Constant-Mean-Curvature Surfaces with Singularities in Minkowski 3-Space

Yuhei Umeda

CONTENTS

We shall investigate spacelike constant-mean-curvature surfaces with singularities in Minkowski 3-space. Recently, Kokubu, Rossman, Saji, Umehara, and Yamada gave useful criteria for cuspidal edges and swallowtails. Applying their criteria, we define a new class of generalized constant-mean-curvature surface and give some examples with cuspidal edges and swallowtails.

1. INTRODUCTION

It is well known that the only complete spacelike maximal (mean-curvature-zero) surface in Minkowski 3-space $\mathbb{L}^3$ is the plane. Estudillo and Romero introduced a notion of generalized maximal surfaces in terms of a holomorphic (Weierstrass-type) representation [Estudillo and Romero 92].

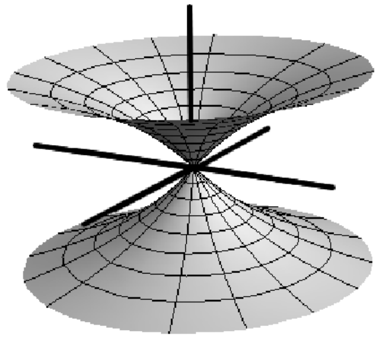
A generalized maximal surface without branch points is called a *maxface*, which was introduced in [Umehara and Yamada 06]; see Figure 1 for typical examples. Umehara and Yamada showed that maxfaces have several interesting global geometric properties. A generic classification of singularities on maxfaces is given in [Fujimori et al. 08].

In contrast to the maximal case, there are many complete spacelike nonzero constant-mean-curvature surfaces (CMC surfaces, for short) in $\mathbb{L}^3$ (see Figure 2). For example, the hyperboloid and the Lorentz cylinder are typical. Moreover, Akutagawa constructed many examples by constructing harmonic maps from the hyperbolic plane to itself [Akutagawa 94]. Furthermore, H. Wang constructed complete surfaces (without singularities) by solving the sinh–Gordon equation [Wang 91] (see Figure 3). On the other hand, there are natural examples of CMC surfaces with singularities, including surfaces of revolution. Since singularities of such surfaces are considered wave-front singularities in 3-dimensional space, it is expected that generic wave-front singularities, that

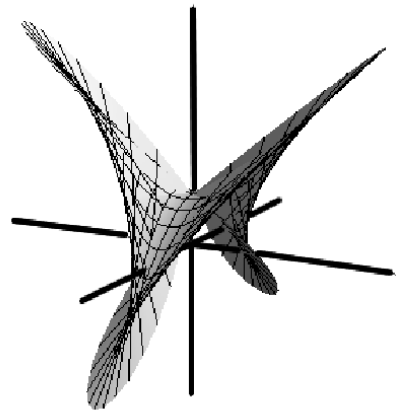
2000 AMS Subject Classification: Primary 53A10; Secondary 53C43, 53C50

Keywords: CMC surface, Gauss map, cuspidal edge, swallowtail, singularity

1058-6458/2009 $ 0.50 per page



Lorentz catenoid



Lorentzian Enneper surface

FIGURE 1. Examples of a maxface with singularities.

is, cuspidal edges and swallowtails, appear in CMC surfaces in $\mathbb{L}^3$. However, the author does not know of any concrete example of CMC surfaces with swallowtails.

Recently, useful criteria for cuspidal edges and swallowtails as wave-front singularities in 3-dimensional space were given in [Kokubu et al. 05].

In this paper, we generalize the notion of CMC surfaces and construct examples with cuspidal edges and swallowtails using their criteria.

Akutagawa and Nishikawa gave a Weierstrass-type representation formula for CMC surfaces in $\mathbb{L}^3$ [Akutagawa and Nishikawa 90] as an analogy of the Kenmotsu formula for CMC surfaces in $\mathbb{R}^3$ [Kenmotsu 79]. The Gauss map g of a nonsingular CMC surface is a harmonic map into the upper or lower connected component of the two-sheeted hyperboloid in $\mathbb{L}^3$. By stereographic projection from $(1,0,0)$ of the hyperboloid to the plane, the Gauss map g can be expressed as a harmonic map into $\mathbb{C}\cup\{\infty\}\setminus\{\zeta\in\mathbb{C} \mid |\zeta|=1\}$. Making use of this result, we construct a class of CMC surfaces with singularities.

This paper is organized as follows. In Section 2, we recall some basic facts on CMC surfaces in $\mathbb{L}^3$. In Section 3, we define a class of CMC surfaces with certain types of singularities, called *extended CMC surfaces*, by extending harmonic maps to the maps into $\mathbb{C}\cup\{\infty\}$, and we characterize the singular set. In Section 4, applying the results on singularities in [Kokubu et al. 05], we investigate singularities of a type of generalized CMC surface and give criteria for a given singular point to be $\mathcal{A}$-equivalent to a cuspidal edge, a swallowtail, or a cuspidal cross cap. In Section 5, we give concrete examples of extended CMC surfaces with cuspidal edges and swallowtails.

2. PRELIMINARIES

Minkowski 3-space $\mathbb{L}^3$ is the 3-dimensional affine space $\mathbb{R}^3$ with the inner product

$$\langle\ ,\ \rangle := -(dx^0)^2 + (dx^1)^2 + (dx^2)^2,$$

where (x^0,x^1,x^2) is the standard coordinate system of $\mathbb{R}^3$. A vector $v\neq 0$ in $\mathbb{L}^3$ is called *spacelike, timelike, null* if $\langle v,v\rangle>0$, $\langle v,v\rangle<0$, $\langle v,v\rangle=0$, respectively. An immersion $f: M^2 \longrightarrow \mathbb{L}^3$ of a 2-manifold M^2 into $\mathbb{L}^3$ is called *spacelike* if the induced metric

$$ds^2 := f^*\langle\ ,\ \rangle = \langle df, df\rangle$$

is positive definite on M^2.

The unit normal vector ν of a spacelike immersion f is a unit timelike vector perpendicular to the tangent plane. Moreover, it can be regarded as a map

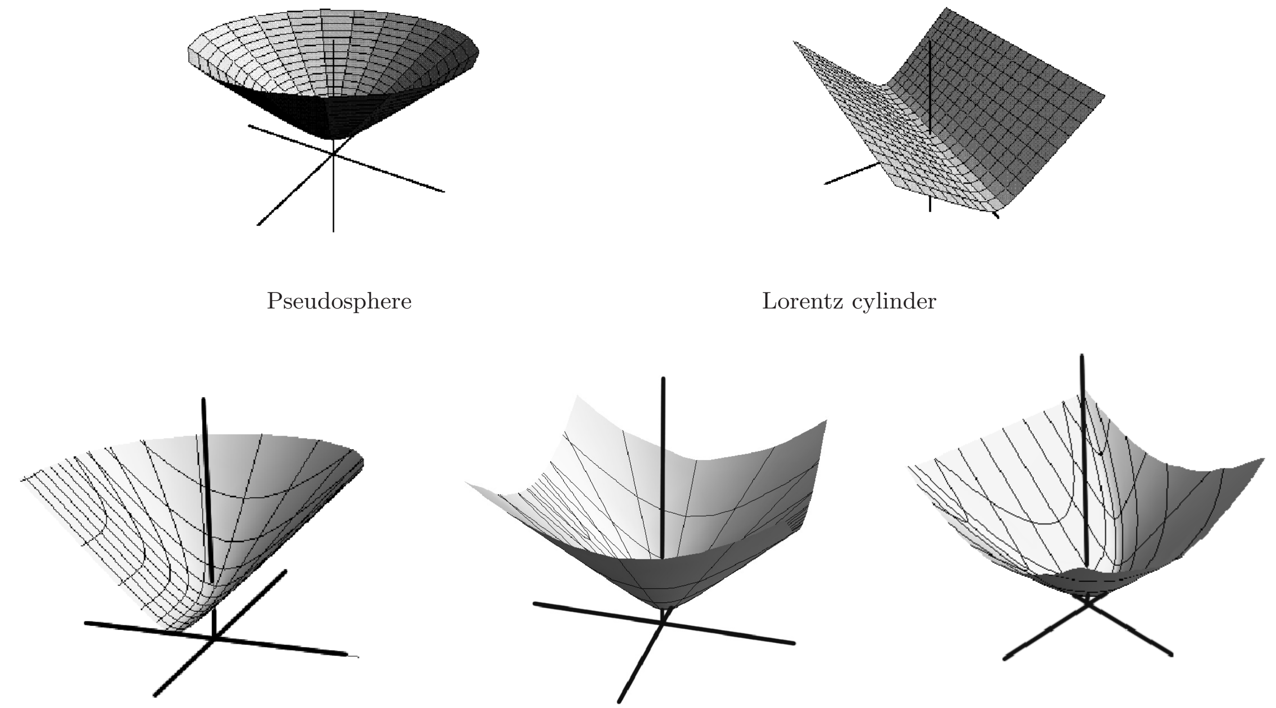
$$\nu : M^2 \longrightarrow \mathbb{H}^2 = \mathbb{H}^2_+ \cup \mathbb{H}^2_-, \tag{2–1}$$

where

$$\mathbb{H}^2_+ := \{\nu=(\nu^0,\nu^1,\nu^2)\in\mathbb{L}^3 \mid \langle\nu,\nu\rangle=-1, \nu^0>0\},$$
$$\mathbb{H}^2_- := \{\nu=(\nu^0,\nu^1,\nu^2)\in\mathbb{L}^3 \mid \langle\nu,\nu\rangle=-1, \nu^0<0\}.$$

The map $\nu : M^2 \longrightarrow \mathbb{H}^2$ is called the *Gauss map* of f. Let $\pi : \mathbb{H}^2 \longrightarrow \mathbb{C}\cup\{\infty\}$ be the stereographic projection from the north pole $(1,0,0)$ given by

$$\pi(\nu^0,\nu^1,\nu^2) = \frac{\nu^1+\sqrt{-1}\nu^2}{1-\nu^0}.$$



A surface of revolution about a spacelike axis

A surface of revolution about a timelike axis

A surface of revolution about a null axis

FIGURE 2. Examples of complete CMC surfaces.

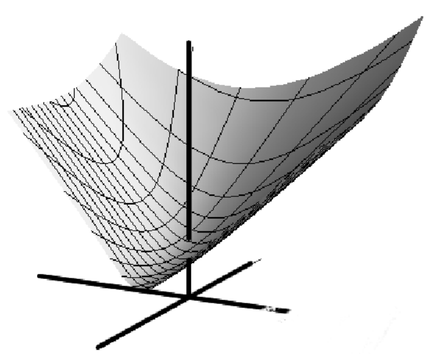


FIGURE 3. A complete surface constructed by H. Wang.

We also call the composition $g := \pi \circ \nu$ the Gauss map of f. Since ν takes values in the set $\mathbb{H}^2$, $|g| \neq 1$ holds on M^2.

Let (u, v) be a local coordinate system of M^2. We write the first fundamental form of f by

$$ds^2 = \langle df, df \rangle = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

and the second fundamental form of f by

$$II = \langle df, d\nu \rangle = L\,du^2 + 2M\,du\,dv + N\,dv^2.$$

Then the mean curvature H of f is given by

$$H = \frac{EN - 2FM + GL}{2(EG - F^2)}.$$

A spacelike immersion $f : M^2 \longrightarrow \mathbb{L}^3$ is said to have *constant mean curvature* if H is equal to a nonzero constant.

In particular, if $H = 0$, the spacelike immersion is said to be *maximal*. A map $g : M^2 \longrightarrow \pi(\mathbb{H}^2)$ defined on a Riemann surface M^2 is called *harmonic* if it satisfies the following condition:

$$\frac{\partial^2 g}{\partial z \partial \bar{z}} + \frac{2\bar{g}}{1-|g|^2}\frac{\partial g}{\partial z}\frac{\partial g}{\partial \bar{z}} = 0, \tag{2–2}$$

where z is a complex coordinate of M^2.

It is well known that the Gauss map of a spacelike constant-mean-curvature immersion is harmonic and the Gauss map of a spacelike maximal immersion is meromorphic (see [Kobayashi 83]; see also [Kobayashi 84]). The unit normal vector field ν of the constant-mean-curvature surface in (2–1) is rewritten as

$$\nu = \frac{1}{1-|g|^2}(-(1+|g|^2), 2\operatorname{Re} g, 2\operatorname{Im} g).$$

For CMC surfaces in $\mathbb{L}^3$, an analogue of the Weierstrass representation formula is given in [Akutagawa and Nishikawa 90].

Theorem 2.1. [Akutagawa and Nishikawa 90] *Let M^2 be a Riemann surface and $f : M^2 \longrightarrow \mathbb{L}^3$ a conformal immersion with nonzero constant mean curvature H. Then there exists a harmonic map g such that*

$$f = \frac{2}{H}\int_{z_0}^{z} \operatorname{Re}\big((-2g, (1+g^2), \sqrt{-1}(1-g^2))\frac{\bar{g}_z}{(1-|g|^2)^2}\big)dz, \tag{2–3}$$

where $z_0 \in M^2$ is a base point and g_z means $\partial g/\partial z$. Conversely, assume that M^2 is simply connected, and take a nonholomorphic harmonic map $g : M^2 \longrightarrow \pi(\mathbb{H}^2)$. Then the integration in (2–3) does not depend on the choice of a path joining z_0 and z, and f in (2–3) is a spacelike immersion of constant mean curvature H with Gauss map g. Furthermore, the induced metric of f is given by

$$ds^2 = \left(\frac{2}{H(1-|g|^2)}|g_{\bar{z}}|\right)^2 |dz|^2,$$

and the Gaussian curvature K is given by

$$K = -H^2\left(\left|\frac{g_z}{g_{\bar{z}}}\right|^2 - 1\right).$$

3. CONSTANT-MEAN-CURVATURE SURFACES WITH SINGULARITIES

In this section, we give the definition of a generalized CMC surface as a constant-mean-curvature surface with singularities. First, we review the definition of maxfaces and generalized maximal surfaces.

A holomorphic map $F = (F^0, F^1, F^2) : M^2 \longrightarrow \mathbb{C}^3$ of a Riemann surface M^2 into the complex Euclidean space $\mathbb{C}^3$ is called *null* if $\sum_{j=1}^{3} F_z^j \cdot F_z^j$ vanishes. We consider the projection

$$\pi_L : \mathbb{C}^3 \ni (\zeta^1, \zeta^2, \zeta^3) \longmapsto \operatorname{Re}\left(-\sqrt{-1}\zeta^3, \zeta^1, \zeta^2\right) \in \mathbb{L}^3.$$

The projection of null holomorphic immersions into $\mathbb{L}^3$ by π_L gives spacelike maximal surfaces with singularities, called *maxfaces* (see [Umehara and Yamada 06] for details). The holomorphic null immersion F as above is called the *holomorphic lift* of the maxface.

The following fact is a generalization of the Weierstrass representation formula for a maximal surface [Kobayashi 83].

Fact 3.1. [Umehara and Yamada 06] *Let M^2 be a simply connected Riemann surface and (g, ω) a pair consisting of a meromorphic function g and a holomorphic 1-form ω on M^2 such that*

$$(1+|g|^2)^2|\omega|^2 \neq 0$$

on U. Then

$$f(z) := \operatorname{Re}\int_{z_0}^{z}\left(-2g, 1+g^2, \sqrt{-1}(1-g^2)\right)\omega \tag{3–1}$$

gives a maxface in $\mathbb{L}^3$. Moreover, all maxfaces are locally obtained in this manner.

The induced metric by f in (3–1) is given by

$$ds^2 = \left(1-|g|^2\right)^2|\omega|^2.$$

In particular, $z \in M^2$ is a singular point of f if and only if $|g| = 1$, and the restriction $f : M^2 \setminus \{z \mid |g(z)| = 1\} \longrightarrow \mathbb{L}^3$ is a spacelike maximal immersion.

If $F : M^2 \longrightarrow \mathbb{C}^3$ is null (not necessarily an immersion) and

$$-|dF^0|^2 + |dF^1|^2 + |dF^2|^2$$

does not identically vanish, then $\pi_L \circ F$ is a generalized maximal surface in the sense of [Estudillo and Romero 92]. The points where F is not immersed are isolated branch points. In this sense, a maxface is a generalized maximal surface without branch points.

We generalize the notion of CMC surfaces in a similar way.

Definition 3.2. A smooth map $g : M^2 \longrightarrow \mathbb{C}\cup\{\infty\}$ of a Riemann surface M^2 into the sphere $\mathbb{S}^2 = \mathbb{C}\cup\{\infty\}$ is called a *regular extended harmonic map* to the hyperbolic plane if

(1) $\omega := \hat{\omega}\, dz$ can be extended to a 1-form of class C^1 across $\{p \in M^2 \mid |g(p)| = 1\}$,

(2) $g_{z\bar{z}} + 2(1-|g|^2)\bar{g} g_z \bar{\hat{\omega}} = 0$ holds,

where

$$\hat{\omega} := \frac{\bar{g}_z}{(1-|g|^2)^2}$$

and z is a complex coordinate of M^2.

Remark 3.3. If $|g| \neq 1$, a regular extended harmonic map g is a harmonic map.

Remark 3.4. The conjugate map $\bar{g}$ of a harmonic map g is also harmonic. However, the conjugate map $\bar{g}$ of the regular extended harmonic map g is not always a regular extended harmonic map, because of the condition (1). Existence of a regular extended harmonic map whose conjugate map is also a regular extended harmonic map is an open problem. This problem is related to the study of surfaces of negative constant Gaussian curvature with singularities (see [Gálvez et al. 03]).

Remark 3.5. Let $g : M^2 \longrightarrow \mathbb{C} \cup \{\infty\}$ be a regular extended harmonic map. Then the integral (2–3) is independent of the choice of path.

Definition 3.6. Let $g : M^2 \longrightarrow \mathbb{C} \cup \{\infty\}$ be a regular extended harmonic map and H a nonzero constant. Then the map $f : M^2 \longrightarrow \mathbb{L}^3$ given explicitly by

$$f = \frac{2}{H} \int_{z_0}^{z} \operatorname{Re}\left(-2g, 1+g^2, \sqrt{-1}(1-g^2)\right) \omega \qquad (3\text{–}2)$$

is called a *generalized constant-mean-curvature (CMC) surface* with mean curvature H. The map g is called the *Gauss map*.

Remark 3.7. The induced metric of the generalized CMC surface f is given by

$$ds^2 = \left(\frac{2|g_{\bar{z}}|}{H(1-|g|^2)}\right)^2 |dz|^2 = \left(\frac{2(1-|g|^2)|\omega|}{H}\right)^2.$$

Proposition 3.8. *Let $f : M^2 \longrightarrow \mathbb{L}^3$ be a generalized CMC surface with Gauss map g. Then a point*

$$p \in M^2 \setminus \left\{q \in M^2 \mid |g(q)| = \infty\right\}$$

is a singular point (that is, $\operatorname{rank} df(p) < 2$*) if and only if $|g(p)| = 1$ or $\hat{\omega} = 0$. In particular, the point p is a singular point with* $\operatorname{rank} df(p) = 0$ *if and only if $\hat{\omega} = 0$. A point $q \in \{p \in M^2 \mid |g(q)| = \infty\}$ is a singular point if and only if $g^2\hat{\omega} = 0$. Moreover, such a singular point p satisfies* $\operatorname{rank} df(p) = 0$.

Proof: We assume that f is written as in (3–2). Let $z = u + \sqrt{-1}v$ be a complex coordinate of M^2 around p. Without loss of generality, we may assume $H = 2$. Then

$$f_z = \frac{1}{2}\left(-2g, 1+g^2, \sqrt{-1}(1-g^2)\right)\hat{\omega},$$
$$f_{\bar{z}} = \frac{1}{2}\left(-2\bar{g}, 1+\bar{g}^2, -\sqrt{-1}(1-\bar{g}^2)\right)\bar{\hat{\omega}}.$$

Thus, we have

$$\begin{aligned}
f_u &= (f_z + f_{\bar{z}}) \\
&= (-2\operatorname{Re}(g\hat{\omega}), \operatorname{Re}((1+g^2)\hat{\omega}), -\operatorname{Im}((1-g^2)\hat{\omega})), \\
f_v &= \sqrt{-1}(f_z - f_{\bar{z}}) \\
&= (2\operatorname{Im}(g\hat{\omega}), -\operatorname{Im}((1+g^2)\hat{\omega}), -\operatorname{Re}((1-g^2)\hat{\omega})), \\
f_u \times f_v &= -2\sqrt{-1} f_z \times f_{\bar{z}} \\
&= (|g|^2-1)|\hat{\omega}|^2(1+|g|^2, 2\operatorname{Re} g, 2\operatorname{Im} g),
\end{aligned}$$

where $\times$ is the Euclidean vector product in $\mathbb{R}^3$. These imply that $\operatorname{rank} df(p) < 2$ if and only if $|g(p)| = 1$ or $\hat{\omega}(p) = 0$ on $p \in M^2 \setminus \{q \in M^2 \mid |g(q)| = \infty\}$. Moreover, $\operatorname{rank} df(p) = 0$ if and only if $\hat{\omega} = 0$. On the other hand, on $\{p \in M^2 \mid |g(p)| = \infty\}$, we have that $\operatorname{rank} df(p) < 2$ if and only if $g^2\hat{\omega} = 0$, and in fact, $\operatorname{rank} df(p) = 0$. □

Definition 3.9. We assume that $\omega = \hat{\omega}\, dz$ never vanishes on $\{p \in M^2 \mid |g(p)| < \infty\}$ and that $g^2\omega = g^2\hat{\omega}\, dz$ does not vanish on $\{p \in M^2 \mid |g(p)| = \infty\}$. A regular extended harmonic map with such a property is called an *extended harmonic map*, and a generalized CMC surface with an extended harmonic map is called an *extended CMC surface*. Moreover, this extended harmonic map is also called the *Gauss map* of the extended CMC surface.

Corollary 3.10. *Let $f : M^2 \longrightarrow \mathbb{L}^3$ be an extended CMC surface with Gauss map g. Then a point $p \in U$ is a singular point if and only if $|g(p)| = 1$.*

4. TYPES OF SINGULARITIES OF EXTENDED CMC SURFACES

In the previous section, we defined generalized CMC surfaces as constant-mean-curvature surfaces with singularities. So it is quite natural to investigate which types of singularities appear on generalized CMC surfaces. In this section, we recall wave fronts in $\mathbb{R}^3$ and give simple criteria for a given singular point on the extended CMC surface to be each of the typical examples of singular points:

cuspidal edges, swallowtails, and cuspidal cross caps using criteria in [Kokubu et al. 05, Fujimori et al. 08].

Let U be a domain in $\mathbb{R}^2$ and $f : U \longrightarrow \mathbb{R}^3$ a C^∞-map from U into Euclidean 3-space $\mathbb{R}^3$. The map f is called a *frontal* if there exists a unit vector field $\boldsymbol{n}$ in $\mathbb{R}^3$ along f such that $\boldsymbol{n}$ is perpendicular to $f_*(TU)$. We call $\boldsymbol{n}$ a *unit normal vector field* of a frontal f. We identify the unit cotangent bundle of $\mathbb{R}^3$ with

$$\mathbb{R}^3 \times S^2 = \{(x, \boldsymbol{n}) \mid x \in \mathbb{R}^3, \boldsymbol{n} \in S^2\}.$$

Then

$$\xi := n_1 dx^1 + n_2 dx^2 + n_3 dx^3,$$
$$x = (x^1, x^2, x^3), \quad \boldsymbol{n} = (n_1, n_2, n_3),$$

gives the canonical contact form and

$$L := (f_L, \boldsymbol{n}) : U \longrightarrow \mathbb{R}^3 \times S^2$$

is called a *Legendrian* if the pullback of the contact form ξ vanishes, that is, if $(f_L)_u$ and $(f_L)_v$ are both perpendicular to $\boldsymbol{n}$, where (u, v) is a local coordinate system of U.

In this terminology, a *frontal* is a projection of a Legendrian map L. If L is a Legendrian immersion, the projection f_L of L into $\mathbb{R}^3$ is called a (*wave*) *front*. Now let $L = (f, \boldsymbol{n}) : U \longrightarrow \mathbb{R}^3 \times S^2$ be a Legendrian immersion. Then a point $p \in U$ where f is not an immersion is called a *singular point* of the front f.

By definition, there exists a smooth function λ on U such that

$$f_u \times f_v = \lambda \boldsymbol{n}, \tag{4–1}$$

where $\times$ is the Euclidean vector product on $\mathbb{R}^3$. A singular point $p \in U$ is called *nondegenerate* if $d\lambda$ does not vanish at p.

It is well known that cuspidal edges and swallowtails are generic singularities of fronts (see [Arnol'd et al. 85, p. 336]), where a cuspidal edge and a swallowtail are respectively a singular point that is $\mathcal{A}$-equivalent at the origin to the C^∞-map germs

$$f_C(u, v) := (u^2, u^3, v),$$
$$f_S(u, v) := (3u^4 + u^2 v, 4u^3 + 2uv, v)$$

(see Figure 4).

Here, two C^∞-map germs $f : (U, p) \longrightarrow \mathbb{R}^3$ and $g : (V, q) \longrightarrow \mathbb{R}^3$ are *$\mathcal{A}$-equivalent* at the points $p \in U$ and $q \in V$ if there are a diffeomorphism germ φ of $\mathbb{R}^2$ with $\varphi(p) = q$ and a local diffeomorphism Φ of $\mathbb{R}^3$ with $\Phi(f(p)) = g(q)$ such that $g = \Phi \circ f \circ \varphi^{-1}$. A *cuspidal cross cap* is a singular point that is $\mathcal{A}$-equivalent to the C^∞-map germ

$$f_{\rm ccc}(u, v) := (u^2, v^2, uv^3)$$

at the origin (see Figure 4, right). A cuspidal cross cap is the generic singularity of frontals, but it is not a front.

Now we identify Minkowski 3-space $\mathbb{L}^3$ with the 3-dimensional affine space $\mathbb{R}^3$. We give the following theorem as analogous to [Umehara and Yamada 06, Theorem 3.1] and [Fujimori et al. 08, Theorem 2.3].

Theorem 4.1. *Let $f : M^2 \longrightarrow \mathbb{L}^3$ be an extended CMC surface with Gauss map g, and set $\hat{\omega} = \hat{g}_z/(1 - |g|^2)^2$, where $g_z = \partial g/\partial z$. Then:*

(1) *A point $p \in U$ is a singular point if and only if $|g| = 1$.*

(2) *f is a front at a singular point p if and only if*

$$\mathrm{Re}\left(\frac{g_z}{g^2 \hat{\omega}}\right) \neq 0$$

holds at p.

(3) *f is $\mathcal{A}$-equivalent to a cuspidal edge at p if and only if*

$$\mathrm{Re}\left(\frac{g_z}{g^2 \hat{\omega}}\right) \neq 0 \quad \textit{and} \quad \mathrm{Im}\left(\frac{g_z}{g^2 \hat{\omega}}\right) \neq 0$$

hold at p.

(4) *f is $\mathcal{A}$-equivalent to a swallowtail at p if and only if*

$$\frac{g_z}{g^2 \hat{\omega}} \in \mathbb{R} \setminus \{0\}$$

and

$$\mathrm{Re}\left\{\frac{g}{g_z}\left(\frac{g_z}{g^2 \hat{\omega}}\right)_z\right\} \neq \mathrm{Re}\left\{\overline{\left(\frac{g}{g_z}\right)}\left(\frac{g_z}{g^2 \hat{\omega}}\right)_{\bar{z}}\right\}$$

hold at p.

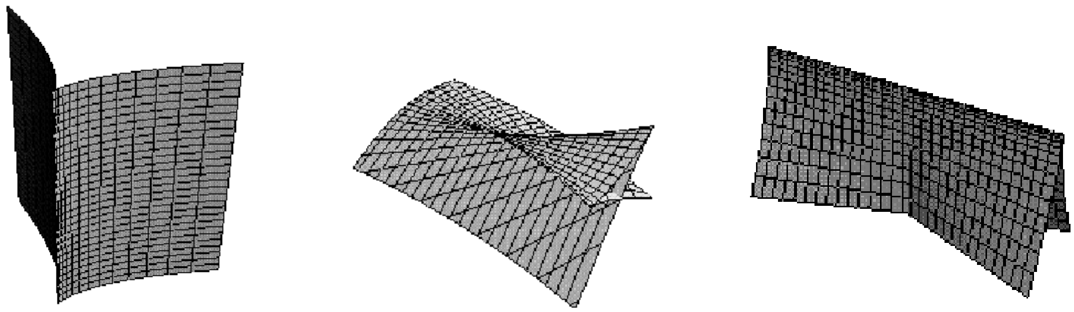
(5) *f is $\mathcal{A}$-equivalent to a cuspidal cross cap at p if and only if*

$$\frac{g_z}{g^2 \hat{\omega}} \in \sqrt{-1}\mathbb{R} \setminus \{0\}$$

and

$$\mathrm{Im}\left\{\frac{g}{g_z}\left(\frac{g_z}{g^2 \hat{\omega}}\right)_z\right\} \neq \mathrm{Im}\left\{\overline{\left(\frac{g}{g_z}\right)}\left(\frac{g_z}{g^2 \hat{\omega}}\right)_{\bar{z}}\right\}$$

hold at p.



(u^2, u^3, v) Cuspidal edge — $(3u^2 + u^2v, 4u^3 + 2uv, v)$ Swallowtail — (u^2, v^2, uv^3) Cuspidal cross cap

FIGURE 4. Examples of singularities.

We assume that p is a nondegenerate singular point. Then there exists a regular curve in a neighborhood of the point p,

$$\gamma = \gamma(t) : (-\epsilon, \epsilon) \longrightarrow U$$

(called the *singular curve*), such that $\gamma(0) = p$ and the image of γ coincides with the set of singular points of f around p.

On the other hand, a nonzero vector $\eta \in TU$ such that $df(\eta) = 0$ is called the *null direction.* For each point $\gamma(t)$, the null direction $\eta(t)$ is determined uniquely up to scalar multiplication. We review the following criteria.

Fact 4.2. [Kokubu et al. 05] *Let $f : U \longrightarrow \mathbb{R}^3$ be a front and $p = \gamma(0) \in U$ a nondegenerate singular point of f.*

(1) *The germ of the image of the front at p is $\mathcal{A}$-equivalent to a cuspidal edge if and only if $\eta(0)$ is not proportional to $\gamma'(0)$, where $' = d/dt$.*

(2) *The germ of the image of the front at p is $\mathcal{A}$-equivalent to a swallowtail if and only if $\eta(0)$ is proportional to $\gamma'(0)$ and*

$$\frac{d}{dt}\det(\gamma'(t), \eta(0))|_{t=0} \neq 0.$$

Fact 4.3. [Fujimori et al. 08] *Let $f : U \longrightarrow \mathbb{R}^3$ be a frontal with unit normal vector field $\boldsymbol{n}$, and $p = \gamma(0) \in U$ a nondegenerate singular point of f. We set $\psi(t) := \det(f_*\gamma', d\boldsymbol{n}(\eta), \boldsymbol{n})$. Then the germ of the image of f at p is $\mathcal{A}$-equivalent to a cuspidal cross cap if and only if $\eta(0)$ is transversal to $\gamma'(0)$, $\psi(0) = 0$, and $\psi'(0) \neq 0$.*

Proof of Theorem 4.1: Though the proofs are parallel to those in [Umehara and Yamada 06, Theorem 3.1], [Umehara and Yamada 06, Lemma 3.3], and [Fujimori et al. 08, Theorem 2.3], we give proofs here for the reader's convenience. We have already proved (1) in Corollary 3.10. We identify $\mathbb{L}^3$ with Euclidean 3-space $\mathbb{R}^3$. Let $f : M^2 \longrightarrow \mathbb{L}^3$ be an extended CMC surface with Gauss map g. Without loss of generality, we may assume $H = 2$. Then

$$\boldsymbol{n} = \frac{1}{\sqrt{(1+|g|^2)^2 + 4|g|^2}}(1+|g|^2, 2\operatorname{Re} g, 2\operatorname{Im} g)$$

is the unit normal vector field of f with respect to the Euclidean metric of $\mathbb{R}^3$. From now on, we assume $|g(p)| = 1$ and $\omega(p) \neq 0$. In the sense of [Umehara and Yamada 06],

$$\eta = \frac{\sqrt{-1}}{g\hat{\omega}}$$

gives the null direction at p. On the other hand, we have

$$d\boldsymbol{n}(p) = \frac{\sqrt{-1}}{2\sqrt{2}}\left(\frac{dg}{g} - \frac{d\bar{g}}{\bar{g}}\right)(0, \operatorname{Im} g, -\operatorname{Re} g).$$

If $dg(p) = 0$, then $(f, \boldsymbol{n})$ is not an immersion at p, because $d\boldsymbol{n} = 0$. So we may assume $dg(p) \neq 0$.

Since $g_{\bar{z}}(p) = 0$ by (1) in Definition 3.2, the null direction of $d\boldsymbol{n}$ at p is proportional to

$$\mu = \overline{\left(\frac{g_z}{g}\right)}.$$

Thus we have (2).

Next, the function λ in (4–1) is calculated as

$$\lambda = (|g|^2 - 1)|\hat{\omega}|^2\sqrt{(1+|g|^2)^2 + 4|g|^2}.$$

By assumption, it is sufficient to consider the case $|g| = 1$ and $\omega(p) \neq 0$. Then we can obtain that if $\operatorname{Re}(g_z/(g^2\hat{\omega})) \neq 0$ at p, then p is nondegenerate.

Assume that $\mathrm{Re}(g_z/(g^2\hat{\omega})) \neq 0$ holds at a singular point p. Then f is a front and p is a nondegenerate singular point. Since the singular set of f is characterized by $g\bar{g} = 1$, the singular curve $\gamma(t)$ with $\gamma(0) = p$ satisfies $g(\gamma(t))\overline{g(\gamma(t))} = 1$. Differentiating this, we get

$$\mathrm{Re}\left(\frac{g_z}{g}\gamma'\right) = 0,$$

because $g_{\bar{z}}(p) = 0$.

This implies that γ' is perpendicular to $\overline{g_z/g}$, that is, proportional to $\sqrt{-1}g_z/g$. Hence we can parameterize γ as

$$\gamma'(t) = \sqrt{-1}\,\overline{\left(\frac{g_z}{g}\right)}(\gamma(t)). \tag{4–2}$$

Thus we have (3).

Next, we assume that $\mathrm{Im}(g_z/(g^2\hat{\omega})) = 0$ holds at the singular point p. In this case,

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} \det(\gamma', \eta)\\
&= \mathrm{Im}\left(\left(\frac{g_z}{g^2\hat{\omega}}\right)_z \frac{d\gamma}{dt} + \left(\frac{g_z}{g^2\hat{\omega}}\right)_{\bar{z}} \frac{d\bar{\gamma}}{dt}\right)\\
&= -\,\mathrm{Re}\left(\frac{g_z}{g^2\hat{\omega}}\right)_z \overline{\left(\frac{g_z}{g}\right)} + \mathrm{Re}\left(\frac{g_z}{g^2\hat{\omega}}\right)_{\bar{z}} \left(\frac{g_z}{g}\right)\\
&= -\left|\frac{g_z}{g}\right|^2 \mathrm{Re}\left[\frac{g}{g_z}\left(\frac{g_z}{g^2\hat{\omega}}\right)_z\right]\\
&\quad + \left|\frac{g_z}{g}\right|^2 \mathrm{Re}\left[\overline{\left(\frac{g}{g_z}\right)}\left(\frac{g_z}{g^2\hat{\omega}}\right)_{\bar{z}}\right].
\end{aligned}$$

Thus we have (4).

On the other hand, if we assume that f is a frontal (not a front), then one can compute ψ as in Fact 4.3 as

$$\psi = \det(f_*\gamma', d\boldsymbol{n}(\eta), \boldsymbol{n}) = \mathrm{Re}\left(\frac{g_z}{g^2\hat{\omega}}\right) \cdot \psi_0,$$

where ψ_0 is a smooth function on a neighborhood of p such that $\psi_0(p) \neq 0$. Then the second and third conditions of Fact 4.3 are written as

$$\mathrm{Re}\left(\frac{g_z}{g^2\hat{\omega}}\right) = 0$$

and

$$\mathrm{Im}\left[\frac{g}{g_z}\left(\frac{g_z}{g^2\hat{\omega}}\right)_z\right] \neq \mathrm{Im}\left[\overline{\left(\frac{g}{g_z}\right)}\left(\frac{g_z}{g^2\hat{\omega}}\right)_{\bar{z}}\right].$$

Here we have used the relation

$$\frac{d}{dt} = \sqrt{-1}\left[\overline{\left(\frac{g_z}{g}\right)}\frac{\partial}{\partial z} - \frac{g_z}{g}\frac{\partial}{\partial \bar{z}}\right],$$

which comes from (4–2). Using the relation $\overline{(g_z/g)} = (g/g_z) \cdot$ (a real-valued function), we obtain (5). □

5. EXAMPLES

We shall introduce two examples.

Example 5.1. (A surface of revolution about a timelike axis.) A surface of revolution about a timelike line, say the x^0-axis, is a surface given by

$$f(s,t) = \left(x^0(s), x^1(s)\cos t, x^1(s)\sin t\right),$$

where $(x^0(s), x^1(s))$ is a profile curve in the x^0x^1-plane. In [Ishihara and Hara 88], the authors give the profile curves of CMC surfaces. If we change the coordinate to an isothermal coordinate for T_3 in [Ishihara and Hara 88, Theorem 1], then this surface is congruent to the surface obtained from the following Gauss map: Let $M^2 = (-\pi, \pi) \times \mathbb{R}$ and let $z = u + \sqrt{-1}v \in M^2$ be a complex coordinate. Then the map $g : M^2 \longrightarrow \mathbb{C} \cup \{\infty\}$ given by

$$g(z) = \frac{\cos u}{1 + \sin u} e^{\sqrt{-1}v}$$

is an extended harmonic map. Therefore, the map g gives an example of an extended CMC surface. The set of singularities is $\{z \mid \mathrm{Re}\, z = 0\}$ and its image is a point in $\mathbb{L}^3$ at which the image is tangent to the light cone (see Figure 5). Such a singularity is called a *conelike singularity*.

Example 5.2. (A surface of revolution about a spacelike axis.) A surface of revolution about a spacelike axis, say the x^1-axis, is a surface given by

$$f(s,t) = (x^0(s)\cosh t, x^1(s), x^0(s)\sinh t),$$

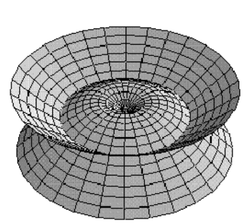
where $(x^0(s), x^1(s))$ is a profile curve in the x^0x^1-plane. If we change the coordinate of S_3 in [Ishihara and Hara 88, Theorem 1], then this surface is congruent to the surface obtained from the following Gauss map: Let $M^2 = \mathbb{C}$ and $z = u + \sqrt{-1}v \in M^2$ be a complex coordinate. Then the map given by

$$g(z) = \frac{\cosh u \sinh v + \sqrt{-1}e^u}{\sinh u + \cosh u \cosh v}$$

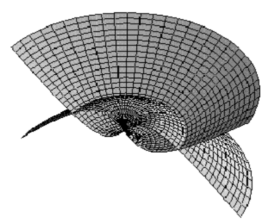
is an extended harmonic map. Therefore, the map g gives an example of an extended CMC surface.

The set of singularities is $\{z \mid \mathrm{Re}\, z = 0\}$, and its image is a conelike singularity (see Figure 6).

Remark 5.3. All incomplete CMC surfaces of revolution in $\mathbb{L}^3$ have only conelike singularities. Moreover, all surfaces of revolution are fronts.



Surface of revolution about a timelike axis



Half cut

FIGURE 5. A surface of revolution about a timelike axis.

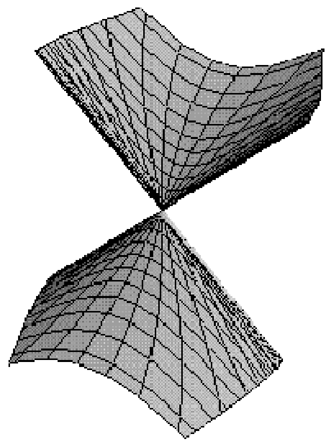


FIGURE 6. Surface of revolution about a spacelike axis.

Next, we introduce two examples with cuspidal edges and swallowtails.

Example 5.4. (A helicoidal surface about a spacelike axis.) A helicoidal surface about a spacelike axis is the surface given by

$$f(s,t) = (x^0(s)\cosh t, x^1(s) + ct, x^0(s)\sinh t),$$

where $(x^0(s), x^1(s))$ is a profile curve in the x^0x^1-plane. In [Beneki et al. 02], the authors give a method to construct helicoidal surfaces about spacelike and timelike axes with prescribed Gaussian and mean curvature. Applying their result to constant-mean-curvature surfaces, we obtain the profile curves of surfaces of constant mean curvature 1 as follows:

$$(x^0(s), x^1(s)) = \left(s, \int \frac{|s^2+c_1|\sqrt{c^2-s^2}}{|s|\sqrt{s^2+(s^2+c_1)^2}}ds\right),$$

where c and c_1 are constants.

To consider such a surface as an extended CMC surface, it is necessary to find an isothermal coordinate of this surface. When $c_1 = 0$, this surface is congruent to the surface obtained from the following Gauss map: we set

$$\xi = \sqrt{\frac{c^2+1+\sqrt{c^2+1}}{2}}u - \sqrt{\frac{c^2+1-\sqrt{c^2+1}}{2}}v,$$

$$\eta = -\sqrt{\frac{c^2+1-\sqrt{c^2+1}}{2}}u + \sqrt{\frac{c^2+1+\sqrt{c^2+1}}{2}}v,$$

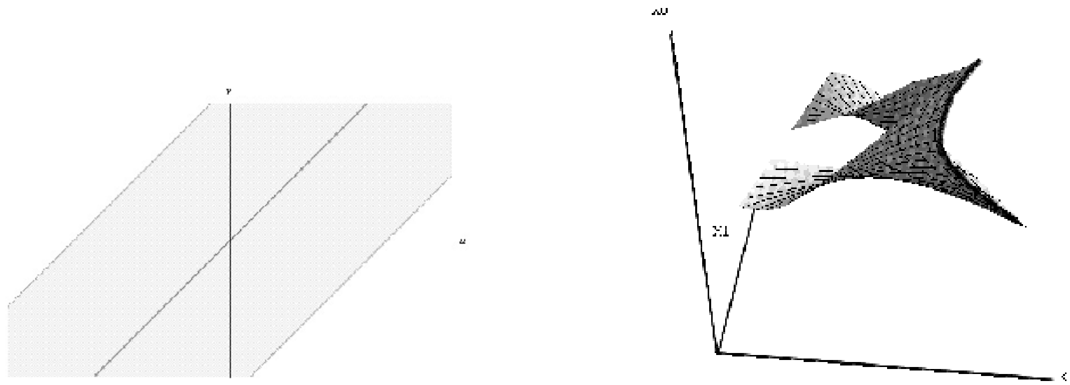
where $z = u + \sqrt{-1}v$ is a complex coordinate. Then the domain of the Gauss map is $M^2 = \{(u,v) \mid -K_1 < \xi < K_1\}$, where K_1 is the complete elliptic integral of the first kind with modulus $k_1 = \sqrt{c^2/(c^2+1)}$, that is,

$$K_1 = K(k_1) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k_1^2\sin\varphi}}$$

(see Figure 7, left).

Then the Gauss map $g : M^2 \longrightarrow \mathbb{C} \cup \{\infty\}$ is given by

$$g(z) = \frac{\hat{c}\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi - \sqrt{-1}(\hat{c}\,\mathrm{dn}\,\xi\sinh\eta + c\,\mathrm{sn}\,\xi\,\mathrm{cn}\,\xi\sinh\eta)}{\mathrm{sn}\,\xi + \hat{c}\,\mathrm{dn}\,\xi\cosh\eta + c\,\mathrm{sn}\,\xi\,\mathrm{cn}\,\xi\sinh\eta},$$



Domain of a helicoidal surface A helicoidal surface with cuspidal edge

FIGURE 7. Helicoidal surface.

where sn, cn, and dn are Jacobi's elliptic functions with modulus k_1, and to save space we have introduced the abbreviation $\hat{c} := \sqrt{c^2+1}$. We get that this map is an extended harmonic map. Therefore, the map g gives an example of an extended CMC surface. The set of singularities is $\{(u,v) \mid \xi(u,v) = 0\}$ (see Figure 7, left). Applying this g to Theorem 4.1(4), we get that the singular points are cuspidal edges. This figure is obtained when $c = 2$.

Example 5.5. (Enneper–Wente-type surface.) CMC surfaces under a curvature line condition are constructed in [Abresch 87] and [Walter 87]. In this example, we construct CMC surfaces in $\mathbb{L}^3$ by an analogy to Walter's method. Here, the curvature line condition is the condition that each u-curve is to lie in some Minkowski sphere, and each v-curve is to lie in some plane where the Minkowski sphere is a quadric surface expressed by

$$\{(x_0, x_1, x_2) \mid -(x_0)^2 + (x_1)^2 + (x_2)^2 = c\}.$$

When the constant c is a positive number, negative number, or zero, the Minkowski sphere is respectively the hyperbolic plane $\mathbb{H}^2$, the de-Sitter plane $\mathbb{S}^2_1$, or the light cone.

Let σ be a solution of the sinh–Gordon equation

$$\frac{\partial^2\sigma}{\partial u^2} + \frac{\partial^2\sigma}{\partial v^2} = \sinh\sigma \tag{5–1}$$

under the relation

$$\sigma_{uv} = \frac{1}{2}\sigma_u\sigma_v \coth\frac{\sigma}{2}. \tag{5–2}$$

Then there is a CMC surface with the following first and second fundamental forms:

$$ds^2 = e^\sigma(du^2 + dv^2),$$
$$II = \frac{e^\sigma + 1}{2}du^2 + \frac{e^\sigma - 1}{2}dv^2,$$

with curvature line condition described above. The solution of (5–1) and (5–2) is written as

$$\sigma = \log\left(\frac{1 + k(u)l(v)}{1 - k(u)l(v)}\right)^2,$$

with functions $k(u)$ and $l(v)$ that satisfy

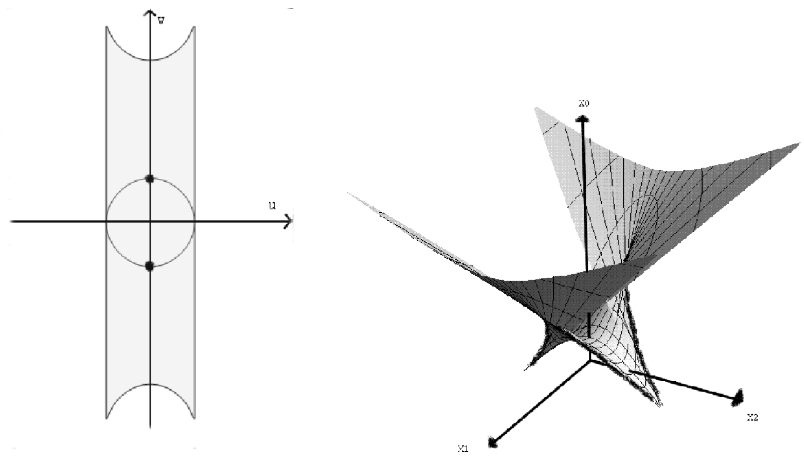
$$2(k'(u))^2 = -pk(u)^4 + 2(1-b)k(u)^2 + q, \tag{5–3}$$
$$2(\dot{l}(v))^2 = -ql(v)^4 + 2bl(v)^2 + p, \tag{5–4}$$

where b, p, q are real constants, $' = d/du$, and $\dot{} = d/dv$. For the sake of simplicity, we set $b = 1/2$, $p = q = 1$. Then we obtain solutions of (5–3) and (5–4) explicitly by

$$k(u) = \sqrt{\frac{1+\sqrt{5}}{2}}\,\mathrm{cn}\left(\sqrt{\frac{\sqrt{5}}{2}}u\right),$$
$$l(v) = \sqrt{\frac{1+\sqrt{5}}{2}}\,\mathrm{cn}\left(\sqrt{\frac{\sqrt{5}}{2}}v\right),$$

where cn is the Jacobi cn function of modulus

$$\sqrt{(5+\sqrt{5})/10}.$$



Domain of Enneper–Wente type Surface with swallowtail

FIGURE 8. Enneper–Wente type surface for $b = \frac{1}{2}$, $p = q = 1$.

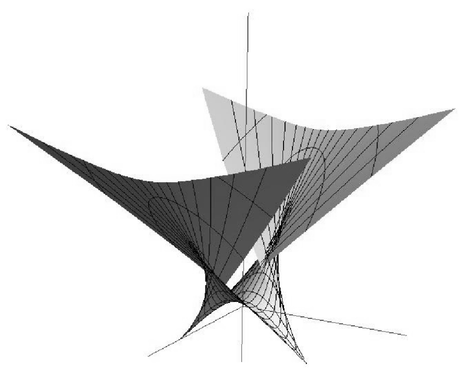


FIGURE 9. Enneper–Wente-type surface for $b = 0$, $p = 2$, $q = 1$.

Applying Walter's construction [Walter 87] to the Minkowski case, we get a CMC surface explicitly by

$$x_0 = \frac{-h'(u)}{\sqrt{h(u)^2 - h'(u)^2}} \sinh \mu(u) + \eta(u,v) \cosh \mu(u),$$

$$x_1 = \frac{2}{\sqrt{5}} \left(\frac{2h(u)h'(v)}{1 - h(u)h(v)} + \int \left(-h(v)^2 + \frac{1}{2} \right) dv \right),$$

$$x_2 = \frac{-h'(u)}{\sqrt{h(u)^2 - h'(u)^2}} \cosh \mu(u) + \eta(u,v) \sinh \mu(u),$$

where

$$h(x) = \sqrt{\frac{1+\sqrt{5}}{2}} \operatorname{cn} \left(\sqrt{\frac{\sqrt{5}}{2}} x \right),$$

$$\mu(u) = \int^u \left(\frac{2 + (1 + \sqrt{5} h(t)^2)}{2 - (1 + \sqrt{5} h(t)^2)} + \frac{h(t)^4 + 1}{h(t)^4 + h(t)^2 - 1} \right) dt,$$

$$h'(x) = \frac{dh}{dx}(x) = -\frac{\sqrt{5+\sqrt{5}}}{2} \operatorname{sn} \left(\sqrt{\frac{\sqrt{5}}{2}} x \right) \operatorname{dn} \left(\sqrt{\frac{\sqrt{5}}{2}} x \right),$$

$$\eta(u,v) = \frac{h(u)(1 + 2h(u)^2) - h(v)(2 - h(u)^2)}{\sqrt{5}(1 - h(u)h(v)) \sqrt{h(u)^2 - h'(u)^2}}.$$

This surface is defined on the following domain M^2:

$$M^2 = \left\{ (u,v) \mid \operatorname{cn} \left(\sqrt{\frac{\sqrt{5}}{2}} u \right) \operatorname{cn} \left(\sqrt{\frac{\sqrt{5}}{2}} v \right) > \frac{1 - \sqrt{5}}{2}, \ \operatorname{cn} \left(\sqrt{\frac{\sqrt{5}}{2}} u \right) > \sqrt{\frac{3 - \sqrt{5}}{2}} \right\}.$$

Then we can calculate the Gauss map $g : M^2 \longrightarrow \mathbb{C} \cup \{\infty\}$ explicitly by

$$g = \frac{1}{\Delta}\Big\{ (h(u)^3h(v)^2 + 2h(u)^3 - 2h(u)h(v)^2 + h(u)) \sinh \mu(u) - \sqrt{5}(1 - h(u)^2h(v)^2)h'(u) \cosh \mu(u) + 4\sqrt{-1}h(u)h'(v)\sqrt{h(u)^2 - h'(u)^2} \Big\},$$

where

$$\Delta = \sqrt{5}\big(1 - h(u)^2h(v)^2\big)\sqrt{h(u)^2 - h'(u)^2} - \sqrt{5}(1 - h(u)^2h(v)^2)h'(u) \sinh \mu(u) - (h(u)^3h(v)^2 + 2h(u)^3 - 2h(u)h(v)^2 + h(u)) \cosh \mu(u).$$

The solution σ of (5–1) diverges to $-\infty$ at some points, but the harmonic map g is defined at these points smoothly. We get that this map is an extended harmonic map. Therefore, the map g gives an example of an extended CMC surface. The set of singularities is

$$\left\{ (u, v) \middle| \operatorname{cn}\left(\sqrt{\frac{\sqrt{5}}{2}}u\right) \operatorname{cn}\left(\sqrt{\frac{\sqrt{5}}{2}}v\right) = \frac{\sqrt{5}-1}{2} \right\}$$

(see Figure 8, left).

By applying this g to Theorem 4.1(4) and (5), we get that the points

$$\left\{ (u, v) \mid \operatorname{cn}\left(\sqrt{\frac{\sqrt{5}}{2}}u\right) = 1,\ \operatorname{cn}\left(\sqrt{\frac{\sqrt{5}}{2}}v\right) = \frac{\sqrt{5}-1}{2} \right\}$$

are swallowtails, and the singular points other than the above points are cuspidal edges.

Remark 5.6. In the Euclidean case, Walter gives some doubly periodic constant-mean-curvature immersions. However, in the Minkowski case, we cannot obtain a doubly periodic constant-mean-curvature immersion by analogy to Walter's method.

Remark 5.7. Under another choice of b, p, q, we get a similar surface. For example, Figure 9 shows the surface when $b = 0$, $p = 2$, $q = 1$.

Finally we conclude this paper with the following open problem.

Open Problem 5.8. Are there extended CMC surfaces in $\mathbb{L}^3$ with cuspidal cross cap?

It is expected that there exists such an example among Enneper–Wente-type surfaces or similar classes.

ACKNOWLEDGMENTS

The author thanks Professors Kotaro Yamada and Wayne Rossman for their useful advice. The author also thanks the referee for a careful reading of this paper and valuable comments.

REFERENCES

[Abresch 87] U. Abresch. "Constant Mean Curvature Tori in Terms of Elliptic Functions." *J. reine angew. Math.* 374 (1987), 169–192.

[Akutagawa 94] K. Akutagawa. "Harmonic Diffeomorphisms of the Hyperbolic Plane." *Trans. Amer. Math. Soc.* 342 (1994), 325–342.

[Akutagawa and Nishikawa 90] K. Akutagawa and S. Nishikawa. "The Gauss Map and Spacelike Surfaces with Prescribed Mean Curvature in Minkowski 3-Space." *Tôhoku Math. J.* 42 (1990), 67–82.

[Arnol'd et al. 85] V. I. Arnol'd, S. M. Gusein-Zade, and A. N. Varchenko. *Singularities of Differentiable Maps*, vol. 1, Monographs in Mathematics 82. Basel: Birkhäuser, 1985.

[Beneki et al. 02] C. C. Beneki, G. Kaimakamis, and B. J. Papantoniou. "Helicoidal Surfaces in Three-Dimensional Minkowski Space." *J. Math. Anal. Appl.* 275 (2002), 586–614.

[Estudillo and Romero 92] F. J. M. Estudillo and A. Romero. "Generalized Maximal Surfaces in Lorentz–Minkowski Space $\mathbb{L}^3$." *Math. Proc. Camb. Phil. Soc.* 111 (1992), 515–524.

[Fujimori et al. 08] S. Fujimori, K. Saji, M. Umehara, and K. Yamada. "Singularities of Maximal Surfaces." *Math. Z.* 259 (2008), 827–848.

[Gálvez et al. 03] J. A. Gálvez, A. Martínez, and F. and Milán. "Complete Constant Gaussian Curvature Surfaces in the Minkowski Space and Harmonic Diffeomorphisms onto the Hyperbolic Plane" *Tôhoku Math. J.* 55 (2003), 467–476.

[Hopf 51] H. Hopf. "Über Flächen mit einer Relation zwischen den Hauptkrümmungen." *Math. Nachr.* 4 (1951), 232–249.

[Ishihara and Hara 88] T. Ishihara and F. Hara. "Surfaces of Revolution in the Lorentzian 3-Space." *J. Math. Tokushima Univ.* 22 (1988), 1–13.

[Kenmotsu 79] K. Kenmotsu. "Weierstrass Formula for Surfaces of Prescribed Mean Curvature." *Math. Ann.* 245 (1979), 89–99.

[Kobayashi 83] O. Kobayashi. "Maximal Surfaces in the 3-Dimensional Minkowski Space L^3." *Tokyo J. Math.* 6 (1983), 297–309.

[Kobayashi 84] O. Kobayashi. "Maximal Surfaces with Cone-like Singularities." *J. Math. Soc. Japan* 36 (1984), 609–617.

[Kokubu et al. 05] M. Kokubu, W. Rossman, K. Saji, M. Umehara, and K. Yamada. "Singularities of Flat Fronts in Hyperbolic 3-Space." *Pacific J. Math.* 221 (2005), 303–351.

[Umehara and Yamada 06] M. Umehara and K. Yamada. "Maximal Surfaces with Singularities in Minkowski Space." *Hokkaido Math. J.* 35 (2006), 13–40.

[Walter 87] R. Walter. "Explicit Examples to the H-Problem of Heinz Hopf." *Geom. Dedicata* 23 (1987), 187–213.

[Wang 91] H. Wang. "The Construction of Surfaces of Constant Mean Curvature in Minkowski 3-Space." In *Differential Geometry (Shanghai, 1991)*, pp. 243–246. River Edge, NJ: World Sci. Publ., 1993.

[Wente 86] H. C. Wente. "Counterexample to a Conjecture of H. Hopf." *Pacific J. Math.* 121 (1986), 193–243.

Yuhei Umeda, Graduate School of Mathematics, Kyushu University, Higashi-ku, Fukuoka, Japan (y-umeda@math.kyushu-u.ac.jp)

Received July 8, 2008; accepted in revised form January 14, 2009.