A Proof of a Recurrence for Bessel Moments

Jonathan M. Borwein and Bruno Salvy

CONTENTS

1. Introduction
2. Existence of a Recurrence
3. Proof of the Main Result
4. Algorithm
5. Another Example
Acknowledgments
References

2000 AMS Subject Classification: Primary 68W30, 47N20, 33E30, 33C10, 33F10

Keywords: Bessel functions, symbolic computation, D-finite functions

We provide a proof of a conjecture in [Bailey et al. 07a] on the existence and form of linear recurrences for moments of powers of the Bessel function K_0.

1. INTRODUCTION

The aim of this note is twofold. First, we prove a conjecture of [Bailey et al. 07b, Bailey et al. 07a] concerning the existence of a recurrence in $k \geq 0$ satisfied by the integrals

$$C_{n,k} := \frac{1}{n!} \int_0^\infty \cdots \int_0^\infty \frac{dx_1 dx_2 \cdots dx_n}{(\cosh x_1 + \cdots + \cosh x_n)^{k+1}},$$

for $n = 1, 2, \ldots$. These integrals naturally arose during the analysis of parts of the Ising theory of solid-state physics [Bailey et al. 07b]. In [Bailey et al. 07a], only the first four cases of Theorem 1.1 below were proven, and the proofs relied on the ability to express the corresponding integrals in (1–3) below as *Meijer G-functions*, something that fails for $n > 4$.

A second aim is to advertise the power of current symbolic computational tools and related algorithmic developments to settle such questions. For this reason we give a quite detailed exposition of the methods entailed.

Our main result (Theorem 1.1) is better phrased in terms of

$$c_{n,k} := \frac{n!\Gamma(k+1)}{2^n} C_{n,k}. \tag{1–1}$$

Theorem 1.1. (Linear recurrence.) *For any fixed $n \in \mathbb{N}$, the sequence $c_{n,k}$ enjoys a linear recurrence with polynomial coefficients of the form*

$$(k+1)^{n+1} c_{n,k} + \sum_{\substack{2 \leq j < n \\ j \text{ even}}} P_{n,j}(k) c_{n,k+j} = 0, \tag{1–2}$$

with $\deg P_{n,j} \leq n + 1 - j$.

1058-6458/2008 $ 0.50 per page

Substituting (1–1) and simplifying by $n!\Gamma(k+1)/2^n$ yields

$$0 = (k+1)^{n+1}C_{n,k} + \sum_{\substack{2\le j<n \\ j \text{ even}}} P_{n,j}(k)(k+j)(k+j-1)\cdots(k+1)C_{n,k+j},$$

which is [Bailey et al. 07a, Conjecture 1], with extra information added on the origin of the linear factors for the recurrence in $C_{n,k}$.

The starting point of our proof is the integral representation [Bailey et al. 07a, equation (8)]

$$c_{n,k} = \int_0^{+\infty} t^k K_0(t)^n \, dt, \qquad (1\text{–}3)$$

where K_0 is the *modified Bessel function*, on which much information is to be found in [Abramowitz and Stegun 92, Chapter 9]. The key properties of K_0 that we use are as follows. First,

$$K_0(t) = \int_0^{\infty} e^{-t\cosh(x)} \, dx,$$

which explains how the integrals in (1–3) arise. Moreover, we have

- a linear differential equation: $(\theta^2 - t^2)K_0(t) = 0$ with $\theta := td/dt$;
- the behavior at the origin: $K_0(t) \sim -\ln t$, $t \to 0$;
- and behavior at infinity: $K_0(t) \sim \sqrt{\pi/2t}e^{-t}$, $t \to +\infty$.

These last two properties show that the integral (1–3) converges for any complex k subject to $\Re k > -1$, where it defines an analytic function of k. The recurrence of Theorem 1.1 then gives the integral a meromorphic continuation to the whole complex plane with poles at the negative integers.

2. EXISTENCE OF A RECURRENCE

The theory of D-finite functions leads to a direct proof of the existence of a recurrence such as (1–2) in a very general setting, together with an algorithm.

Recall that a function is called *D-finite* when it satisfies a linear differential equation with polynomial coefficients. A good introduction to the basic properties of these functions is given in [Stanley 99]. What makes these functions appealing from the algorithmic point of view is that they live in finite-dimensional vector spaces, and thus many of their properties can be explicitly computed by linear algebra in finite dimensions. In this setting, the following proposition is easily obtained. It is a generalization of our main theorem (Theorem 1.1) except for the absence of degree bounds.

Proposition 2.1. *Assume that $f(t)$ obeys a homogeneous linear differential equation*

$$a_r(t)f^{(r)}(t) + \cdots + a_0(t)f(t) = 0,$$

with polynomial coefficients $a_i(t)$ in $\mathbb{C}[t]$. For a fixed $n \in \mathbb{N}\setminus\{0\}$, let Γ be a path in $\mathbb{C}$ such that for any $k, j \in \mathbb{N}$ the integrals

$$I_{k,j} := \int_\Gamma t^k (f(t)^n)^{(j)} \, dt$$

converge and the limits of the integrand at both endpoints coincide. Then the sequence $\{I_{k,0}\}$ obeys a linear recurrence with coefficients that are polynomial in n and k and that can be computed given the coefficients a_i.

We give the proof in two steps. The first one is classical and can be found, for instance, in [Stanley 99, Theorem 6.4.9].

Lemma 2.2. *D-finite functions form an algebra over the rational functions.*

This means that any polynomial in D-finite functions with rational-function coefficients defines a function that is itself D-finite. In particular, K_0^n satisfies a linear differential equation.

Proof: The proof is effective. The difficult part is the product. The derivatives of two D-finite functions f and g live in finite-dimensional vector spaces generated by $f, \dots, f^{(r-1)}$ and $g, \dots, g^{(s-1)}$. Therefore by repeated differentiation, the derivatives of a product $h := fg$ can be rewritten as linear combinations of the terms $f^{(i)}g^{(j)}$, $0 \le i < r$, $0 \le j < s$, which generate a vector space of dimension at most rs. It follows that the $rs+1$ successive $h^{(k)}$, $k = 0, \dots, rs$, are linearly dependent. A linear dependency between them can be found as the kernel of the linear map $(\lambda_0, \dots, \lambda_{rs}) \mapsto \lambda_0 h + \cdots + \lambda_{rs}h^{(rs)}$. Any such linear dependency is a linear differential operator annihilating fg. □

The corresponding algorithm is implemented, among other places, in the Maple package `gfun` [Salvy and Zimmermann 94].

Example 2.3. Here is how the function gfun[poltodiffeq] is invoked to compute a differential equation for K_0^4:

```
> eqK0:=t*diff(t*diff(y(t),t),t)-t^2*y(t);
```

$$eqK0 := t\left(t\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t)\right) - t^2 y(t).$$

```
> gfun[poltodiffeq](y(t)^4,[eqK0],[y(t)],y(t))=0;
```

$$t^4\frac{d^5}{dt^5}y(t) + 10t^3\frac{d^4}{dt^4}y(t) - (20t^4 - 25t^2)\frac{d^3}{dt^3}y(t) - (120t^3 - 15t)\frac{d^2}{dt^2}y(t) + (64t^4 - 152t^2 + 1)\frac{d}{dt}y(t) + (128t^3 - 32t)y(t) = 0.$$

Example 2.4. Here are the corresponding steps of the calculation for the smaller example K_0^2:

$$\begin{aligned} h &= K_0^2, \quad h' = 2K_0K_0', \\ h'' &= 2K_0'^2 - 2t^{-1}K_0K_0' + 2K_0^2, \\ h^{(3)} &= -6t^{-1}K_0'^2 + 4(2+t^{-2})K_0K_0' - 2t^{-1}K_0^2, \end{aligned}$$

where whenever possible we have replaced K_0'' by $K_0 - t^{-1}K_0'$. Then we find the vector $(-4t, 1-4t^2, 3t, t^2)$ in the kernel of

$$\begin{pmatrix} 1 & 0 & 2 & -2t^{-1} \\ 0 & 2 & -2t^{-1} & 4(2+t^{-2}) \\ 0 & 0 & 2 & -6t^{-1} \end{pmatrix}.$$

This vector then produces a differential equation satisfied by K_0^2:

$$t^2y^{(3)} + 3ty'' + (1-4t^2)y' - 4ty = 0.$$

Proof (continued).: The second step of the proof of Proposition 2.1 starts by expanding the differential equation for $h := f^n$ as

$$\sum_{i,j} d_{i,j}t^ih^{(j)} = 0,$$

for scalars $d_{i,j}$. This is then multiplied by t^k and integrated along Γ. Use of the convergence hypotheses then allows us to deduce that

$$\sum_{i,j} d_{i,j}\int_\Gamma t^{k+i}h^{(j)}\,dt = 0. \tag{2–1}$$

Now integration by parts gives

$$\begin{aligned} \int_\Gamma t^{k+i}h^{(j)}\,dt &= \underbrace{t^{k+i}h^{(j-1)}\Big|_\Gamma}_{0} - (k+i)\int_\Gamma t^{k+i-1}h^{(j-1)}\,dt, \\ &= (-k-i)(-k-i+1)\cdots(-k-i+j-1) \\ &\quad \times I_{k+i-j}, \end{aligned}$$

the latter equality following by induction. Adding the contributions of all the terms in (2–1) finally yields the desired recurrence over I_k. □

Example 2.5. For $h := K_0^2$, the computation gives

$$\int_0^{+\infty} t^{k+2}h^{(3)} + 3t^{k+1}h'' + (t^k - 4t^{k+2})h' - 4t^{k+1}h\,dt = 0,$$

whence

$$(-k-2)(-k-1)(-k)c_{2,k-1} + 3(-k-1)(-k)c_{2,k-1} + (-k)c_{2,k-1} - 4(-k-2)c_{2,k+1} - 4c_{2,k+1} = 0.$$

Once simplified, this reduces to

$$4(k+1)c_{2,k+1} = k^3c_{2,k-1}. \tag{2–2}$$

Example 2.6. Quantum-field theorist David Broadhurst has recently studied [Broadhurst 07] the *vacuum-diagram* integrals for $n \geq 0$:

$$V(n,a,b) := \int_0^\infty x^{2n+1}K_0^a(x)(xK_0'(x))^b\,dx,$$

and he provides the recurrence

$$2(n+1)V(n,a,b) + aV(n,a-1,b+1) + bV(n+1,a+1,b-1) = 0,$$

which *preserves* $N := a+b > 0$ and allows one to reduce to V values with $ab = 0$. Note that $K_0^{'} = -K_1$. Proposition 2.1 applies to $n \mapsto V(n,a,b)$ for each a and b and as in Section 4 below, leads to very efficient code for the recurrence. The difficult question of understanding the initial values is discussed in [Broadhurst 07] and [Bailey et al. 08].

2.1 Mellin Transform

As the proof indicates, Proposition 2.1 is not restricted to integer values of k. In particular, the method gives a difference equation for the *Mellin transform*

$$h^\star(s) := \int_0^{+\infty} t^{s-1}h(t)\,dt,$$

provided that the appropriate convergence properties are satisfied. This difference equation then gives a meromorphic continuation in the whole complex plane. The most basic example is $\Gamma(s)$: starting from the elementary differential equation $y' + y = 0$ for $h(t) = \exp(-t)$ leads to the classical functional equation $\Gamma(s+1) = s\Gamma(s)$.

2.2 Coefficients

The path Γ can also be a closed contour. For instance, if h is analytic at the origin, then the kth coefficient of its Taylor series at the origin is given by the Cauchy integral

$$\frac{1}{2\pi i}\oint \frac{h(t)}{t^{k+1}}\,dt,$$

where the contour encloses the origin and no other singularity of h. The algebraic manipulations are the same as in the previous case, followed by replacing k by $-k-1$ and the sequence c_k by the sequence u_{-k-1}.

For instance, if we apply this transform to the functional equation for Γ, we get first $c_{-s} = -sc_{-s-1}$ and then $u_{s-1} = -su_s$, which is the expected recurrence for the sequence of coefficients $u_s = (-1)^s/s!$ of $\exp(-t)$.

Similarly, starting from (2–2), we obtain the mirror recurrence

$$4k\tilde{c}_{k-1} = (k+1)^3\tilde{c}_{k+1}.$$

Observe that this is obeyed by the coefficients of $\ln^2(t)$ in the series expansion

$$\begin{aligned} K_0^2(t) &= \ln^2(t)\left(1 + \frac{1}{2}t^2 + \frac{3}{32}t^4 + \frac{5}{576}t^6 + \cdots\right) \\ &\quad + \ln(t)\left(2\gamma - 2\ln 2 + (\gamma - \ln 2 - \frac{1}{2})t^2 + \cdots\right) \\ &\quad + \left((\ln 2 - \gamma)^2 + \cdots\right), \quad t \to 0^+. \end{aligned}$$

The Frobenius computation of expansions of solutions of linear differential equations at regular singular points (see, e.g., [Ince 56]) explains why this is so.

2.3 Further Generalizations

The sequence $\{t^k\}$ in the integral of the proposition can be replaced by more general sequences of functions that satisfy both a linear recurrence in k and a linear differential equation in t. Provided that proper analytic conditions are satisfied at the endpoints of the path, the same result will hold. This makes it possible to compute, for instance, recurrences for the Fourier coefficients with respect to various bases such as orthogonal polynomials and Bessel functions.

The algorithm that applies in this case is called *creative telescoping*. It was discovered by Zeilberger [Zeilberger 90] and further automated in [Chyzak and Salvy 98, Chyzak 00]. Again, the computation boils down to linear algebra in a suitably constructed finite-dimensional vector space.

In summary, all these algorithms succeed in making effective and efficient the familiar method of differentiation under the integral sign and integration by parts.

3. PROOF OF THE MAIN RESULT

If A is a linear differential operator, the operator of minimal order annihilating the nth power of every solution of A is called its nth *symmetric power*. Because of its role in algorithms for differential Galois theory [van der Put and Singer 02], there has been interest in efficient algorithms for computing symmetric powers. In the case of *second-order* operators, such an algorithm has been found in [Bronstein et al. 97]. We state it in terms of the derivation $\theta := td/dt$ in order to get better control over the coefficients of the resulting recurrence; but the statement and proof hold for any derivation.

Lemma 3.1. (Linear differential equation) [Bronstein et al. 97]. *Let $A = \theta^2 + a(t)\theta + b(t)$ be a linear differential operator with rational-function coefficients $a(t)$ and $b(t)$. Let $L_0 = 1$, $L_1 = \theta$, and for $k = 1, 2, \ldots, n$ define the operator L_{k+1} by*

$$L_{k+1} := (\theta + ka)L_k + bk(n-k+1)L_{k-1}. \tag{3–1}$$

Then for $k = 0, \ldots, n+1$ and for an arbitrary solution y of $Ay = 0$,

$$L_k y^n = n(n-1)\cdots(n-k+1)y^{n-k}(\theta y)^k,$$

and in particular, $L_{n+1}y^n = 0$.

(This recurrence can be viewed as an efficient computation of the kernel that was described in the previous section, taking advantage of the special structure of the current matrix.)

Proof: The proof is a direct verification by induction. For $k = 0$ and $k = 1$ the identity reduces respectively to $y^n = y^n$ and $\theta y^n = ny^{n-1}\theta(y)$, which are obviously true for any function y. Assuming the identity to hold up to $k \geq 1$, the heart of the induction is the rule for differentiation of a product $\theta(uv) = \theta(u)v + u\theta(v)$:

$$\begin{aligned} \theta(y^{n-k}(\theta y)^k) &= \theta(y^{n-k})(\theta y)^k + y^{n-k}\theta((\theta y)^k) \\ &= (n-k)y^{n-k-1}(\theta y)^{k+1} + ky^{n-k}(\theta y)^{k-1}(\theta^2 y) \\ &= (n-k)y^{n-k-1}(\theta y)^{k+1} + ky^{n-k}(\theta y)^{k-1}(a\theta y + by). \end{aligned}$$

Reorganizing terms concludes the induction. $\square$

Example 3.2. In the case of K_0, we have $a = 0$ and $b = -t^2$. For $n = 4$, starting with $L_0 = 1$ and $L_1 = \theta$, the recurrence of Lemma 3.1 gives

$$\begin{aligned}
L_2 &= \theta^2 - 4t^2,\\
L_3 &= \theta^3 - 10t^2\theta - 8t^2,\\
L_4 &= \theta^4 - 16t^2\theta^2 - 28t^2\theta + 8t^2(3t^2-2),\\
L_5 &= \theta^5 - 20t^2\theta^3 - 60t^2\theta^2 + 8t^2(8t^2-9)\theta + 32t^2(4t^2-1).
\end{aligned}$$

The operator L_5 annihilates K_0^4. It is a rewriting in terms of θ of the equation of Example 2.3.

Some of the patterns that emerge in this example can be proved in the general case.

Lemma 3.3. (Closed form.) *With the same notation as in Lemma 3.1, when $A = \theta^2 - t^2$, L_k may be written as*

$$L_k = \theta^k + \sum_{j=0}^{k-2} a_j^{(k)}(t)\theta^j,$$

where each $a_j^{(k)}$ is a polynomial in t^2 divisible by t^2, and $\deg a_j^{(k)} \le k - j$.

Proof: Again, the proof is by induction. For $k = 0$ and $k = 1$ we recover the definition of L_0 and L_1. For larger k, the recurrence (3–1) simplifies to

$$L_{k+1} := \theta L_k - k(n-k+1)t^2 L_{k-1}.$$

If the property holds up to $k \ge 1$, then this shows that the degree of L_{k+1} in θ is $k+1$, with leading coefficient 1, and also that the coefficient of θ^k in L_{k+1} is 0. Extracting the coefficient of θ^j then gives

$$a_j^{(k+1)} = \begin{cases} a_{j-1}^{(k)} + \theta(a_j^{(k)}) - k(n-k+1)t^2 a_j^{(k-1)}, \\ \qquad 0 \le j \le k-2, \\ -k(n-k+1)t^2, \qquad j = k-1. \end{cases}$$

These last two identities give the desired degree bound and divisibility property for the coefficients $a_j^{(k+1)}$, $0 \le j \le k-1$. □

We may now complete the proof of the main result.

Proof of Theorem 1.1.: Lemma 3.3 shows that the coefficients of L_{n+1} can be rewritten as

$$L_{n+1} = \theta^{n+1} + \sum_{\substack{2\le j<n\\ j \text{ even}}} t^j Q_j(\theta), \tag{3–2}$$

where the polynomials Q_j satisfy $\deg Q_j \le n + 1 - j$.

Thanks to the properties of K_0 recalled in the introduction, an integration by parts yields

$$\int_0^{+\infty} t^{k+j}\theta^m(K_0^n(t))\,dt = (-1-k-j)^m c_{n,k+j}, \tag{3–3}$$

for each m. We now multiply $L_{n+1}K_0^n$ from (3–2) by t^k and integrate from zero to infinity:

$$\int_0^{\infty} \Big\{t^k\theta^{n+1}K_0^n(t) + \sum_{\substack{2\le j<n\\ j \text{ even}}} t^{k+j}Q_j(\theta)K_0^n(t)\Big\}\,dt = 0.$$

Integrating term by term and using (3–3) finally gives the recurrence

$$(-k-1)^{n+1}c_{n,k} + \sum_{\substack{2\le j<n\\ j \text{ even}}} Q_j(-1-k-j)c_{n,k+j} = 0,$$

which is the desired one, up to renaming and sign changes. □

4. ALGORITHM

In summary, we have a relatively straightforward algorithm to compute the linear recurrences for the $c_{n,k}$ or $C_{n,k}$ for given n.

First, the operators L_k can be computed as *commutative* polynomials $\tilde{L}_k$ as follows:

$$\tilde{L}_{k+1} := t\frac{\partial \tilde{L}_k}{\partial t} + \theta\tilde{L}_k - k(n-k+1)t^2\tilde{L}_{k-1}, \quad 1 \le k \le n, \tag{4–1}$$

with initial values $\tilde{L}_0 := 1$ and $\tilde{L}_1 := \theta$. These polynomials $\tilde{L}_k$ coincide with the operators L_k when the powers of θ are written on the *right* of the monomials in t and θ.

By collecting coefficients of t in $\tilde{L}_{n+1}$, we recover (3–2). Substituting $-1-k-j$ for θ in the coefficient of t^j then produces the desired recurrence for $c_{n,k}$, while replacing $c_{n,k+j}$ by $(k+1)\cdots(k+j)C_{n,k+j}$ for all j produces one for $C_{n,k}$.

Example 4.1. We illustrate the process for $n = 4$. The last operator in Example 3.2 may be rewritten as

$$L_5 = \theta^5 - 4t^2(5\theta^3 + 15\theta^2 + 18\theta + 8) + 64t^4(\theta + 2).$$

It annihilates $K_0^4(t)$. Substituting $-1-k-j$ for θ in the coefficient of t^j for $j = 0, 2, 4$ gives

$$\begin{aligned}
&-(k+1)^5 c_{4,k} + 4(k+2)(5k^2+20k+23)c_{4,k+2}\\
&\qquad - 64(k+3)c_{4,k+4} = 0.
\end{aligned}$$

Since $c_{4,k} = \frac{3}{2}\Gamma(k+1)C_{4,k}$, this is equivalent to

$$-\frac{3}{2}(k+1)^4 C_{4,k} + 6(k+2)^2(5k^2+20k+23)C_{4,k+2} \\ - 96(k+4)(k+3)^2(k+2)C_{4,k+4} = 0,$$

which was proven by different methods in [Bailey et al. 07a].

Here is the corresponding Maple code:

```
compute_Q:=proc(n,theta,t)
local k, L;
  L[0]:=1; L[1]:=theta;
  for k to n do
    L[k+1]:=expand(series(
    t*diff(L[k],t)+L[k]*theta-k*(n-k+1)*t^2*L[k-1],
       theta,infinity))
  od;
  series(convert(L[n+1],polynom),t,infinity)
end:

rec_c:=proc(c::name,n::posint,k::name)
local Q,theta,t,j;
  Q:=compute_Q(n,theta,t);
  add(factor(subs(theta=-1-k-j,coeff(Q,t,j)))
    *c(n,k+j),j=0..n+1)=0
end:

rec_C:=proc(C::name,n::posint,k::name)
local Q,theta,t,j,ell;
  Q:=compute_Q(n,theta,t);
  (-1)^(n+1)*(k+1)^n*C(n,k)+
    add(factor(subs(theta=-1-k-j,coeff(Q,t,j))
      *mul(k+1+ell,ell=1..j-1))*C(n,k+j),j=1..n+1)=0
end:
```

On a reasonably recent personal computer, all recurrences for n up to 100 can be obtained in less than five minutes (further time could be saved by not factoring the coefficients). For example, the recurrences for $c_{4,k}$ and $C_{4,k}$ may be determined thus:

```
> rec_c(c, 4, k);
```

$$-(k+1)^5 c_{4,k} + 4(k+2)(5k^2+20k+23)c_{4,k+2} \\ -(64k+192)c_{4,k+4} = 0$$

```
> rec_C(C, 4, k);
```

$$-(k+1)^4 C_{4,k} + 4(k+2)^2(5k^2+20k+23)C_{4,k+2} \\ -64(k+4)(k+3)^2(k+2)C_{4,k+4} = 0$$

The first six cases for $C_{n,k}$ are

$$\begin{aligned}
0 &= (k+1)C_{1,k} - (k+2)C_{1,k+2}\\
0 &= (k+1)^2C_{2,k} - 4(k+2)^2C_{2,k+2}\\
0 &= (k+1)^3C_{3,k} - 2(k+2)\left(5(k+2)^2+1\right)C_{3,k+2}\\
&\quad + 9(k+2)(k+3)(k+4)C_{3,k+4} \qquad (4\text{–}2)\\
0 &= (k+1)^4C_{4,k} - 4(k+2)^2(5(k+2)^2+3)C_{4,k+2}\\
&\quad + 64(k+2)(k+3)^2(k+4)C_{4,k+4} \qquad (4\text{–}3)\\
0 &= (k+1)^5C_{5,k}\\
&\quad - (k+2)\\
&\qquad \times \left(35k^4+280k^3+882k^2+1288k+731\right)C_{5,k+2}\\
&\quad + (k+2)(k+3)(k+4)\\
&\qquad \times \left(259k^2+1554k+2435\right)C_{5,k+4}\\
&\quad - 225(k+2)(k+3)(k+4)(k+5)(k+6)C_{5,k+6}\\
0 &= (k+1)^6C_{6,k}\\
&\quad - 8(k+2)^2\left(7k^4+56k^3+182k^2+280k+171\right)\\
&\qquad \times C_{6,k+2}\\
&\quad + 16(k+2)(k+3)^2(k+4)\left(49k^2+294k+500\right)\\
&\qquad \times C_{6,k+4}\\
&\quad - 2304(k+2)(k+3)(k+4)^2(k+5)(k+6)C_{6,k+6},
\end{aligned}$$

as given in [Bailey et al. 07a], but in which only the first four were proven (see also [Ouvry 05] for an earlier proof up to $n = 4$).

Many more recurrences were determined empirically using integer relation methods—which relied on being able to compute the integrals in (1–3) to very high precision—and led to the now-proven conjecture. The versions of these recurrences in terms of $c_{n,k}$ instead of $C_{n,k}$ were also determined empirically for $n = 1, \ldots, 6$ in [Guttmann and Prellberg 93, equations (11a–e)] for the enumeration of staircase polygons.

Implicit in this algorithm is an explicit recurrence for the polynomial coefficients of each recurrence. In the case of (4–2) and (4–3), these recurrences lead to new continued fractions for $L_{-3}(2)$ and $\zeta(3)$ respectively [Bailey et al. 07b, Bailey et al. 07a]. These rely additionally on the facts that

$$C_{3,1} = L_{-3}(2), \qquad C_{3,3} = 2\,L_{-3}(2)/9 - 4/27,$$
$$C_{4,1} = 7\,\zeta(3)/12, \qquad C_{4,3} = 7\,\zeta(3)/288 - 1/48$$

[Bailey et al. 07a]. Corresponding continued fractions arising from $C_{3,2}/C_{3,0}$ and $C_{4,2}/C_{4,0}$ are determined in [Bailey et al. 08].

5. ANOTHER EXAMPLE

In [Bailey et al. 07c] (to which we refer for motivation and references), the following "box integrals" have been considered

$$B_n(s) = \int_0^1 \cdots \int_0^1 (r_1^2 + \cdots + r_n^2)^{s/2}\, dr_1 \cdots dr_n,$$
$$\Delta_n(s) = \int_0^1 \cdots \int_0^1 ((r_1 - q_1)^2 + \cdots + (r_n - q_n)^2)^{s/2}\, dr_1 \cdots dr_n dq_1 \cdots dq_n.$$

As in the case of the $C_{n,k}$ we have considered here, a good starting point is provided by alternative integral representations for $\Re s > 0$:

$$B_n(-s) = \frac{2}{\Gamma(s/2)} \int_0^\infty u^{s-1} b(u)^n\, du,$$
$$b(u) = \frac{\sqrt{\pi}\,\mathrm{erf}(u)}{2u},$$
$$\Delta_n(-s) = \frac{2}{\Gamma(s/2)} \int_0^\infty u^{-s-1} d(u)^n\, du,$$
$$d(u) = \frac{e^{-u^2} - 1 + \sqrt{\pi} u\,\mathrm{erf}(u)}{u^2}.$$

The first one is given explicitly as [Bailey et al. 07c, (33)], and the second one can be derived similarly. From classical properties of the error functions, the functions $b(u)$ and $d(u)$ satisfy the linear differential equations

$$ub''(u) + 2(1 + u^2)b'(u) + 2ub(u) = 0$$

and

$$2u^2 d'''(u) + 4u(3 + u^2)d''(u) + 4(3 + 4u^2)d'(u) + 8ud(u) = 0.$$

This is exactly the setup of our Proposition 2.1. We thus deduce the existence of linear difference equations (with respect to s) for both B_n and Δ_n. The fast computation of the difference equation for B_n follows directly from the algorithm of the previous section, and for instance, we get

$$\begin{aligned}
&(s + 9)(s + 10)(s + 11)(s + 12)B_4(s + 8)\\
&\quad - 10(s + 8)^2(s + 9)(s + 10)B_4(s + 6)\\
&\quad + (s + 6)(s + 8)(35s^2 + 500s + 1792)B_4(s + 4)\\
&\quad - 2(25s + 148)(s + 4)(s + 6)^2 B_4(s + 2)\\
&\quad + 24(s + 2)(s + 4)^2(s + 6)B_4(s) = 0.
\end{aligned}$$

The recurrence holds for all s by meromorphic continuation. A result on the shape of this recurrence for arbitrary n could be obtained along the lines of Lemma 3.3.

ACKNOWLEDGMENTS

The authors wish to express their thanks to David Broadhurst for directing them to several relevant references and for his many incisive comments.

Jonathan Borwein's research was supported in part by NSERC and the Canada Research Chair Programme. Bruno Salvy's research was supported in part by the French *Agence Nationale pour la Recherche* (ANR Gecko) and the Joint Inria-Microsoft Research Centre.

REFERENCES

[Abramowitz and Stegun 92] Milton Abramowitz and Irene A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, reprint of the 1972 edition. New York: Dover, 1992.

[Bailey et al. 07a] D. H. Bailey, D. Borwein, J. M. Borwein, and R. E. Crandall. Hypergeometric Forms for Ising-Class Integrals." *Experimental Mathematics* 3 (2007), 257–276.

[Bailey et al. 07b] D. H. Bailey, J. M. Borwein, and R. E. Crandall. "Integrals of the Ising Class." *Journal of Physics A* 39 (2007), 12271–12302.

[Bailey et al. 07c] D. H. Bailey, J. M. Borwein, and R. E. Crandall. "Box Integrals." *Journal of Computational and Applied Mathematics* 206 (2007) 196–208.

[Bailey et al. 08] D. H. Bailey, J. M. Borwein, D. M. Broadhurst, and L. Glasser. "Elliptic Integral Representation of Bessel Moments." *Journal of Physics A: Mathematical and Theoretical* 41 (2008). DOI:10.1088/1751-8113/41/20/205203.2008.

[Broadhurst 07] David J. Broadhurst. "Reciprocal PSLQ and the Tiny Nome of Bologna." Talk presented at the Zentrum für interdisziplinäre Forschung in Bielefeld, June 2007. Available online (http://www.physik.uni-bielefeld.de/igs/schools/ZiF2007/Broadhurst.pdf), 2007.

[Bronstein et al. 97] Manuel Bronstein, Thom Mulders, and Jacques-Arthur Weil. "On Symmetric Powers of Differential Operators." In *ISSAC '97*, edited by Wolfgang W. Küchlin, pp. 156–163. Kihei, Maui, Hawaii: ACM Press, 1997.

[Chyzak 00] Frédéric Chyzak. "An Extension of Zeilberger's Fast Algorithm to General Holonomic Functions." *Discrete Mathematics* 217 (2000), 115–134.

[Chyzak and Salvy 98] Frédéric Chyzak and Bruno Salvy. "Non-commutative Elimination in Ore Algebras Proves Multivariate Holonomic Identities." *Journal of Symbolic Computation* 26 (1998), 187–227.

[Guttmann and Prellberg 93] A. J. Guttmann and T. Prellberg. "Staircase Polygons, Elliptic Integrals, Heun Functions, and Lattice Green Functions." *Physical Review E* 47 (1993), 2233–2236.

[Ince 56] E. L. Ince. *Ordinary Differential Equations*, reprint of the 1926 edition. New York: Dover, 1956.

[Ouvry 05] Stéphane Ouvry. "Random Aharonov–Bohm Vortices and Some Exactly Solvable Families of Integrals." *Journal of Statistical Mechanics: Theory and Experiment* 1 (2005). DOI:10.1088/1742-5468/2005/09/P09004.

[Salvy and Zimmermann 94] "Gfun: A Maple Package for the Manipulation of Generating and Holonomic Functions in One Variable." *Transactions on Mathematical Software* 2 (1994), 163–177.

[Stanley 99] Richard P. Stanley. *Enumerative Combinatorics*. Cambridge: Cambridge University Press, 1999.

[van der Put and Singer 02] Marius van der Put and Michael F. Singer. *Galois Theory of Linear Differential Equations*, second edition, Grundlehren der Mathematischen Wissenschaften 328. New York: Springer, 2002.

[Zeilberger 90] Doron Zeilberger. "A Holonomic Systems Approach to Special Functions Identities." *Journal of Computational and Applied Mathematics* 32 (1990), 321–368.

Jonathan M. Borwein, Faculty of Computer Science, Dalhousie University, Halifax, NS, B3H 2W5, Canada (jborwein@cs.dal.ca)

Bruno Salvy, Algorithms Project, Inria Paris-Rocquencourt, 78153 Le Chesnay Cedex, France (Bruno.Salvy@inria.fr)

Received June 7, 2007; accepted January 21, 2008.