Base-Tangle Decompositions of n-String Tangles with $1 < n < 10$

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CONTENTS

This study describes the program bTd, which was developed for the decomposition of any n-tangle with $1 < n < 10$ into base n-tangles using the Skein relation. The program enables us to compute HOMFLY polynomials of knots and links with a large number of crossing points within a matter of hours (see Examples 4.4 and 4.5). This contrasts with the results of attempting computations using Hecke algebras $H(q, n)$ with $18 \geq n$. Such a computation did not complete even after a period of thirty days in a recent examination by the first author and F. Kako [Imafuji and Ochiai 02, Murakami 89, Ochiai and Murakami 94, Ochiai and Kako 95]. In this paper, we first introduce two new concepts: an oriented ordered tangle and a subdivision of a tangle. We then present some examples of base-tangle decompositions achieved using the present program along with the corresponding computational times.

1. INTRODUCTION

Let T be a 2-string tangle with four endpoints $0, 1, 2, 3$ as shown in Figures 1 and 3 [Conway 70, Murasugi 96]. Then T is said to be a *base tangle* if the following conditions are satisfied:

1. Every string of T is a line segment connecting its endpoints.
2. Every crossing point is a double point.
3. If any crossing points exist, they have a plus sign.

For example, all the 2-tangles shown in Figure 1 are base tangles.

Let T be a 2-string tangle. Then T has a base-tangle decomposition by the following Skein relation [Freyd et al. 85, Imafuji and Ochiai 02]:

$$xP(T_+; x, y) + yP(T_-; x, y) = P(T_\infty; x, y).$$

Let $P(T; x, y) = \alpha A + \beta B$ be a base-tangle decomposition, where A and B are base tangles and α and β are HOMFLY polynomials obtained by the Skein relation.

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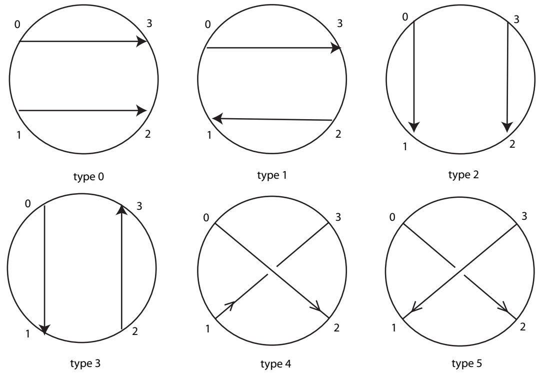

FIGURE 1. Base 2-tangles.

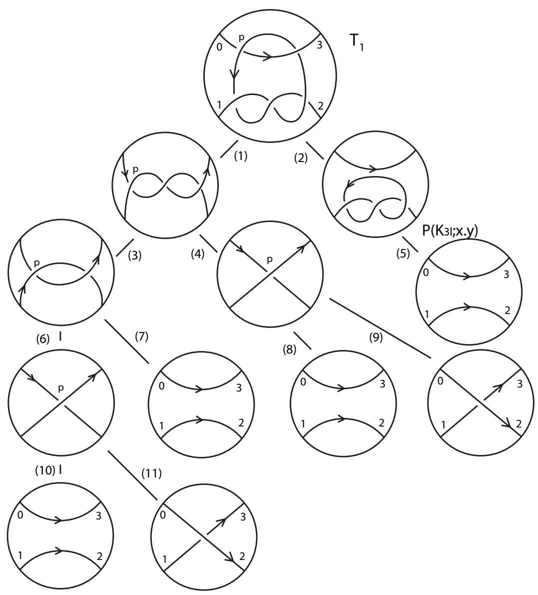

FIGURE 2. A resolution tree.

For example, Figure 2 gives a resolution tree of a 2-string tangle T_1, where p is the resolution point in each step. Furthermore, (1) in the figure denotes $1/x$, while (3), (6), (8), (10) denote $1/y$ and (2), (4), (7), (9), (11) denote $-x/y$; finally, (5) denotes $P(K_{3l};x,y)(1/y^2 - x^2/y^2 - 2x/y)$, where $P(K_{3l};x,y)$ is the HOMFLY polynomial of the clover knot.

Let K_{cw} be the Conway knot and $K_{\mathrm{cw}} = T_1 + T_2$ a 2-tangle decomposition of K_{cw} given by Figure 3.

Then we have

$$P(T_1;x,y) = \alpha_1 A_1 + \beta_1 B_1,$$

where A_1 is a base tangle of type 0, B_1 is a base tangle of type 4,

$$\alpha_1 = 2 + \frac{1}{x\,y^3} - \frac{2}{y^2} - \frac{1}{x\,y} + \frac{x}{y}, \quad \beta_1 = -y^{-3} + \frac{x}{y^2},$$

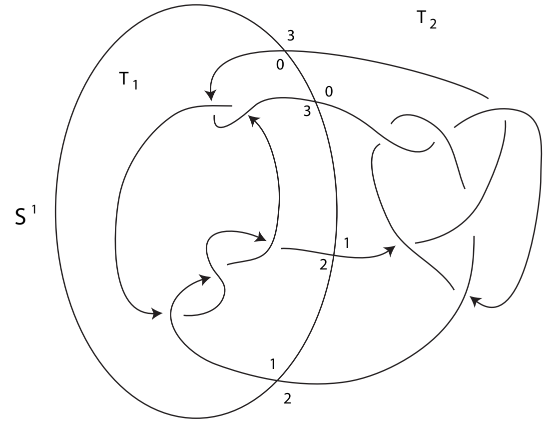

FIGURE 3. Conway knot.

and

$$P(T_2;x,y) = \alpha_2 A_2 + \beta_2 B_2,$$

where A_2 is a base tangle of type 4, B_2 is a base tangle of type 0,

$$\alpha_2 = \frac{1}{x^2\,y^2} - \frac{1}{x\,y} - \frac{x}{y}, \quad \beta_2 = \frac{1}{x} - \frac{1}{x\,y^2} + \frac{2}{y} - \frac{1}{x^2\,y},$$

and

$$\begin{aligned} P(K_{\mathrm{cw}};x,y) &= \alpha_1\alpha_2(A_1 \cup A_2) + \alpha_2\beta_1(A_2 \cup B_1) \\ &\quad + \alpha_1\beta_2(A_1 \cup B_2) + \beta_1\beta_2(B_1 \cup B_2), \end{aligned}$$

where $A_1 \cup A_2$, $A_2 \cup B_1$, $A_1 \cup B_2$, and $B_1 \cup B_2$ are links obtained by attaching two base tangles along their endpoints in the standard manner.

This leads to the following result:

$$\begin{aligned} &P(K_{\mathrm{cw}};x,y)(x+y)\alpha_1\alpha_2 + \alpha_2\beta_1 + \alpha_1\beta_2 \\ &\quad + \left(\frac{1}{x} - \frac{y\,(x+y)}{x}\right)\beta_1\beta_2 \\ &\quad\; = 7 - \frac{3}{x^2} + y^{-4} - \frac{1}{x^2\,y^4} - \frac{1}{x^3\,y^3} + \frac{6}{x\,y^3} - \frac{3\,x}{y^3} \\ &\qquad - \frac{11}{y^2} + \frac{6}{x^2\,y^2} + \frac{2\,x^2}{y^2} + \frac{1}{x^3\,y} - \frac{11}{x\,y} + \frac{6\,x}{y} + \frac{2\,y}{x}. \end{aligned}$$

2. ORIENTED ORDERED TANGLES

A general base-tangle decomposition is considered in this section. An *oriented ordered tangle* is defined as an n-tangle with n ordered oriented strings $s_1, s_2, \ldots, s_n$, each of which has two endpoints $p_{2(i-1)}$ and $p_{2(i-1)+1}$, $i = 1, 2, \ldots, n$, with

$$\min\{p_{2(j-1)}, p_{2(j-1)+1}\} < \min\{p_{2j}, p_{2j+1}\},$$

$j = 1, 2, \ldots, n-1$.

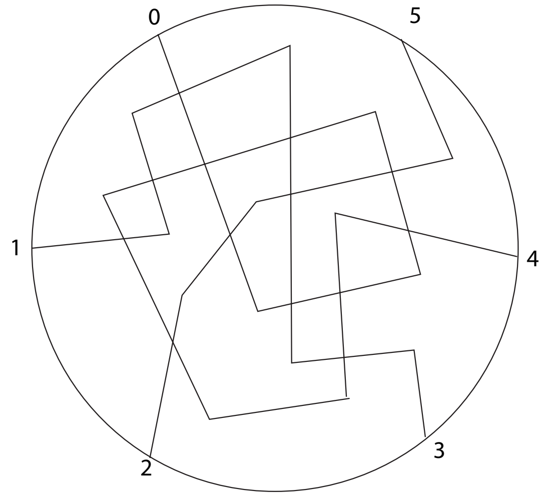

FIGURE 4. An ordered 3-tangle.

Let T be an oriented ordered n-tangle. Then we may represent T by $[T; p_0, p_1, p_2, p_3, \ldots, p_{2n-2}, p_{2n-1}]$, or, more simply, $[p_0, p_1, p_2, p_3, \ldots, p_{2n-2}, p_{2n-1}]$.

Figure 4 depicts an example of an ordered 3-tangle T. If the first string in the figure is directed from 0 to 4, then the second goes from 1 to 3, the third goes from 2 to 5, and the oriented ordered tangle is $[T; 0, 4, 1, 3, 2, 5]$. But if the first string has the opposite direction, namely from 4 to 0, then the oriented ordered tangle is $[T; 4, 0, 1, 3, 2, 5]$.

An n-tangle T is a base n-tangle if it satisfies the following conditions:

1. Every string of T is a line segment connecting its endpoints.
2. Every crossing point is a double point.
3. Each string s_i has only overcrossing points or only undercrossing points to s_j, $i = 1, \ldots, n-1, j = i+1, \ldots, n$.

It may therefore be noted that in particular, every base n-tangle has at most $n(n-1)/2$ crossing points.

An oriented ordered tangle T has a base-tangle decomposition given by the Skein relation

$$P(T; x, y) = \sum_{i=1}^{n!} \alpha_i A_i,$$

where α_i is a 2-variable polynomial determined by the resolution and A_i is a base tangle.

If the first string of every base tangle A_i always has only overcrossing points (or undercrossing points), then we say that the resolution has 0-*baserule* (1-*baserule*). For example, the 2-tangle of type 4 in Figure 1 is a base tangle having 0-baserule but not 1-baserule. The base-tangle decomposition in Figure 2 is also a base-tangle decomposition according to the above definition, having 0-baserule.

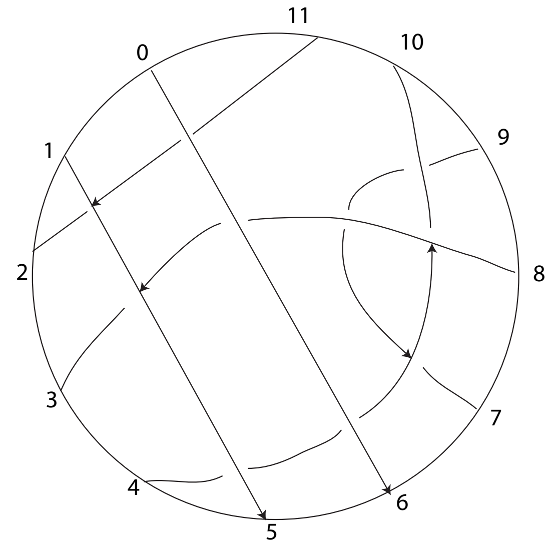

FIGURE 5. An unreduced 6-tangle.

When a base n-tangle decomposition is carried out via a computer program, the following problems need to be solved:

1. recognizing whether a tangle is a base tangle;
2. reconstructing a base tangle from an ordered number sequence of length $2n$.

For example, the 6-tangle shown in Figure 5 is equivalent to the base 6-tangle shown in Figure 6, because the former can be transformed into the latter by Reidemeister moves keeping its endpoints fixed.

Those transformations, however, can never be completely implemented by a computer program [Ochiai 90]. When the i-th string s_i in a tangle T has only overcrossings (or undercrossings) other than its nontrivial self-intersections (that is, local knots K), K can be removed from s_i by $P(T; x, y) = P(T'; x, y)\ P(K; x, y)$, where T' is a tangle obtained by removing K from T. As a result, s_i can be deleted from T to avoid Reidemeister moves by maintaining $\{p_{2i-2}, p_{2i-1}\}$ for later reconstructions of base tangles, where p_{2i-2} and p_{2i-1} are endpoints of s_i. Following the completion of a base-tangle decomposition, at most $n!$ oriented ordered base tangles $[p_0, p_1, p_2, p_3, \ldots, p_{2n-2}, p_{2n-1}]$ are obtained as number sequences. Suppose that $2n$ endpoints, $0, 1, \ldots, 2n-1$ are equidistantly positioned in counterclockwise order on a unit circle.

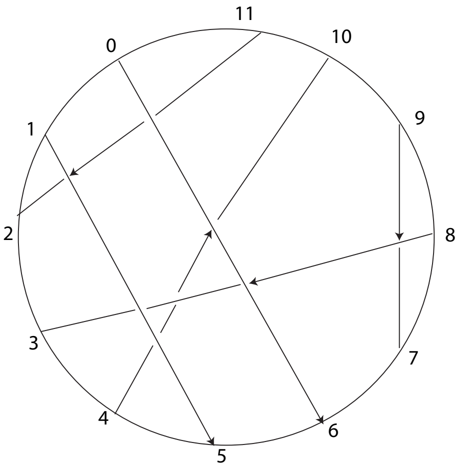

FIGURE 6. A reduced base 6-tangle.

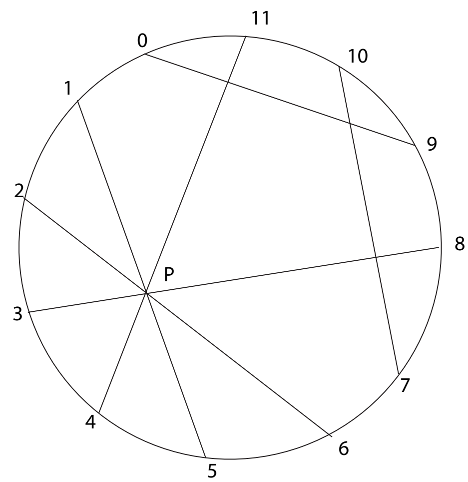

FIGURE 7. A 4-multiple point.

On connecting these endpoints with line segments according to the number sequences

$$\{p_0, p_1, p_2, p_3, \ldots, p_{2n-2}, p_{2n-1}\},$$

an oriented ordered base n-tangle is obtained. In most cases, more than two strings cross at one point. In other words, r-multiple crossing points with $r > 2$ occur. In order to reconstruct a base tangle, each r-multiple crossing point must be transformed into $r(r-1)/2$ double points. For example, a base 6-tangle $[0, 9, 1, 5, 2, 6, 3, 8, 4, 11, 7, 10]$ has a line-segment diagram as depicted in Figure 7. There is a 4-multiple point P in the segment, which can be locally transformed into six double points under 0-baserule, as seen in Figure 8.

A substitute model of $r(r-1)/2$ double points is created beforehand for each r-multiple crossing point with

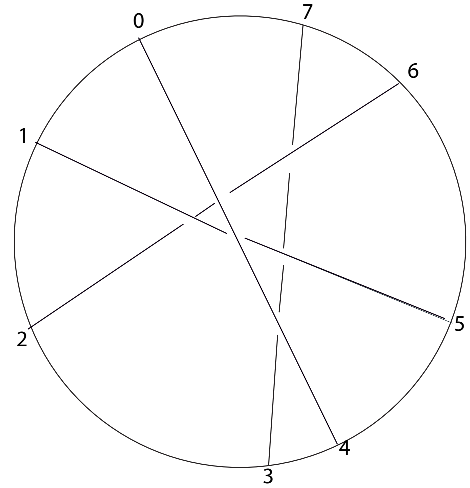

FIGURE 8. A 4-tangle with baserule 0.

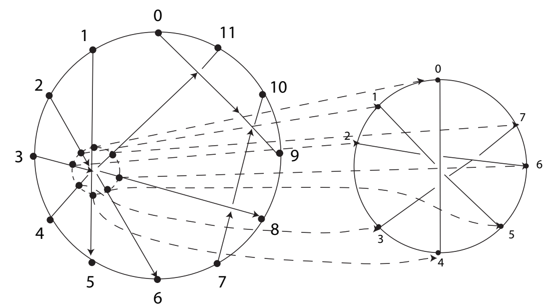

FIGURE 9. A mapping of a 4-multiple point.

$r \geq 3$. The model is locally mapped on each r-multiple crossing point as illustrated in Figure 9.

3. SUBDIVISIONS OF TANGLES

Generally, constructing base-tangle decompositions of n-tangles possessing many crossing points using the method described above is difficult: it is very time-consuming. Thus the subdivision of a tangle is now considered in more detail.

It is well known that HOMFLY polynomials of 3-parallel versions of knots can usually distinguish two distinct mutant knots [Imafuji and Ochiai 02, Ochiai and Murakami 94, Ochiai and Kako 95].

Let L^3_{15} be a link with 135 crossing points (see Figure 10). It will be noticed that L^3_{15} is a link 3-paralleled along a knot with 15 crossings.

First, L^3_{15} is decomposed into two 6-tangles T_1 and T_2. While a base-tangle decomposition of T_2 is achieved relatively easily, that of T_1 is difficult. For this reason, T_1 is subdivided into three tangles $T_{11} + T_{12} + T_{13}$ such that

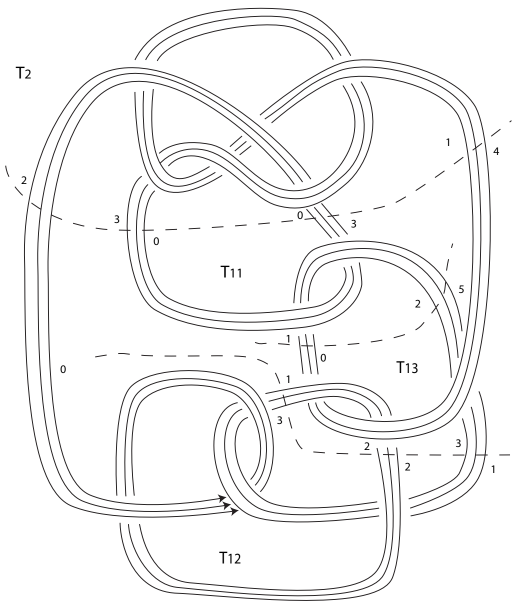

FIGURE 10. A subdivision.

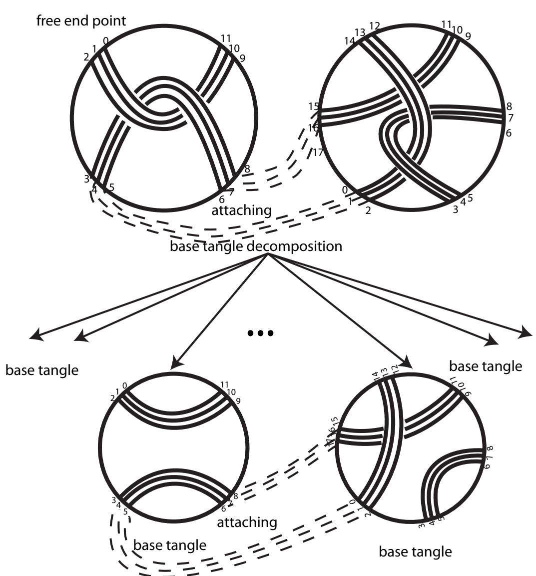

FIGURE 11. A composition of two base tangles.

T_{11}, T_{12}, and T_{13} are respectively a 6-tangle, a 6-tangle, and a 9-tangle.

These three base-tangle decompositions are easily accomplished. A base-tangle decomposition of T_1 is obtained as $T_{11} + (T_{12} + T_{13})$ by attaching (1) each base 6-tangle and base 9-tangle along the original nine attachment points between T_{12} and T_{13} and (2) each base 6-tangle of T_{11} and base 6-tangle of $T_{12} + T_{13}$ along the original six attachment points between T_{11} and T_{13}, as shown in Figure 11.

It may be noted that when composing two base tangles along a subset of their endpoints, a tangle may occur with free loops. The notation $(s, t; u)$ denotes that a tangle is subdivided into an s-tangle and a t-tangle along u cutting points. This subdivision can be carried out as many times as desired. The following describes subdivisions of n-string tangles in the current version of `bTd`, where n is $6, 8, 9$:

$$(6,6;12),\ (6,9;6),\ (6,9;9),\ (6,6;6),\ (9,9;18),\ (9,9;12),\ (9,9;9),\ (8,8;16),\ (8,8;8).$$

Any knot K can generally be decomposed into two n-tangles subdividing K into $2n$ points on K, and similarly, those two tangles can be divided into smaller pieces. The value of n, however, should be no more than 9 from a practical point of view. Note that the time complexity of decompositions depends on string numbers but not on crossing numbers if subdivisions are small enough.

4. COMPUTATIONAL RESULTS

This section shows some examples of base-tangle decompositions using the present software `bTd` along with the corresponding computational times. We used Linux machines running on 3.0-GHz Pentium-4 hardware. In order to speed up the computation, our software uses a first-string selection strategy for the baserule. First, the number c_0 of overcrossings and the number c_1 of undercrossings other than self-intersections on the first string are compared. If $c_0 \geq c_1$, then the 0-baserule must be selected; otherwise, the 1-baserule. Then, applying Skein's relation reduces the number of crossing points under the same baserule. Note that at the beginning of a base-tangle decomposition, the user can also impose a change to a baserule.

Example 4.1. Let K_T^2 be a link 2-paralleled along the Kinoshita–Terasaka knot shown in Figure 12. It can calculate $P(K_T^2; x, y)$ within a time on the order of tens of seconds, and of course it is found to be equal to $P(K_{\mathrm{cw}}^2; x, y)$, where K_{cw}^2 is a link 2-paralleled along the Conway knot [Lickorish and Lipson 87].

Example 4.2. Let K_O^2 be a link 2-paralleled along the trivial knot shown in Figure 13 [Ochiai 90]. The present program can also calculate $P(K_O^2; x, y)$ within the same time scale. It may be noted that obtaining $P(K_O^2; x, y)$ using only Skein's relation would, in contrast, be difficult,

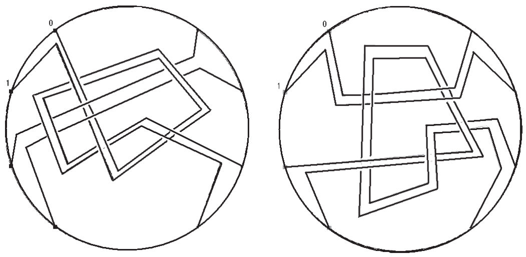

FIGURE 12. A tangle decomposition of 2-paralleled version of Kinoshita–Terasaka knot.

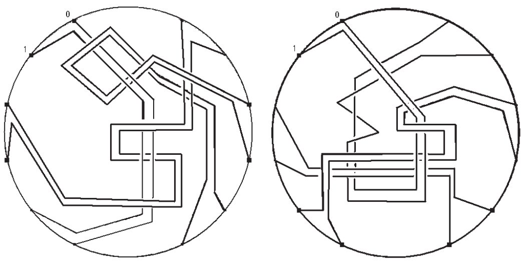

FIGURE 13. A tangle decomposition of the trivial knot.

requiring a significantly higher computational time than that required by our software.

Example 4.3. Let K_T^3 be a link 3-paralleled along the Kinoshita–Terasaka knot [Imafuji and Ochiai 02, Ochiai and Murakami 94, Ochiai and Kako 95]. It can calculate the first tangle in about 2120 seconds, the second one in about 1940 seconds, and $P(K_T^3; x, y)$ from these results in a further 2000 seconds. It is found that $P(K_T^3; x, y)$ is not equal to $P(m(K_K^3); x, y)$, where $m(K_T^3)$ is a 180-mutant link of K_T^3 [Ochiai and Murakami 94, Ochiai and Kako 95].

Example 4.4. Let L_{15}^3 be the link mentioned in the previous section. Let $m(L_{15}^3)$ denote a 180-mutant link of L_{15}^3, and let $h(L_{15}^3)$ and $v(L_{15}^3)$ denote a horizontal-mutant and vertical-mutant link respectively. Note that the base-tangle decomposition of T_2 (respectively $h(T_2)$) accords exactly with that of $m(T_2)$ (respectively $v(T_2)$) under the same baserule, but the base-tangle decompositions of T_2 and $h(T_2)$ are not equal.

As seen in Thistlethwaite's knot table,[1] there exist two distinct knots with fifteen crossing points whose HOMFLY polynomials are $P(L_{15}; x, y)$. The computation time for $P(L_{15}^3; x, y)$ was almost the same as for $P(K_T^3; x, y)$, whereas it would take around 12 days using only a base-tangle decomposition of T_1. The following are the computational times required for each step of base-tangle decomposition and tangle composition in order to obtain $P(L_{15}^3; x, y)$:

- T_{12}: a few seconds,
- T_{13}: a few seconds,
- $T_{23} = T_{12} + T_{13}$: about ten minutes,
- T_{11}: a few seconds,
- $T_1 = T_{11} + T_{23}$: about 21 minutes,
- T_2: about 28 minutes,
- $T_1 + T_2$: about 118 minutes.

Example 4.5. Let $L_{15_1}^3$ be the link shown in Figure 14, let $L_{15_1}^3 = T_1 + T_2$ be a tangle decomposition of this link, and let $T_1 = T_{11} + T_{12}$ be a subdivision of T_1. Then the following times are necessary for the calculation of $P(L_{15_1}^3; x, y)$:

- T_{11}: about 38 minutes,
- T_{12}: a few seconds,
- $T_1 = T_{11} + T_{12}$: about ten minutes,
- T_2: about 19 minutes,
- $T_1 + T_2$: about 120 minutes.

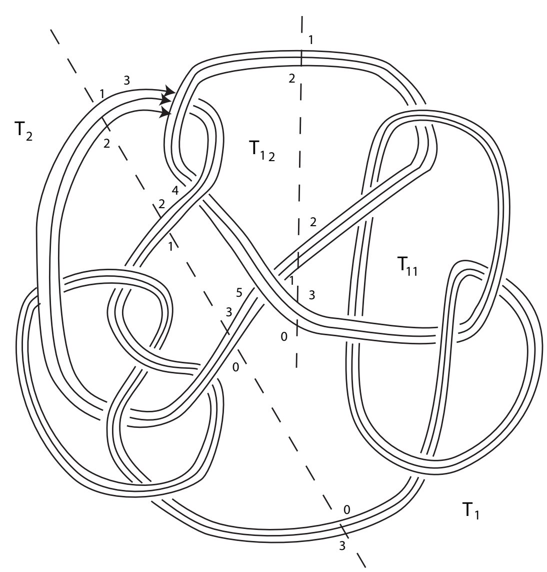

FIGURE 14. A subdivision.

[1] Available at http://www.math.utk.edu/~morwen/download.

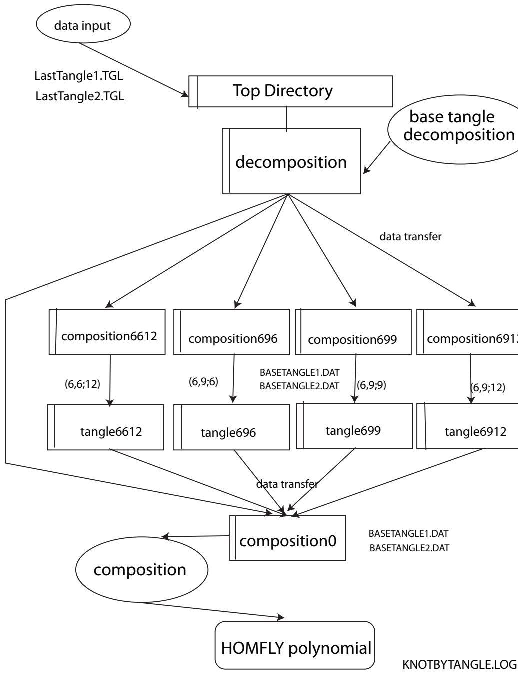

FIGURE 15. The placement of the working directory.

It will be noticed that $P(L^3_{15_1}; x, y)$ is shown to be not equal to $P(m(L^3_{15_1}); x, y)$ by our software.

Figure 15 indicates the flow of data through subdivisions due to the application of the present software. For example, the computational results of two base-tangle decompositions are stored in the directory `/decomposition` as two files: `BASETANGLE1.DAT` and `BASETANGLE2.DAT`. These two files must then be moved to the directory `/composition0` to obtain a HOMFLY polynomial using $(6, 6; 12)$.

5. FINAL REMARKS

The first author together with N. Imafuji developed a program, `K2K`, that assists research in knot theory. We introduced this program at the international symposium KNOT2000, held in Korea [Imafuji and Ochiai 02]. One year later, a revised version of `K2K` with a function of base 2-tangle decompositions[2] was made available to the public.

In 2003, we presented work relating to base 3-tangle decompositions, and in 2004, work relating to base n-tangle decompositions with $6 \geq n$, both in seminars at the Tokyo Institute of Technology.

[2]Available at http://amadeus.ics.nara-wu.ac.jp/~ochiai/.

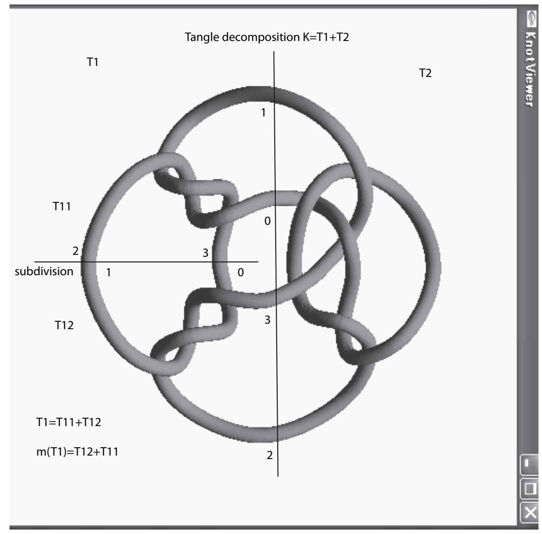

FIGURE 16. A mutant knot.

In 2005, we introduced a new program, `bTd`, for base n-tangle decompositions with $9 \geq n$ in a seminar in Osaka. Version 1.0 of this program was made available to the public on the web site mentioned above in January 2006.

The current version, 1.1, can subdivide an n-tangle into two tangles along only six, eight, nine, or twelve cutting points. The next version, 2.0, will be able to compute base-tangle decompositions using `PVM`, a software tool for parallel networking of computers, which was developed by the University of Tennessee and Oak Ridge National Laboratory [Ochiai and Kadobayashi 07].

Recently, the first author discovered a 2-tangle T_1 with $4n + 2$, $n > 0$, crossings such that its mutation image is in agreement with its mirror image and such that a base-tangle decomposition of a 6-tangle T_1^3 3-paralleled along T_1 completely agrees with that of a 6-tangle $m(T_1^3)$ 3-paralleled along $m(T_1)$ (see Figure 16).

Let T^r be a $2r$-tangle r-paralleled along a 2-tangle T, and let $B(T^r; x, y : b)$ be a base-tangle decomposition of T^r under a baserule b. We verified by computations that

$$B(T_1^3; x, y : 0) = B(m(T_1^3); x, y : 0),$$
$$B(T_1^3; x, y : 0) = B(m(T_1^3); y, x : 1),$$
$$B(T_1^4; x, y : 0) = B(m(T_1^4); y, x : 1),$$

but $B(T_1^4; x, y : 0) \neq B(m(T_1^4); x, y : 0)$. In particular, $B(T_1^4; x, y : 0)$ and $B(m(T_1^4); x, y : 0)$ have 40320 bases, and they are different in only 5420 bases. For example, the difference $B(m(T_1^4); x, y : 0) - B(T_1^4; x, y : 0)$ has as part $x^2y^{-8} - xy^{-9} - y^{-6} + x^{-1}y^{-7}$ about a base $(0, 15, 1, 13, 2, 11, 3, 10, 4, 9, 5, 14, 6, 8, 7, 12)$.

It may be noted that $P(T_1^3+T_2^3;x,y)$ and $P(m(T_1^3)+T_2^3;x,y)$ agree, even though T_1+T_2 and $m(T_1)+T_2$ are different knots by Thistlethwaite's knot table.[3]

Note that the latest version of `bTd` can calculate

$$P(T_1^3;x,y), \quad P(m(T_1^3);x,y)$$

in a few minutes, and $P(T_1^4;x,y)$, $P(m(T_1^4);x,y)$ in around ten days using subdivisions and base conversions, while the latest version of `K2K` can calculate only certain one-variable polynomials, which are induced from HOMFLY polynomials of closed braids by restricting W-graphs to their subgraphs, but not $P(T_1^3+T_2^3;x,y)$ and $P(m(T_1^3)+T_2^3;x,y)$ themselves [Murakami 89, Ochiai and Kako 95].

Unfortunately, `bTd` failed also to calculate $B(T_2^4;x,y:b)$, because we need huge memory to store 12! base 12-tangles in the worst case of subdivisions of T_2^4.

We remark further that K. Murasugi asked the first author whether there is any periodicity in n to satisfy the equality $B(T_1^n;x,y:0)=B(m(T_1^n);x,y:0)$, $n\geq 3$. Though this is a very interesting problem, there appears to be no tool to calculate it at present.

6. ACKNOWLEDGMENTS

We are grateful to Professors J. Murakami, F. Kako, and T. Kanenobu for useful discussions on a preliminary version of this software, and also to Ms. A. Emoto for writing a program to generate base n-tangles with $12\geq n$ from number sequences in conjunction with the first author.

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[3]Recently, F. Kako created a knot table that includes all alternating knots with n crossings ($16\leq n\leq 18$). It is available at http://kako.ics.nara-wu.ac.jp/~kako/research/knot/index.html.