Sampling Lissajous and Fourier Knots

Adam Boocher, Jay Daigle, Jim Hoste, and Wenjing Zheng

CONTENTS

A *Lissajous knot* is one that can be parameterized as

$$K(t) = (\cos(n_x t + \phi_x), \cos(n_y t + \phi_y), \cos(n_z t + \phi_z))\,,$$

where the *frequencies* n_x, n_y, and n_z are relatively prime integers and the *phase shifts* ϕ_x, ϕ_y, and ϕ_z are real numbers. Lissajous knots are highly symmetric, and for this reason, not all knots are Lissajous. We prove several theorems that allow us to place bounds on the number of Lissajous knot types with given frequencies and to efficiently sample all possible Lissajous knots with a given set of frequencies. In particular, we systematically tabulate all Lissajous knots with small frequencies and as a result substantially enlarge the tables of known Lissajous knots.

A *Fourier-*(i, j, k) *knot* is similar to a Lissajous knot except that the x, y, and z coordinates are now each described by a sum of i, j, and k cosine functions, respectively. According to Lamm, every knot is a Fourier-$(1, 1, k)$ knot for some k. By randomly searching the set of Fourier-$(1, 1, 2)$ knots we find that all 2-bridge knots with up to 14 crossings are either Lissajous or Fourier-$(1, 1, 2)$ knots. We show that all twist knots are Fourier-$(1, 1, 2)$ knots and give evidence suggesting that all torus knots are Fourier-$(1, 1, 2)$ knots.

As a result of our computer search, several knots with relatively small crossing numbers are identified as potential counterexamples to interesting conjectures.

1. INTRODUCTION

A *Lissajous* knot K in $\mathbb{R}^3$ is a knot that has a parameterization $K(t) = (x(t), y(t), z(t))$ given by

$$\begin{aligned} x(t) &= \cos(n_x t + \phi_x), \\ y(t) &= \cos(n_y t + \phi_y), \\ z(t) &= \cos(n_z t + \phi_z), \end{aligned}$$

where $0 \le t \le 2\pi$; n_x, n_y, and n_z are integers; and $\phi_x, \phi_y, \phi_z \in \mathbb{R}$.

Lissajous knots were first studied in [Bogle et al. 94], where some of their elementary properties were established. Most notably, Lissajous knots enjoy a high degree of symmetry. In particular, if the three *frequencies* n_x,

2000 AMS Subject Classification: Primary 57M25

Keywords: Lissajous knot, Fourier knot, 2-bridge knot, Alexander polynomial

n_y, and n_z (which must be pairwise relatively prime; see [Bogle et al. 94]) are all odd, then the knot is strongly plus amphicheiral. If one of the frequencies is even, then the knot is 2-periodic, with the additional property that it links its axis of rotation once. These symmetry properties imply (strictly) weaker properties such as the fact that the Alexander polynomial of a Lissajous knot must be a square modulo 2, which in turn implies that its Arf invariant must be zero. See [Bogle et al. 94, Hoste and Zirbel 07, Lamm 96] for details. Thus for example, the trefoil and figure-eight knots are not Lissajous, since their Arf invariants are one. In fact, "most" knots are not Lissajous.

To date it is unknown whether every knot that is strongly plus amphicheiral or 2-periodic (and links its axis of rotation once) is Lissajous. Several knots with relatively few crossings exist that meet these symmetry requirements and for which it is unknown whether they are Lissajous. For example, according to [Hoste et al. 98], there are only three prime knots with 12 or fewer crossings that are strongly plus amphicheiral: 10a103 (10_{99}), 10a121 (10_{123}), and 12a427. Here we have given knot names in both the Dowker–Thistlethwaite ordering of the Hoste–Thistlethwaite–Weeks table [Hoste et al. 98] and, in parentheses, the Rolfsen ordering (for knots with ten or fewer crossings) [Rolfsen 90]. Symmetries of the knots in the Hoste–Thistlethwaite–Weeks table were computed using SnapPea as described in [Hoste et al. 98]. Of these three knots, only 10a103 (10_{99}) was previously reported as Lissajous. (See [Lamm 96].) However, we find 12a427 to be Lissajous. (See Section 5 of this paper.) This leaves open the case of 10a121. As a further example, there are exactly four 8-crossing knots that are 2-bridge, 2-periodic, and link their axis of rotation once. Despite our extensive searching (see Section 5), only one of these knots turned up as Lissajous (and it had already been reported as such in [Lamm 96]). Whether the other three are Lissajous remains unknown.

Lissajous knots are a subset of the more general class of *Fourier* knots. A Fourier-(i, j, k) knot is one that can be parameterized as

$$\begin{aligned} x(t) &= A_{x,1}\cos(n_{x,1}t + \phi_{x,1}) + \cdots + A_{x,i}\cos(n_{x,i}t + \phi_{x,i}),\\ y(t) &= A_{y,1}\cos(n_{y,1}t + \phi_{y,1}) + \cdots + A_{y,j}\cos(n_{y,j}t + \phi_{y,j}),\\ z(t) &= A_{z,1}\cos(n_{z,1}t + \phi_{z,1}) + \cdots + A_{z,k}\cos(n_{z,k}t + \phi_{z,k}). \end{aligned}$$

Because any function can be closely approximated by a sum of cosines, every knot is a Fourier knot for some (i, j, k). But a remarkable theorem of Lamm [Lamm 99] states that in fact, every knot is a Fourier-$(1, 1, k)$ knot for some k. While k cannot equal 1 for all knots (these are the Lissajous knots, and not all knots are Lissajous), could k possibly be less than some universal bound M for all knots? This seems unlikely, with the more reasonable outcome being that k depends on the specific knot K. Yet no one has found a knot for which k must be bigger than 2!

If K is a Fourier-$(1, 1, k)$ knot, then its bridge number is less than or equal to the minimum of n_x and n_y. (The bridge number of a knot K can be defined as the smallest number of extrema on K with respect to a given direction in $\mathbb{R}^3$, taken over all representations of K and with respect to all directions. See [Burde and Zieschang 03] or [Rolfsen 90] for more details.) Moreover, Lamm's proof is constructive and explicitly shows that if K has bridge number b, then K is a Fourier-$(1, 1, k)$ knot for some k and with $n_x = b$. This raises several interesting questions. For any knot K, when expressed as a Fourier-$(1, 1, k)$ knot, can the minimum values of n_x and k be simultaneously realized? In particular, can a knot that is Lissajous and with bridge index b be realized as a Lissajous knot with $n_x = b$?

Let $\mathcal{L}(n_x, n_y, n_z)$ be the set of all Lissajous knots with frequencies n_x, n_y, n_z. (Throughout this paper we consider a knot and its mirror image to be equivalent.) One of the main goals of this paper is to investigate the set $\mathcal{L}(n_x, n_y, n_z)$. By a simple change of variables, $t \mapsto t + c$, we may alter the phase shifts. Therefore we will assume that $\phi_x = 0$ in all that follows. This leaves the pair of parameters (ϕ_y, ϕ_z), which vary within the *phase torus* $[0, 2\pi] \times [0, 2\pi]$. In Section 2, we will examine the phase torus and identify a finite number of regions in which the phase shifts must lie, with each region corresponding to a single knot type. We further show that a periodic pattern of knot types is produced as one traverses the phase torus. This allows us to prove the following theorem.

Theorem 1.1. *Let $|\mathcal{L}(n_x, n_y, n_z)|$ be the number of distinct Lissajous knots with frequencies (n_x, n_y, n_z). Then*

$$|\mathcal{L}(n_x, n_y, n_z)| \leq 2n_xn_y.$$

If furthermore $n_x = 2$, then

$$|\mathcal{L}(2, n_y, n_z)| \leq 2n_y + 1.$$

There is also a periodicity that exists across frequencies, and in Section 2 we also prove the following theorem.

Theorem 1.2. *$\mathcal{L}(n_x, n_y, n_z) \subseteq \mathcal{L}(n_x, n_y, n_z+2n_xn_y)$, with equality if $n_z \geq 2n_xn_y - n_y$.*

Our analysis of the phase torus together with these theorems allows us to efficiently sample (with the aid of a computer) all possible Lissajous knots having two of the three frequencies bounded. Even with relatively small frequencies, the three natural projections of a Lissajous knot into the three coordinate planes can have a large number of crossings. (The projection into the xy-plane has $2n_xn_y - n_x - n_y$ crossings.) With frequencies of 10 or more, diagrams with hundreds of crossings result and many, if not most, knot invariants are computationally out of reach. Thus it becomes extremely difficult to compare different Lissajous knots with large frequencies, or to try to locate them in existing knot tables. However, if one frequency is 2, the knot is 2-bridge, and even with hundreds of crossings it is relatively simple to compute the identifying fraction p/q by which 2-bridge knots are classified.

In Section 2 we recall basic facts about Lissajous knots and prove several theorems, including the two already mentioned, that will allow us to efficiently sample all Lissajous knots with two given frequencies. In Section 4 we then recall some basic facts about 2-bridge knots. Using these results we then report in Section 5 on our computer experiments. Theorems similar to those given in Section 2 but for Fourier-$(1, 1, k)$ knots would necessarily be much more complicated, and we begin the analysis of the phase torus for Fourier-$(1, 1, 2)$ knots in Section 3. Without the analogous results, we have not been able to rigorously sample Fourier knots. Instead, we have proceeded by two methods: random sampling and a sampling based on first forming a bitmap image of the phase torus and its singular curves. However, even without exhaustive sampling, our data show that all 2-bridge knots up to 14 crossings are Fourier-$(1, 1, k)$ knots with $k \leq 2$ and with $n_x = 2$.

An early version of this paper contained 18 tables of Lissajous and Fourier knots, which were trimmed considerably for this publication. The original tables are available in [Boocher et al. 07].

2. THE PHASE TORUS—LISSAJOUS KNOTS

Suppose $K(t)$ is a Lissajous knot and consider its diagram in the xy-plane. Each crossing in this diagram corresponds to a double point in the xy-projection given by a pair of times (t_1, t_2), where $x(t_1) = x(t_2)$ and $y(t_1) = y(t_2)$. It is straightforward to prove the following lemma. Essentially equivalent formulations are given in [Jones and Przytycki 98] and [Bogle et al. 94]. (Unfortunately, [Bogle et al. 94] contains typographical errors.)

Lemma 2.1. *Let $K(t)$ be a Lissajous knot. There are two types of time pairs (t_1, t_2) that give double points in the xy-projection:*

Type I:

$$(t_1, t_2) = \left(\left(-\frac{k}{n_x} + \frac{j}{n_y}\right)\pi - \frac{\phi_y}{n_y}, \left(\frac{k}{n_x} + \frac{j}{n_y}\right)\pi - \frac{\phi_y}{n_y}\right),$$
$$1 \leq k \leq n_x - 1,$$
$$1 + \left\lfloor \frac{n_y}{n_x}k + \frac{\phi_y}{\pi} \right\rfloor \leq j \leq \left\lfloor 2n_y - \frac{n_y}{n_x}k + \frac{\phi_y}{\pi} \right\rfloor.$$

Type II:

$$(t_1, t_2) = \left(\left(-\frac{k}{n_y} + \frac{j}{n_x}\right)\pi - \frac{\phi_x}{n_x}, \left(\frac{k}{n_y} + \frac{j}{n_x}\right)\pi - \frac{\phi_x}{n_x}\right),$$
$$1 \leq k \leq n_y - 1,$$
$$1 + \left\lfloor \frac{n_x}{n_y}k + \frac{\phi_x}{\pi} \right\rfloor \leq j \leq \left\lfloor 2n_x - \frac{n_x}{n_y}k + \frac{\phi_x}{\pi} \right\rfloor.$$

There are $n_xn_y - n_y$ double points of Type I, and $n_xn_y - n_x$ double points of Type II.

Figure 1 shows a Lissajous knot with frequencies $(3, 5, 7)$ and corresponding phase shifts $(0, \pi/4, \pi/12)$. Since all the frequencies are odd, this knot is symmetric through the origin. It is not hard to show that in general, the Type-I crossings line up in sets of size n_y on $n_x - 1$ horizontal lines, while the Type-II crossings line up in sets of size n_x on $n_y - 1$ vertical lines. If $n_x = 2$, there is a single row of Type-I crossings, all of which lie on the x-axis, and $n_y - 1$ columns of Type-II crossings with each column consisting of two crossings.

Not all phase shift pairs will generate curves that are knots. Assuming $\phi_x = 0$, the knot $K(t)$ will intersect itself—and thus fail to be a knot—exactly when the phase shifts satisfy

$$\phi_z = \frac{n_z}{n_y}\phi_y + l\frac{\pi}{n_y} \tag{2–1}$$

or

$$\phi_z = l\frac{\pi}{n_x} \tag{2–2}$$

or

$$\phi_y = l\frac{\pi}{n_x}. \tag{2–3}$$

for some integer l. Crossings of Type I become singular precisely when (2–1) holds, while crossings of Type II

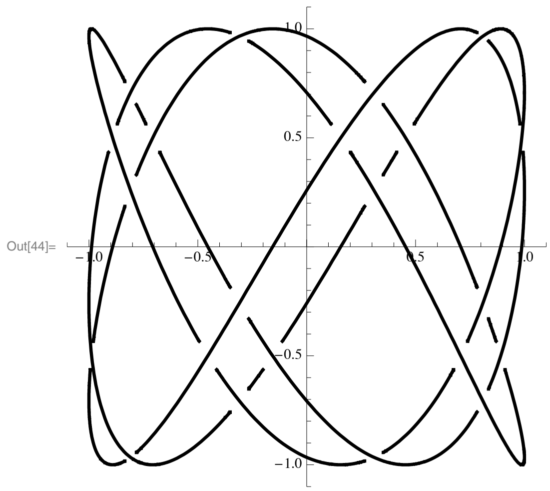

FIGURE 1. A Lissajous knot with frequencies $(3, 5, 7)$ and corresponding phase shifts $(0, \pi/4, \pi/12)$. The Type-I crossings appear in two rows with five crossings each, and the Type-II crossings appear in four columns with three crossings each.

become singular precisely when when (2–2) holds. When (2–3) holds, the entire xy-projection degenerates to an arc. While this alone does not imply that the knot has points of self-intersection, this is indeed the case. See [Jones and Przytycki 98, Bogle et al. 94, Hoste and Zirbel 07] for more details. In Proposition 2.3, we specifically identify which crossings become singular as the phase shifts move across these lines.

The slanted horizontal and vertical lines given in (2–1)–(2–3) obviously divide the phase torus into regions with each region defining one knot type. Thus there is only a finite number of knot types possible for a given set of frequencies. There is, however, a great deal of repetition in knot types as one traverses the phase torus due to the periodicity of the cosine function. The following theorem describes a nice choice of "fundamental domain" on the phase torus to which we may restrict our attention.

Theorem 2.2. *Any knot in $\mathcal{L}(n_x, n_y, n_z)$ can be represented with $\phi_x = 0$ and using some phase shift pair (ϕ_y, ϕ_z) in $[0, \frac{\pi}{n_x}] \times [0, \pi]$.*

Proof: Define an equivalence relation $\sim$ on the phase torus for $\mathcal{L}(n_x, n_y, n_z)$ by $(\phi_y, \phi_z) \sim (\phi'_y, \phi'_z)$ if the Lissajous knot with phase shifts $(0, \phi_y, \phi_z)$ is the same as the knot with phase shifts $(0, \phi'_y, \phi'_z)$, or its mirror image. Clearly

$$(\phi_y, \phi_z) \sim (\phi_y \pm \pi, \phi_z) \sim (\phi_y, \phi_z \pm \pi). \tag{2–4}$$

If $K \in \mathcal{L}(n_x, n_y, n_z)$, a change of variable $t \mapsto t + \frac{\pi}{n_x}$ shows that K is also parameterized as

$$\begin{aligned} x &= -\cos(n_x t), \\ y &= \cos\left(n_y t + \phi_y + \frac{n_y \pi}{n_x}\right), \\ z &= \cos\left(n_z t + \phi_z + \frac{n_z \pi}{n_x}\right). \end{aligned}$$

Therefore we also have

$$(\phi_y, \phi_z) \sim \left(\phi_y + \frac{n_y \pi}{n_x}, \phi_z + \frac{n_z \pi}{n_x}\right). \tag{2–5}$$

Since n_x and n_y are relatively prime, there are integers k and l with

$$0 \le \phi_y + \frac{k n_y \pi}{n_x} - l\pi < \frac{\pi}{n_x}.$$

Repeatedly using (2–4) and (2–5), we obtain

$$(\phi_y, \phi_z) \sim \left(\phi_y + \frac{k n_y \pi}{n_x} - l\pi, \phi_z + \frac{k n_z \pi}{n_x}\right).$$

The first coordinate is already in $[0, \frac{\pi}{n_x}]$; we can shift the second coordinate by multiples of π until it is in $[0, \pi]$. Thus an arbitrary point (ϕ_y, ϕ_z) is equivalent to some point in $[0, \frac{\pi}{n_x}] \times [0, \pi]$, as desired. $\square$

Figure 2 shows the fundamental domain on the phase torus for $\mathcal{L}(2, 3, 5)$. The singular lines divide the domain into regions, with each region determining a single knot type. Since these knots are all 2-bridges, we identify each with its classifying fraction p/q.

Our next result specifically describes what happens as we cross a singular line of the type given in (2–1) or (2–2).

Proposition 2.3. *Let K and K' be two Lissajous knots with frequencies (n_x, n_y, n_z) and phase shifts (ϕ_y, ϕ_z) and (ϕ'_y, ϕ'_z) respectively.*

(i) *Suppose (ϕ_y, ϕ_z) and (ϕ'_y, ϕ'_z) lie in two adjacent regions of the phase torus separated by a diagonal line L given by $\phi_z = \frac{n_z}{n_y}\phi_y + l\frac{\pi}{n_y}$. Then K and K' differ by changing all Type-I crossings (k, j) such that $jn_z + l \equiv 0 \mod n_y$. The number of such crossings is $n_x - 1$.*

(ii) *Suppose (ϕ_y, ϕ_z) and (ϕ'_y, ϕ'_z) lie in two adjacent regions of the phase torus separated by a horizontal line L given by $\phi_z = l\frac{\pi}{n_x}$. Then K and K' differ by changing all Type-II crossings with parameters (k, j) such that $jn_z + l \equiv 0 \mod n_x$. The number of such crossings is $n_y - 1$.*

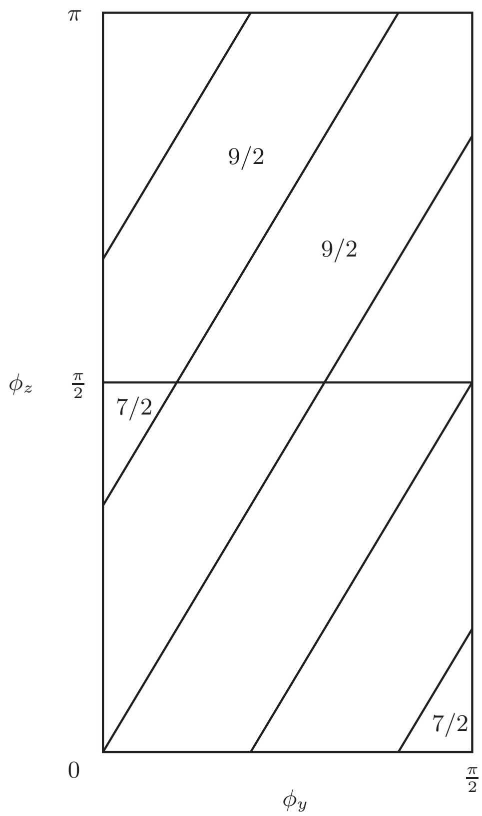

FIGURE 2. The fundamental domain of the phase torus for $\mathcal{L}(2,3,5)$. Each region defines a single 2-bridge knot that is identified by its classifying fraction p/q. Unlabeled regions define unknots.

Proof: If (t_1,t_2) is a Type-I crossing with parameters (k,j), then $z(t_1)=z(t_2)$ if and only if

$$\cos(n_z t_1+\phi_z)=\cos(n_z t_2+\phi_z),$$

which will occur if and only if

$$n_z(t_1-t_2)=2m\pi \quad \text{or} \quad n_z(t_1+t_2)+2\phi_z=2m'\pi \quad (2\text{–}6)$$

for some integers m,m'. For Type-I crossings,

$$t_1-t_2=-\frac{2k}{n_x}\pi \quad \text{and} \quad t_1+t_2=\frac{2j}{n_y}\pi-\frac{2\phi_y}{n_y}.$$

If (2–6) is to hold, then in the first case, we have

$$-\frac{2kn_z}{n_x}\pi=2m\pi,$$

which is equivalent to $-kn_z=mn_x$. This is impossible, since n_x and n_z are relatively prime and $1\le k\le n_x-1$.

In the second case, we have

$$n_z\left(2\frac{j}{n_y}\pi-2\frac{\phi_y}{n_y}\right)+2\phi_z=2m'\pi,$$

which is equivalent to

$$\phi_z=\frac{n_z}{n_y}\phi_y+(m'n_y-jn_z)\frac{\pi}{n_y}.$$

Thus Type-I crossings become singular only on lines of the form given in (2–1) with $l=mn_y-jn_z$.

If $\phi_z=\frac{n_z}{n_x}\phi_y+l\frac{\pi}{n_y}+\varepsilon$ and $jn_z+l=mn_y$ for some integer m, then it is straightforward to check that

$$z(t_1)-z(t_2)=(-1)^m 2\sin\varepsilon\sin\frac{kn_z\pi}{n_x}.$$

Hence, as we move across the line L by letting ε go from a small positive value to a small negative value, the difference $z(t_1)-z(t_2)$ changes sign. Thus the Type-I crossings with parameters (k,j) actually change from over to under or vice versa, rather than simply becoming singular and then "rebounding" to their original positions.

Finally, note that once l is fixed, this does not necessarily uniquely determine j and thus the corresponding Type-I crossing. If both $jn_z+l\equiv 0 \bmod n_y$ and $j'n_z+l\equiv 0 \bmod n_y$, then $j\equiv j' \bmod n_y$, since n_y and n_z are relatively prime. If $n_x=2$, then $k=1$ and j lies in an interval of length n_y. Thus with $n_x=2$ we have that j is uniquely determined by l, and a single crossing changes as we move across L. But if $n_x>2$ and $k=1$, then j lies in an interval of length greater than n_y. Hence two admissible values, j and $j+n_y$, are possible. Using j, we must have $1\le k\le \left\lfloor \frac{n_x}{n_y}(j-\frac{\phi_y}{\pi})\right\rfloor$, while for $j+n_y$ we must have $1\le k\le \left\lfloor \frac{n_x}{n_y}(-j+n_y+\frac{\phi_y}{\pi})\right\rfloor$. Thus the total number of possible points (k,j) is

$$\left\lfloor \frac{n_x}{n_y}(j-\frac{\phi_y}{\pi})\right\rfloor+\left\lfloor \frac{n_x}{n_y}(-j+n_y+\frac{\phi_y}{\pi})\right\rfloor=n_x-1.$$

A similar discussion handles the Type-II crossings. $\square$

Corollary 2.4. *Suppose K and K' are Lissajous knots with frequencies (n_x,n_y,n_z) and phase shifts that belong to regions separated by $2n_y$ singular lines of the type given in (2–1). Then all Type-I crossings are the same for both knots.*

Proof: From Proposition 2.3 we know that crossing the line $\phi_z=\frac{n_z}{n_y}\phi_y+l\frac{\pi}{n_y}$ changes exactly those Type-I crossings with parameters (k,j) for which $jn_z+l\equiv 0 \bmod n_y$. Thus if we cross the singular line corresponding to l and then later cross the line corresponding to $l+n_y$, the same set of Type-I crossings will first be changed and then changed back again. Hence, after crossing over $2n_y$ such lines, all Type-I crossings will be restored to their original position. $\square$

If $n_x = 2$, there is even more repetition due to additional symmetry, as is shown in the following result.

Proposition 2.5. *Let K and K' be Lissajous knots with frequencies $(2, n_y, n_z)$ and phase shifts (ϕ_y, ϕ_z) and (ϕ'_y, ϕ'_z) respectively. If (ϕ_y, ϕ_z) and (ϕ'_y, ϕ'_z) are symmetric with respect to the point $(\pi/4, \pi/4)$ or the point $(\pi/4, 3\pi/4)$, then K and K' are equivalent.*

Proof: Suppose (ϕ_y, ϕ_z) and (ϕ'_y, ϕ'_z) are symmetric with respect to the point $(\pi/4, \pi/4)$. Then $\phi'_y = \pi/2 - \phi_y$ and $\phi'_z = \pi/2 - \phi_z$. Thus

$$\begin{aligned} &K'(-t+\pi/2) \\ &= \big(\cos(-2t+\pi), \cos(-n_y t + n_y\pi/2 + \pi/2 - \phi_y), \\ &\quad \cos(-n_z t + n_z\pi/2 + \pi/2 - \phi_z)\big) \\ &= \big(-\cos(2t), (-1)^{(n_y+1)/2}\cos(n_y t + \phi_y), \\ &\quad (-1)^{(n_z+1)/2}\cos(n_z t + \phi_z)\big), \end{aligned}$$

which is either $K(t)$ or its mirror image $\overline{K}(t)$.

The second case follows similarly. □

We may now prove Theorem 1.1.

Proof of Theorem 1.1.: The fundamental domain is divided into n_x "boxes" of the form $[0, \frac{\pi}{n_x}] \times [k\frac{\pi}{n_x}, (k+1)\frac{\pi}{n_x}]$ for $0 \le k \le n_x - 1$. Within each box all the knots have the same Type-II crossings, and hence by Corollary 2.4, there are at most $2n_y$ different knot types in that box. Since there are n_x boxes we obtain at most $2n_x n_y$ different knots.

If $n_x = 2$, there is the additional rotational symmetry in each box given by Proposition 2.5. The center of each box lies either on a slanted singular line or midway between two such lines. Moreover, one of the two boxes will satisfy the first condition and the other box will satisfy the other. There are at most n_y knot types in the box where the center of the box lies on a singular line, and there are at most $n_y + 1$ knot types in the box otherwise. Thus there are at most $2n_y + 1$ knot types altogether. □

Our results thus far allow us to efficiently sample all Lissajous knots with a given set of frequencies (n_x, n_y, n_z). We can easily pick one set of phase shifts from each region on the phase torus, and Corollary 2.4, and Proposition 2.5 in the case $n_x = 2$, allow us to further restrict the regions that we must sample. However, once n_x, n_y, and ϕ_y are given, the xy-projection has been fixed, and it is natural to ask whether all possible choices for n_z are necessary. Theorem 1.2, which is stated in the introduction, shows that in fact, only a finite number of values for n_z are needed to produce all possible knots.

Proof of Theorem 1.2.: Suppose that $K \in \mathcal{L}(n_x, n_y, n_z)$, $K' \in \mathcal{L}(n_x, n_y, n_z + 2n_x n_y)$, and that both knots have the same phase shifts. We will show first that each knot has its Type-II crossings arranged the same way.

Let (t_1, t_2) be a Type-II crossing with parameters (k, j) and let

$$\begin{aligned} &\Delta_{\mathrm{II}}(n_x, n_y, n_z, \phi_y, \phi_z, k, j) \\ &\quad = \cos(n_z t_1 + \phi_z) - \cos(n_z t_1 + \phi_z) \\ &\quad = 2\sin\left(n_z\left(\frac{t_1+t_2}{2}\right) + \phi_z\right)\sin\left(n_z\left(\frac{t_1-t_2}{2}\right)\right) \\ &\quad = -2\sin\left(n_z\frac{j\pi}{n_x} + \phi_z\right)\sin\left(n_z\frac{k\pi}{n_y}\right) \end{aligned}$$

be the height difference between the two points on the knot directly above the crossing.

It is easy to verify that

$$\begin{aligned} &\Delta_{\mathrm{II}}(n_x, n_y, n_z, \phi_y, \phi_z, k, j) \\ &\quad = \Delta_{\mathrm{II}}(n_x, n_y, n_z + 2n_x n_y, \phi_y, \phi_z, k, j) \end{aligned}$$

for all k, j. Thus if n_z is increased by $2n_x n_y$, not only do all Type-II crossings remain unchanged, they each maintain the same height difference between the upper and lower strands.

We now shift our focus to Type-I crossings. Let K have phase shifts $(\phi_y, \frac{n_z}{n_y}\phi_y - \varepsilon)$ and choose ε small enough that K corresponds to the region just below the singular line $\phi_z = \frac{n_z}{n_y}\phi_y$. Let K' correspond to the "same" region, that is, let K' have phase shifts $(\phi_y, \frac{n_z+2n_x n_y}{n_y}\phi_y - \varepsilon)$. As before, let (t_1, t_2) be a Type-I crossing with parameters (k, j) and let

$$\begin{aligned} &\Delta_{\mathrm{I}}(n_x, n_y, n_z, \phi_y, \phi_z, k, j) \\ &\quad = \cos(n_z t_1 + \phi_z) - \cos(n_z t_1 + \phi_z) \\ &\quad = 2\sin\left(n_z\left(\frac{t_1+t_2}{2}\right) + \phi_z\right)\sin\left(n_z\left(\frac{t_1-t_2}{2}\right)\right) \\ &\quad = -2\sin\left(n_z\frac{j\pi}{n_y} - \frac{n_z\phi_y}{n_y} + \phi_z\right)\sin\left(n_z\frac{k\pi}{n_x}\right) \end{aligned}$$

be the height difference between the two points on the knot directly above the crossing. It is easy to check that

$$\begin{aligned} &\Delta\left(n_x, n_y, n_z, \phi_y, \frac{n_z}{n_y}\phi_y - \varepsilon, k, j\right) \\ &\quad = \Delta\left(n_x, n_y, n_z + 2n_x n_y, \phi_y, \frac{n_z + 2n_x n_y}{n_y}\phi_y - \varepsilon, k, j\right). \end{aligned}$$

Thus K and K' are the same knot, since both the Type-I and Type-II crossings are arranged the same way in each.

If the phase shifts for K are now changed by moving into an adjacent region, and if the phase shifts for K' are changed in the same way, then the same set of crossings is changed for both K and K', and hence K and K' remain the same knot. Therefore

$$\mathcal{L}(n_x, n_y, n_z) \subseteq \mathcal{L}(n_x, n_y, n_z + 2n_x n_y). \qquad (2\text{–}7)$$

According to Corollary 2.4, the pattern of knot types in each square $[0, \pi/n_x] \times [k\pi/n_x, (k+1)\pi/n_x]$, as we move from the upper left corner to the lower right corner, is periodic with period $2n_y$. Thus if each box contains at least $2n_y$ regions, the inclusion in (2–7) is equality. Now the distance between successive singular lines of the type given in (2–1) is $\frac{\pi}{\sqrt{n_y^2+n_z^2}}$, and the distance between lines of slope $\frac{n_z}{n_y}$ containing opposite corners of the square is $\frac{(n_2+n_3)\pi}{n_1\sqrt{n_y^2+n_z^2}}$. Thus there are at least $\left\lfloor \frac{n_y+n_z}{n_x} \right\rfloor$ regions in each square. Hence the inclusion in (2–7) is equality if $2n_y \leq \left\lfloor \frac{n_y+n_z}{n_x} \right\rfloor$. It is easy to check that this is true if $n_z \geq 2n_x n_y - n_y$. □

Figure 3 illustrates the periodicity in n_z described in Theorem 1.2. The phase tori for $\mathcal{L}(2, 5, 9)$, $\mathcal{L}(2, 5, 29)$, and $\mathcal{L}(2, 5, 49)$ are shown. Each region of each phase torus defines a 2-bridge knot, color-coded to its defining fraction. White regions define unknots. Notice that $\mathcal{L}(2, 5, 29)$ contains one more knot type than $\mathcal{L}(2, 5, 9)$.

Because of Theorem 1.2, it makes sense to let $\mathcal{L}(n_x, n_y) = \bigcup \mathcal{L}(n_x, n_y, n_z)$, where n_x and n_y are relatively prime integers and the union is taken over all n_z prime to both n_x and n_y.

Theorem 2.6. *Let n_x, n_y be relatively prime integers. Then*

$$|\mathcal{L}(n_x, n_y)| \leq 4n_x n_y (n_x - 1)(n_y - 1).$$

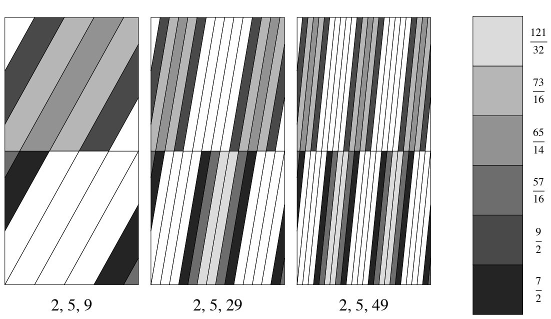

FIGURE 3. Fundamental domains of the phase tori for $\mathcal{L}(2, 5, 9)$, $\mathcal{L}(2, 5, 29)$, and $\mathcal{L}(2, 5, 49)$. White regions define unknots. Shaded regions define corresponding 2-bridge knots.

If furthermore $n_x = 2$, then

$$|\mathcal{L}(2, n_y)| \leq 2(n_y - 1)(2n_y + 1).$$

Proof: For fixed n_x, n_y we need only consider $2n_x n_y$ consecutive values of n_z. To count the number of values that are relatively prime to both n_x and n_y we first subtract $2n_y$ multiples of n_x that lie in that range as well as $2n_x$ multiples of n_y and then add back in the two multiples of $n_x n_y$. Thus the number of possible values of n_z is bounded above by $2n_x n_y - 2n_x - 2n_y + 2 = 2(n_x - 1)(n_y - 1)$, and applying Theorem 1.1 yields the result. □

3. THE PHASE TORUS—FOURIER-$(1, 1, 2)$ KNOTS

The phase torus of a Fourier-$(1, 1, k)$ is, in general, $(k+2)$-dimensional, although we may set any one phase shift equal to zero and drop to a $(k+1)$-dimensional space. If $k = 2$, $\phi_x = 0$, and we fix ϕ_y, then we may again think of the two-dimensional phase torus associated with the pair $(\phi_{z,1}, \phi_{z,2})$. The singular curves are now much more complicated than in the Lissajous case, but can still be carefully described.

Suppose K is a Fourier-$(1, 1, 2)$ knot with parameterization

$$\begin{aligned} x(t) &= \cos(n_x t), \\ y(t) &= \cos(n_y t + \phi_y), \\ z(t) &= \cos(n_{z,1} t + \phi_{z,1}) + A_{z,2} \cos(n_{z,2} t + \phi_{z,2}). \end{aligned} \qquad (3\text{–}1)$$

Note that by rescaling we may assume that three of the four amplitudes are 1.

In the Lissajous case, we require that the three frequencies be pairwise relatively prime. The same proof (see [Bogle et al. 94]) can be used now to conclude that the three integers n_x, n_y, and $\gcd(n_{z,1}, n_{z,2})$ must be pairwise relatively prime. This rules out several of the 16 cases that arise by considering all possible parities for the frequencies. Some of the remaining cases still give rise to highly symmetric knots, such as when all the frequencies are odd. In this case the knot is strongly plus amphicheiral just as in the Lissajous setting. But some of the parity cases produce knots with no apparent symmetry, suggesting that the set of Fourier-$(1, 1, 2)$ knots is much richer than the set of Lissajous knots.

We will not undertake an exhaustive analysis of the phase torus of Fourier-$(1, 1, 2)$ knots. Instead we offer a glimpse of the situation in the following proposition, which could be stated much more precisely. In particular, the constants in the statement of the proposition all depend on the pair of indices (k, j) associated with either a Type-I or Type-II crossing. The interested reader can easily determine the constants by going through the details of the proof. Results analogous to Propositions 2.3 and 2.5 seem much harder.

Proposition 3.1. *Let K be a Fourier-$(1,1,2)$ knot with parameterization as given in (3–1). Then the singular curves on the phase torus are of four possible types:*

1. *lines of the form $\phi_{z,2} = c$,*
2. *lines of the form $\phi_{z,1} = c$,*
3. *lines of the form $\phi_{z,2} = \pm\phi_{z,1} + c$,*
4. *curves with the shape of $\sin(\phi_{z,2}) = c\sin(\phi_{z,1})$,*

where c is a constant that in the last case is neither 0 nor ± 1.

Proof: Suppose that (t_1, t_2) is a pair of times that produce a double point in the xy-projection of K. Using the identity $\cos x - \cos y = -2\sin(\frac{x+y}{2})\sin(\frac{x-y}{2})$, we obtain

$$\begin{aligned} &z(t_1) - z(t_2) \\ &\quad = -2\sin\left(n_{z,1}\frac{t_1+t_2}{2} + \phi_{z,1}\right)\sin\left(n_{z,1}\frac{t_1-t_2}{2}\right) \\ &\qquad - 2A\sin\left(n_{z,2}\frac{t_1+t_2}{2} + \phi_{z,2}\right)\sin\left(n_{z,2}\frac{t_1-t_2}{2}\right). \end{aligned}$$

We are interested in those values of $\phi_{z,1}$ and $\phi_{z,2}$ that make this difference zero.

Suppose now that (t_1, t_2) define a Type-II crossing with indices (k, j). Then

$$\frac{t_1+t_2}{2} = \frac{j\pi}{n_x}, \quad \frac{t_1-t_2}{2} = -\frac{k\pi}{n_y},$$

and the crossing is singular if

$$\begin{aligned} &\sin\left(\frac{n_{z,1}j\pi}{n_x} + \phi_{z,1}\right)\sin\left(\frac{n_{z,1}k\pi}{n_y}\right) \\ &\quad = -A\sin\left(\frac{n_{z,2}j\pi}{n_x} + \phi_{z,2}\right)\sin\left(\frac{n_{z,2}k\pi}{n_y}\right). \end{aligned} \tag{3–2}$$

We are now led to several cases.

Case I: $n_y \mid n_{z,1}k$.

If n_y divides $n_{z,1}$, then we must have that $\sin\left(\frac{n_{z,2}j\pi}{n_x} + \phi_{z,2}\right) = 0$, since $k < n_y$, and $n_y, n_{z,1}$ and $n_{z,2}$ cannot have a common factor. This means that

$$\phi_{z,2} = m\pi - \frac{n_{z,2}j\pi}{n_x}$$

for some integer m.

Case II: $n_y \mid n_{z,2}k$.

This is similar to Case I, leading to

$$\phi_{z,1} = m\pi - \frac{n_{z,1}j\pi}{n_x}$$

for some integer m.

If the first two cases do not occur, then we may rewrite (3–2) as

$$\sin\left(\frac{n_{z,1}j\pi}{n_x} + \phi_{z,1}\right) = C\sin\left(\frac{n_{z,2}j\pi}{n_x} + \phi_{z,2}\right),$$

where

$$C = -A\sin\left(\frac{n_{z,2}k\pi}{n_y}\right) / \sin\left(\frac{n_{z,1}k\pi}{n_y}\right).$$

Case III: $|C| = 1$.

In this case we must have

$$\left(\frac{n_{z,1}j\pi}{n_x} + \phi_{z,1}\right) \pm \left(\frac{n_{z,2}j\pi}{n_x} + \phi_{z,2}\right) = m\pi \tag{3–3}$$

for some integer m, where the parity of m depends on the sign of C and whether we are forming a sum or difference in (3–3). Thus $\phi_{z,2} = \pm\phi_{z,1} + c$ for some constant c.

Case IV: $|C| \neq 1$.

In this case we are left with a translate of the curve

$$\sin(\phi_{z,1}) = C\sin(\phi_{z,2}).$$

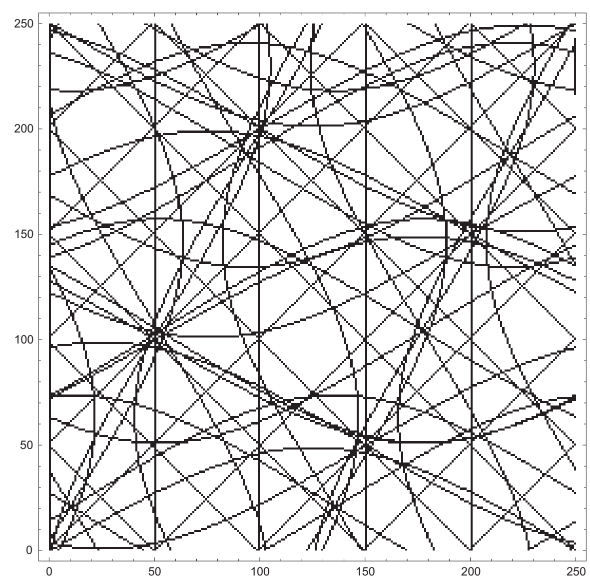

FIGURE 4. The phase torus for the Fourier-$(1,1,2)$ knot with $n_x = 5, n_y = 6, n_{z,1} = 1, n_{z,2} = 2, \phi_x = 0, \phi_y = \pi/4$ and $A_{z,1} = 1$, shown for $0 \leq \phi_{z,1} \leq \pi$ and $0 \leq \phi_{z,2} \leq \pi$.

This is an interesting curve, which at first glance appears much like a sine curve. It is oriented either vertically or horizontally depending on the value of $|C|$.

The analysis of a Type-I crossing is similar and is left to the reader. □

In Figure 4 we give an example showing a 250×250 pixel bitmap image of the phase torus for a specific set of parameters. Even with relatively small frequencies, one can begin to appreciate the difficulty of systematically sampling each region of the phase torus for an arbitrary Fourier-$(1, 1, 2)$ knot.

4. 2-BRIDGE KNOTS

Every 2-bridge knot can be classified by a pair of relatively prime integers (p, q) such that p is odd and $0 < q < p$. We will often write the pair (p, q) as the fraction p/q. If $K_{p/q}$ and $K_{p'/q'}$ are two 2-bridge knots with corresponding fractions p/q and p'/q', then they are equivalent knots if and only if $p = p'$ and $\pm q'q^{\pm 1} \equiv 1 \bmod p$. The reader is referred to [Burde and Zieschang 03] for details.

If K is a Fourier-$(1, 1, 2)$ knot with $n_x = 2$, then K is a 2-bridge knot. We may recover the fraction p/q from the Lissajous projection in the xy-plane as follows. This projection is a 4-plat diagram. As we move in the x-direction from left to right we see a single Type-I crossing on the x-axis, then a pair of Type-II crossings that are symmetric with respect to the x-axis, then another Type-I crossing on the x-axis, and so on. Let $\eta_1, \eta_2, \ldots$ be the signs of the Type-I crossings from left to right along the x-axis. Let $\{\varepsilon_1^1, \varepsilon_1^2\}, \{\varepsilon_2^1, \varepsilon_2^2\}, \ldots$ be the signs of the pairs of Type-II crossings from left to right. Proceeding in a fashion similar to that given in [Rolfsen 90, pp. 300–303], we obtain that p/q is given by the continued fraction

$$\begin{aligned} p/q &= [\eta_1, \varepsilon_1^1 + \varepsilon_1^2, \eta_2, \varepsilon_2^1 + \varepsilon_2^2, \ldots, \eta_{n_y}] \qquad (4\text{–}1) \\ &= \eta_1 + \cfrac{1}{\varepsilon_1^1 + \varepsilon_1^2 + \cfrac{1}{\eta_2 + \cdots + \cfrac{1}{\eta_{n_y}}}} \end{aligned}$$

Note that if K is Lissajous, then it is rotationally symmetric with respect to the x-axis and each pair of Type-II crossings has the same sign. In this case each $\varepsilon_i^1 + \varepsilon_i^2$ can be replaced with $2\varepsilon_i^1$. Using this formula, it is easy to determine the 2-bridge knot given by a Fourier-$(1, 1, 2)$ representation with $n_x = 2$. Hence, when we sample Lissajous and Fourier-$(1, 1, 2)$ knots with $n_x = 2$, even if we obtain knots with hundreds of crossings, it is a simple matter to distinguish them.

Since every Lissajous knot with $n_x = 2$ is 2-bridge, a good question is this: What 2-bridge knots are Lissajous with $n_x = 2$? As mentioned in the introduction, every Lissajous knot is either strongly plus amphicheiral, or 2-periodic and linking its axis of rotation once. It is known that a 2-bridge knot cannot be strongly plus amphicheiral [Hartley and Kawauchi 79]. It is also known (and will be shown below) that every 2-bridge knot is 2-periodic, but may or may not link its axis of rotation once. The following theorem makes it easy to identify which 2-bridge knots might be Lissajous.

Theorem 4.1. *Let K be a 2-bridge knot. Then the following are equivalent:*

1. *K has a symmetry of period 2 with axis A such that A is disjoint from K and $|\operatorname{lk}(A, K)| = 1$*
2. *$\Delta_K(t)$ is a square modulo 2.*
3. *$\Delta_K(t) \equiv 1 \bmod 2$.*

Proof: As already mentioned in the introduction, it follows from a result of Murasugi (see [Murasugi 71]) that statement 1 implies statement 2, and clearly statement 3 implies statement 2.

Before proving that statement 2 implies both 1 and 3, we make some preparatory remarks.

Suppose K is a 2-bridge knot given by the pair of relatively prime integers (p, q) with p odd and $0 < q < p$. There is a unique continued fraction expansion of the form

$$p/q = [2a_1, -2a_2, \ldots, (-1)^{n+1} 2a_n].$$

Corresponding to this expansion is a Seifert surface made from plumbing together twisted bands as shown in Figure 5. Notice that the Seifert surface, and hence K, is rotationally symmetric around the axis A. Thus every 2-bridge knot has a symmetry of period 2 with axis disjoint from the knot. However, the linking number of A and K need not be ± 1 in general. From the plumbing picture we also see that the number of bands, n, must be even in order to get a knot. If n is odd, we obtain a 2-bridge link (of two components).

The axis A meets the Seifert surface S transversely in $n + 1$ points, and we may compute the linking number of A and K by counting the signed intersection points. Let ϵ_i be the sign of the intersection point that occurs between band i and band $i + 1$. Let ϵ_0 be the sign of the leftmost intersection point in the figure and choose orientations so that $\epsilon_0 = 1$. It is easy to see that $\epsilon_{i+1} = \epsilon_i$

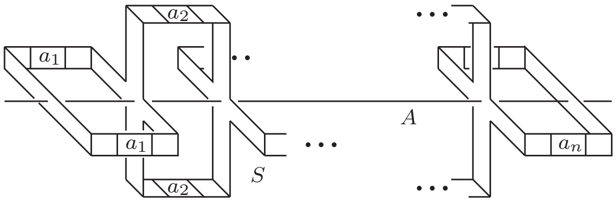

FIGURE 5. The Seifert surface S and axis A for the 2-bridge knot $K_{p/q}$. Each a_i represents a_i right-handed half-twists in the band.

if a_i is odd and $\epsilon_{i+1} = -\epsilon_i$ if a_i is even. Thus the sequence $\{a_1, a_2, \ldots, a_n\}$ determines the sequence $\{\epsilon_0, \epsilon_1, \ldots, \epsilon_n\}$, which in turn determines the linking number $\Sigma\epsilon_i$ between A and K.

From the Seifert surface we may obtain the Seifert matrix V and compute the Conway polynomial $\nabla(z) = \det(t^{-1/2}V - t^{1/2}V^T)$, where $z = t^{1/2} - t^{-1/2}$. It is a straightforward calculation to show that

$$\nabla_K(z) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -a_1 z & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -a_2 z & 1 \\ 1 & 0 \end{pmatrix} \cdots \times \begin{pmatrix} -a_n z & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(See [Cromwell 04, p. 207].)

Finally, we leave as an exercise for the reader the fact that an Alexander polynomial of the form

$$\Delta(t) = b_0 + b_1(t + t^{-1}) + b_2(t^2 + t^{-2}) + \cdots + b_m(t^m + t^{-m})$$

is a square modulo 2 if and only if $b_{2k+1} \equiv 0 \bmod 2$ for all k.

We now show that statement 2 implies both statement 1 and statement 3 by induction on n.

If $n = 2$, the Conway polynomial is

$$\nabla(z) = 1 + a_1 a_2 z^2$$

and the Alexander polynomial is

$$\Delta(t) = \nabla(t^{1/2} - t^{1/2}) = a_1 a_2 t^{-1} + (1 - 2a_1 a_2) + a_1 a_2 t.$$

Assuming that $\Delta(t)$ is a square modulo 2 implies that at least one of a_1 and a_2 is even. It now follows that $\sum \epsilon_i = \pm 1$ and that $\Delta(t) \equiv 1 \bmod 2$.

Suppose now that $n > 2$ and that K is a knot with $\Delta_K(t)$ a square modulo 2. We first show that a_i is even for at least one value of i. If a_i were odd for every i, then replacing each a_i with -1 would not change any a_i modulo 2, and hence would not change $\nabla_K(z)$ modulo 2 or $\Delta_K(t)$ modulo 2. But if $a_i = -1$ for all i, it is not difficult to prove (by induction on n) that

$$\Delta(t) = 1 - t + t^2 - t^3 + \cdots + t^n,$$

and it follows that $\Delta(t)$ is not a square modulo 2. Thus at least one a_i is even. Replacing this a_i with zero transforms K into a knot J with the same Alexander polynomial modulo 2 but with a Seifert surface having two fewer bands. Our inductive hypothesis now gives that $\mathrm{lk}(J, A) = \pm 1$ and that $\Delta_J(t) \equiv 1 \bmod 2$. Thus $\Delta_K(t) \equiv 1 \bmod 2$, and because a_i is even, K must also link A once. □

5. SAMPLING LISSAJOUS AND FOURIER KNOTS

Using the results of Section 2 we are now in a position to efficiently sample Lissajous knots. In the case $n_x = 2$, we obtain 2-bridge knots and can take advantage of this to compare knots in our sample. For the more general case of Fourier knots, we have not carried out a complete analysis of the phase torus, a task that seems much more difficult. Hence, we have not attempted to sample Fourier knots rigorously. Instead, we have relied on two methods: random sampling and an algorithm that first "draws" a bitmap image of the phase torus (as in Figure 4) and then picks one point from each "white" region. This latter approach is fraught with difficulty, since, for example, some white regions may be smaller than a single pixel and be missed. Our samples naturally fall into four cases, which we describe in turn in this section.

Finally, in Tables 6 and 7 we summarize our data for all prime knots to nine crossings. As mentioned earlier, these summary data were culled from a much larger set of tables that appeared in the original version of this paper [Boocher et al. 07]. In this section we will sometimes refer to the tables in that work.

5.1 Lissajous Knots with $2 = n_x < n_y < n_z$

We have determined all knots in $\mathcal{L}(2, n_y)$ for $3 \le n_y \le 105$. For a given value of n_y we let n_z run from $3n_y + 2$ to $7n_y$. These values of n_z are sufficient to guarantee that we obtain all possible knots in $\mathcal{L}(2, n_y)$. Since each of these knots is 2-bridge, we were able to use (4–1) to identify the associated pair (p, q) and thus compare knots in the output.

The total number of knots in $\mathcal{L}(2, n_y)$ is given in Table 1 for each value of n_y. It is interesting to compare these numbers with the upper bound given by Theorem 2.6. Depending on n_y, the actual number of knots found is roughly between five and ten percent of the upper bound. The discrepancy is almost certainly due to the presence of huge numbers of unknots. The xy-projection of a Lissajous knot with $n_x = 2$ and $n_y = 99$ has $(2)(2)(99) - 2 - 99 = 295$ crossings, and knots in

$\mathcal{L}(2, 99)$ have crossing numbers ranging from 5 to 293. Of course, the bound of 78,008 given by Theorem 2.6 for $n_x = 2$ and $n_y = 99$ is well below the upper bound of 2^{295} obtained by considering all possible crossing arrangements!

The total number of knots in Table 1 is 135,061, far too many to describe one by one. In Table 2 we list the knots in $\mathcal{L}(2, 3)$, $\mathcal{L}(2, 5)$, $\mathcal{L}(2, 7)$, and $\mathcal{L}(2, 9)$. In [Boocher et al. 07, Tables 4–7] we list all knots in $\mathcal{L}(2, n_y)$, grouped by crossing number, for $3 \leq n_y \leq 15$.

Several interesting things can be seen in these tables. The same knot often appears in $\mathcal{L}(2, n_y)$ for many different values of n_y. For example, $K_{7/2}$ (which is the twist knot 5_2 in [Rolfsen 90]) is contained in $\mathcal{L}(2, n_y)$ for $3 \leq n_y \leq 105$. This is also true for $K_{9/2}$. The knot $K_{15/4}$ first appears for $n_y = 3$, misses a few values of n_y, and then is contained in $\mathcal{L}(2, n_y)$ for $23 \leq n_y \leq 105$.

Similar patterns hold for the other small-crossing knots, suggesting that if $K \in \mathcal{L}(2, n_y)$ for some n_y then there exists N such that $K \in \mathcal{L}(2, n_y)$ for all $n_y \geq N$.

A second observation is that several small-crossing knots are already conspicuously absent. In particular, there are exactly four 8-crossing knots with Alexander polynomial congruent to 1 modulo 2 (and hence possibly Lissajous). These are $K_{17/4}$, $K_{23/7}$, $K_{25/9}$, and $K_{31/12}$, only one of which, $K_{31/12}$, appears to be Lissajous. While [Boocher et al. 07, Tables 4–7] display only a small fraction of our total sample, it is in fact true that the other three 8-crossing knots do not appear for any n_y up to 105.

Question 5.1. Does there exist a 2-bridge knot K with $\Delta_K(t) \equiv 1 \bmod 2$ that is not Lissajous (with or without one frequency equal to 2)? In particular, are any of the 8-crossing 2-bridge knots $K_{17/4}$, $K_{23/7}$, $K_{25/9}$, or any of the 9-crossing 2-bridge knots $K_{23/4}$, $K_{33/10}$, $K_{39/16}$, $K_{41/12}$, $K_{41/16}$ Lissajous?

In Table 3 we list the numbers of 2-bridge knots, 2-bridge knots with Alexander polynomial congruent to 1 mod 2, and finally, the number of these that are Lissajous knots with $n_x = 2$ and $3 \leq n_y \leq 105$. The table has entries for each crossing number from 3 to 16. Very quickly we see that many 2-bridge knots with the required symmetry are not Lissajous, at least not with $n_x = 2$ and $3 \leq n_y \leq 105$.

It seems unlikely that choosing $n_y > 105$ will yield more 2-bridge knots in the 3-to-16 crossing range. On the other hand, perhaps letting the even frequency be more than 2 will yield more 2-bridge knots with small crossing number. We examine this further in Section 5.2.

In [Boocher et al. 07, Tables 8 and 9] we list all 2-bridge knots with crossings from 3 to 16 that are Lissajous knots with $n_x = 2$ and $3 \leq n_y \leq 105$. For each knot, the given value of n_y is minimal. However, since our search let n_z run from $3n_y + 2$ to $7n_y$, it might be possible for a given knot to be represented with a smaller value of n_z. Data from [Boocher et al. 07, Tables 8 and 9], for knots up to nine crossings, appear in Tables 6 and 7.

As a check against errors, we took all the 2-bridge knots in the data set that have Lissajous diagrams with fewer than 50 crossings (the built-in limit for Knotscape) and crossing number less than 17, and looked them up in the Knotscape table of knots in two different ways. First we converted their Lissajous diagrams to Dowker–Thistlethwaite code (the input format for Knotscape) and then used the "Locate in Table" feature. Next we converted the defining fraction p/q into DT code and again used the "Locate in Table" routine. Happily, the results matched.

5.2 Lissajous Knots with $2 < n_x < n_y < n_z$

Our goal in this section is simply to find as many Lissajous knots in the 3-to-16 crossing range as we can. We may still use the results of Section 2 to efficiently sample Lissajous knots with all frequencies greater than 2, but it is more difficult to tabulate the output. This is because even with relatively small frequencies, knots with very large crossing number can result, and we can no longer use the classification of 2-bridge knots to sort them out. Therefore, we limited ourselves to producing diagrams with at most 49 crossings, the limit of what can be input to Knotscape. Assuming that $2 < n_x < n_y$, and that $\gcd(n_x, n_y) = 1$, we are left with the following (n_x, n_y) pairs:

$$\{(3,4),(3,5),(3,7),(3,8),(3,10),(4,5),(4,7),(5,6)\}.$$

For each of these pairs we let n_z run from $2n_xn_y - n_x - n_y$ to $4n_xn_y - n_x - n_y - 1$, a range sufficient to produce all possible Lissajous knots. We obtained a total of 6352 knots, 1428 of which Knotscape identified as unknots. The remaining 4924 knots fell into four categories:

1. knots identified as composites by Knotscape,
2. knots that Knotscape located in the Hoste–Thistlethwaite–Weeks table,
3. knots that Knotscape simplified to alternating projections with more than 16 crossings, and
4. knots that Knotscape simplified to nonalternating projections with more than 16 crossings.

n_y	$\lvert\mathcal{L}(2,n_y)\rvert$	n_y	$\lvert\mathcal{L}(2,n_y)\rvert$	n_y	$\lvert\mathcal{L}(2,n_y)\rvert$	n_y	$\lvert\mathcal{L}(2,n_y)\rvert$
3	3	29	645	55	1854	81	3761
5	11	31	737	57	1727	83	5805
7	28	33	533	59	2859	85	4654
9	37	35	684	61	3062	87	4195
11	78	37	1075	63	1946	89	6707
13	109	39	772	65	2639	91	5647
15	93	41	1339	67	3708	93	4805
17	203	43	1473	69	2593	95	5892
19	258	45	904	71	4191	97	7984
21	195	47	1782	73	4433	99	5208
23	390	49	1688	75	2584	101	8699
25	390	51	1365	77	3933	103	9036
27	387	53	2287	79	5248	105	4425

TABLE 1. The number of distinct Lissajous knots with $n_x = 2$ as a function of n_y.

Lissajous set	2-bridge knots
$\mathcal{L}(2,3)$	7/2; 9/2; 15/4
$\mathcal{L}(2,5)$	7/2; 9/2; 17/5; ; ; 17/2, 49/20, 57/16; 65/14, 73/16, 97/26; 121/32; 209/56
$\mathcal{L}(2,7)$	7/2; 9/2; 15/4, 17/5; ; 15/2, 31/7; 17/2, 49/20, 55/12, 57/16; 73/16; 169/50; 239/71; 25/2, 89/36, 289/118; 151/20, 319/144, 359/82, 463/130; 529/114, 593/130, 777/208; 975/274, 983/260, 1351/362; 1681/450; 2911/780
$\mathcal{L}(2,9)$	7/2; 9/2; 17/5; ; 31/7; 17/2, 57/16; 49/9, 65/14, 73/16; 167/46; ; 25/2, 89/36, 289/118, 409/121; 441/101, 463/130; 593/130; ; 33/2, 129/52, 529/214, 1681/696, 2321/622; 273/32, 673/78, 1961/800, 2001/898, 3329/989; 3761/1056; 4297/926, 4305/944, 4817/1056; 7921/2224, 7985/2112, 10865/2912; 18817/5042; 23409/6272; 40545/10864

TABLE 2. The sets $L(2, n_y)$ for $3 \leq n_y \leq 9$ given by 2-bridge fraction p/q. Within each set, knots are ordered by crossing number with each semicolon indicating a crossing number increase of 1.

	crossing number													
	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2-bridge	1	1	2	3	7	12	24	45	91	176	352	693	1387	2752
$\Delta(t) \equiv 1$	0	0	1	1	2	4	8	13	26	51	97	185	365	705
$\mathcal{L}(2,n_y)$	0	0	1	1	2	1	3	4	8	5	9	7	15	15

TABLE 3. The number of 2-bridge knots, 2-bridge knots with Alexander polynomial congruent to 1 modulo 2, and the number of these that are Lissajous with $n_x = 2$ and $3 \leq n_y \leq 105$, as a function of crossing number.

In Table 4 we list all knots in the first category. In Tables 6 and 7 we list all knots in the second category up to 9 crossings, and in [Boocher et al. 07, Tables 10 and 11] we continue up to 16 crossings. We note that while Knotscape can identify a knot as a composite, it identifies the summands only up to mirror image. In order to properly identify the composites in Table 4, we compared their Jones polynomials to the Jones polynomials of all possible composites using the given summands or their mirror images in all possible ways.

The third category cannot include knots in the Hoste–Thistlethwaite–Weeks table, and we make no attempt to list them here. The fourth category *might* include knots with 16 or fewer crossings that Knotscape simply failed to simplify correctly. To investigate this we first computed the Jones polynomial of each knot and eliminated knots whose Jones polynomial had a span of 17 or more. (Recall that the crossing number of a knot is bounded below by the span of the Jones polynomial.) This left a total of 78 knots. Of these, only 5 shared the same Jones polynomial with prime knots having fewer than 17 crossings and furthermore having an Alexander polynomial that is a square modulo 2. In each of these five cases either the Alexander polynomial or the Kauffman 2-variable polynomial was sufficient to show that the knots did indeed have crossing numbers of 17 or more.

Thus, barring clerical errors, Table 4 and [Boocher et al. 07, Tables 10 and 11] provide a complete list of all

knot	n_x	n_y	n_z	ϕ_x	ϕ_y	ϕ_z
$3a1\#3a1$	3	4	23	0	0.25210	1.84229
$3a1\#\overline{3a1}$	3	5	29	0	0.23099	2.91059
$5a1\#5a1$	3	7	50	0	0.50522	1.58916
$5a1\#\overline{5a1}$	3	5	29	0	0.26179	1.83259
$6a1\#6a1$	4	5	37	0	0.18699	2.95459
$6a3\#6a3$	3	8	47	0	0.23799	0.80919
$6a3\#\overline{6a3}$	3	5	29	0	0.29259	0.75459
$3a1\#3a1\#5a1$	4	5	39	0	0.16064	2.19554
$3a1\#3a1\#\overline{5a1}$	4	7	55	0	0.13934	2.21684
$3a1\#3a1\#8a2$	5	6	59	0	0.11116	2.40211
$5a1\#5a1\#\overline{5a1}$	4	7	55	0	0.15201	1.41878
$6a3\#6a3\#\overline{6a3}$	4	7	55	0	0.16468	0.62071
$3a1\#3a1\#3a1\#3a1$	5	6	59	0	0.10149	1.78345

TABLE 4. Small-crossing composite Lissajous knots. A bar over a knot name indicates mirror image. Knot names are as in Knotscape.

	crossing number											
(n_x, n_y)	5	6	7	8	9	10	11	12	13	14	15	16
$(3,4)$	**1**	**1**		2		2		1	2	2		1
$(3,5)$					1	1		3		4		5
$(3,7)$			**1**			2				7		8
$(3,8)$									1	4	1	
$(3,10)$								2	1			
$(4,5)$							1	2	1	2	2	2
$(4,7)$							1			1	1	
$(5,6)$								1			1	

TABLE 5. Number of prime Lissajous knots with given x and y frequencies through 16 crossings. Only three of these, 5a1, 6a3, and 7a6 (which correspond to bold entries) are in fact 2-bridge knots.

Lissajous knots with x and y frequencies of $(3,4)$, $(3,5)$, $(3,7)$, $(3,8)$, $(3,10)$, $(4,5)$, $(4,7)$, or $(5,6)$ that are either composite, or prime with 16 or fewer crossings. Again, Tables 6 and 7 of this paper contain data only to nine crossings. In Table 5 we list the number of prime knots in this set by crossing number.

As mentioned in the introduction, there are exactly three prime knots with 12 or fewer crossings that are strongly plus amphicheiral: 10a103 (10_{99}), 10a121 (10_{123}), and 12a427. The knots 10a103 and 12a427 are Lissajous and are listed in [Boocher et al. 07, Table 10]. A natural question is the following.

Question 5.2. Is the strongly plus amphicheiral knot 10a121 Lissajous?

The knot 10a121 is one member of a family of knots known as *Turk's head* knots. These knots are conjectured not to be Lissajous by Przytycki. See [Przytycki 98].

It is easy to see that every composite knot of the form $K\#\overline{K}$ is strongly plus amphicheiral, while composites of the form $K\#K$ are 2-periodic and link their axis of rotation once. Several knots of this form appear in Table 4. Thus another good question is the following.

Question 5.3. Is every composite knot of the form $K\#K$ or $K\#\overline{K}$ Lissajous?

5.3 Fourier-$(1,1,2)$ Knots with $2 = n_x < n_y$

Rather than trying to choose one point in each region of the phase torus for a Fourier-$(1,1,2)$ knot algorithmically, we chose instead to sample points from the phase torus randomly. Fixing $n_x = 2$, $\phi_x = 0$, and $A_{z,1} = 1$, we then let n_y take on odd values from 3 to 99. For each value of n_y the remaining parameters were then chosen at random such that

$$\phi_y = \frac{k}{7}\pi, \quad k \in \{1,2,3,4,5,6\},$$
$$0 < n_{z,1} < n_{z,2} < 301,$$
$$0 \le \phi_{z,1} \le \pi,$$
$$0 \le \phi_{z,2} \le 2\pi,$$
$$0 \le A_{z,2} \le 2.$$

knot	p/q	n_x	n_y	$n_{z,1}$	$n_{z,2}$	ϕ_x	ϕ_y	$\phi_{z,1}$	$\phi_{z,2}$
3a1^t	3/1	2	3	2	1	0	$\pi/4$	$\pi/2$	$\pi/4$
4a1	5/2	2	3	1	3	0	$\pi/4$	1.62773	5.79254
5a1	**7/2**	2	3	11	–	0	0.56099	2.58059	–
5a2^t	5/1	2	5	2	3	0	$\pi/4$	$\pi/2$	$\pi/4$
6a1	13/5	2	5	1	5	0	$\pi/4$	0.03573	2.53353
6a2	11/3	2	7	1	7	0	$\pi/4$	1.90655	5.01637
6a3	**9/2**	2	3	11	–	0	0.67319	0.89759	–
7a1	21/8	2	5	1	5	0	$\pi/4$	1.60021	5.52412
7a2	19/7	2	7	3	7	0	$\pi/4$	1.66835	6.11271
7a3	**17/5**	2	5	17	–	0	0.49979	2.64179	–
7a4	11/2	2	7	3	7	0	$\pi/4$	1.60853	6.27384
7a5	13/3	2	9	4	7	0	$\pi/4$	1.57817	4.41032
7a6	**15/4**	2	3	11	–	0	0.78539	2.35619	–
7a7^t	7/1	2	7	2	5	0	$\pi/4$	$\pi/2$	$\pi/4$
8a1	**31/12**	2	11	41	–	0	0.39269	2.74889	–
8a2	–	3	4	23	–	0	0.29088	2.85070	–
8a3	–	3	5	6	13	0	$\pi/6$	0.56548	2.03575
8a4	*25/9*	2	7	1	5	0	$\pi/4$	2.04720	5.29197
8a5	29/12	2	5	3	5	0	$\pi/4$	1.59453	2.05821
8a6	23/5	2	9	1	9	0	$\pi/4$	0.35397	2.65710
8a7	29/8	2	9	1	5	0	$\pi/4$	1.47451	2.10447
8a8	17/3	2	11	3	10	0	$\pi/4$	0.39241	5.09182
8a9	27/8	2	9	1	5	0	$\pi/4$	2.03830	2.05668
8a10	*23/7*	2	7	1	9	0	$\pi/4$	0.48400	5.18915
8a11	13/2	2	5	3	5	0	$\pi/4$	1.58524	0.24531
8a12	–	3	4	3	5	0	$\pi/6$	1.04300	0.80424
8a13	–	3	4	1	9	0	$\pi/6$	0.26389	1.58336
8a14	–	3	4	7	14	0	$\pi/6$	1.28176	1.78442
8a15	–	3	7	1	10	0	$\pi/6$	1.64619	2.31221
8a16	25/7	2	7	3	7	0	$\pi/4$	0.04412	2.25248
8a17	19/4	2	9	3	7	0	$\pi/4$	1.92077	6.06457
8a18	*17/4*	2	7	1	5	0	$\pi/4$	1.42912	1.98797
8n1	–	3	4	1	14	0	$\pi/6$	1.94778	2.76460
8n2	–	3	4	37	–	0	0.49805	2.64353	–
8n3^t	–	3	4	3	1	0	$\pi/6$	$\pi/2$	$5\pi/48$

TABLE 6. Fourier and Lissajous descriptions for all prime knots to eight crossings. Knot names are as in Knotscape. The classifying fraction p/q is given for 2-bridge knots. All amplitudes are 1. Boldface entries are Lissajous. Italic entries are 2-bridge with $\Delta(t) \equiv 1$ mod 2, and hence might be Lissajous. The "t" superscript denotes torus knots.

For each value of n_y, random sampling in batches of 10,000 took place until no new knots were found. If a knot was produced that had already been found, the one with the lexicographically smallest set $\{n_x, n_y, n_{z,1}, n_{z,2}\}$ was kept.

This tended to produce knots with fairly small values of $\{n_x, n_y, n_{z,1}\}$ but with $n_{z,2}$ often in the hundreds. Furthermore, only knots with fewer than 17 crossings were kept in the sample.

After a modest amount of searching, we turned up all 2-bridge knots with 14 or fewer crossings, and nearly all 15- and 16-crossing ones as well. (We found 1386 out of 1387 15-crossing knots and 2731 out of 2752 16-crossing knots.) We believe that the following conjecture is reasonable.

Conjecture 5.4. *Every 2-bridge knot can be expressed as a Fourier-$(1,1,k)$ knot with $n_x = 2$ and $k \leq 2$.*

Additional evidence for this conjecture is provided by the twist knots. The twist knot T_m, which is the 2-bridge knot $K_{\frac{2m+1}{2}}$, is shown in Figure 6. The mirror image of T_m is the twist knot T_{-1-m}. Thus it suffices to consider $m > 1$. It is shown in [Hoste and Zirbel 07] that T_m is

knot	p/q	n_x	n_y	$n_{z,1}$	$n_{z,2}$	ϕ_x	ϕ_y	$\phi_{z,1}$	$\phi_{z,2}$
9a1	–	3	5	7	10	0	$\pi/6$	2.29964	0.03769
9a2	–	3	7	1	6	0	$\pi/6$	0.35185	1.05557
9a3	*41/16*	2	7	3	5	0	$\pi/4$	1.88608	4.98854
9a4	–	3	4	1	14	0	$\pi/6$	1.33203	2.27451
9a5	–	3	8	1	8	0	$\pi/6$	0.65345	1.70902
9a6	–	3	7	1	10	0	$\pi/6$	1.88495	2.43787
9a7	–	3	4	1	14	0	$\pi/6$	2.03575	2.37504
9a8	**31/11**	2	11	41	–	0	0.48332	1.08747	–
9a9	–	3	7	4	15	0	$\pi/6$	1.15610	0.76654
9a10	*39/16*	2	5	1	7	0	$\pi/4$	2.08636	5.17367
9a11	–	3	5	9	14	0	$\pi/6$	1.04300	1.33203
9a12	49/18	2	17	1	9	0	$\pi/4$	0.60289	4.56332
9a13	55/21	2	7	1	7	0	$\pi/4$	0.00613	5.43134
9a14	39/14	2	9	1	9	0	$\pi/4$	2.13083	5.89035
9a15	47/13	2	9	1	9	0	$\pi/4$	1.74912	1.92322
9a16	45/19	2	9	1	5	0	$\pi/4$	1.93719	4.94328
9a17	37/8	2	5	2	9	0	$\pi/4$	0.19367	2.96295
9a18	–	3	4	2	7	0	$\pi/6$	2.37504	2.03575
9a19	41/11	2	15	3	10	0	$\pi/4$	2.62089	0.60844
9a20	33/7	2	17	5	7	0	$\pi/4$	0.10752	5.13086
9a21	43/12	2	5	3	9	0	$\pi/4$	2.14045	5.61205
9a22	35/8	2	5	7	9	0	$\pi/4$	1.67486	1.65979
9a23	27/5	2	13	4	5	0	$\pi/4$	1.36285	5.38881
9a24	*41/12*	2	11	1	7	0	$\pi/4$	0.44828	2.24339
9a25	–	3	5	28	–	0	0.26973	1.82466	–
9a26	29/9	2	11	8	9	0	$\pi/4$	1.42268	3.50098
9a27	**15/2**	2	7	25	–	0	0.49087	1.07992	–
9a28	–	3	7	4	5	0	$\pi/6$	1.28176	0.45238
9a29	–	4	7	2	13	0	$\pi/8$	0.26389	2.07345
9a30	–	3	7	8	9	0	$\pi/6$	0.23876	0.77911
9a31	–	3	4	10	11	0	$\pi/6$	1.38230	1.87238
9a32	–	3	7	4	13	0	$\pi/6$	0.46495	1.20637
9a33	**31/7**	2	7	23	–	0	0.47123	2.67035	–
9a34	37/10	2	9	3	5	0	$\pi/4$	1.98367	5.56618
9a35	21/4	2	11	9	10	0	$\pi/4$	0.35932	5.18305
9a36	*23/4*	2	9	3	5	0	$\pi/4$	1.65102	3.04593
9a37	–	3	4	2	11	0	$\pi/6$	1.04300	2.62637
9a38	19/3	2	13	1	7	0	$\pi/4$	1.93386	2.02910
9a39	*33/10*	2	11	3	7	0	$\pi/4$	2.16159	2.03213
9a40	–	3	7	4	13	0	$\pi/6$	1.06814	1.06814
$9a41^t$	9/1	2	9	2	7	0	$\pi/4$	$\pi/2$	$\pi/4$
9n1	–	3	4	1	14	0	$\pi/6$	0.05026	0.27646
9n2	–	3	5	4	7	0	$\pi/6$	0.15079	1.99805
9n3	–	3	8	1	6	0	$\pi/6$	0.76654	0.95504
9n4	–	3	4	2	11	0	$\pi/6$	0.35185	2.51327
9n5	–	3	4	1	4	0	$\pi/6$	1.33203	2.09858
9n6	–	3	4	2	13	0	$\pi/6$	0.08796	2.48814
9n7	–	3	7	2	9	0	$\pi/6$	2.62637	1.05557
9n8	–	3	7	4	13	0	$\pi/6$	0.05026	0.18849

TABLE 7. Fourier and Lissajous descriptions for all prime knots with nine crossings. Knot names are as in Knotscape. The classifying fraction p/q is given for 2-bridge knots. All amplitudes are 1. Boldface entries are Lissajous. Italic entries are 2-bridge with $\Delta(t) \equiv 1 \bmod 2$, and hence might be Lissajous. The "t" superscript denotes torus knots.

Lissajous if and only if $m \equiv 0 \bmod 4$ or $m \equiv 3 \bmod 4$. If this is not the case, the knot does not have the required symmetry to be Lissajous.

However, in these cases, the following examples show that K_m is a Fourier-$(1, 1, 2)$ knot. Thus all twist knots are Fourier-$(1, 1, k)$ knots with $k \leq 2$.

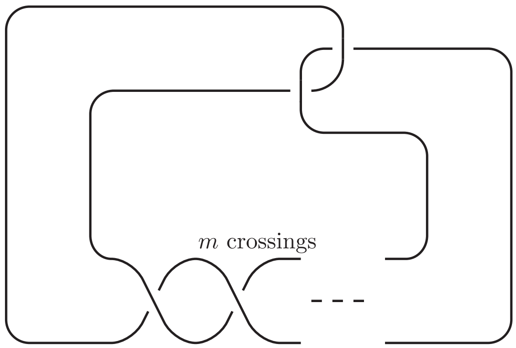

FIGURE 6. The twist knot T_m.

Theorem 5.5. *Twist knots that are not Lissajous may be expressed as Fourier-*$(1,1,2)$ *knots as follows:*

1. *The twist knot* T_{4n+1} *can be expressed as the Fourier-*$(1,1,2)$ *knot with* $n_x = 2, \phi_x = 0, n_y = 8n+3, \phi_y = 1/2, n_{z,1} = 2, \phi_{z,1} = \pi/4, n_{z,2} = 8n+1, \phi_{z,2} = \frac{8n+1+(8n+5)\pi}{2(8n+3)}$ *and* $A_{z,2} = 1$ *for all* $n \geq 1$.

2. *The twist knot* T_{2n} *can be expressed as the Fourier-*$(1,1,2)$ *knot with* $n_x = 2, \phi_x = 0, n_y = 2n+1, \phi_y = 1/2, n_{z,1} = 2, \phi_{z,1} = \pi/4, n_{z,2} = 2n+3, \phi_{z,2} = \frac{2n+3-3\pi}{2(2n+1)}$ *and* $A_{z,2} = 1$ *for all* $n \geq 1$.

The proof is similar to the proof of [Hoste and Zirbel 07, Theorem 4] and relies on very carefully determining the sign of each crossing in the diagram. The details are quite long and not particularly insightful. We leave this as a rather complicated exercise for the reader.

Our sample of all 2-bridge knots to 16 crossings expressed as Fourier-$(1,1,2)$ knots is too large to reproduce here. Instead, in Tables 6 and 7, we include Fourier descriptions for all 2-bridge knots to 9 crossings. Table 13 of [Boocher et al. 07] continues to 10 crossings. To generate this table we again undertook a random sample but this time sharply reduced the range of the parameters. In particular, we kept all amplitudes equal to one, set $\phi_y = \pi/4$, and allowed z-frequencies only as large as 10. An interesting variation on Conjecture 5.4 would be to require that all amplitudes be 1. Knots appearing in Tables 6 and 7 (and in Tables 12 and 13 of [Boocher et al. 07]) that are known to be Lissajous are shown in boldface, while those 2-bridge knots that have the necessary symmetry to be Lissajous, and hence *might* be Lissajous, are shown in italics.

5.4 Fourier-$(1,1,2)$ Knots with $2 < n_x < n_y$

We made only a modest attempt to sample Fourier-$(1,1,2)$ knots with x and y frequencies greater than two. Rather than sampling at random as in Section 5.3, we now chose one sampling point from each region of the phase torus by first creating a bitmap image as in Figure 4 and then taking the centroid of each white region. Sometimes the centroid fell outside of the region, and in this case an arbitrary point of the region was selected. Because of the large crossing numbers that result, and the consequent difficulty in identifying these knots, we again restricted our sample to x and y frequencies of $(3,4), (3,5), (3,7), (3,8), (3,10), (4,5), (4,7), (5,6)$. We further restricted the z-frequencies to be less than 15 and somewhat arbitrarily fixed all amplitudes at 1.

Using Knotscape to identify the resulting knots, and keeping only knots with 16 or fewer crossings, we found several thousand prime knots. Tables 6 and 7 include these knots with 9 or fewer crossings. Tables 15–17 in [Boocher et al. 07] continue to 10 crossings.

All knots through 9 crossings were found, and all but 20 alternating 10-crossing knots were found. We suspect that limiting the z-frequencies to fewer than 15 is a severe restriction.

We did however, find all torus knots up to 16 crossings. It is shown in [Kauffman 98] that every torus knot is a Fourier-$(1,3,3)$ knot. Interestingly, we found that up to 16 crossings, the torus knot $T_{p,q}$ can be represented by a Fourier-$(1,1,2)$ knot with $n_x = p$ and $n_y = q$. By carefully analyzing these examples, the third author was able to prove the following theorem. The proof can be found in [Hoste 09].

Theorem 5.6. *The torus knot* $T_{p,q}$*, with* $0 < p < q$ *and* $\gcd(p,q) = 1$*, is equivalent to the Fourier-*$(1,1,2)$ *knot given by*

$$x(t) = \cos(pt),$$
$$y(t) = \cos\left(qt + \pi/(2p)\right),$$
$$z(t) = \cos\left(pt + \pi/2\right) + \cos\left((q-p)t + \pi/(2p) - \pi/(4q)\right)$$

Furthermore, if p *is even, we may replace* $\phi_{z,2}$ *with* $\pi/(2p)$.

It would be interesting to undertake a large-scale sampling of Fourier-$(1,1,2)$ knots with x and y frequencies greater than two to see whether *every* knot with 16 or fewer crossings turns up. Such a study might shed light on the following question.

Question 5.7. Is there a knot that cannot be expressed as a Fourier-$(1, 1, k)$ knot for $k \leq 2$?

ACKNOWLEDGMENTS

This research was carried out at the Claremont College's REU program in the summer of 2006. The authors thank the National Science Foundation and the Claremont Colleges for their generous support.

REFERENCES

[Bogle et al. 94] M. G. V. Bogle, J. E. Hearst, V. F. R. Jones, and L. Stoilov. "Lissajous Knots." *J. Knot Theory Ramifications* 3:2 (1994), 121–140.

[Boocher et al. 07] A. Boocher, J. Daigle, J. Hoste, and W. Zheng. "Sampling Lissajous and Fourier Knots." arXiv:0707.4210, 2007.

[Burde and Zieschang 03] Gerhard Burde and Heiner Zieschang. *Knots*, de Gruyter Studies in Mathematics 5. Berlin: Walter de Gruyter & Co., 2003.

[Cromwell 04] Peter Cromwell. *Knots and Links*. Cambridge UK: Cambridge University Press, 2004.

[Hartley and Kawauchi 79] Richard Hartley and Akio Kawauchi. "Polynomials of Amphicheiral Knots." *Math. Ann.* 243:1 (1979), 63–70.

[Hoste 09] Jim Hoste. "Torus Knots Are Fourier-$(1, 1, 2)$ Knots." *J. Knot Theory Ramifications*. 18:2 (2009), 265–270.

[Hoste and Thistlethwaite 98] Jim Hoste and Morwen Thistlethwaite. Knotscape. Available online (http://www.math.utk.edu/~morwen), 1998.

[Hoste and Zirbel 07] Jim Hoste and Laura Zirbel. "Lissajous Knots and Knots with Lissajous Projections." *Kobe J. Math.* 24 (2007), 87–106.

[Hoste et al. 98] Jim Hoste, Morwen Thistlethwaite, and Jeff Weeks. "The First 1,701,936 Knots." *Math. Intelligencer* 20:4 (1998), 33–48.

[Jones and Przytycki 98] Vaughan F. R. Jones and Józef H. Przytycki. "Lissajous Knots and Billiard Knots." *Banach Center Publications* 42 (1998), 145–163.

[Kauffman 98] Louis H. Kauffman. "Fourier Knots." In *Ideal Knots*, Ser. Knots Everything 19, pp. 364–373. River Edge, NJ: World Sci. Publ., 1998.

[Lamm 96] Christoph Lamm. "There Are Infinitely Many Lissajous Knots." *Manuscripta Math.* 93 (1996), 29–37.

[Lamm 99] Christoph Lamm. "Cylinder Knots and Symmetric Unions" (Zylinder-Knoten und symmetrische Vereinigungen), Ph.D. thesis, Bonner Mathematische Schriften 321, Universität Bonn, 1999.

[Murasugi 71] Kunio Murasugi. "On Periodic Knots." *Comment. Math. Helv.* 46 (1971), 162–174.

[Przytycki 98] Józef H. Przytycki. "Symmetric Knots and Billiard Knots." In *Ideal Knots*, Ser. Knots Everything 19, pp. 374–414. River Edge, NJ: World Sci. Publ., 1998.

[Rolfsen 90] Dale Rolfsen. *Knots and Links*, Mathematics Lecture Series 7. (Corrected reprint of the 1976 original.) Houston: Publish or Perish Inc., 1990.

Adam Boocher, Mathematics Department, University of California, Berkeley, Evans Hall, Berkeley, CA 94720 (aboocher@gmail.com)

Jay Daigle, California Institute of Technology, Mail Code 253–37, Pasadena, CA 91125 (jdaigle@caltech.edu)

Jim Hoste, Pitzer College, Claremont, CA 91711 (jhoste@pitzer.edu)

Wenjing Zheng, Mathematics Department, University of California, Berkeley, Evans Hall, Berkeley, CA 94720 (wzheng@math.berkely.edu)

Received October 5, 2007; accepted in revised form August 20, 2008.