{N}(\nu)<-m/p$. So negative exponents may derive only from the factor $\mathbf{e}(\rho_W\tau_1+\overline{\rho_W}\tau_2)$ and $\nu\in\mathfrak{o}/\sqrt{p}$ with $(W,\nu)>0$ and

$$\mathrm{N}(\nu)=\nu_1^2-p\nu_2^2\geq -m/p \quad\Longleftrightarrow\quad \nu_2^2\leq\frac{\nu_1^2}{p}+\frac{m}{p^2}.$$

The combination of both conditions gives

$$\begin{aligned}
|\nu_1|^2\frac{(y_1+y_2)^2}{|y_1-y_2|^2p}&<\nu_2^2\leq\frac{\nu_1^2}{p}+\frac{m}{p^2}\\
&\Longrightarrow\frac{|\nu_1|^2}{p}\left(\frac{(y_1+y_2)^2}{|y_1-y_2|^2}-1\right)<\frac{m}{p^2}\\
&\Longrightarrow|\nu_1|^2\frac{(y_1+y_2)^2-(y_1-y_2)^2}{(y_1-y_2)^2}<\frac{m}{p}\\
&\Longrightarrow|\nu_1|^2<\frac{m}{p}\frac{(y_1-y_2)^2}{4y_1y_2}.
\end{aligned}$$

Since $s(-m)a(-m) = 1$, every factor $(1-q^{2\nu_1}h^{2p\nu_2})$ with negative q-exponent occurs once, so by Lemma 5.8, we need every factor in the product expansion of Ψ_m with precision g^{N+k}. It remains to show that the geometric series can be applied for negative exponents. Since $\nu\,\mathrm{Im}\,(\tau_1) + \overline{\nu}\,\mathrm{Im}\,(\tau_2) > 0$, by $(W,\nu) > 0$ and

$$|\mathbf{e}(\nu\tau_1 + \overline{\nu}\tau_2)| = e^{-2\pi\nu\,\mathrm{Im}(\tau_1)+\overline{\nu}\,\mathrm{Im}(\tau_2)} < 1,$$

the geometric series converges. □

Remark 5.10. Some results of these calculations can be found in Tables 10, 11, and 12. The full data and the corresponding Maple worksheets can be found at http://www.matha.rwth-aachen.de.

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