The Rational Homology Group of $\mathrm{Out}(F_n)$ for $n \leq 6$

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CONTENTS

We determine the rational homology group of $\mathrm{Out}(F_n)$ for $n \leq 6$. Combining this result with results of Conant and Vogtmann proves that the first two Morita classes are nontrivial. We conclude that these classes generate the nontrivial part of the rational homology in the range $n \leq 6$.

1. INTRODUCTION

1.1 Basic Notation and Historical Background

We denote by F_n a free group of rank n. In this paper, we always assume that $n \geq 2$. Let $\mathrm{Aut}(F_n)$ be the automorphism group of F_n. The outer automorphism group of F_n, denoted by $\mathrm{Out}(F_n)$, is the quotient of $\mathrm{Aut}(F_n)$ by the inner automorphism group $\mathrm{Inn}(F_n)$ of F_n.

In the fundamental paper [Culler and Vogtmann 86], the authors constructed a space called the *Outer Space* on which the group $\mathrm{Out}(F_n)$ acts properly discontinuously. Here we briefly recall those of their results that will be needed in this paper. By a *graph* we mean a 1-dimensional finite CW-complex. The 0-cells are called *vertices*, and the 1-cells are called *edges*. The *valency* of a vertex is the number of half-edges adjoining the vertex. A graph is called *minimal* if it is connected and has no univalent or bivalent vertex. We denote by R_n a graph with one vertex and n edges, and call it a *rose* with n petals.

A *tree* is a subgraph with no cycles, and a *forest* is a disjoint union of trees. Consider a graph G and an edge e. We denote by $G - e$ the new graph obtained by removing e from G. We also denote by G_e the new graph obtained by collapsing the edge e in G. An edge e of a graph G is called a *separating edge* or a *bridge* if removing e makes the graph disconnected.

A *metric graph* is a graph equipped with lengths on the edges. The *volume* of a metric graph is the sum of the lengths of the edges.

A *marking* of a metric graph G is a homotopy equivalence

$$g \colon R_n \to G.$$

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Two markings $g_1\colon R_n \to G_1$ and $g_2\colon R_n \to G_2$ are *equivalent* if there is an isometry $i\colon G_1 \to G_2$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} R_n & \xrightarrow{g_1} & G_1 \\ \| & & \downarrow i \\ R_n & \xrightarrow{g_2} & G_2 \end{array}$$

Using the notation above, the Outer Space X_n was defined in [Culler and Vogtmann 86] as follows. A point in X_n is an equivalence class of pairs (g, G), where $g\colon R_n \to G$ is a marking of a metric graph G whose volume is 1. The Outer Space X_n is a topological space, where a neighborhood of a point (g, G) in X_n is obtained by changing the metric of G slightly. The space X_n is represented by a set of open simplices. However, it does not have the structure of a simplicial complex. We define a subspace Y_n of X_n to be the subspace obtained from X_n by deleting all the graphs with separating edges. Then the whole space X_n can be deformed onto the subspace Y_n by collapsing separating edges uniformly.

We now define a subspace K_n to be a subcomplex of the barycentric subdivision of Y_n as follows. Since K_n is an equivariant deformation retract of Y_n, it is called a *spine* of Y_n.

A vertex of K_n is an equivalence class of pairs (g, G), where G is a minimal graph with no separating edges and

$$g\colon R_n \to G$$

is a marking. This vertex can be interpreted as a point in X_n by regarding G to have the same length on each edge. In other words, it is a barycenter of a corresponding simplex in X_n. As mentioned above, the spine K_n and the subspace Y_n are deformation retracts of the Outer Space X_n.

A k-dimensional simplex of K_n is given by a triple

$$(g, G, \Phi_1 \subset \cdots \subset \Phi_k),$$

where (g, G) is a vertex of K_n and each Φ_i is a nonempty forest (subgraph with no cycles) of G. A 0-face of this simplex is represented by

$$(g, G),\ \ (c_1 \circ g, G_{\Phi_1}),\ \ \ldots,\ \ (c_k \circ g, G_{\Phi_k}),$$

where $c_i\colon G \to G_{\Phi_i}$ is the collapse of the forest Φ_i in G. The $(k-1)$-faces are obtained by removing one forest from the sequence.

The boundary map of each cell of K_n is defined by

$$\begin{aligned} &\partial(g, G, \Phi_1 \subset \cdots \subset \Phi_k) \\ &= (c_1 \circ g, G_{\Phi_1}, c_1(\Phi_2) \subset \cdots \subset c_1(\Phi_k)) \\ &\quad + \sum_{i=1}^{k} (-1)^i (g, G, \Phi_1 \subset \cdots \subset \Phi_{i-1} \subset \Phi_{i+1} \\ &\qquad\qquad \subset \cdots \subset \Phi_k). \end{aligned}$$

The spine K_n also has the natural structure of a cubical complex. The k-dimensional cube in K_n is given by the triple (g, G, Φ), where Φ is a forest that contains exactly k edges. An orientation on the cube is specified by the ordering of the edges of Φ up to even permutation. The boundary map is

$$\partial(g, G, \Phi) = \sum_{e_i \in \Phi} (-1)^i ((g, G, \Phi - e_i) - (g, G_{e_i}, \Phi_{e_i})).$$

The former definition of K_n is the subdivision of the latter.

The following theorem is proved in [Culler and Vogtmann 86].

Theorem 1.1. [Culler and Vogtmann 86] *The Outer Space X_n and the spine K_n are contractible.*

Fix an identification $\pi_1(R_n) \cong F_n$. For each $\varphi \in \mathrm{Out}(F_n)$, there exists a homotopy equivalence $f\colon R_n \to R_n$ that satisfies $f_* = \varphi$. The mapping f is unique up to homotopy. The action of $\mathrm{Out}(F_n)$ on K_n is defined by

$$(g, G, \Phi) \cdot \varphi = (g \circ f, G, \Phi).$$

According to [Culler and Vogtmann 86], this action is properly discontinuous and cocompact, so that the stabilizers are finite. It follows that the rational homology of $\mathrm{Out}(F_n)$ can be computed as the quotient of K_n by $\mathrm{Out}(F_n)$; namely, we have

$$H_*(\mathrm{Out}(F_n); \mathbb{Q}) \cong H_*(K_n/\mathrm{Out}(F_n); \mathbb{Q}).$$

The quotient $Q_n = K_n/\mathrm{Out}(F_n)$ is a cell complex, but no longer a cubical complex.

The quotient space $X_n/\mathrm{Out}(F_n)$ of the whole Outer Space X_n by $\mathrm{Out}(F_n)$ is called the *moduli space* of graphs. The Outer Space and the moduli space of graphs are analogues for the Teichmüller space and the moduli space of Riemann surfaces. In the case of Riemann surfaces, the mapping class group M_g acts on the contractible Teichmüller space T_g properly discontinuously, so that the homology of the moduli space T_g/M_g is the same as that of M_g rationally. Similarly, the rational homology of Q_n computes the rational homology of $\mathrm{Out}(F_n)$.

1.2 Known Results

In the case of $\mathrm{Aut}(F_n)$, $H_i(\mathrm{Aut}(F_n);\mathbb{Q})$ for $i \leq 6$ and all n is computed in [Hatcher and Vogtmann 98]:

Theorem 1.2. [Hatcher and Vogtmann 98]

$$\widetilde{H}_i(\mathrm{Aut}(F_n);\mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & \textit{if } i = n = 4, \\ 0, & \textit{otherwise } (i \leq 6). \end{cases}$$

Thus the authors concluded that there is a single nontrivial class in

$$H_4(\mathrm{Aut}(F_4);\mathbb{Q})$$

up to degree $i \leq 6$. Their strategy is also useful for computation of the rational homology of $\mathrm{Out}(F_n)$. However, it becomes more complicated than the case of $\mathrm{Aut}(F_n)$ because of the problem of base points in graphs.

The rational homology of $\mathrm{Out}(F_n)$ has a stable range. It is proved in [Hatcher and Vogtmann 04] that $H_i(\mathrm{Out}(F_n);\mathbb{Q})$ is independent of n when $n \geq 2i+4$ (see also [Hatcher et al. 06]). Therefore we can speak of the stable homology of $\mathrm{Out}(F_n)$. Very recently, it was shown in [Galatius 06] that this stable homology is trivial.

Theorem 1.3. [Galatius 06]

$$\lim_{n\to\infty} \widetilde{H}_i(\mathrm{Out}(F_n);\mathbb{Q}) = 0 \quad \textit{for all } i.$$

On the other hand, the unstable homology $H_i(\mathrm{Out}(F_n);\mathbb{Q})$ for $n \leq 5$ is now known from [Vogtmann 02, Vogtmann 06, Gerlits 02, Ohashi 05]. The result is

$$H_i(\mathrm{Out}(F_n);\mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & \text{if } i = 0, \\ \mathbb{Q}, & \text{if } i = n = 4 \text{ (Vogtmann)}, \\ 0, & \text{otherwise.} \end{cases}$$

S. Morita defined the trace map as a mapping from some Lie algebra to a certain polynomial algebra [Morita 93]. Furthermore, Kontsevich ([Kontsevich 93], [Kontsevich 94]; see also [Conant and Vogtmann 03]) defined a certain series of homology classes of $\mathrm{Out}(F_n)$, namely

$$\mu_i \in H_{4i}(\mathrm{Out}(F_{2i+2});\mathbb{Q}).$$

Conant and Vogtmann interpreted these classes graphically as cycles in the Outer Space [Conant and Vogtmann 04]. Whether these classes are 0 is an interesting problem. However, the answer is unknown except for the first two classes $\mu_1 \in H_4(\mathrm{Out}(F_4);\mathbb{Q})$ and $\mu_2 \in H_8(\mathrm{Out}(F_6);\mathbb{Q})$. The first class, μ_1, was shown to be nonzero in [Morita 93] and confirmed geometrically by Conant and Vogtmann (see also [Ohashi 05]). The second class, μ_2, was shown to be nonzero in [Conant and Vogtmann 04].

In this paper, $H_i(\mathrm{Out}(F_n);\mathbb{Q})$ was computed for $n \leq 6$. From this and the known results above, we can conclude that the Morita classes $\{\mu_i\}$ generate the nontrivial part of the rational homology of $\mathrm{Out}(F_n)$ in the range $n \leq 6$.

2. A CHAIN COMPLEX THAT COMPUTES $H_*(\mathrm{Out}(F_n);\mathbb{Q})$

2.1 Definitions

As mentioned before, the spine K_n of the Outer Space X_n is a locally finite contractible simplicial complex. The action of $\mathrm{Out}(F_n)$ on K_n is properly discontinuous and cocompact, so that the stabilizers are finite. It follows that the rational homology of $\mathrm{Out}(F_n)$ can be computed as the quotient of K_n by $\mathrm{Out}(F_n)$; namely, we have

$$H_*(\mathrm{Out}(F_n);\mathbb{Q}) \cong H_*(K_n/\mathrm{Out}(F_n);\mathbb{Q}).$$

The action of $\mathrm{Out}(F_n)$ on each cell of K_n changes only the marking. We note that if $(G,\Phi) \cong (G',\Phi')$, then there exists a certain element $\varphi \in \mathrm{Out}(F_n)$ such that φ maps (g,G,Φ) to (g',G',Φ'). Therefore we can use pairs $\{(G,\Phi)\}$ for a set of representatives of cells in

$$Q_n = K_n/\mathrm{Out}(F_n).$$

Each element of $\mathrm{Aut}(G,\Phi)$ induces a permutation of the edges of Φ. If there is an odd permutation, the forested graph (G,Φ) is said to have odd symmetry. In this case, the cube (g,G,Φ) in K_n is folded under the action of $\mathrm{Out}(F_n)$, so that the cube can be eliminated in the computation of the rational homology of Q_n.

The rational homology of Q_n is computed by the chain complex $\{C_p\}$, with C_* a vector space spanned by the isomorphism classes of (G,Φ) with a relation $(G,\Phi) = -(G,-\Phi)$, which implies that a graph with odd symmetry is equal to 0. The dimension of (G,Φ) is the number of edges in Φ. The boundary operator is naturally defined.

2.2 The Filtration of Kontsevich

For a graph G, we define the degree of G to be $\sum(|v|-3)$, where $|v|$ represents the valency of the vertex v and we take the sum over all vertices of G. The filtration of Kontsevich $\{F_pK_n\}_p$ is defined as follows. The pth term

F_pK_n is the subcomplex of K_n consisting of cells (g, G, Φ) such that the degree of G_Φ is at most p. We have

$$F_0K_n \subset F_1K_n \subset \cdots \subset F_{2n-3}K_n = K_n.$$

Note that the complex F_pK_n has dimension p. Since the action of $\mathrm{Out}(F_n)$ on K_n changes only the marking, this filtration on K_n induces a filtration on Q_n and so on C_*.

The boundary map on our chain complex $\{C_*\}$ is represented by the sum of the two maps

$$\partial^C_{p,q} \colon F_pC_{p+q} \to F_pC_{p+q-1}$$

and

$$\partial^R_{p,q} \colon F_pC_{p+q} \to F_{p-1}C_{p+q-1}.$$

More precisely,

$$\partial^C(G, \Phi) = \sum_{e_i \in \Phi} (-1)^{i-1}(G_{e_i}, \Phi_{e_i}),$$
$$\partial^R(G, \Phi) = \sum_{e_i \in \Phi} (-1)^{i}(G, \Phi - e_i).$$

The filtration F_* induces a spectral sequence $E^r_{p,q}$ that converges to $H_*(Q_n; \mathbb{Q})$. The boundary operator ∂^C induces the d^0-map of $E^0_{p,q}$.

2.3 Convergence

Kontsevich proved, among other things, the following result [Kontsevich 93, Kontsevich 94]; see also [Conant and Vogtmann 03], [Vogtmann 90, Proposition 2.4].

Proposition 2.1.

$$E^1_{p,q} = H_{p+q}(F_pQ_n, F_{p-1}Q_n; \mathbb{Q}) = 0 \quad \textit{for } q \neq 0.$$

This means that

$$E^r_{p,q} = 0 \quad \text{for } r \geq 2,\ q \neq 0,$$

so that

$$d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$$

is the 0-map if $r \geq 2$. Therefore the spectral sequence degenerates at degree 2.

The associated homology group $E^2_{p,0}$ gives the desired homology group, namely

$$H_p(\mathrm{Out}(F_n); \mathbb{Q}) \cong H_p(Q_n; \mathbb{Q}) \cong E^2_{p,0}.$$

3. COMPUTATIONS

3.1 The First Step of the Computation

By the result of the preceding section, we have

$$E^2_{p,0} = \frac{F_pC_p \cap \mathrm{Ker}\partial}{F_pC_p \cap \partial(F_{p+1}C_{p+1})} = \frac{\mathrm{Ker}\partial^C_{p,0} \cap \mathrm{Ker}\partial^R_{p,0}}{\partial^R_{p+1,0}(\mathrm{Ker}\partial^C_{p+1,0})}.$$

From this, we should first compute $\mathrm{Ker}\partial^C_{p,0}$. Note that a generator of F_pC_p is a forested graph (G, Φ) such that G is a trivalent graph.

3.2 Ker $\partial^C_{p,0}$

We divide F_pC_p into components. Consider the map

$$c_p \colon F_pC_p \to F_pC_0,$$

which takes a generator (G, Φ) to $(G_\Phi, \varnothing)$, where the symbol $\varnothing$ denotes the empty set. We define the direct sum decomposition of F_pC_p so that the generators of each component in this decomposition are mapped to the same image under the mapping c_p. This implies that the mapping c_p is the direct sum of the restriction of c_p to each component. The following result follows immediately from the above definition.

Lemma 3.1. $\mathrm{Ker}\partial^C_{p,0}$ *is the direct sum of the kernels of* $\partial^C_{p,0}$ *restricted to each component.*

See Figure 1 for an example. The third forested graph is in a different component from that of the first two.

Definition 3.2. Let (G, Φ) be a trivalent graph with a forest, and (G', Φ') a degree-1 graph with a forest. If there is $e \in \Phi$ such that $(G_e, \Phi_e) = (G', \Phi')$, we call (G, Φ) a *parent* of (G', Φ') and call (G', Φ') a *child* of (G, Φ).

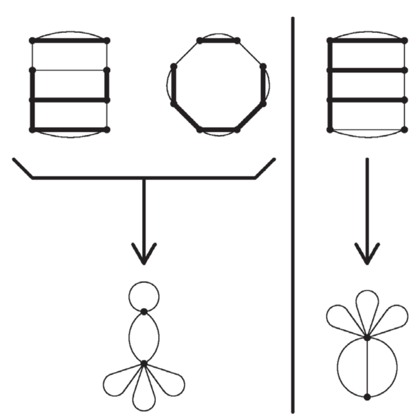

FIGURE 1. Decomposition of ∂^C.

Lemma 3.3. *Suppose (G', Φ') is a child of a forested trivalent graph (G, Φ). We denote by $\pi_{(G',\Phi')}$ a projection with respect to (G', Φ'). Then the composition $\pi_{(G',\Phi')} \circ \partial^C(G, \Phi)$ is nonzero unless $(G, \Phi) = 0$ or $(G', \Phi') = 0$.*

Proof: Fix an ordering of edges in Φ such that it represents the orientation of (G, Φ). We denote it by $\{e_1, \ldots, e_k\}$. We define

$$E = \{i = 1, 2, \ldots, k \mid (G_{e_i}, \Phi_{e_i}) = \pm(G', \Phi')\}.$$

Since (G', Φ') is a child of (G, Φ), we have that $\#E \ge 1$. We claim that

$$\forall i, j \in E,\ (-1)^i (G_{e_i}, \Phi_{e_i}) = (-1)^j (G_{e_j}, \Phi_{e_j}).$$

If this holds, the lemma follows immediately from the definition of ∂^C.

Now we consider the process of edge collapsing more precisely by taking care of the orientation. Consider the situation in which we collapse an edge $\widetilde{e}$ of a forested graph $(\widetilde{G}, \widetilde{\Phi})$. If $\widetilde{e}$ is the first edge in the sequence $\widetilde{\Phi}$, the orientation of the forest $\widetilde{\Phi}_{\widetilde{e}}$ is obtained simply by deleting $\widetilde{e}$ from the sequence of $\widetilde{\Phi}$. If $\widetilde{e}$ is at the ith position ($i \ne 1$), then we have to apply the transposition $(1, i)$ to the sequence $\widetilde{\Phi}$ to move $\widetilde{e}$ to the first position and multiply the resulting forested graph by (-1). We may erase $\widetilde{e}$ from the sequence $\widetilde{\Phi}$ and multiply by $(-1)^{i-1}$ instead.

Note that (G, Φ) does not have odd symmetry, because $(G, \Phi) \ne 0$. For $i < j \in E$, there is an even symmetry $\varphi \in \mathrm{Aut}(G, \Phi)$ such that $\varphi(e_i) = e_j$. Then

$$\begin{aligned}(-1)^j (G_{e_j}, \Phi_{e_j}) &= (-1)^j (G_{\varphi(e_i)}, \Phi_{\varphi(e_i)}) \\ &= (-1)^j (G_{e_i}, \varphi^{-1}(\Phi)_{e_i}).\end{aligned}$$

Now think of $\varphi^{-1}(\Phi)_{e_i}$. Since the permutation φ^{-1} is even and the collapsed edge e_i is at the jth position in the sequence $\varphi^{-1}(\Phi)$, we have

$$\begin{aligned}(-1)^j (G_{e_j}, \Phi_{e_j}) &= (-1)^j (G_{e_i}, \varphi^{-1}(\Phi)_{e_i}) \\ &= (-1)^j (-1)^{(i-1)-(j-1)} (G_{e_i}, \Phi_{e_i}) \\ &= (-1)^i (G_{e_i}, \Phi_{e_i}).\end{aligned}$$

□

Some generators do not contribute to the kernel of $\partial^C_{p,0}$. For example, a trivalent forested graph (G, Φ) does not if G has double edges and Φ contains one of these edges. This is because a component of the image of the

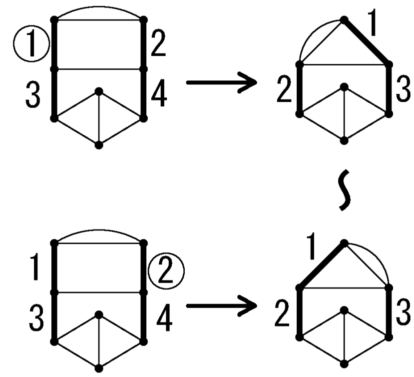

FIGURE 2. Isomorphic faces.

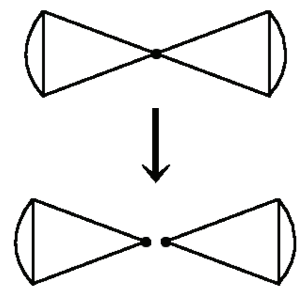

FIGURE 3. Wedge summands.

forested graph has a valence-4 vertex two of whose adjoining half-edges compose a loop. Since K_n does not contain graphs with separating edges, no other trivalent forested graphs have the same component in the image of ∂^C. In other words, (G, Φ) has a child that has no parents other than (G, Φ). Lemma 3.3 guarantees that there is no (G, Φ) that satisfies $\partial^C_{p,0}(G, \Phi) = 0$ and contributes to the kernel of $\partial^C_{p,0}$. Hence this pair does not contribute to the kernel.

More generally, we can prove the following lemma. First, we recall some technical terminology from [Hatcher and Vogtmann 98].

For a graph G and its vertex v, a *wedge summand* with respect to v is one of the connected components obtained by dividing G at v.

Lemma 3.4. *(1) Let G be a minimal trivalent nonseparating graph. If we remove a neighborhood of an edge e of G, either the graph remains connected or it is divided into two components. In the latter case, each endpoint of e is adjacent to the both components.*

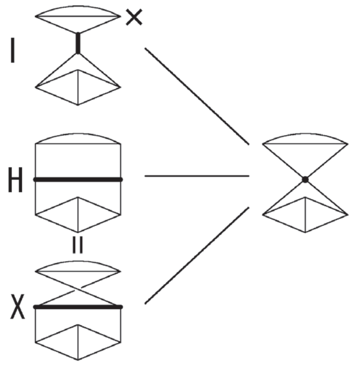

FIGURE 4. The I-parent has a separating edge, and the H-parent is equal to the X-parent.

(2) Let (G, Φ) be a degree-1 graph with a forest. Suppose that if we remove the unique valence-4 vertex, then G becomes disconnected and one of the wedge summands with respect to the vertex has the symmetry of interchanging the edges adjoining the vertex. In this situation, (G, Φ) has only one parent and therefore does not contribute to the kernel of ∂^C.

Proof: (1) Since G is trivalent, removing e from G formally divides the graph into four components (see Figure 5), where some of these components may be the same.

Since e is nonseparating, then exchanging A with C and B with D if necessary, A and B denote the same component. Similarly, the component C is equal to D or $A = B$, and the component D is equal to C or $A = B$. Therefore $C = D$, and this proves the lemma.

(2) Suppose $(G, \Phi) \in F_p C_{p-1}$ is of degree 1 and hence has one vertex with valency 4. If we remove a neighborhood of this vertex, either the graph remains connected or it is divided into two components. In the latter case, suppose one of the components has a symmetry switching the adjoining edges. Then there is only one generator $(G', \Phi') \in F_p C_p$ that has the ∂^C-image on (G, Φ), because K_n contains no graphs with separating edges.

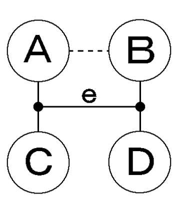

FIGURE 5. A neighborhood of an edge of a trivalent graph.

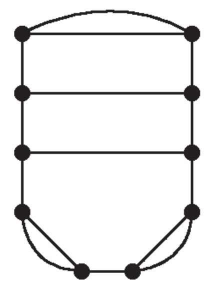

FIGURE 6. An example of a graph.

Therefore (G, Φ) and (G', Φ') do not contribute to the kernel of ∂^C. $\square$

3.3 Implementation on Computers

To perform calculations on computers, we must represent graphs and forests in some numerical form. The method of this section is taken from [Hatcher and Vogtmann 98].

In the situation in which vertices are labeled by distinct integers, we can represent the set of edges by the set of the pairs of endpoint integers. Since the vertices of our graphs have valency at least 3, the set of edges completely determines the graph, so that this set contains all the information about the graph with each vertex labeled. To compare these, we store the set in lexicographically ordered form.

Given a specific graph G, consider all the labelings of the vertices of G such that the labels are taken from the set $\{1, 2, \ldots, |v(G)|\}$. Here $|v(G)|$ is the number of vertices in the graph G. Represent these in numerical form as above and take the first one in lexicographic order. We call this *normal form*. Note that two graphs that are isomorphic have the same normal form.

For example, the normal form of the graph in Figure 6 is

$$\big\{\{1,2\},\{1,2\},\{1,3\},\{2,4\},\{3,4\},\{3,5\},\{4,6\},\{5,6\},\\ \{5,7\},\{6,8\},\{7,9\},\{7,9\},\{8,10\},\{8,10\},\{9,10\}\big\}.$$

The automorphism group of a graph G induces a subgroup of the symmetric group of $\{1, 2, \ldots, |v(G)|\}$. We represent a forest of a graph G by the subset of the normal form of G. We are concerned only with the isomorphism classes of the pair (G, Φ). Therefore, this subset is not uniquely determined. However, it is unique up to the action of the automorphism group of G. We take the first one with respect to the lexicographic order for its representative.

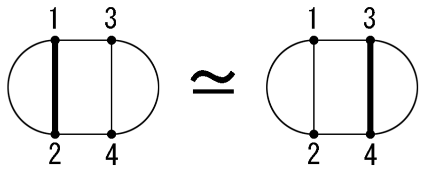

FIGURE 7. Normal form of forests.

Figure 7 is an example that contains isomorphic forested graphs with different edge sets. In this case, the normal form of the graph is

$$\{\{1,2\},\{1,2\},\{1,3\},\{2,4\},\{3,4\},\{3,4\}\},$$

and the edge sets of the forests are

$$\{\{1,2\}\} \text{ and } \{\{3,4\}\}.$$

We take $\{\{1,2\}\}$ for the normal form in this example.

By definition, we have to take all the labelings on vertices to compute the normal form of a graph. As n increases, this takes a large amount of time.

Lemma 3.5. *(1) Fix a graph G. For each vertex labeled by $k \neq 1$, there is a vertex labeled by a smaller number ℓ such that the edge $\{\ell, k\}$ is in the normal form of G.*

(2) Fix a graph G. In the normal form, if a vertex v of G has shorter distance from the vertex labeled by 1 than that of another vertex w, then v is labeled by a smaller number than w, where the distance between two vertices is the minimum number of edges of paths adjoining the vertices.

Proof: (1) We denote by $n(w)$ the number attached to a vertex w in normal form. Take a vertex v labeled by the largest number that is not connected to vertices labeled by smaller numbers. If there is no such vertex except for the vertex labeled by 1, then the lemma is proved.

Choose a vertex v_1 that is connected to v. Since v_1 is labeled by a larger number than that of v, v_1 is connected to vertices labeled by smaller numbers than that of v_1. Take the vertex labeled by the smallest in these and call it v_2. If v_2 is labeled by a larger number than that of v, v_2 is connected to vertices labeled by smaller numbers than that of v_2. Take the vertex labeled by the smallest in these and call it v_3. Repeat this procedure until v_{k+1} is labeled by a smaller number than that of v. Now consider a labeling that is the same as the normal form except that labels of v and v_k are exchanged. Then the pair $\{n(v_{k+1}), n(v_k)\}$ is replaced by $\{n(v_{k+1}), n(v)\}$, and pairs that are smaller in lexicographic order do not

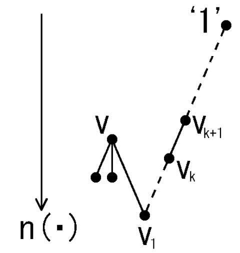

FIGURE 8. The case in which there is a vertex not connected to vertices with smaller numbers.

change. Since $\{n(v_{k+1}), n(v_k)\} > \{n(v_{k+1}), n(v)\}$, this contradicts that the original labeling represents the normal form.

(2) Take a pair (v, w) such that the distance from the vertex labeled by 1 to v is shorter than that of w, and $n(v) > n(w)$ such that $(n(w), n(v))$ is the smallest in lexicographic order among pairs satisfying the above condition. Note that $n(w)$ is not equal to 1.

By (1), w is connected to a vertex w' labeled by a smaller number, namely $n(v) > n(w) > n(w')$. From the smallest condition, w' has shorter distance to the vertex 1 than w, and v does not have shorter distance than w'. This implies that v is connected to a vertex v' labeled by a smaller number than the numbering of $n(w')$. This means that if we exchange the numbering of w and v, we obtain a smaller form in the lexicographic order. This contradicts that the original form is normal. □

Fix a graph G. Suppose that the numbering 1 is given on a specified vertex. We divide $V(G)$, the set of vertices in G, into components $\{V_k(G)\}$ such that $V_k(G)$ consists of vertices that are k edges away from the vertex 1. Lemma 3.5 implies that the set of numberings on vertices in $V_k(G)$ is

$$\left\{ \sum_{i=0}^{k-1} \#V_i(G) + 1,\ \sum_{i=0}^{k-1} \#V_i(G) + 2,\ \ldots,\ \sum_{i=0}^{k} \#V_i(G) \right\}$$

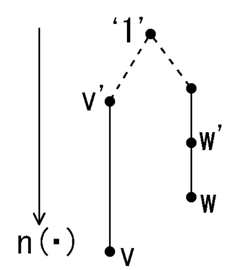

FIGURE 9. The case in which there is a pair of vertices that does not satisfy the condition of the lemma.

in normal form. Therefore we can apply the symmetric group "distancewise," namely on the set of numberings in each $V_k(G)$, to obtain the normal form.

3.4 Trivalent Graphs

We obtain trivalent graphs by induction. For convenience, in this section we denote by T_n the set of trivalent nonseparating graphs in whose fundamental group is the free group of rank n. In the case of n=2, T_n consists of only one trivalent graph, whose normal form is

$$\{\{1,2\},\{1,2\},\{1,2\}\}.$$

From T_{n-1}, the elements in T_n are obtained by attaching an edge to a graph in T_{n-1}. More precisely, choose two edges, placing new vertices in the middle of the edges, and join the vertices by a new edge. The new graph is also trivalent, and its fundamental group is the free group of rank n, which is easily verified by considering the Euler number. The next lemma guarantees that all minimal trivalent nonseparating graphs are obtained by this procedure.

Lemma 3.6. *Suppose $n \geq 3$. Each graph in T_n is obtained by attaching a new edge to some element in T_{n-1}.*

Proof: To prove that any graph in T_n is obtained from some graph in T_{n-1}, we have to show that each element in T_n has an edge that if removed, leaves the graph nonseparating.

Fix a graph $G \in T_n$. If G has a double edge, our claim is proved by removing one of the double edges. Hence we assume that G has no double edges.

First, we choose an edge e of G. We denote by G' the new graph obtained from $G - e$ by removing valence-2 vertices. The new graph G' may contain separating edges. However, separating edges in G' are also edges in G, because the unified edges that are adjacent to e in G are not separating by Lemma 3.4(1) and Figure 5.

Separating edges in the edge-removed graph are not adjacent to each other. Suppose that there are two separating edges in G' adjacent to each other. In the notation in Figure 10, suppose $\{u,v\}$ and $\{u,w\}$ are separating edges.

The vertices u, v, w are in distinct connected components if the edges $\{u,v\}$ and $\{u,w\}$ are removed. The edge e joins at most two of these in the original graph G and is not attached to $\{u,v\}$ or to $\{u,w\}$. Therefore, $\{u,v\}$ or $\{u,w\}$ is a separating edge also in G, which contradicts that G is nonseparating.

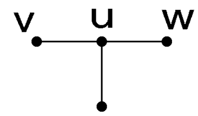

FIGURE 10. Separating edges are not adjacent to each other.

Now remove separating edges in G' to obtain a new nonempty graph. The connected components of the new graph are trivalent graphs, a loop, or a single vertex. If one of the connected components were a loop, then G would have a double edge. Also, if one of the connected components were a single vertex, then there would be separating edges adjacent to each other in G'. Therefore the connected components are trivalent nonseparating graphs. Induction on the rank of the fundamental group of the graph guarantees that there is an edge that if removed, leaves the component nonseparating in a connected component of the new graph. Since this edge is in a connected component of the new graph, it can be interpreted as an edge in G. The lemma is proved. Note that a nonseparating edge stays nonseparating if other edges are attached. □

Lists of all the trivalent graphs for $n \leq 6$ are given in Tables 1–5.

We also need the list of all the forested graphs on a specified graph G. It is obtained by listing all the subsets of $E(G)$ and the set of edges in G, and erasing subgraphs containing cycles. Whether a subgraph contains cycles is verified by computing the Euler number and counting the number of connected components.

3.5 Process of Computation

Now we are ready to compute the desired homology. The computation goes through the following steps.

(1) Enumerate all the minimal trivalent nonseparating graphs whose fundamental groups are the free groups of rank n.

(2) Enumerate all the degree-1 graphs whose fundamental groups are the free groups of rank n.

(3) Enumerate all the forests of the graphs obtained in (1) and (2).

(4) Compute the automorphism groups of forested graphs and erase forested graphs with odd symmetry from the list.

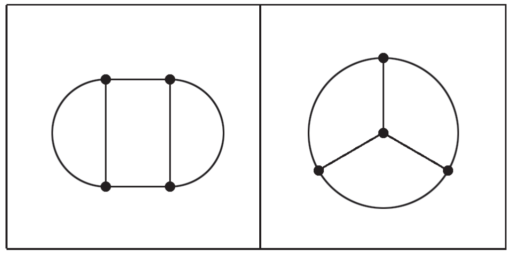

TABLE 1. Trivalent graphs of $n = 3$.

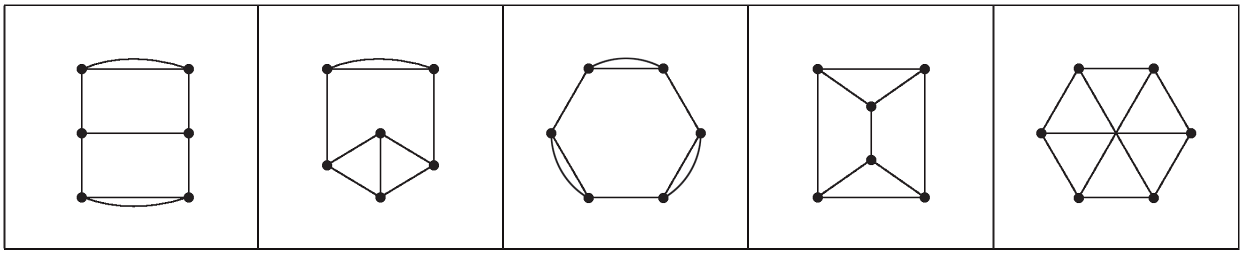

TABLE 2. Trivalent graphs of $n = 4$.

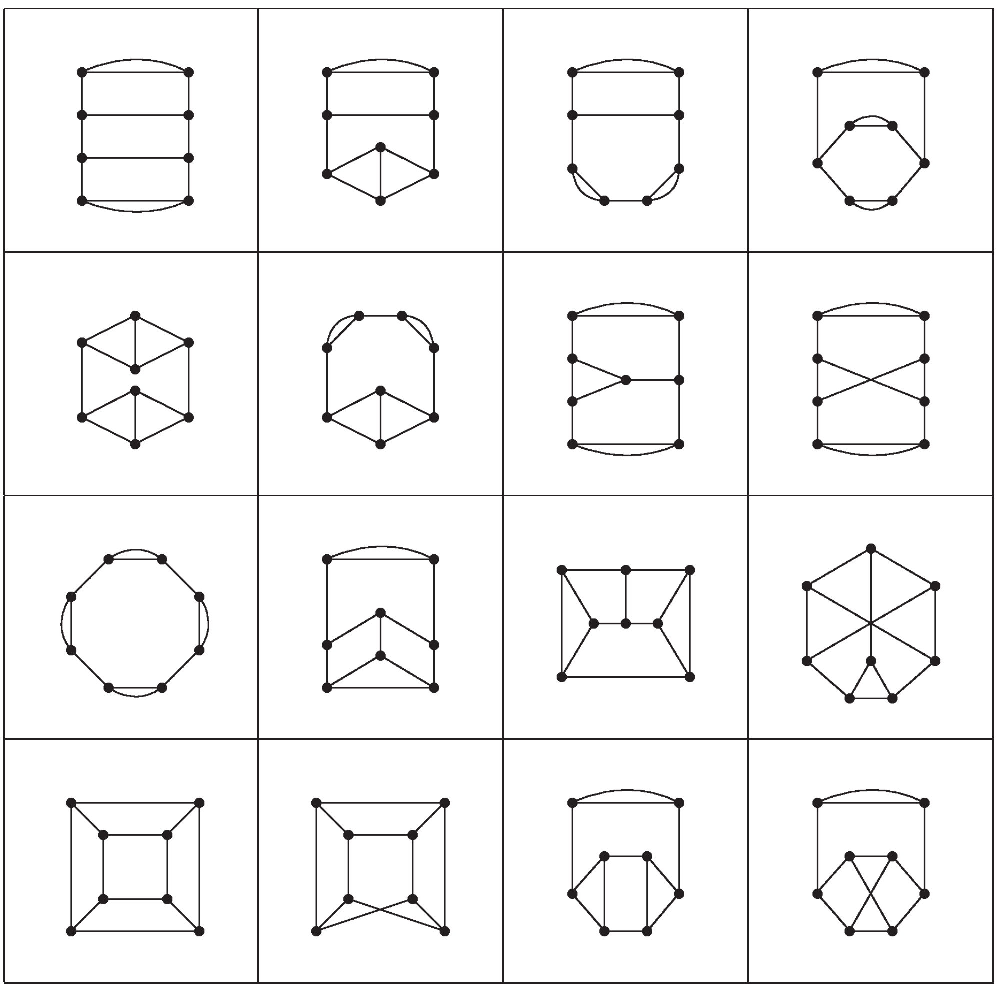

TABLE 3. Trivalent graphs of $n = 5$.

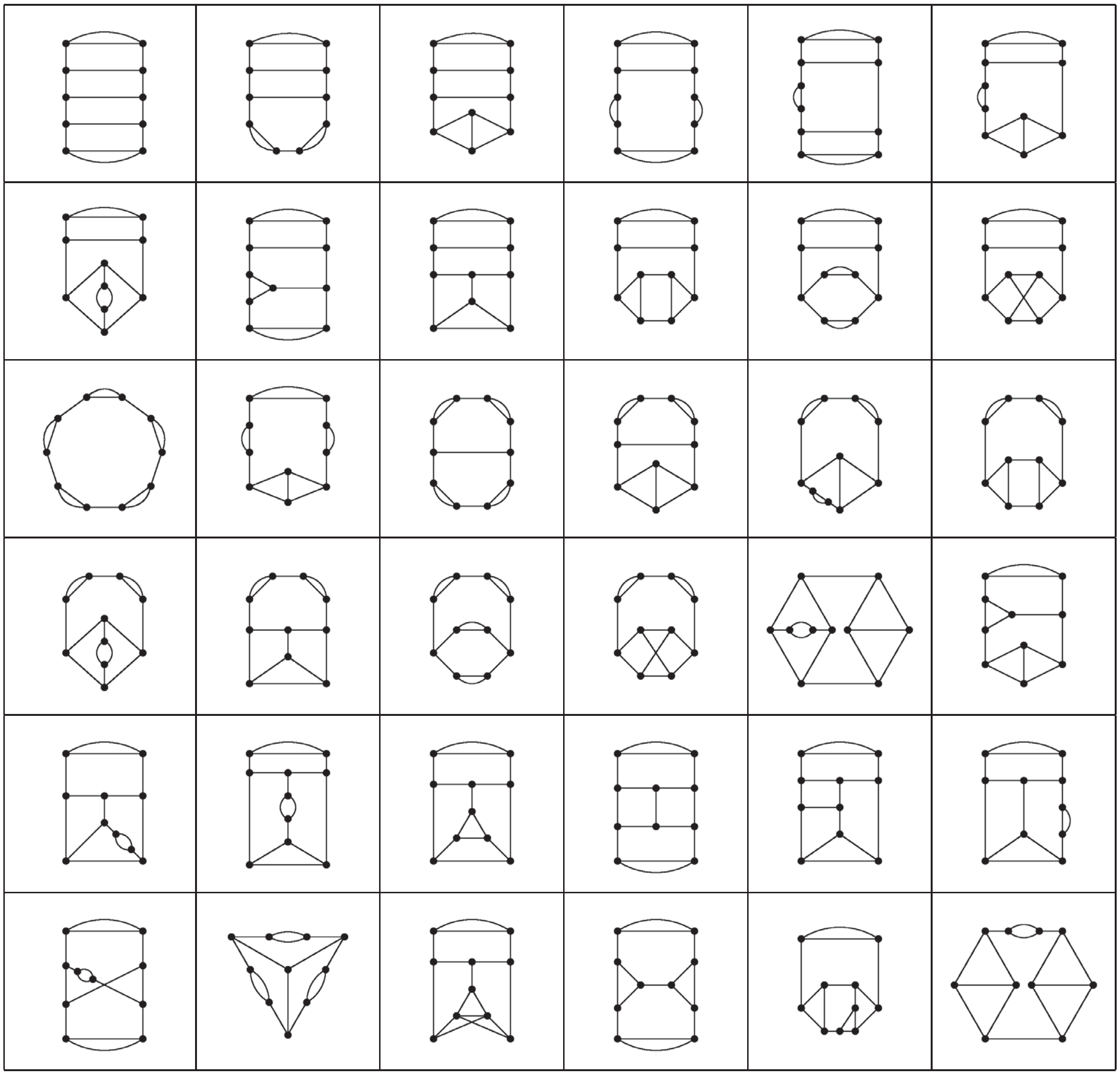

TABLE 4. Trivalent graphs of $n = 6$ (part 1).

(5) Erase all the pairs that satisfy the condition of Lemma 3.4.

(6) Classify forested graphs with respect to the image of c_p.

(7) Construct the matrices that represent ∂^C for each component in (6).

(8) Compute the kernel of the matrices in (7).

(9) Construct the matrices that represent $\partial^R|_{\mathrm{Ker}\partial^C}$.

(10) Compute the kernel of the matrices in (9).

To compute the kernels of these matrices, we use the usual method of Gauss–Jordan elimination. Since the boundary operator ∂^C is represented by a sparse matrix, the most important technique for reducing the computation is to manage the pivoting in a given matrix appropriately.

4. MAIN RESULTS

By making an explicit computation according to the recipe given above, we obtain the rank of the kernels as shown in Table 6.

The dimension of

$$E^2_{p,0} = \frac{\mathrm{Ker}\partial^C_{p,0} \cap \mathrm{Ker}\partial^R_{p,0}}{\partial^R_{p+1,0}(\mathrm{Ker}\partial^C_{p+1,0})}$$

is obtained from $\dim\left(\mathrm{Ker}\partial^C_{p,0} \cap \mathrm{Ker}\partial^R_{p,0}\right)$ and $\dim \mathrm{Ker}\partial^C_{p,0}$, because the dimension theorem about

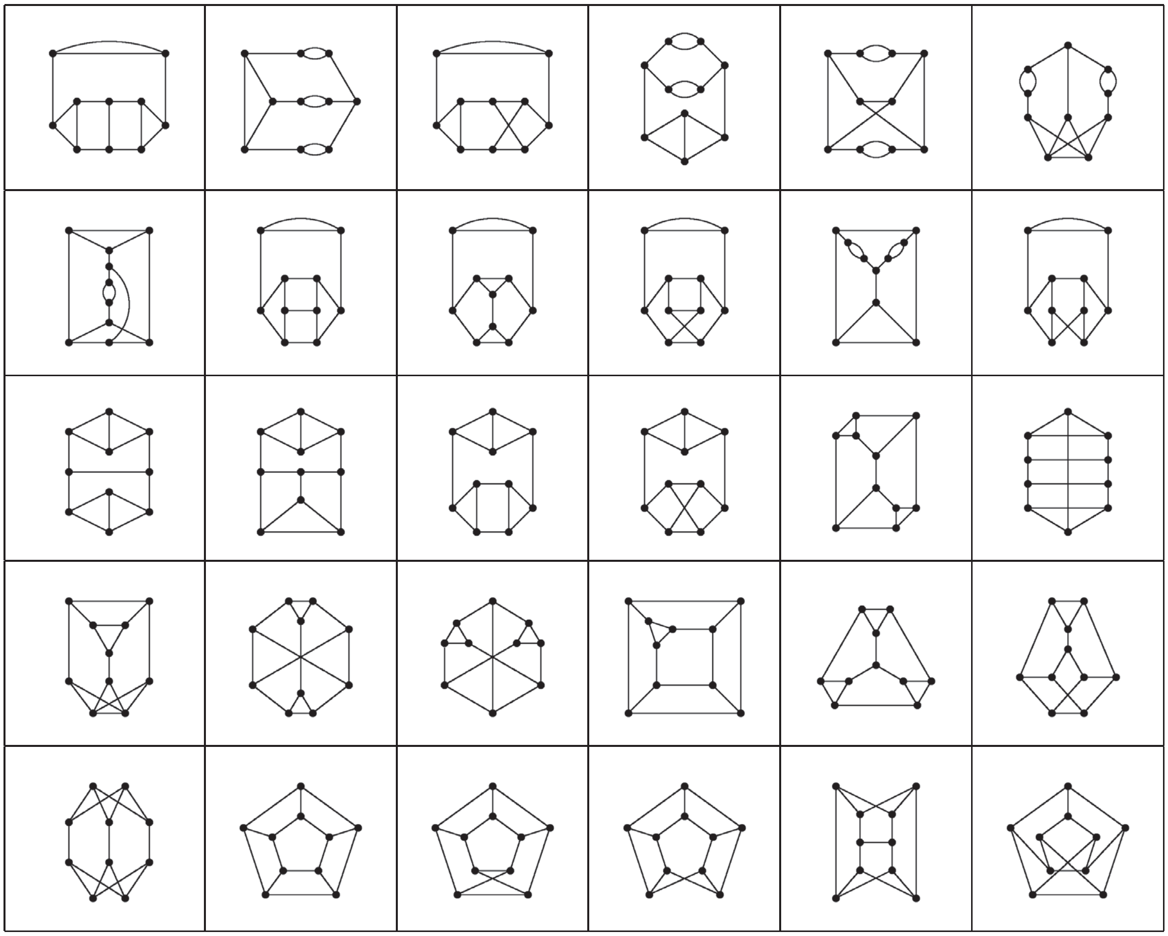

TABLE 5. Trivalent graphs of $n = 6$ (part 2).

the denominator says that

$$\begin{aligned}
&\dim \partial^R_{p+1,0}(\mathrm{Ker}\partial^C_{p+1,0}) \\
&\quad = \dim \mathrm{Im}\ \partial^R_{p+1,0}\big|_{\mathrm{Ker}\partial^C_{p+1,0}} \\
&\quad = \dim \mathrm{Ker}\partial^C_{p+1,0} - \dim \mathrm{Ker}\ \partial^R_{p+1,0}\big|_{\mathrm{Ker}\partial^C_{p+1,0}} \\
&\quad = \dim \mathrm{Ker}\partial^C_{p+1,0} - \dim \left(\mathrm{Ker}\partial^C_{p+1,0} \cap \mathrm{Ker}\partial^R_{p+1,0}\right).
\end{aligned}$$

Tables 7, 8, and 9 contain the dimensions of the kernels of the boundary maps. Note that the number of trivalent graphs for each n also appears on the top left of the table, because trivalent graphs are 0-cells.

n	Number of Trivalent Graphs
2	1
3	2
4	5
5	16
6	66

TABLE 6. Number of trivalent graphs.

Since the rank of the homology group is computed through

$$\begin{aligned}
H_p(\mathrm{Out}(F_n);\mathbb{Q}) &\cong H_p(Q_n;\mathbb{Q}) = E^2_{p,0} \\
&= \frac{\mathrm{Ker}\partial^C_{p,0} \cap \mathrm{Ker}\partial^R_{p,0}}{\partial^R_{p+1,0}(\mathrm{Ker}\partial^C_{p+1,0})},
\end{aligned}$$

we obtain the following theorem.

Theorem 4.1. *For $n \leq 6$, the rational homology of $\mathrm{Out}(F_n)$ is given as follows:*

$$H_p(\mathrm{Out}(F_n);\mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & \text{if } p = 0, \\ \mathbb{Q}, & \text{if } p = n = 4, \\ \mathbb{Q}, & \text{if } p = 8, n = 6, \\ 0, & \text{otherwise.} \end{cases}$$

p	$\dim \mathrm{Ker}\partial^C_{p,0}$	$\dim (\mathrm{Ker}\partial^C_{p,0} \cap \mathrm{Ker}\partial^R_{p,0})$	$\dim \partial^R_{p+1,0}(\mathrm{Ker}\partial^C_{p+1,0})$	$\dim E^2_{p,0}$
0	5	5	4	1
1	4	0	0	0
2	0	0	0	0
3	1	1	1	0
4	4	3	2	1
5	2	0	0	0

TABLE 7. The ranks of the kernels in the case of $n = 4$.

p	$\dim \mathrm{Ker}\partial^C_{p,0}$	$\dim (\mathrm{Ker}\partial^C_{p,0} \cap \mathrm{Ker}\partial^R_{p,0})$	$\dim \partial^R_{p+1,0}(\mathrm{Ker}\partial^C_{p+1,0})$	$\dim E^2_{p,0}$
0	16	16	15	1
1	26	11	11	0
2	20	9	9	0
3	37	28	28	0
4	69	41	41	0
5	53	12	12	0
6	12	0	0	0
7	0	0	0	0

TABLE 8. The ranks of the kernels in the case $n = 5$.

p	$\dim \mathrm{Ker}\partial^C_{p,0}$	$\dim (\mathrm{Ker}\partial^C_{p,0} \cap \mathrm{Ker}\partial^R_{p,0})$	$\dim \partial^R_{p+1,0}(\mathrm{Ker}\partial^C_{p+1,0})$	$\dim E^2_{p,0}$
0	66	66	65	1
1	193	128	128	0
2	372	244	244	0
3	807	563	563	0
4	1389	826	826	0
5	1440	614	614	0
6	889	275	275	0
7	399	124	124	0
8	160	36	35	1
9	35	0	0	0

TABLE 9. The ranks of the kernels in the case $n = 6$.

Since the first two Morita classes $\mu_1 \in H_4(\mathrm{Out}(F_4); \mathbb{Q})$ and $\mu_2 \in H_8(\mathrm{Out}(F_6); \mathbb{Q})$ are known to be nontrivial, as mentioned above, we have the following corollary.

Corollary 4.2. *For $n \leq 6$, Morita classes generate the nontrivial part of $H_*(\mathrm{Out}(F_n); \mathbb{Q})$.*

Remark 4.3. The cycle representing μ_1 has a very simple form, as depicted in Figure 11. It is a multiple of the cycle $z(\gamma)$ described in [Conant and Vogtmann 06]. The description of the 4-dimensional boundaries is also simple. However, the boundary image related to μ_2 is at present very complicated. In fact, the size of our file that contains the boundaries is more than two megabytes.

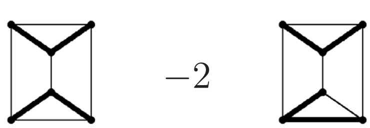

FIGURE 11. Cycle representing μ_1 .

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