On Symmetry of Flat Manifolds

Rafał Lutowski

CONTENTS

2000 AMS Subject Classification: Primary 20H15;
Secondary 20F34, 57S30

Keywords: Bieberbach groups, directly indecomposable groups

We give an example of a Bieberbach group Γ for which $\mathrm{Out}(\Gamma)$ is a cyclic group of order 3. We also calculate the outer automorphism group of a direct product of n copies of a Bieberbach group with trivial center, for $n \in \mathbb{N}$. As a corollary we get that every symmetric group can be realized as an outer automorphism group of some Bieberbach group.

1. INTRODUCTION

Let X be a compact, connected, flat Riemannian manifold (flat manifold for short) and let Γ be the fundamental group of X. Then Γ is a Bieberbach group, i.e., a torsion-free group defined by a short exact sequence

$$0 \longrightarrow M \longrightarrow \Gamma \longrightarrow G \longrightarrow 1, \qquad (1\text{–}1)$$

where G is a finite group, called the holonomy group of Γ, and M is free abelian of finite rank and the maximal abelian subgroup of Γ.

Up to affine equivalence, X is determined by Γ (see [Charlap 86, Chapter II]). The set $\mathrm{Aff}(X)$ of affine self-equivalences of X is a Lie group. Let $\mathrm{Aff}_0(X)$ denote its identity component. Then $\mathrm{Aff}_0(X)$ is a torus whose dimension equals the first Betti number of X, and $\mathrm{Aff}(X)/\mathrm{Aff}_0(X)$ is isomorphic to $\mathrm{Out}(\Gamma)$, the outer automorphism group of Γ (see [Charlap 86, Chapter V]).

From the above, if $\mathrm{Aff}(X)$ is finite, then the first Betti number of X is equal to zero. Hence the center of Γ is trivial and

$$\mathrm{Aff}(X) \cong \mathrm{Out}(\Gamma).$$

Let H be a finite group. In this article we want to consider the following question: Does H occur as an outer automorphism group of some Bieberbach group with a trivial center (see [Szczepański 06, Problem 6])?

To give a more explicit description of $\mathrm{Out}(\Gamma)$, let N be the normalizer of G in $\mathrm{Aut}(M) \cong \mathrm{GL}_n(\mathbb{Z})$, and let $\delta \in H^2(G, M)$ be the cohomology class defining (1–1). There is a natural action of N on $H^2(G, M)$ (see [Charlap 86, page 168]) and a short exact sequence

$$0 \longrightarrow H^1(G, M) \longrightarrow \mathrm{Out}(\Gamma) \longrightarrow N_\delta/G \longrightarrow 1, \qquad (1\text{–}2)$$

where N_δ is the stabilizer of δ in N (see [Charlap 86, Theorem V.1.1]). Moreover, the center of Γ equals $M^G = \{m \in M \mid g \cdot m = m \ \forall_{g\in G}\}$ and Γ is torsion-free if and only if δ is special, i.e., $\mathrm{res}^G_U(\delta)$ is nonzero for every representative U of conjugacy classes of subgroups of G of prime order (see [Waldmüller 03, Section 1], with references).

There are examples of Bieberbach groups with trivial center and outer automorphism group isomorphic to the trivial group [Waldmüller 03], C_2 (cyclic group of order 2) and $C_2 \times (C_2 \wr F)$, where $F \subset S_{2k+1}$ is a cyclic group generated by the cycle $(1, 2, \ldots, 2k+1), k \geq 2$ [Hiss and Szczepański 97], $C_2^k, k \geq 2$ [Lutowski 09].

We would like to mention that an analogous problem for hyperbolic manifolds was recently solved by Belolipetsky and Lubotzky [Belolipetsky and Lubotzky 05].

In Section 2 we give an example of a flat manifold with group of affinities isomorphic to C_3, the cyclic group of order 3. In Section 3 we show that if Γ is a directly indecomposable Bieberbach group with trivial center, then the outer automorphism group of

$$\Gamma^n = \underbrace{\Gamma \times \cdots \times \Gamma}_{n}$$

is isomorphic to $\mathrm{Out}(\Gamma) \wr S_n$, the wreath product of $\mathrm{Out}(\Gamma)$ by S_n, the symmetric group on n letters. Hence, using the example from [Waldmüller 03], for every $n \in \mathbb{N}$, we get a flat manifold X with $\mathrm{Aff}(X) \cong S_n$.

All data needed for the calculations given in Section 2 can be found in the online supplement [Lutowski 08] to this article.

2. A FLAT MANIFOLD WITH ODD-ORDER GROUP OF SYMMETRIES

Let $G = \mathrm{M}_{11}$ be the Mathieu group on 11 letters. Then G has a presentation

$$G = \langle a, b \mid a^2,\ b^4,\ (ab)^{11},\ (ab^2)^6,\ ababab^{-1}abab^2ab^{-1}abab^{-1}ab^{-1} \rangle.$$

A representative of the conjugacy class of subgroups of order 2 is $\langle a \rangle$ and that of order 3 is $\langle (ab^2)^2 \rangle$ (see [Wilson et al. 06, Waldmüller 03]). Since $|G| = 7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$, subgroups of G of orders 5 and 11 are the Sylow subgroups. Let M_1, M_3, M_4 be integral representations of G from [Waldmüller 03] of degree respectively 20, 44, and 45. Let M_2 be a sublattice of index 3 of the lattice of degree 32 given in [Waldmüller 03], i.e., it is given by the G-orbit of the vector

$$(\underbrace{2, 1, \ldots, 1}_{32}).$$

The lattices have the following properties:

1. The character afforded by M_1 is $\chi + \overline{\chi}$, where χ is one of the two nonreal irreducible characters of G of degree 10; $H^1(G, M_1) = 0$ and $H^2(G, M_1) = C_6$. For $\delta_1 \in H^2(G, M_1)$ we pick one of the two cohomology classes of order 6. We get $\mathrm{res}^G_{\langle (ab^2)^2 \rangle} \delta_1 \neq 0$.

2. The character afforded by M_2 is $\chi + \overline{\chi}$, where χ is one of the two nonreal irreducible characters of G of degree 16; $H^1(G, M_2) = C_3$ and $H^2(G, M_2) = C_5$. For $\delta_2 \in H^2(G, M_2)$ we pick any of the cohomology classes of order 5. Restriction of δ_2 to any subgroup of order 5 is nonzero.

3. The character afforded by M_3 is the irreducible character of G of degree 44; $H^1(G, M_3) = 0$ and $H^2(G, M_3) = C_6$. For $\delta_3 \in H^2(G, M_3)$ we pick one of the two cohomology classes of order 6. We get $\mathrm{res}^G_{\langle a \rangle} \delta_3 \neq 0$.

4. The character afforded by M_4 is the irreducible character of G of degree 45; $H^1(G, M_4) = 0$ and $H^2(G, M_4) = C_{11}$. For $\delta_4 \in H^2(G, M_4)$ we pick any of the cohomology classes of order 11. Restriction of δ_4 to any subgroup of order 11 is nonzero.

Thus $\delta := \delta_1 + \cdots + \delta_4 \in H^2(G, M)$ is a special element, where $M := M_1 \oplus \cdots \oplus M_4$. Let Γ be an extension of M by G defined by δ. Then Γ is torsion-free, and since $M^G = \bigoplus_{i=1}^4 M_i^G = 0$, it has trivial center. Moreover, $H^1(G, M) = C_3$.

We will show that $N_{\mathrm{Aut}(M)}(G)_\delta = G$. Since G is simple and $\mathrm{Out}(G) = 1$, we have

$$N_{\mathrm{Aut}(M)}(G) = C_{\mathrm{Aut}(M)}(G) \cdot G.$$

We claim that

$$C_{\mathrm{Aut}(M)}(G) = C_{\mathrm{Aut}(M_1)}(G) \times \cdots \times C_{\mathrm{Aut}(M_4)}(G) = (\mathbb{Z}_2)^4,$$

i.e.,

$$C_{\mathrm{Aut}(M_i)}(G) = \{\pm 1\}$$

for $1 \leq i \leq 4$. The calculations of $C_{\mathrm{Aut}(M_i)}(G)$, for $i = 1, 3, 4$, can be found in [Waldmüller 03]. We have that $C_{\mathrm{Aut}(M_2)}(G)$ is the unit group of $\mathrm{End}_{\mathbb{Z}G}(M_2)$, and this ring is generated by I_{32}, the identity matrix of degree 32, and a matrix B such that $(9I_{32} + 2B)^2 = -99I_{32}$.

Hence $\mathrm{End}_{\mathbb{Z}G}(M_2)$ is isomorphic to $\mathbb{Z}\left[\left(3\sqrt{-11}-1\right)/2\right]$, and the claim follows.

Now it is obvious that $C_{\mathrm{Aut}(M)}(G)_\delta = 1$, since none of the classes δ_i, for $1 \leq i \leq 4$, has order 2. Thus $N_{\mathrm{Aut}(M)}(G)_\delta = C_{\mathrm{Aut}(M)}(G)_\delta \cdot G = G$.

Theorem 2.1. *Let X be a flat manifold with fundamental group Γ. Then $\mathrm{Aff}(X) \cong \mathrm{Out}(\Gamma) \cong C_3$ is a group of order* 3.

The computations in this example have been performed with GAP[GAP 2007] and CARAT[Opgenorth et al. 03].

Remark 2.2. Recall the short exact sequence (1–2). In all the above examples the group N_δ/G has even order or is trivial. If this holds in general, then since the group $H^1(G,M)$ is abelian, we may suspect that any nonabelian group of odd order cannot be realized as a group of affinities of a flat manifold.

3. (OUTER) AUTOMORPHISMS OF DIRECT PRODUCTS OF BIEBERBACH GROUPS

Recall that a group is directly indecomposable if it cannot be expressed as a direct product of its nontrivial subgroups (see [Suzuki 82, page 129]). The following lemma is a corollary of [Golowin 39, Theorem 1].

Lemma 3.1. *Let Γ be a directly indecomposable Bieberbach group with trivial center, $n \in \mathbb{N}$, and $\varphi \in \mathrm{Aut}(\Gamma^n)$. Then*

$$\exists_{\sigma \in S_n} \forall_{1 \leqslant i \leqslant n}\ \varphi(\Gamma_i) = \Gamma_{\sigma(i)},$$

where $\Gamma_i := \{1\}^{i-1} \times \Gamma \times \{1\}^{n-i} < \Gamma^n$, for $1 \leq i \leq n$.

Corollary 3.2. *Let Γ be a directly indecomposable Bieberbach group with trivial center and $n \in \mathbb{N}$. Then*

$$\mathrm{Aut}(\Gamma^n) = \mathrm{Aut}(\Gamma) \wr S_n,$$

and hence

$$\mathrm{Out}(\Gamma^n) = \mathrm{Out}(\Gamma) \wr S_n.$$

Since the holonomy group of the Bieberbach group given in [Waldmüller 03] is directly indecomposable, it follows that the Bieberbach group is directly indecomposable itself, and by Corollary 3.2 we get a family of finite groups that can be realized as groups of affinities of flat manifolds.

Corollary 3.3. *For every $n \in \mathbb{N}$ there exists a flat manifold with group of affine self-equivalences isomorphic to the symmetric group S_n.*

Using again [Golowin 39, Theorem 1], we get a generalization of Corollary 3.2.

Theorem 3.4. *Let Γ_i, $i = 1, \ldots, k$, be mutually nonisomorphic directly indecomposable Bieberbach groups with trivial center. Let $n_i \in \mathbb{N}$, $i = 1, \ldots, k$. Then*

$$\mathrm{Out}(\Gamma_1^{n_1} \times \cdots \times \Gamma_k^{n_k}) \cong \mathrm{Out}(\Gamma_1) \wr S_{n_1} \times \cdots \times \mathrm{Out}(\Gamma_k) \wr S_{n_k}.$$

ACKNOWLEDGMENTS

I would like to thank A. Szczepański and G. Hiss for their advice and the referees for helpful comments.

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Rafał Lutowski, Institute of Mathematics, University of Gdańsk, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland (rlutowsk@math.univ.gda.pl)

Received March 26, 2008; accepted in revised form August 7, 2008.