Lens Spaces Given from L-Space Homology 3-Spheres

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CONTENTS

We consider the problem of when an L-space homology sphere gives rise to lens spaces. We will show that when a knot in an L-space homology sphere Y yields $L(p, q)$ by an integral Dehn surgery, then the slope p is bounded by the genus of the knot and the correction term of Y, and we will demonstrate that many lens spaces are obtained from an L-space homology sphere whose correction term is equal to 2. 6

1. INTRODUCTION

Let K be a knot in a homology sphere Y. If an integral Dehn surgery over Y is homeomorphic to a lens space, then we say that K admits *lens surgery on* Y or simply *lens surgery*. The main problems on lens surgery are to determine when a lens space is obtained from Dehn surgery of a knot and when a knot K admits lens surgery.

J. Berge has defined the notion of a doubly primitive knot [Berge 90]. A doubly primitive knot K is defined to be a knot in S^3 such that K lies on the boundary of a genus-2 Heegaard surface Σ of S^3; for the Heegaard decomposition $V_1 \cup_\Sigma V_2$, K induces primitive elements in both $\pi_1(V_1)$ and $\pi_1(V_2)$. Berge conjectured that any knot admitting lens surgery on S^3 must be a doubly primitive knot. This conjecture remains open.

Berge divided the doubly primitive knots into several types, but it is unknown whether his classification is complete. One of our motivations has been to classify lens spaces obtained from the Poincaré homology sphere in analogy to Berge's classification, but we have thus far found no proof of completeness.

Throughout this paper we denote by $Y_r(K)$ Dehn surgery with slope r of a knot K in a 3-manifold Y. We define a lens space $L(p, q)$ to be the $-p/q$ Dehn surgery of the unknot U, namely $L(p, q) = S^3_{-p/q}(U)$. When we perform Dehn surgery on Y along K with slope r, the *dual knot* of K in $Y_r(K)$ is defined to be the core circleof

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the newly attached solid torus, and we denote the dual knot by $\widetilde{K}$.

For the classification of knots yielding lens spaces, the Alexander polynomial is used effectively. Indeed, the author showed in [Tange 07b] that the doubly primitive knots that yield $-L(p,q)$ can be distinguished by the Alexander polynomials. There exist examples that can determine knot type from the form of the Alexander polynomial.

For example, if $K \subset S^3$ admits lens surgery and $\Delta_K(t)$ is equal to 1, then K is the unknot (see [Kronheimer et al. 07]). Secondly, if $K \subset S^3$ admits lens surgery and the degree of $\Delta_K(t)$ is 1, then K is the trefoil knot (see [Goda and Teragaito 00]).

In this way, it seems that the condition that a knot admits lens surgery determines the isotopy type of knot from the Alexander polynomial. The degree of the Alexander polynomial, which is equal to the Seifert genus of the knot in this case, is the first invariant arising from the polynomial and must be studied first.

P. Kronheimer et al. proved in [Kronheimer et al. 07] that if $-L(p,q)$ is obtained from Dehn surgery of K in S^3, then the slope satisfies the lower bound $2g(K)-1 \leq p$. On the other hand, J. Rasmussen has proven in [Rasmussen 04] that the slope for lens surgery on S^3 satisfies the upper bound $p \leq 4g(K)+3$. We will provide a lower bound and an upper bound for the slopes of lens surgery on L-space homology spheres.

We call a rational homology sphere Y an *L-space* when $\widehat{HF}(Y,\mathfrak{s}) \cong \mathbb{Z}$ holds for any spin^c structure $\mathfrak{s}$. The only known examples of L-space homology spheres are S^3 and connected sums of several copies of the Poincaré homology spheres with the standard orientation or with the reverse orientation.

The lower bound of the slope for lens surgery on an L-space homology sphere is stated as $2g(K)-1 \leq p$ by the same argument as the proof of [Ozsváth and Szabó 03, Theorem 7.2] and [Kronheimer et al. 07, Corollary 8.5]. The upper bound is given in the following theorem.

Theorem 1.1. *Let Y be an L-space homology sphere. Suppose that $Y_p(K)$ is a lens space and K is a nontrivial knot in Y. Then $g(K)+2d(Y) > 0$, and the following bound holds:*

$$p < \frac{4g(K)(g(K)+1)}{g(K)+2d(Y)}. \tag{1–1}$$

This is proven in Section 3. If $Y = S^3$, then the bound recovers a result in [Rasmussen 04]. From the lower and upper bounds we obtain the following corollary:

Corollary 1.2. *Let Y be an L-space homology sphere. If $Y_p(K) = -L(p,q)$ and this is nontrivial surgery, then $2d(Y) \leq g+3$.*

Proof: From the upper and lower bounds for the slope, we have

$$2g(K)-1 < \frac{4g(K)(g(K)+1)}{g(K)+2d(Y)}$$
$$\Leftrightarrow g(K)+2d(Y) < \frac{4g(K)(g(K)+1)}{2g(K)-1}$$
$$\Leftrightarrow 2d(Y) < g+3+\frac{3}{2g-1}.$$

Here if $g=1$, then from Theorem 1.1 we have $2d(Y) > -1$ and $1 \leq p < \frac{8}{2d(Y)+1}$. Thus we have $d(Y)=0$ or 2. Therefore $2d(Y) < g+4$ holds. If $g \geq 2$, then $\frac{3}{2g-1} \leq 1$ holds. In this case we have $2d(Y) < g(K)+4$. □

There exist lens spaces coming from L-space homology spheres other than S^3. For example, $-L(22,3)$ is the 22 Dehn surgery on the Poincaré homology sphere with the standard orientation. In the case that an L-space homology sphere Y satisfies $d(Y)=2$, then as we will demonstrate in Section 5, we can construct many lens spaces by Dehn surgeries on Y (see Theorem 5.1 and Lemma 5.2). This construction is the second main result of this paper.

In fact, these lens spaces can be constructed from knots in the Poincaré homology sphere $\Sigma(2,3,5)$, and the dual knots are 1-bridge simple knots in the lens spaces. Moreover, except for one example, those examples appear as quadratic families such as Berge's sporadic families.

On the other hand, when $d(Y) \neq 0,2$, it is unlikely that Y constructs lens spaces by positive integral Dehn surgeries. As evidence, no homology sphere satisfying $p < 1000$ and $2 < |d(Y)| \leq 40$ can construct any lens spaces by a Maple computation. We conjecture the following:

Conjecture 1.3. *Let Y be an L-space homology sphere with $d(Y) \neq 0,2$. None of knots in Y constructs any lens space by positive integral Dehn surgery.*

The author proved this conjecture in the case that Y is the Poincaré homology sphere with the reverse orientation (see [Tange 07a]).

Furthermore, in Section 5, we shall show that there exists a lens space given as Dehn surgery on both S^3 and $\Sigma(2,3,5)$. Our example corresponds to the case in which the parameter ℓ in Lemma 5.2 is 0 and the two

Alexander polynomials coincide in $\mathbb{Z}[t,t^{-1}]/(t^p-1)$. It can be concluded that the dual knots of the two knots are homologous in the lens space.

2. THE EXACT TRIANGLE AND THE ALEXANDER POLYNOMIAL

In this section we review invariants concerning lens space surgery and L-space surgery. Let Y be an L-space homology sphere. We identify $\mathrm{Spin}^c(-L(p,q))$ with $\mathbb{Z}/p\mathbb{Z}$ after the canonical ordering in [Ozsváth and Szabó 03], with $\mathrm{Spin}^c(Y_0(K))$ identified with $\mathbb{Z}$ in the obvious way.

If a positive p Dehn surgery $Y_p(K)$ along knot K is a lens space $-L(p,q)$, then we have the following short exact sequence for every $0\neq i\in\mathbb{Z}/p\mathbb{Z}$:

$$0\to \bigoplus_{j\equiv i \bmod p} HF^+(Y_0,j)\to HF^+(Y_p(K),Q(i))$$
$$\to HF^+(Y)\to 0,$$

where $Q:\mathrm{Spin}^c(Y_0(K))\to\mathrm{Spin}^c(Y_p(K))$ is the correspondence induced from surgery cobordism by a 4-dimensional 2-handle. For any integer i with $i\equiv 0 \bmod p$ we have

$$0\to HF^+(Y)\to \bigoplus_{j\equiv 0 \bmod p} HF^+(Y_0,j)$$
$$\to HF^+(Y_p(K),Q(i))\to 0.$$

From these exact sequences the formulas

$$d(Y)-d(Y_p(K),Q(i))+d(-L(p,1),i)=2t_i(K), \quad (2\text{–}1)$$

are extracted as in [Ozsváth and Szabó 03], where the invariant $t_i(K)$ is the ith Turaev torsion of $Y_0(K)$. The Turaev torsion is nonnegative if $Y_p(K)$ is an L-space (see [Ozsváth and Szabó 03]).

Let $\tilde{K}$ be the dual knot of K, and C the core circle of a handlebody of the genus-one Heegaard decomposition of $-L(p,q)$, and let h be the integer satisfying $[\tilde{K}]=h[C]$, where $[\,*\,]$ stands for the homology class of $*$.

The set of classes $\mathcal{H}(p,K):=\{\pm h^{\pm}\}\subset\mathbb{Z}/p\mathbb{Z}$ is an invariant of lens surgery as stated in [Berge 90]. We always regard any element h in this set as the integer that satisfies $0\leq h<p$. The function $Q(i)$ can be written as $hi+c$, where $c=(h+1+p)(h-1)/2$ (see [Tange 09a]). If we change h to another element in $\mathcal{H}(p,K)$, we have to recalculate c, but by the same formula we can compute the same value $t_i(K)$.

By taking the summation of (2–1) over $i\in\mathbb{Z}/p\mathbb{Z}$, we obtain

$$p\left(d(Y)+\lambda(-L(p,q))-\lambda(-L(p,1))\right) \quad (2\text{–}2)$$
$$=2\sum_{i\in\mathbb{Z}}t_i(K)=2\sum_{i\geq 1}i^2a_i(K)=\Delta_K''(1),$$

where the Alexander polynomial satisfies $\Delta_K(t^{-1})=\Delta_K(t)$. The Casson–Walker invariant λ for a rational homology sphere W is computed by Rustamov's formula:

$$\sum_{\mathfrak{s}\in\mathrm{Spin}^c(W)}\left(\chi(HF_{\mathrm{red}}(W,\mathfrak{s}))-\frac{1}{2}d(W,\mathfrak{s})\right)$$
$$=\frac{|H_1(W,\mathbb{Z})|}{2}\lambda(W),$$

where λ multiplies the definition in [Ozsváth and Szabó 05] by 2. The Casson–Walker invariant of $L(p,q)$ is $-s(q,p)$, where $s(q,p)$ is the Dedekind sum (see, for example, [Walker 90]).

In [Tange 09a] we have computed the coefficients of the Alexander polynomial of the knot admitting lens surgery. Let $[\alpha,\beta]_{\mathbb{Z}}$ be the interval $[\alpha,\beta]\cap\mathbb{Z}$, and $[\gamma]_p$ the reduction of γ in $\mathbb{Z}/p\mathbb{Z}$ satisfying $0\leq[\gamma]_p<p$. We define $\Phi^k_{p,q}(h)$ to be

$$\#\{j\in[1,h']_{\mathbb{Z}}\,|\,[qj-k]_p\in[1,h]_{\mathbb{Z}}\},$$

where h and h' satisfy

$$h=[h]_p,\quad h'=[h^{-1}]_p,\quad h^2\equiv q \bmod p.$$

When $-L(p,q)=Y_p(K)$, for each class $i\in\mathbb{Z}/p\mathbb{Z}$ the sum

$$\tilde{a}_i(K):=\sum_{j\equiv i \bmod p}a_j(K)$$

is equal to

$$-m+\Phi^{hi+c}_{p,q}(h), \quad (2\text{–}3)$$

where m is the integer $\frac{hh'-1}{p}$.

3. THE UPPER BOUND FOR THE SLOPE OF LENS SURGERY

In this section we prove Theorem 1.1. We begin with a couple of lemmas.

Lemma 3.1. *Let Y be an L-space homology sphere and $Y_p(K)$ an L-space. The degree of the Alexander polynomial $\Delta_K(t)$ coincides with the Seifert genus $g(K)$.*

Lemma 3.2. *Let Y be an L-space homology sphere and $Y_p(K)$ an L-space. Then $\Delta_K(t)$ has the form*

$$\Delta_K(t)=(-1)^k+\sum_{j=1}^{k}(-1)^{k-j}(t^{n_j}+t^{-n_j})$$

for some increasing sequence of positive integers $0<n_1<n_2<\cdots<n_k$.

When Y and $Y_p(K)$ satisfy the same condition as that of Lemmas 3.1 and 3.2, the proof of Lemma 3.2 is immediately derived from an application of [Kronheimer et al. 07] and [Ozsváth and Szabó 05]. The homology of the top degree of $\widehat{HFK}(Y, K)$, which is equal to

$$\max\left\{j|\widehat{HFK}(Y, K, j) \neq 0\right\},$$

is $\mathbb{Z}$. This implies that $g(K)$ and the top degree coincide by [Ni 06, Theorem 1.1]. Therefore $g(K)$ and the degree of $\Delta_K(t)$ also coincide.

Here we compute the Alexander polynomials of knots admitting lens surgery on L-space homology spheres. From the estimate $2g(K)-1 \leq p$, the Alexander polynomial of K satisfying the condition of Lemma 3.2 has one of the following forms:

(I) $\sum_{|i|<\frac{p}{2}} \tilde{a}_i(K)t^i$ if $2g(K) < p$;

(II) $\sum_{|i|<\frac{p}{2}} \tilde{a}_i(K)t^i + t^{p/2} + t^{-p/2}$ if $2g(K) = p$;

(III) $\sum_{|i|<\frac{p}{2}} \tilde{a}_i(K)t^i - \left(t^{(p-1)/2} + t^{-(p-1)/2}\right) + \left(t^{(p+1)/2} + t^{-(p+1)/2}\right)$ if $2g(K) = p+1$.

We note that $\tilde{a}_i(K) = 0$, ± 1, or 2 and if $\tilde{a}_i(K) = 2$, then $\Delta_K(t)$ satisfies (II) and $2i = p$ holds.

Next we compute the coefficient $\tilde{a}_{-h'(c+1)}(K)$.

Proposition 3.3. *Let Y be an L-space homology sphere. If $Y_p(K) = -L(p, q)$, then $\tilde{a}_{-h'(c+1)}(K)$ is 0 or ± 1. In particular, if $\tilde{a}_{-h'(c+1)}(K) = 1$, then p is even and $q = 1$.*

Proof: Using (2–3), we obtain

$$\begin{aligned}
\tilde{a}_{-h'(c+1)}(K) &= -m + \Phi^{-1}_{p,q}(h) \\
&= -m + \#\{j \in [1, h']_{\mathbb{Z}} | [qj + 1]_p \in [1, h]_{\mathbb{Z}}\} \\
&= -m + \#\{j \in [1, h']_{\mathbb{Z}} | [qj]_p \in [0, h-1]_{\mathbb{Z}}\} \\
&= -m + \Phi^0_{p,q}(h) - 1 \\
&= \tilde{a}_{-h'c}(K) - 1.
\end{aligned}$$

Using Lemma 3.2, we see that $\tilde{a}_{-h'(c+1)}(K) = 0$ or ± 1. If $\tilde{a}_{-h'(c+1)}(K) = 1$ holds, then $\tilde{a}_{-h'c}(K)$ is 2, p is even, and c is $\frac{p}{2}$. On the other hand, c is $\frac{p+q-1}{2}$ or $\frac{q-1}{2}$. Hence we have $q = 1$. □

To prove Theorem 1.1 we essentially use the following proposition from [Rasmussen 04] and two lemmas.

Proposition 3.4. [Rasmussen 04, Proposition 2.4] *Assume that*

$$s(q, p) - s(1, p) \leq \frac{1}{4}\left(\frac{p}{4} - 1\right).$$

Then q is 1, 2, or 3.

Lemma 3.5. *Let Y be an L-space homology sphere. If $Y_p(K) = -L(p, 2)$, then $p^2 + 8$ or $p^2 - 8p + 8$ is a perfect square.*

Lemma 3.6. *If $Y_p(K) = -L(p, 3)$, then one of the following holds for an integer i satisfying $i = 0$ or 1:*

(i) $p^2 + 4(3i - 3)p + 12$ *is a perfect square;*

(ii) $p^2 + 4(3i - 4)p + 12$ *is a perfect square and* $p \equiv 1 \bmod 3$*;*

(iii) $p^2 + 4(3i - 2)p + 12$ *is a perfect square and* $p \equiv 2 \bmod 3$.

Lemmas 3.5 and 3.6 will be proved in Section 4.

Proof of Theorem 1.1: From Frøyshov's inequality in [Rasmussen 04] and (2–2), we have

$$\begin{aligned}
p(s(q, p) - s(1, p)) &= p(\lambda(-L(p, q)) - \lambda(-L(p, 1))) \\
&= 2\sum_{i \geq 1} i^2 a_i(K) - pd(Y) \\
&\leq g(K)(g(K) + 1) - pd(Y).
\end{aligned}$$

We assume that

$$p \geq \frac{4g(K)(g(K) + 1)}{g(K) + 2d(Y)}. \tag{3–1}$$

Then we have

$$\begin{aligned}
&g(K)(g(K) + 1) - pd(Y) \leq \frac{p}{4}\left(\frac{p}{4} - 1\right) \\
&\Leftrightarrow p^2 + (16d(Y) - 4)p - 16g(K)(g(K) + 1) \geq 0 \\
&\Leftrightarrow p \geq \sqrt{(8d(Y) - 2)^2 + 16g(K)(g(K) + 1)} - 8d(Y) + 2 \\
&\quad = \frac{16g(K)(g(K) + 1)}{\sqrt{(8d(Y) - 2)^2 + 16g(K)(g(K) + 1)} + 8d(Y) - 2} \\
&\Leftarrow p \geq \frac{16g(K)(g(K) + 1)}{\sqrt{4 + 16g(K)(g(K) + 1)} + 8d(Y) - 2} \\
&\quad = \frac{4g(K)(g(K) + 1)}{g(K) + 2d(Y)}.
\end{aligned}$$

Thus by Proposition 3.4, q must be 1, 2, or 3. Next we consider the case $q = 1$, 2, or 3.

(A) The case $q = 1$: Then we have $h = h'$. We may assume that $2h < p$ by replacing h with $p - h$. From (2–3) we have $\tilde{a}_i(K) = -m$ and $\tilde{a}_j(K) = -m + h$ for integers i, j. By Lemma 3.2, h is 1, 2, or 3. The integer h is equal to 1 if and only if $m = 0$. By the definition of m, we have $h^2 = mp + 1$.

(a) The case $h = 1$: From (2–1) we have $d(Y) = 0$ or 2.

(0) The case $d(Y) = 0$: Y is S^3 and K is the unknot.

(2) The case $d(Y) = 2$: $\Delta_K(t)$ is $t^{-(p+1)/2} - t^{-(p-1)/2} + 1 - t^{(p-1)/2} + t^{(p+1)/2}$.

(b) The case $h = 2$: There is no h satisfying $p \mid h^2 - 1$ and $2h < p$.

(c) The case $h = 3$: From $p \mid h^2 - 1$ and $2h < p$ we have $p = 8$. By (2–1) we have $d(Y) = 2$ and $\Delta_K(t) = t^{-4} - t^{-3} + t^{-1} - 1 + t - t^3 + t^4$.

(B) The case $q = 2$: From Lemma 3.5, the only possibility for p is 7. Then we have $d(Y) = 0$ or 2.

(a) The case $d(Y) = 0$: $\Delta_K(t)$ is $t^{-1} - 1 + t^{-1}$.

(b) The case $d(Y) = 2$: $\Delta_K(t)$ is $t^{-4} - t^{-3} + t^{-1} - 1 + t - t^3 + t^4$.

(C) The case $q = 3$: From Lemma 3.6, the possibilities for p are 11, 13, and 22.

(a) The case $p = 11$: We have $d(Y) = 0$ or 2.

(0) The case $d(Y) = 0$: $\Delta_K(t)$ is $t^{-2} - t^{-1} + 1 - t + t^2$.

(2) The case $d(Y) = 2$: $\Delta_K(t)$ is $t^{-6} - t^{-5} + t^{-2} - t^{-1} + 1 - t + t^2 - t^5 + t^{-6}$.

(b) The case $p = 13$: We have $d(Y) = 0$ or 2.

(0) The case $d(Y) = 0$: $\Delta_K(t)$ is $t^{-3} - t^{-2} + 1 - t^2 + t^3$.

(2) The case $d(Y) = 2$: $\Delta_K(t)$ is $t^{-7} - t^{-6} + t^{-3} - t^{-2} + 1 - t^2 + t^3 - t^{-6} + t^{-7}$.

(c) The case $p = 22$: We have $d(Y) = 2$ and $\Delta_K(t)$ is $t^{-11} - t^{-10} + t^{-6} - t^{-5} + t^{-2} - 1 + t^2 - t^5 + t^6 - t^{10} + t^{11}$.

None of the cases satisfies inequality (3–1). This proves inequality (1–1). □

4. PROOFS OF LEMMAS 3.5 AND 3.6

In this section, h satisfies $0 < h < p$ and $\gcd(h, p) = 1$, and h' is the inverse of $h \bmod p$ with $0 < h' < p$. To prove Lemmas 3.5 and 3.6, we first recall the following result in [Tange 09b].

Proposition 4.1. [Tange 09b, Proposition 2.2] *Let p, q be a pair of coprime integers with $0 < q < p$. The integer h is one of the solutions to $x^2 \equiv q \bmod p$. Let w be the integer with $qh' = h + pw$. Then we have*

$$\Phi^{-1}_{p,q}(h) = -2 \sum_{j=1}^{w} \left\lfloor \frac{pj}{q} \right\rfloor + (w+1)(h'-1). \qquad (4\text{–}1)$$

We now prove Lemmas 3.5 and 3.6.

Proof of Lemma 3.5: The integer p is odd, because $\gcd(p, 2) = 1$. We can see easily that $2h' = h$ or $2h' = h + p$.

If $2h' = h$, namely $w = 0$, from Proposition 4.1 we have $\Phi^{-1}_{p,2}(h) = h' - 1$. From Proposition 3.3 we have $\tilde{a}_{-h'c-h'}(K) = -m + h' - 1 = 0$ or -1. Here for an integer i with $i = 0$ or 1, we have $m = h' - 1 + i$. Since by the definition of m we have $mp = hh' - 1 = 2h'^2 - 1$, h' is the solution of the quadratic equation

$$2x^2 - px + (1-i)p - 1 = 0.$$

The discriminant $p^2 + 8(i-1)p + 8$ has to be a perfect square.

If $2h' = h + p$, namely $w = 1$, from Proposition 4.1 we have

$$\begin{aligned}\Phi^{-1}_{p,2}(h) &= -2\left\lfloor \frac{p}{2} \right\rfloor + 2(h'-1) = -(p-1) + 2(h'-1) \\ &= -p + 2h' - 1.\end{aligned}$$

From Proposition 3.3 we have $\tilde{a}_{-h'c-h'}(K) = -m - p + 2h' - 1 = 0$ or -1. In the same way as above, $p^2 + 8(i-1)p + 8$ is a perfect square for an integer i with $i = 0$ or 1. □

Proof of Lemma 3.6: The integer p is congruent to 1 mod 3 or to 2 mod 3, because $\gcd(p, 3) = 1$. We can see easily that $3h'$ is h, $h + p$, or $h + 2p$.

If $3h' = h$, namely $w = 0$, from Proposition 4.1 we have $\Phi^{-1}_{p,3}(h) = h' - 1$. From Proposition 3.3 we have $\tilde{a}_{-h'c-h'}(K) = -m + h' - 1 = 0$ or -1. Here for an integer i with $i = 0$ or 1 we have $m = h' - 1 + i$. Similarly, $p^2 + 12(i-1)p + 12$ is a perfect square.

If $3h' = h + p$ or $h + 2p$, namely $w = 1$ or 2, from Proposition 4.1 we can derive square conditions as in Table 1. Therefore the stated condition holds. □

w	condition on p	square condition
0	$p = 1, 2 \bmod 3$	$X^2 = p^2 + 4(3i-3)p + 12$
1	$p = 1 \bmod 3$	$X^2 = p^2 + 4(3i-4)p + 12$
1	$p = 2 \bmod 3$	$X^2 = p^2 + 4(3i-2)p + 12$
2	$p = 1, 2 \bmod 3$	$X^2 = p^2 + 4(3i-3)p + 12$

TABLE 1. The square conditions in the case $q = 3$.

5. A TABLE OF LENS SURGERIES ON Y WITH $d(Y) = 2$

In this section we shall assume that Y is an L-space homology sphere with $d(Y) = 2$. For example, Y is $\Sigma(2,3,5)$. We restrict our attention to lens surgery satisfying $2g(K) - 1 < p$, because if a lens surgery $Y_p(K) = -L(p,q)$ satisfies $2g(K) - 1 = p$, then the situation is slightly subtle and difficult. In fact, we can construct a lens space from both S^3 and $\Sigma(2,3,5)$ whose dual knots give the same homology class.

For example, $-L(3,1)$ is given from the unknot K_1 in S^3, and $-L(3,1)$ is also given from the knot K_2 in $\Sigma(2,3,5)$ as in Figure 1. The duals to these knots are $\tilde{K}_1$ and $\tilde{K}_2$. The genera of K_1 and K_2 are 0 and 2, respectively. To avoid such Dehn surgeries we establish the condition $2g(K) - 1 < p$.

All the data (p,q,h,g') in Tables 2–5 are obtained in the following way. The pairs (p,h) with $1 \le p \le 5000$ are the solutions satisfying the following: a_i and t_i are computed by (2–1), (2–3), and $2g - 1 < p$; the integers a_i satisfy Lemma 3.2, and the integers t_i are nonnegative. The parameter q satisfies $0 < q < p$ and $q = h_1^2 \bmod p$. Here h_1 is the minimal value in $\{h_1, h_1', p - h_1, p - h_1'\}$. The fourth integer, g', is defined to be $2g - p - 1$, where

$$g = \max\left\{ i \in \left\{0, 1, \ldots, \left\lfloor \frac{p}{2} \right\rfloor \right\} \mid -m + \Phi_{p,q}^{hi+c}(h) \neq 0 \right\}.$$

Conjecture 1.3 is based on this computation for $d(Y) \neq 0, 2$.

Here we say that $K \subset -L(p,q)$ is a 0-*bridge knot* if K is isotopic to a knot that lies on a Heegaard surface of the genus-one Heegaard splitting, and a 1-*bridge knot* if K is not a 0-bridge knot and K is the union of two proper arcs embedded in the handlebodies of the genus-one Heegaard splitting.

Moreover, the knot is said to be *simple* if the arcs are embedded in meridian disks of the handlebodies. This definition is based on [Berge 90]. Any triple (p,q,h) uniquely determines either a 0-bridge knot or a 1-bridge simple knot in $-L(p,q)$.

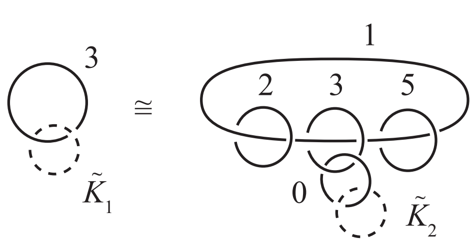

FIGURE 1. Lens space $-L(3,1)$ constructed from both S^3 and $\Sigma(2,3,5)$.

Theorem 5.1. *The lens spaces $-L(p,q)$ in Tables 2–5 are constructed by p Dehn surgery on knots in $\Sigma(2,3,5)$. Moreover, the dual knots are 1-bridge simple knots in $-L(p,q)$.*

Before presenting the proof of this theorem we prove the following lemma.

Lemma 5.2. *Every lens space in Tables 2–5 appears in Table 6 for some $\ell \in \mathbb{Z} \setminus \{0\}$.*

Proof: By direct computation we can prove that each pair (p,q,h) in Tables 2–5 is covered by the twenty families in Table 6. The computation of the genus of K is due to Lemma 3.1. □

Proof of Theorem 5.1: To each pair (p,q,h) in Lemma 5.2 we can take the 1-bridge simple knot $\tilde{K}$ in $-L(p,q)$, because if K were a 0-bridge, then K would be a torus knot in S^3, which is inconsistent with $d(Y) = 2$.

The $-\tilde{a}_{-h'(c+1)}(K)^*$-surgery in the sense of [Saito 07] yields a homology sphere Y. Thus we can find a knot $K \subset Y$ satisfying $Y_p(K) = -L(p,q)$. The presentation of $\pi_1(Y)$ is the following according to [Tange 07b]:

$$\left\langle x_1, x_2 \,\middle|\, \prod_{i=1}^{p} x_1 x_2^{E_h(qi+1)}, \Big(\prod_{i=1}^{h'-1} x_1 x_2^{E_h(qi+1)} \Big) x_1 x_2^{-\tilde{a}_{h'c-h}} \right\rangle, \qquad (5\text{–}1)$$

where $E_h : \mathbb{Z}/p\mathbb{Z} \to \{0,1\}$ is defined by

$$E_h(k) = \begin{cases} 1 & \text{if } 1 \le [k]_p \le h, \\ 0 & \text{otherwise.} \end{cases}$$

Transforming the group presentation (5–1) for each of the lens spaces in the twenty families of Table 6, we can easily show that the fundamental group is isomorphic to

$$\langle x, y \mid (xy)^2 = x^3 = y^5 \rangle. \qquad (5\text{–}2)$$

In Lemma 5.2 we show the existence of the isomorphism in all cases satisfying $\ell \ge 1$. That the case $\ell \le -1$ is satisfied can be also proven in the same way. From these isomorphisms and the celebrated resolution of the Poincaré conjecture by G. Perelman in [Perelman 02], Y is homeomorphic to $\Sigma(2,3,5)$. The orientation of $\Sigma(2,3,5)$ is the usual one because $d(Y) = 2$. □

In Tables 7 through 26 we show that the group presentation (5–1) is isomorphic to $\pi_1(\Sigma(2,3,5))$ in the case of $\ell \ge 1$ in Lemma 5.2.

p	q	h	g'	p	q	h	g'	p	q	h	g'
8	1	3	−1	221	127	41	−4	442	157	77	−3
22	3	5	−1	228	61	17	−3	445	186	39	−4
38	7	7	−1	239	67	28	−2	445	84	23	−4
40	9	7	−1	243	133	43	−4	449	80	23	−4
43	15	12	−2	244	45	17	−3	450	79	23	−3
53	11	8	−2	246	43	17	−3	463	211	40	−6
67	14	9	−2	247	134	58	−2	469	107	24	−4
68	13	9	−1	249	94	29	−4	497	79	24	−4
70	11	9	−1	250	39	17	−3	509	116	25	−4
71	38	16	−2	253	141	30	−4	513	112	25	−4
87	13	10	−2	263	61	18	−4	514	139	41	−3
100	29	27	−1	275	49	18	−2	517	108	25	−4
101	21	18	−2	294	67	19	−3	521	201	42	−4
102	19	11	−1	297	64	19	−4	532	93	25	−3
103	18	11	−2	298	13	19	−3	532	309	85	−3
105	16	11	−2	298	67	31	−3	537	337	64	−6
106	37	19	−3	301	176	64	−2	547	295	44	−6
113	31	12	−2	303	115	32	−4	555	121	26	−4
125	19	12	−2	311	168	49	−4	571	202	66	−6
134	39	21	−1	312	49	19	−3	578	151	27	−5
137	30	13	−2	316	65	33	−5	583	93	26	−4
138	31	13	−3	329	71	20	−4	599	139	44	−4
139	30	13	−2	337	188	51	−4	610	351	91	−3
141	37	22	−2	353	97	34	−2	625	241	46	−6
145	51	44	−2	376	145	71	−3	633	151	28	−6
148	85	23	−3	379	159	36	−4	638	93	47	−7
159	37	14	−2	383	101	22	−4	673	473	72	−6
179	39	24	−2	386	211	37	−5	676	181	47	−3
187	69	50	−2	411	73	22	−4	706	135	29	−5
191	34	15	−2	424	157	37	−3	709	251	49	−6
197	51	26	−4	428	89	23	−5	710	131	29	−5
217	39	16	−2	441	121	38	−4	711	493	74	−6

TABLE 2. Lens spaces with $p \leq 711$ which homology spheres with $d(Y) = 2$ yield.

p	q	h	g'	p	q	h	g'	p	q	h	g'
715	199	98	−4	1103	291	60	−4	1552	849	145	−5
736	393	51	−7	1129	240	37	−6	1563	640	73	−8
739	161	30	−6	1135	234	37	−6	1583	334	110	−10
767	133	30	−4	1141	421	62	−6	1618	149	75	−11
773	181	50	−4	1162	253	125	−5	1634	427	73	−5
789	172	31	−6	1163	149	38	−8	1641	340	112	−10
790	171	31	−5	1168	201	37	−5	1653	283	44	−6
796	165	31	−5	1171	321	63	−8	1717	307	152	−6
805	211	104	−4	1173	814	95	−8	1727	389	46	−8
813	211	32	−6	1191	253	38	−6	1742	283	45	−7
823	340	53	−6	1198	631	65	−9	1758	451	47	−9
828	133	31	−5	1223	848	97	−8	1772	925	79	−11
841	107	54	−8	1226	257	63	−5	1779	337	46	−8
873	151	32	−4	1243	201	38	−6	1783	427	76	−6
878	129	33	−7	1276	291	131	−5	1803	406	47	−8
893	237	54	−4	1285	336	66	−8	1807	309	46	−6
919	379	56	−6	1298	223	39	−5	1811	1247	118	−10
925	519	112	−4	1331	135	68	−10	1841	561	78	−8
938	151	33	−5	1376	361	67	−5	1849	360	47	−8
953	505	58	−8	1377	223	40	−6	1853	189	48	−10
975	181	34	−6	1379	302	41	−8	1855	319	158	−6
991	265	87	−8	1403	361	42	−8	1857	352	47	−8
999	226	35	−6	1408	273	41	−7	1873	1289	120	−10
1004	233	57	−5	1414	267	41	−7	1887	406	80	−10
1021	301	58	−6	1426	783	139	−5	1900	289	47	−7
1027	189	35	−6	1437	589	70	−8	1933	163	82	−12
1027	573	118	−4	1447	317	42	−8	1963	511	80	−6
1033	192	35	−6	1471	771	72	−10	1985	416	49	−8
1037	271	89	−8	1488	169	43	−9	1993	408	49	−8
1057	239	36	−6	1513	285	70	−6	2001	721	82	−8
1072	121	61	−9	1526	323	43	−7	2031	796	83	−10
1088	281	37	−7	1534	315	43	−7	2035	269	166	−6

TABLE 3. Lens spaces with $712 \leq p \leq 2035$ which homology spheres with $d(Y) = 2$ yield.

p	q	h	g'	p	q	h	g'	p	q	h	g'
2067	433	50	−8	2703	661	58	−12	3357	739	64	−12
2101	1093	86	−12	2747	617	58	−10	3361	1366	107	−12
2126	511	83	−7	2752	401	193	−7	3449	889	106	−8
2153	551	52	−10	2773	1090	97	−12	3473	1681	110	−16
2185	1179	172	−6	2823	541	58	−10	3501	214	164	−14
2217	487	52	−10	2843	389	96	−8	3532	401	65	−11
2221	906	87	−10	2843	1471	100	−14	3542	683	65	−11
2222	379	51	−7	2875	489	58	−8	3577	1310	220	−8
2258	551	53	−11	2901	901	98	−10	3578	911	67	−13
2269	589	86	−6	2911	570	59	−10	3587	2447	166	−14
2276	1093	89	−13	2921	560	59	−10	3612	613	65	−9
2303	487	53	−10	2926	463	199	−7	3664	441	109	−9
2313	403	133	−12	2947	1159	100	−12	3697	1905	114	−16
2325	379	52	−8	2992	489	59	−9	3713	911	68	−14
2350	459	53	−10	3008	777	99	−7	3718	771	67	−11
2358	451	53	−9	3046	1471	103	−15	3730	759	67	−11
2377	969	90	−10	3063	781	62	−12	3743	613	66	−10
2380	361	179	−7	3077	523	60	−8	3775	2001	226	−8
2383	688	135	−12	3081	640	61	−10	3777	847	68	−12
2400	409	53	−7	3091	630	61	−10	3829	1509	114	−14
2444	361	89	−7	3101	1101	102	−10	3838	651	67	−9
2458	1275	93	−13	3151	584	206	−8	3851	991	112	−8
2502	523	55	−9	3175	1291	104	−12	3889	872	69	−12
2507	409	54	−8	3181	1652	156	−14	3928	1089	117	−17
2512	513	55	−9	3183	661	62	−10	3973	651	68	−10
2542	1179	185	−7	3188	781	63	−13	4030	599	233	−9
2587	443	141	−12	3198	523	61	−9	4033	1590	117	−14
2588	661	57	−11	3209	777	102	−8	4078	991	115	−9
2647	624	96	−14	3253	716	63	−12	4107	793	70	−12
2653	596	57	−10	3256	961	107	−15	4133	1051	72	−14
2654	687	93	−7	3263	2123	158	−14	4166	1053	121	−17
2661	1822	143	−12	3337	1563	212	−8	4187	269	179	−16

TABLE 4. Lens spaces with $2036 \leq p \leq 4187$ which homology spheres with $d(Y) = 2$ yield.

p	q	h	g	p	q	h	g
4201	1321	118	−12	4539	937	74	−12
4213	828	71	−12	4553	2054	187	−16
4225	816	71	−12	4578	751	73	−11
4240	2001	239	−9	4589	493	122	−10
4278	1051	73	−15	4609	1016	75	−14
4281	1288	181	−16	4651	3164	189	−16
4299	1744	121	−14	4663	2395	128	−18
4348	657	119	−9	4683	793	74	−10
4411	2143	124	−18	4728	841	77	−15
4417	912	73	−12	4732	569	253	−9
4429	900	73	−12	4798	1231	125	−9
4433	751	72	−10	4832	457	75	−11
4441	1561	122	−12	4883	1201	78	−16
4487	989	74	−14	4922	1829	131	−19
4510	661	247	−9	4954	975	77	−13
4515	1831	124	−14	4966	963	77	−13

TABLE 5. Lens spaces with $4088 \leq p \leq 5000$ which homology spheres with $d(Y) = 2$ yield.

	p	h	$2g - p - 1$
A_1	$14\ell^2 + 7\ell + 1$	$\pm(7\ell + 2)^{\pm 1} \mod p$	$-\lvert\ell\rvert$
A_2	$20\ell^2 + 15\ell + 3$	$\pm(5\ell + 2)^{\pm 1} \mod p$	$-\lvert\ell\rvert$
B	$30\ell^2 + 9\ell + 1$	$\pm(6\ell + 1)^{\pm 1} \mod p$	$-\lvert\ell\rvert$
C_1	$42\ell^2 + 23\ell + 3$	$\pm(7\ell + 2)^{\pm 1} \mod p$	$-\lvert\ell\rvert$
C_2	$42\ell^2 + 47\ell + 13$	$\pm(7\ell + 4)^{\pm 1} \mod p$	$-\lvert\ell\rvert$
D_1	$52\ell^2 + 15\ell + 1$	$\pm(13\ell + 2)^{\pm 1} \mod p$	$-\lvert\ell\rvert$
D_2	$52\ell^2 + 63\ell + 19$	$\pm(13\ell + 8)^{\pm 1} \mod p$	$-\lvert\ell\rvert$
E_1	$54\ell^2 + 15\ell + 1$	$\pm(27\ell + 4)^{\pm 1} \mod p$	$-\lvert\ell\rvert$
E_2	$54\ell^2 + 39\ell + 7$	$\pm(27\ell + 10)^{\pm 1} \mod p$	$-\lvert\ell\rvert$
F_1	$69\ell^2 + 17\ell + 1$	$\pm(23\ell + 3)^{\pm 1} \mod p$	$-2\lvert\ell\rvert$
F_2	$69\ell^2 + 29\ell + 3$	$\pm(23\ell + 5)^{\pm 1} \mod p$	$-2\lvert\ell\rvert$
G_1	$85\ell^2 + 19\ell + 1$	$\pm(17\ell + 2)^{\pm 1} \mod p$	$-2\lvert\ell\rvert$
G_2	$85\ell^2 + 49\ell + 7$	$\pm(17\ell + 5)^{\pm 1} \mod p$	$-2\lvert\ell\rvert$
H_1	$99\ell^2 + 35\ell + 3$	$\pm(11\ell + 2)^{\pm 1} \mod p$	$-2\lvert\ell\rvert$
H_2	$99\ell^2 + 53\ell + 7$	$\pm(11\ell + 3)^{\pm 1} \mod p$	$-2\lvert\ell\rvert$
I_1	$120\ell^2 + 16\ell + 1$	$\pm(12\ell + 1)^{\pm 1} \mod p$	$-2\lvert\ell\rvert$
I_2	$120\ell^2 + 20\ell + 1$	$\pm(20\ell + 2)^{\pm 1} \mod p$	$-2\lvert\ell\rvert$
I_3	$120\ell^2 + 36\ell + 3$	$\pm(12\ell + 2)^{\pm 1} \mod p$	$-2\lvert\ell\rvert$
J	$120\ell^2 + 104\ell + 22$	$\pm(12\ell + 5)^{\pm 1} \mod p$	$-\lvert 2\ell + 1\rvert$
K	191	15	−2

TABLE 6. The families of lens spaces obtained from homology spheres with $d(Y) = 2$.

$$\begin{aligned}
\pi_1(Y) &\cong \left\langle x,y \mid x^2yx^{2\ell-1}y[(x^{2\ell+1}y)^6x^{2\ell-1}y]^{\ell}, x^2yx^{2\ell-1}y[(x^{2\ell+1}y)^6x^{2\ell-1}y]^{\ell-1}x^2\right\rangle\\
&\cong \left\langle x,y \mid x^{-2\ell+1}yx^{-2}y[y^6x^{-2}y]^{\ell}, x^{-2\ell+1}yx^{-2}y[y^6x^{-2}y]^{\ell-1}x^2\right\rangle\\
&\cong \left\langle x,y \mid x^{-2\ell+1}yx^{-2}y[y^6x^{-2}y]^{\ell-1}x^2, x^{-2}(y^6x^{-2}y)\right\rangle\\
&\cong \left\langle x,y \mid x^{-2\ell+1}yx^{-2}yx^{2\ell}, x^{-2}y^6x^{-2}y\right\rangle\\
&\cong \left\langle x,y \mid xyx^{-2}y, x^{-2}(y^6x^{-2}y)\right\rangle\\
&\cong \left\langle x,y \mid (xy)^2=x^3, y^5=(xy)^2\right\rangle \quad \text{by } x^{-2}y=(xy)^{-1}
\end{aligned}$$

TABLE 7. The case A_1 $(\ell \geq 1)$.

$$\begin{aligned}
\pi_1(Y) &\cong \left\langle x,y \mid x^4yx^{4\ell-1}y[(x^{4\ell+3}y)^4x^{4\ell-1}y]^{\ell},\ x^4yx^{4\ell-1}y[(x^{4\ell+3}y)^4x^{4\ell-1}y]^{\ell-1}x^4\right\rangle\\
&\cong \left\langle x,y \mid x^{-4\ell+1}yx^{-4}y[y^4x^{-4}y]^{\ell},\ x^{-4\ell+1}yx^{-4}y[y^4x^{-4}y]^{\ell-1}x^4\right\rangle\\
&\cong \left\langle x,y \mid x^{-4\ell+1}yx^{-4}y[y^4x^{-4}y]^{\ell-1}x^4,\ x^{-4}(y^4x^{-4}y)\right\rangle\\
&\cong \left\langle x,y \mid x^{-4\ell+1}yx^{-4}yx^{4\ell},\ x^{-4}(y^4x^{-4}y)\right\rangle\\
&\cong \left\langle x,y \mid xyx^{-4}y,\ x^{-4}(y^4x^{-4}y)\right\rangle\\
&\cong \left\langle x,y \mid (xy)^2=x^5,\ y^3=(xy)^2\right\rangle \quad \text{by } x^{-4}y=(xy)^{-1}
\end{aligned}$$

TABLE 8. The case A_2 $(\ell \geq 1)$.

$$\begin{aligned}
\pi_1(Y) &\cong \left\langle x,y \mid \left[x^5yx^{5\ell-1}y[(x^{5\ell+4}y)^2x^{5\ell-1}y]^{\ell-1}\right]^2(x^{5\ell+4}y)^2x^{5\ell-1}y, \left[x^5yx^{5\ell-1}y[(x^{5\ell+4}y)^2x^{5\ell-1}y]^{\ell-1}\right]^2x^5\right\rangle\\
&\cong \left\langle x,y \mid \left[x^5yx^{5\ell-1}y[(x^{5\ell+4}y)^2x^{5\ell-1}y]^{\ell-1}\right]^2x^5,\ x^{-5}(x^{5\ell+4}y)^2x^{5\ell-1}y\right\rangle\\
&\cong \left\langle x,y \mid \left[x^{-5\ell+6}y^2[(x^5y)^2y]^{\ell-1}\right]^2x^5,\ x^{-5}(x^5y)^2y\right\rangle\\
&\cong \left\langle x,y \mid \left[x^{-5\ell+6}y^2x^{5\ell-5}\right]^2x^5,\ y^3=x^5\right\rangle\\
&\cong \left\langle x,y \mid (xy^2)^2x^5,\ y^{-3}=x^5\right\rangle \quad \text{by } [x^5,y]=e\\
&\cong \left\langle x,y \mid (xy^{-1})^2=y^{-3},\ y^{-3}=x^5\right\rangle
\end{aligned}$$

TABLE 9. The case B $(\ell \geq 1)$.

$$\begin{aligned}
\pi_1(Y) &\cong \left\langle x,y \mid [x^{6\ell+5}y(x^{6\ell-1}y)^2]^2x^{12\ell+4}y(x^{6\ell-1}y)^2[x^{6\ell+5}y(x^{6\ell-1}y)^2x^{6\ell+5}y(x^{6\ell-1}y)^3]^{\ell-1},\ [x^{6\ell+5}y(x^{6\ell-1}y)^2]^2x^{6\ell+5}y\right\rangle\\
&\cong \left\langle x,y \mid (x^6y^3)^2x^{6\ell+5}y^3(x^6y^3x^6y^4)^{\ell-1},\ (x^6y^3)^2x^6y\right\rangle\\
&\cong \left\langle x,y \mid y^{-1}x^{6\ell-1}y^3(y^{-1}x^{-6}y)^{\ell-1},\ (x^6y^3)^2x^6y\right\rangle\\
&\cong \left\langle x,y \mid x^{6\ell-1}y^2x^{-6\ell+6},\ (x^6y^3)^2x^6y\right\rangle\\
&\cong \left\langle x,y \mid x^5=y^{-2},\ (x^6y^3)^2x^6y\right\rangle\\
&\cong \left\langle x,y \mid x^5=y^{-2},\ (xy)^3=x^{-5}\right\rangle
\end{aligned}$$

TABLE 10. The case C_1 $(\ell \geq 1)$.

$$\begin{aligned}\pi_1(Y) &\cong \big\langle x, y \mid (x^{6\ell+5}y)^2(x^{6\ell-1}yx^{6\ell+5}y)^2x^{12\ell+4}y(x^{6\ell+5}yx^{6\ell-1}y)^2[x^{6\ell+5}y(x^{6\ell+5}yx^{6\ell-1}y)^3]^{\ell-1},\\ &\qquad (x^{6\ell+5}y)^2(x^{6\ell-1}yx^{6\ell+5}y)^2x^{6\ell+5}y\big\rangle\\ &\cong \big\langle x, y \mid y^2(x^{-6}y^2)^2x^{6\ell-1}y(yx^{-6}y)^2[y(yx^{-6}y)^3]^{\ell-1},\ y^2(x^{-6}y^2)^2y\big\rangle\\ &\cong \big\langle x, y \mid y^2(x^{-6}y^2)^2y,\ y^{-1}x^{6\ell-1}y(yx^{-6}y)^2[y(yx^{-6}y)^3]^{\ell-1}\big\rangle\\ &\cong \big\langle x, y \mid y^3(x^{-6}y^2)^2,\ y^{-1}x^{6\ell-1}y^{-2}[y^2(x^{-6}y^2)^3y^{-1}]^{\ell-1}\big\rangle\\ &\cong \big\langle x, y \mid y^3(x^{-6}y^2)^2,\ y^{-1}x^{6\ell-1}y^{-2}(y^{-1}x^{-6}y^2y^{-1})^{\ell-1}\big\rangle\\ &\cong \big\langle x, y \mid y^3(x^{-6}y^2)^2,\ y^{-1}x^{6\ell-1}y^{-2}(y^{-1}x^{-6\ell+6}y)\big\rangle\\ &\cong \big\langle x, y \mid y^3(x^{-6}y^2)^2,\ y^{-3}x^5\big\rangle \cong \big\langle x, y \mid y^3(x^{-1}y^{-1})^2,\ y^{-3}x^5\big\rangle\end{aligned}$$

TABLE 11. The case C_2 $(\ell \geq 1)$.

$$\begin{aligned}\pi_1(Y) &\cong \big\langle x, y \mid [x^{4\ell+3}y(x^{4\ell-1}y)^2]^4x^{8\ell+2}y(x^{4\ell-1}y)^2[(x^{4\ell+3}y(x^{4\ell-1}y)^2)^4x^{4\ell-1}y]^{\ell-1},\ [x^{4\ell+3}y(x^{4\ell-1}y)^2]^4x^{4\ell+3}y\big\rangle\\ &\cong \big\langle x, y \mid [x^4y^3]^4x^{4\ell+3}y^3[(x^4y^3)^4y]^{\ell-1},\ [x^4y^3]^4x^4y\big\rangle\\ &\cong \big\langle x, y \mid (x^4y)^{-1}x^{4\ell+3}y^3[(x^4y)^{-1}y]^{\ell-1},\ [x^4y^3]^4x^4y\big\rangle\\ &\cong \big\langle x, y \mid y^{-1}x^{4\ell-1}y^3[y^{-1}x^{-4}y]^{\ell-1},\ [x^4y^3]^4x^4y\big\rangle\\ &\cong \big\langle x, y \mid x^3y^2,\ [x^4y^3]^4x^4y\big\rangle\\ &\cong \big\langle x, y \mid x^3y^2,\ (xy)^5x^3\big\rangle\end{aligned}$$

TABLE 12. The case D_1 $(\ell \geq 1)$.

$$\begin{aligned}\pi_1(Y) &\cong \big\langle x, y \mid x^{4\ell+3}y[(x^{4\ell+3}y)^3x^{4\ell-1}y]^2(x^{4\ell+3}y)^3x^{8\ell+2}y\{(x^{4\ell+3}y)^3x^{4\ell-1}y\}^2[x^{4\ell+3}y\{(x^{4\ell+3}y)^3x^{4\ell-1}y\}^3]^{\ell-1},\\ &\qquad x^{4\ell+3}y[(x^{4\ell+3}y)^3x^{4\ell-1}y]^2(x^{4\ell+3}y)^4\big\rangle\\ &\cong \big\langle x, y \mid x^4y[(x^4y)^3y]^2(x^4y)^3x^{4\ell+3}y\{(x^4y)^3y\}^2[x^4y\{(x^4y)^3y\}^3]^{\ell-1},\ x^4y[(x^4y)^3y]^2(x^4y)^4\big\rangle\\ &\cong \big\langle x, y \mid (x^4y)^{-1}x^{4\ell+3}y\{(x^4y)^3y\}^2[(x^4y)^{-1}y]^{\ell-1},\ x^4y[(x^4y)^3y]^2(x^4y)^4\big\rangle\\ &\cong \big\langle x, y \mid y^{-1}x^{4\ell-1}y\{(x^4y)^3y\}^2y^{-1}x^{-4\ell+4}y,\ x^4y[(x^4y)^3y]^2(x^4y)^4\big\rangle\\ &\cong \big\langle x, y \mid x^3y\{(x^4y)^3y\}^2y^{-1},\ x^4y[(x^4y)^3y]^2(x^4y)^4\big\rangle\\ &\cong \big\langle x, y \mid x^3y\{(x^4y)^3y\}^2y^{-1},\ xy(x^4y)^4\big\rangle\\ &\cong \big\langle x, y \mid x^{-1}y(y^3x^{-4}y)^2(x^{-4}y)^{-1},\ x^{-3}y^5\big\rangle\\ &\cong \big\langle x, y \mid x^{-1}(y^4x^{-4})^2x^4,\ x^{-3}y^5\big\rangle \cong \big\langle x, y \mid y^3(y^{-1}x^{-1})^2,\ y^{-3}x^5\big\rangle\end{aligned}$$

TABLE 13. The case D_2 $(\ell \geq 1)$.

$$\begin{aligned}
\pi_1(Y) &\cong \big\langle x, y \mid [x^{2\ell+1}y(x^{2\ell+1}yx^{2\ell-1}y)^2]^5 x^{2\ell+1}yx^{4\ell}y(x^{2\ell+1}yx^{2\ell-1}y)^2[(x^{2\ell+1}y(x^{2\ell+1}yx^{2\ell-1}y)^2)^5x^{2\ell+1}yx^{2\ell-1}y]^{\ell-1},\\
&\qquad [x^{2\ell+1}y(x^{2\ell+1}yx^{2\ell-1}y)^2]^5(x^{2\ell+1}y)^2\big\rangle\\
&\cong \big\langle x, y \mid [x^2y(x^2y^2)^2]^5x^2yx^{2\ell+1}y(x^2y^2)^2[(x^2y(x^2y^2)^2)^5x^2y^2]^{\ell-1},\ [x^2y(x^2y^2)^2]^5(x^2y)^2\big\rangle\\
&\cong \big\langle x, y \mid (x^2y)^{-1}x^{2\ell+1}y(x^2y^2)^2[(x^2y)^{-2}x^2y^2]^{\ell-1},\ [x^2y(x^2y^2)^2]^5(x^2y)^2\big\rangle\\
&\cong \big\langle x, y \mid y^{-1}x^{2\ell-1}y(x^2y^2)^2y^{-1}x^{-2\ell+2}y,\ [x^2y(x^2y^2)^2]^5(x^2y)^2\big\rangle\\
&\cong \big\langle x, y \mid y^{-1}xy(x^2y^2)^2,\ [x^2y(x^2y^2)^2]^5(x^2y)^2\big\rangle\\
&\cong \big\langle x, y \mid x(yx^2y)^2,\ (xy)^5(x^2y)^2\big\rangle\\
&\cong \big\langle x, y \mid x(x^{-2}y^2)^2,\ (x^{-1}y)^5y^2\big\rangle \cong \big\langle x, y \mid x^{-1}(x^2y^2)^2,\ (xy)^5y^2\big\rangle\\
&\cong \big\langle x, y \mid x^3y^4,\ (xy)^5y^2\big\rangle \cong \big\langle x, y \mid (xy^2)^3y^{-2},\ (xy)^5y^2\big\rangle \cong \big\langle x, y \mid x^3y^2,\ (xy)^5y^{-2}\big\rangle
\end{aligned}$$

TABLE 14. The case E_1 $(\ell \geq 1)$.

$$\begin{aligned}
\pi_1(Y) &\cong \big\langle x, y \mid [x^{2\ell+1}y\{(x^{2\ell+1}y)^4x^{2\ell-1}y\}^2]^2(x^{2\ell+1}y)^4x^{4\ell}y[(x^{2\ell+1}y)^4x^{2\ell-1}y]^2\\
&\qquad \times [\{x^{2\ell+1}y\{(x^{2\ell+1}y)^4x^{2\ell-1}y\}^2\}^2(x^{2\ell+1}y)^4x^{2\ell-1}y]^{\ell-1},\ [x^{2\ell+1}y\{(x^{2\ell+1}y)^4x^{2\ell-1}y\}^2]^2(x^{2\ell+1}y)^5\big\rangle\\
&\cong \big\langle x, y \mid [x^2y\{(x^2y)^4y\}^2]^2(x^2y)^4x^{2\ell+1}y[(x^2y)^4y]^2[\{x^2y\{(x^2y)^4y\}^2\}^2(x^2y)^4y]^{\ell-1},\ [x^2y\{(x^2y)^4y\}^2]^2(x^2y)^5\big\rangle\\
&\cong \big\langle x, y \mid (x^2y)^{-1}x^{2\ell+1}y[(x^2y)^4y]^2[(x^2y)^{-1}y]^{\ell-1},\ [x^2y\{(x^2y)^4y\}^2]^2(x^2y)^5\big\rangle\\
&\cong \big\langle x, y \mid xy[(x^2y)^4y]^2y^{-1},\ [x^2y\{(x^2y)^4y\}^2]^2(x^2y)^5\big\rangle\\
&\cong \big\langle x, y \mid xy[(x^2y)^4y]^2y^{-1},\ (xy)^2(x^2y)^5\big\rangle\\
&\cong \big\langle x, y \mid x[y(x^2y)^4]^2,\ (xy)^2(x^2y)^5\big\rangle\\
&\cong \big\langle x, y \mid x(x^{-2}y^5)^2,\ (x^{-1}y)^2y^5\big\rangle \cong \big\langle x, y \mid (xy^5)^3y^{-5},\ (xy^6)^2y^{-5}\big\rangle \cong \big\langle x, y \mid x^3y^{-5},\ (xy)^2y^{-5}\big\rangle
\end{aligned}$$

TABLE 15. The case E_2 $(\ell \geq 1)$.

$$\begin{aligned}
\pi_1(Y) &\cong \big\langle x, y \mid [(x^{3\ell+1}y)^3x^{3\ell-2}y]^5(x^{3\ell+1}y)^2x^{6\ell-1}y(x^{3\ell+1}y)^2x^{3\ell-2}y[\{(x^{3\ell+1}y)^3x^{3\ell-2}y\}^5](x^{3\ell+1}y)^2x^{3\ell-2}y]^{\ell-1},\\
&\qquad [(x^{3\ell+1}y)^3x^{3\ell-2}y]^5(x^{3\ell+1}y)^3\big\rangle\\
&\cong \big\langle x, y \mid [(x^3y)^3y]^5(x^3y)^2x^{3\ell+1}y(x^3y)^2y[\{(x^3y)^3y\}^5](x^3y)^2y]^{\ell-1},\ [(x^3y)^3y]^5(x^3y)^3\big\rangle\\
&\cong \big\langle x, y \mid (x^3y)^{-1}x^{3\ell+1}y(x^3y)^2y[(x^3y)^{-1}y]^{\ell-1},\ [(x^3y)^3y]^5(x^3y)^3\big\rangle\\
&\cong \big\langle x, y \mid xy(x^3y)^2,\ [(x^3y)^3y]^5(x^3y)^3\big\rangle\\
&\cong \big\langle x, y \mid x^{-2}y^3,\ (y^3x^{-3}y)^5y^3\big\rangle\\
&\cong \big\langle x, y \mid x^{-2}y^3,\ (x^{-1}y)^5y^3\big\rangle \cong \big\langle x, y \mid (xy^{-1})^2y^3,\ x^5y^3\big\rangle
\end{aligned}$$

TABLE 16. The case F_1 $(\ell \geq 1)$.

$$\begin{aligned}\pi_1(Y) &\cong \big\langle x,y \mid [(x^{3\ell+1}y)^5x^{3\ell-2}y]^3(x^{3\ell+1}y)^4x^{6\ell-1}y(x^{3\ell+1}y)^4x^{3\ell-2}y[\{(x^{3\ell+1}y)^5x^{3\ell-2}y\}^3(x^{3\ell+1}y)^4x^{3\ell-2}y]^{\ell-1},\\ &\qquad [(x^{3\ell+1}y)^5x^{3\ell-2}y]^3(x^{3\ell+1}y)^5\big\rangle\\ &\cong \big\langle x,y \mid [(x^3y)^5y]^3(x^3y)^4x^{3\ell+1}y(x^3y)^4y[\{(x^3y)^5y\}^3(x^3y)^4y]^{\ell-1},\ [(x^3y)^5y]^3(x^3y)^5\big\rangle\\ &\cong \big\langle x,y \mid (x^3y)^{-1}x^{3\ell+1}y(x^3y)^4y[(x^3y)^{-1}y]^{\ell-1},\ [(x^3y)^5y]^3(x^3y)^5\big\rangle\\ &\cong \big\langle x,y \mid xy(x^3y)^4,\ [(x^3y)^5y]^3(x^3y)^5\big\rangle\\ &\cong \big\langle x,y \mid x^{-2}y^5,\ (y^5x^{-3}y)^3y^5\big\rangle\\ &\cong \big\langle x,y \mid x^{-2}y^5,\ (x^{-1}y)^3y^5\big\rangle \cong \big\langle x,y \mid (xy^{-1})^2y^5,\ x^3y^5\big\rangle \cong \big\langle x,y \mid (xy)^2y^{-5},\ x^3y^{-5}\big\rangle\end{aligned}$$

TABLE 17. The case F_2 $(\ell \geq 1)$.

$$\begin{aligned}\pi_1(Y) &\cong \big\langle x,y \mid [(x^{5\ell+2}y)^2x^{5\ell-3}y]^5x^{5\ell+2}yx^{10\ell-1}yx^{5\ell+2}yx^{5\ell-3}y[\{(x^{5\ell+2}y)^2x^{5\ell-3}y\}^5x^{5\ell+2}yx^{5\ell-3}y]^{\ell-1},\\ &\qquad [(x^{5\ell+2}y)^2x^{5\ell-3}y]^5(x^{5\ell+2}y)^2\big\rangle\\ &\cong \big\langle x,y \mid [(x^5y)^2y]^5x^5yx^{5\ell+2}yx^5y^2[\{(x^5y)^2y\}^5x^5y^2]^{\ell-1},\ [(x^5y)^2y]^5(x^5y)^2\big\rangle\\ &\cong \big\langle x,y \mid (x^5y)^{-1}x^{5\ell+2}yx^5y^2[(x^5y)^{-1}y]^{\ell-1},\ [(x^5y)^2y]^5(x^5y)^2\big\rangle\\ &\cong \big\langle x,y \mid x^2yx^5y,\ [(x^5y)^2y]^5(x^5y)^2\big\rangle\\ &\cong \big\langle x,y \mid x^{-3}y^2,\ (y^2x^{-5}y)^5y^2\big\rangle\\ &\cong \big\langle x,y \mid x^{-3}y^2,\ (xy^{-1})^5y^2\big\rangle \cong \big\langle x,y \mid x^{-3}(xy)^2,\ y^5(xy)^{-2}\big\rangle\end{aligned}$$

TABLE 18. The case G_1 $(\ell \geq 1)$.

$$\begin{aligned}\pi_1(Y) &\cong \big\langle x,y \mid [(x^{5\ell+2}y)^5x^{5\ell-3}y]^2(x^{5\ell+2}y)^4x^{10\ell-1}y(x^{5\ell+2}y)^4x^{5\ell-3}y[\{(x^{5\ell+2}y)^5x^{5\ell-3}y\}^2(x^{5\ell+2}y)^4x^{5\ell-3}y]^{\ell-1},\\ &\qquad [(x^{5\ell+2}y)^5x^{5\ell-3}y]^2(x^{5\ell+2}y)^5\big\rangle\\ &\cong \big\langle x,y \mid [(x^5y)^5y]^2(x^5y)^4x^{5\ell+2}y(x^5y)^4y[\{(x^5y)^5y\}^2(x^5y)^4y]^{\ell-1},\ [(x^5y)^5y]^2(x^5y)^5\big\rangle\\ &\cong \big\langle x,y \mid (x^5y)^{-1}x^{5\ell+2}y(x^5y)^4y[(x^5y)^{-1}y]^{\ell-1},\ [(x^5y)^5y]^2(x^5y)^5\big\rangle\\ &\cong \big\langle x,y \mid x^2y(x^5y)^4,\ [(x^5y)^5y]^2(x^5y)^5\big\rangle\\ &\cong \big\langle x,y \mid x^{-3}y^5,\ (x^{-2}y)^2y^5\big\rangle\\ &\cong \big\langle x,y \mid x^{-3}y^5,\ (x^{-2}y^6)^2y^{-5}\big\rangle \cong \big\langle x,y \mid x^{-3}y^5,\ (xy)^2y^{-5}\big\rangle\end{aligned}$$

TABLE 19. The case G_2 $(\ell \geq 1)$.

$$\begin{aligned}\pi_1(Y) &\cong \big\langle x,y \mid [(x^{9\ell+4}y)^2x^{9\ell-5}y]^3x^{9\ell+4}yx^{18\ell-1}yx^{9\ell+4}yx^{9\ell-5}y[\{(x^{9\ell+4}y)^2x^{9\ell-5}y\}^3x^{9\ell+4}yx^{9\ell-5}y]^{\ell-1},\\ &\qquad [(x^{9\ell+4}y)^2x^{9\ell-5}y]^3(x^{9\ell+4}y)^2\big\rangle\\ &\cong \big\langle x,y \mid [(x^9y)^2y]^3x^9yx^{9\ell+4}yx^9y^2[\{(x^9y)^2y\}^3x^9y^2]^{\ell-1},\ [(x^9y)^2y]^3(x^9y)^2\big\rangle\\ &\cong \big\langle x,y \mid (x^9y)^{-1}x^{9\ell+4}yx^9y^2[(x^9y)^{-1}y]^{\ell-1},\ [(x^9y)^2y]^3(x^9y)^2\big\rangle\\ &\cong \big\langle x,y \mid x^4yx^9y,\ [(x^9y)^2y]^3(x^9y)^2\big\rangle\\ &\cong \big\langle x,y \mid x^{-5}y^2,\ (y^2x^{-9}y)^3y^2\big\rangle\\ &\cong \big\langle x,y \mid x^{-5}y^2,\ (x^{-4}y)^3y^2\big\rangle \cong \big\langle x,y \mid x^{-5}y^2,\ (xy^{-1})^3y^2\big\rangle \cong \big\langle x,y \mid x^{-5}(xy)^2,\ y^3(xy)^{-2}\big\rangle\end{aligned}$$

TABLE 20. The case H_1 $(\ell \geq 1)$.

$$\begin{aligned}\pi_1(Y) &\cong \big\langle x,y \mid [(x^{9\ell+4}y)^3x^{9\ell-5}y]^2(x^{9\ell+4}y)^2x^{18\ell-1}y(x^{9\ell+4}y)^2x^{9\ell-5}y[\{(x^{9\ell+4}y)^3x^{9\ell-5}y\}^2(x^{9\ell+4}y)^2x^{9\ell-5}y]^{\ell-1},\\ &\qquad [(x^{9\ell+4}y)^3x^{9\ell-5}y]^2(x^{9\ell+4}y)^3\big\rangle\\ &\cong \big\langle x,y \mid [(x^9y)^3y]^2(x^9y)^2x^{9\ell+4}y(x^9y)^2y[\{(x^9y)^3y\}^2(x^9y)^2y]^{\ell-1},\ [(x^9y)^3y]^2(x^9y)^3\big\rangle\\ &\cong \big\langle x,y \mid (x^9y)^{-1}x^{9\ell+4}y(x^9y)^2y[(x^9y)^{-1}y]^{\ell-1},\ [(x^9y)^3y]^2(x^9y)^3\big\rangle\\ &\cong \big\langle x,y \mid x^4y(x^9y)^2,\ [(x^9y)^3y]^2(x^9y)^3\big\rangle\\ &\cong \big\langle x,y \mid x^{-5}y^3,\ (y^3x^{-9}y)^2y^3\big\rangle\\ &\cong \big\langle x,y \mid x^{-5}y^3,\ (x^{-4}y)^2y^3\big\rangle \cong \big\langle x,y \mid x^{-5}y^3,\ (xy^{-2})^2y^3\big\rangle \cong \big\langle x,y \mid x^{-5}y^3,\ (xy)^2y^{-3}\big\rangle\end{aligned}$$

TABLE 21. The case H_2 ($\ell \geq 1$).

$$\begin{aligned}\pi_1(Y) &\cong \big\langle x,y \mid [(x^{10\ell+3}y)^2x^{5\ell-1}y(x^{10\ell+3}yx^{15\ell+2}yx^{10\ell+3}yx^{5\ell-1}y)^{\ell-1}]^3x^{10\ell+3}yx^{15\ell+2}yx^{10\ell+3}yx^{5\ell-1}y,\\ &\qquad [(x^{10\ell+3}y)^2x^{5\ell-1}y(x^{10\ell+3}yx^{15\ell+2}yx^{10\ell+3}yx^{5\ell-1}y)^{\ell-1}]^3(x^{10\ell+3}y)^2\big\rangle\\ &\cong \big\langle x,y \mid [y^2x^{-5\ell-4}y(yx^{5\ell-1}y^2x^{-5\ell-4}y)^{\ell-1}]^3yx^{5\ell-1}y^2x^{-5\ell-4}y,\ [y^2x^{-5\ell-4}y(yx^{5\ell-1}y^2x^{-5\ell-4}y)^{\ell-1}]^3y^2\big\rangle\\ &\cong \big\langle x,y \mid y^{-1}x^{5\ell-1}y^2x^{-5\ell-4}y,\ [y^2x^{-5\ell-4}y(yx^{5\ell-1}y^2x^{-5\ell-4}y)^{\ell-1}]^3y^2\big\rangle\\ &\cong \big\langle x,y \mid x^{-5}y^2,\ (y^2x^{-5\ell-4}y^{2\ell-1})^3y^2\big\rangle\\ &\cong \big\langle x,y \mid x^{-5}y^2,\ (y^2xy^{-3})^3y^2\big\rangle \cong \big\langle x,y \mid x^{-5}y^2,\ (xy^{-1})^3y^2\big\rangle \cong \big\langle x,y \mid x^{-5}(xy)^2,\ y^3(xy)^{-2}\big\rangle\end{aligned}$$

TABLE 22. The case I_1 ($\ell \geq 1$).

$$\begin{aligned}\pi_1(Y) &\cong \big\langle x,y \mid [(x^{6\ell+1}y)^5x^{3\ell-1}y\{(x^{6\ell+1}y)^4x^{9\ell}y(x^{6\ell+1}y)^4x^{3\ell-1}y\}^{\ell-1}]^2(x^{6\ell+1}y)^4x^{9\ell}y(x^{6\ell+1}y)^4x^{3\ell-1}y,\\ &\qquad [(x^{6\ell+1}y)^5x^{3\ell-1}y\{(x^{6\ell+1}y)^4x^{9\ell}y(x^{6\ell+1}y)^4x^{3\ell-1}y\}^{\ell-1}]^2(x^{6\ell+1}y)^5\big\rangle\\ &\cong \big\langle x,y \mid [y^5x^{-3\ell-2}y\{y^4x^{3\ell-1}y^5x^{-3\ell-2}y\}^{\ell-1}]^2y^4x^{3\ell-1}y^5x^{-3\ell-2}y,\ [y^5x^{-3\ell-2}y\{y^4x^{3\ell-1}y^5x^{-3\ell-2}y\}^{\ell-1}]^2y^5\big\rangle\\ &\cong \big\langle x,y \mid y^{-1}x^{3\ell-1}y^5x^{-3\ell-2}y,\ [y^5x^{-3\ell-2}y\{y^4x^{3\ell-1}y^5x^{-3\ell-2}y\}^{\ell-1}]^2y^5\big\rangle\\ &\cong \big\langle x,y \mid x^{-3}y^5,\ (y^5x^{-3\ell-2}y^{5\ell-4})^2y^5\big\rangle \cong \big\langle x,y \mid x^{-3}y^5,\ (y^5xy^{-9})^2y^5\big\rangle\\ &\cong \big\langle x,y \mid x^{-3}y^5,\ (xy^{-4})^2y^5\big\rangle \cong \big\langle x,y \mid x^{-3}y^5,\ (xy)^2y^{-5}\big\rangle\end{aligned}$$

TABLE 23. The case I_2 ($\ell \geq 1$).

$$\begin{aligned}\pi_1(Y) &\cong \big\langle x,y \mid [(x^{10\ell+3}y)^3x^{5\ell-1}y\{(x^{10\ell+3}y)^2x^{15\ell+2}y(x^{10\ell+3}y)^2x^{5\ell-1}y\}^{\ell-1}]^2(x^{10\ell+3}y)^2x^{15\ell+2}y(x^{10\ell+3}y)^2x^{5\ell-1}y,\\ &\qquad [(x^{10\ell+3}y)^3x^{5\ell-1}y\{(x^{10\ell+3}y)^2x^{15\ell+2}y(x^{10\ell+3}y)^2x^{5\ell-1}y\}^{\ell-1}]^2(x^{10\ell+3}y)^3\big\rangle\\ &\cong \big\langle x,y \mid [y^3x^{-5\ell-4}y\{y^2x^{5\ell-1}y^3x^{-5\ell-4}y\}^{\ell-1}]^2y^2x^{5\ell-1}y^3x^{-5\ell-4}y,\ [y^3x^{-5\ell-4}y\{y^2x^{5\ell-1}y^3x^{-5\ell-4}y\}^{\ell-1}]^2y^3\big\rangle\\ &\cong \big\langle x,y \mid y^{-1}x^{5\ell-1}y^3x^{-5\ell-4}y,\ [y^3x^{-5\ell-4}y(y^2x^{5\ell-1}y^3x^{-5\ell-4}y)^{\ell-1}]^2y^3\big\rangle\\ &\cong \big\langle x,y \mid x^{-5}y^3,\ (y^3x^{-5\ell-4}y^{3\ell-2})^2y^3\big\rangle\\ &\cong \big\langle x,y \mid x^{-5}y^3,\ (y^3xy^{-5})^2y^3\big\rangle \cong \big\langle x,y \mid x^{-5}y^3,\ (xy^{-2})^2y^3\big\rangle \cong \big\langle x,y \mid x^{-5}y^3,\ (xy)^2y^{-3}\big\rangle\end{aligned}$$

TABLE 24. The case I_3 ($\ell \geq 1$).

$$\begin{aligned}\pi_1(Y) &\cong \langle x, y \mid [(x^{5\ell+1}yx^{10\ell+7}yx^{15\ell+8}yx^{10\ell+7}y)^\ell x^{5\ell+1}y(x^{10\ell+7}y)^2]^2(x^{5\ell+1}yx^{10\ell+7}yx^{15\ell+8}yx^{10\ell+7}y)^{\ell-1}x^{5\ell+1}y(x^{10\ell+7}y)^2,\\ &\quad x^{5\ell+1}yx^{10\ell+7}yx^{5\ell+1}\rangle\\ &\cong \langle x, y \mid [(x^{-5\ell-6}y^2x^{5\ell+1}y^2)^\ell x^{-5\ell-6}y^3]^2(x^{-5\ell-6}y^2x^{5\ell+1}y^2)^{\ell-1}x^{-5\ell-6}y^3,\ x^{-5}y^2\rangle\\ &\cong \langle x, y \mid (y^{2\ell}x^{-5\ell-6}y^3)^2y^{2\ell-2}x^{-5\ell-6}y^3,\ x^{-5}y^2\rangle\\ &\cong \langle x, y \mid (x^{-1}y)^2y^{-4}x^{-1}y^3,\ x^{-5}y^2\rangle\\ &\cong \langle x, y \mid (x^{-1}y)^2x^{-1}y^{-1},\ x^{-5}y^2\rangle \cong \langle x, y \mid (x^{-1}y)^3y^{-2},\ x^{-5}y^2\rangle\\ &\cong \langle x, y \mid y^3(xy)^{-2},\ x^{-5}(xy)^2\rangle\end{aligned}$$

TABLE 25. The case J ($\ell \geq 1$).

$$\begin{aligned}\pi_1(Y) &\cong \langle x, y \mid x^6yx^{11}yx^{17}yx^{11}y(x^{17}y)^2x^{11}yx^{17}yx^{11}yx^6yx^{11}y(x^{17}yx^{11}y)^2,\ x^6yx^{11}yx^{17}yx^{11}yx^6\rangle\\ &\cong \langle x, y \mid x^{-5}y^2x^6y^2(x^6y)^2yx^6y^2x^{-5}y^2(x^6y^2)^2,\ x^{-5}y^2x^6y^2x^6\rangle\\ &\cong \langle x, y \mid x^6y^2x^6y^2x^{-5}y^2(x^6y^2)^2y,\ x^{-5}y^2x^6y^2x^6\rangle\\ &\cong \langle x, y \mid x^6y^2x^6y^5,\ x^{-5}y^2x^6y^2x^6\rangle \cong \langle x, y \mid x^6y^2x^6y^5,\ xy^2x^6y^2\rangle\\ &\cong \langle x, y \mid x^6x^{-1}y^{-2}y^5,\ xy^2x^6y^2\rangle \cong \langle x, y \mid x^5y^3,\ xy^2x^6y^2\rangle\\ &\cong \langle x, y \mid x^5y^3,\ xy^{-1}xy^2\rangle \cong \langle x, y \mid x^5y^{-3},\ y^{-3}(xy)^2\rangle\end{aligned}$$

TABLE 26. The case K ($\ell \geq 1$).

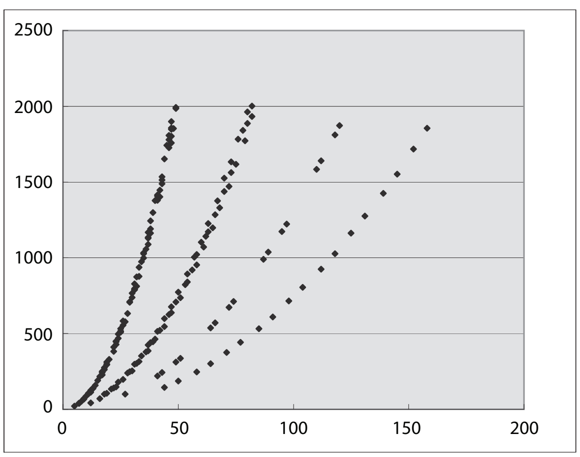

FIGURE 2. The h-p graph in the case $\Sigma(2, 3, 5)$.

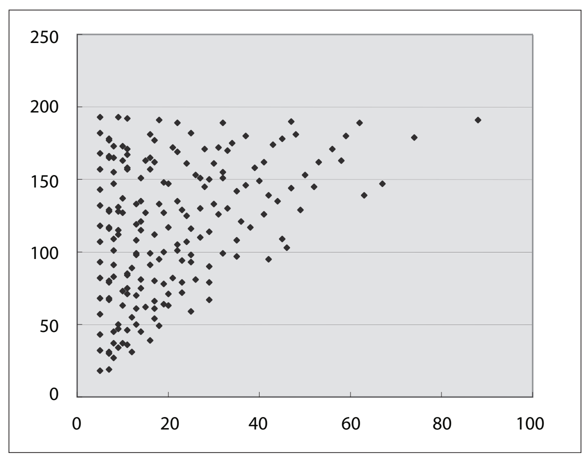

FIGURE 3. The h-p graph in the case S^3.

The author expects that lens spaces obtained from the Poincaré homology sphere are contained in a quadratic family. The type K may be an annoyance for the classification, but further calculation may reveal a new sequence.

In Figures 2 and 3, the horizontal and vertical axes represent the parameters h and p respectively. Each point in Figure 2 represents a lens surgery over $\Sigma(2, 3, 5)$ with slope $p \leq 2007$, while each point in Figure 3 represents hyperbolic lens surgery over S^3 plotted in the order of the slope from the smallest to that of the same cardinality as in the plots in Figure 2.

The plots in Figure 2 are less dense than those in Figure 3. The two right-hand families in Figure 2 are of types E and F. The rest of the plots are of A, B, C, D, G, H, I, J, and K. To draw Figure 3, we referred to the last table in [Berge 90]. Tables 2 through 5 give a rough conjecture on the basis of Figure 2.

Conjecture 5.3. *Suppose* $\Sigma(2,3,5)_p(K) = -L(p,q)$. *For* $h \in \mathcal{H}(p,K)$, *one of the following six cases holds:*

(i) $L(p,q) = L(54\ell^2 + 15\ell + 1, 27\ell^2 + 21\ell + 3)$ *for* $\ell \in \mathbb{Z} \setminus \{0\}$,

(ii) $L(p,q) = L(54\ell^2 + 39\ell + 7, 27\ell^2 + 33\ell + 9)$ *for* $\ell \in \mathbb{Z} \setminus \{0\}$,

(iii) $L(p,q) = L(69\ell^2 + 17\ell + 1, 46\ell^2 + 19\ell + 2)$ *for* $\ell \in \mathbb{Z} \setminus \{0\}$,

(iv) $L(p,q) = L(69\ell^2 + 29\ell + 3, 46\ell^2 + 27\ell + 4)$ *for* $\ell \in \mathbb{Z} \setminus \{0\}$,

(v) $3.21 \leq \frac{h^2}{p} \leq 3.61$,

(vi) $1.15 \leq \frac{h^2}{p} \leq 1.28$.

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