Exceptional Regions and Associated Exceptional Hyperbolic 3-Manifolds

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with an appendix by Alan W. Reid

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A closed hyperbolic 3-manifold is exceptional if its shortest geodesic does not have an embedded tube of radius $\ln(3)/2$. D. Gabai, R. Meyerhoff, and N. Thurston identified seven families of exceptional manifolds in their proof of the homotopy rigidity theorem. They identified the hyperbolic manifold known as Vol3 in the literature as the exceptional manifold associated with one of the families. It is conjectured that there are exactly six exceptional manifolds. We find hyperbolic 3-manifolds, some from the SnapPea census of closed hyperbolic 3-manifolds, associated with five other families. Along with the hyperbolic 3-manifold found by Lipyanskiy associated with the seventh family, we show that any exceptional manifold is covered by one of these manifolds. We also find their group coefficient fields and invariant trace fields.

1. INTRODUCTION

A closed hyperbolic 3-manifold is *exceptional* if its shortest geodesic does not have an embedded tube of radius $\ln(3)/2$. Exceptional manifolds arise in the proof of the rigidity theorem proved by D. Gabai, R. Meyerhoff, and N. Thurston in [Gabai et al. 03]. It is conjectured that there are exactly seven exceptional manifolds.

Let N be a closed hyperbolic 3-manifold and δ the shortest geodesic in N. If δ does not have an embedded tube of radius $\ln(3)/2$, then there is a two-generator subgroup G of $\pi_1(N)$ such that $\mathbb{H}^3/G$ also has this property. Assume that G is generated by f and w, where $f \in \pi_1(N)$ is a primitive hyperbolic isometry whose fixed axis $\delta_0 \in \mathbb{H}^3$ projects to δ and $w \in \pi_1(N)$ is a hyperbolic isometry that takes δ_0 to its nearest translate. Thus, it is necessary to study two-generator subgroups Γ of $\mathrm{PSL}(2, \mathbb{C})$ with the property that one of the generators is the shortest geodesic in $\mathbb{H}^3/\Gamma$ and the distance from its nearest translate is less than $\ln(3)$.

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Parameter	Range $\Re$(Parameter)	Range $\Im$(Parameter)
L'	0.58117 to 0.58160	-3.31221 to -3.31190
D'	1.15644 to 1.15683	-2.75628 to -2.75573
R'	1.40420 to 1.40454	-1.17968 to -1.17919

TABLE 1. Parameter ranges for the region X_3.

The space of two-generator subgroups of $\mathrm{PSL}(2,\mathbb{C})$ is analyzed in the proof of the rigidity theorem of [Gabai et al. 03]. The rigidity theorem is proved using Gabai's theorem [Gabai 03], which states that the rigidity theorem is true if some closed geodesic has an embedded tube of radius $\ln(3)/2$. The authors of [Gabai et al. 03] show that this holds for all but seven exceptional families of closed hyperbolic 3-manifolds.

These seven families are handled separately. The seven families are obtained by parameterizing the space of two-generator subgroups of $\mathrm{PSL}(2,\mathbb{C})$ by a subset of $\mathbb{C}^3$, dividing the parameter space into about a billion regions and eliminating all but seven regions. These seven regions correspond to the seven exceptional families and are known as *exceptional regions*. They are denoted by X_i for $i = 0,\ldots,6$ and described as boxes in $\mathbb{C}^3$. For example, for the region X_3 see Table 1.

A *quasi-relator* in a region is a word in f, w, f^{-1}, and w^{-1} that is close to the identity throughout the region and experimentally appears to converge to the identity at some point. Table 2 gives the two quasi-relators specified for each region X_i in [Gabai et al. 03]. The group $G_i = \langle f,\ w | r_1(X_i),\ r_2(X_i)\rangle$, where $r_1(X_i),\ r_2(X_i)$ are the quasi-relators for X_i, is called the *marked group* for the region X_i. It follows from [Gabai et al. 03] that any exceptional manifold has a two-generator subgroup of its fundamental group whose parameter lies in one of the exceptional regions.

Let $\rho(x,y)$ denote the hyperbolic distance between x and y in $\mathbb{H}^3$. For an isometry f of $\mathbb{H}^3$ define $\mathrm{Relength}(f) = \inf\{\rho(x, f(x)) \mid x \in \mathbb{H}^3\}$. Let T consist of those parameters corresponding to the groups $\{G,f,w\}$ such that $\mathrm{Relength}(f)$ is the shortest element of G and the distance between the axis of f and its nearest translate is less than $\ln(3)$. Let $S = \exp(T)$. In [Gabai et al. 03] the authors made the following conjecture.

Conjecture 1.1. [Gabai et al. 03] *Each subbox X_i, $0 \le i \le 6$, contains a unique element s_i of S. Further, if $\{G_i, f_i, w_i\}$ is the marked group associated with s_i, then $N_i = \mathbb{H}^3/G_i$ is a closed hyperbolic 3-manifold with the following properties:*

(i) *N_i has fundamental group $\langle f,\ w|r_1(X_i),\ r_2(X_i)\rangle$, where r_1 and r_2 are the quasi-relators associated with the box X_i.*

(ii) *N_i has a Heegaard genus-2 splitting realizing the above group presentation.*

(iii) *N_i nontrivially covers no manifold.*

(iv) *N_6 is isometric to N_5.*

(v) *If (L_i, D_i, R_i) is the parameter in T corresponding to s_i, then L_i, D_i, R_i are related as follows: For X_0, X_5, X_6 we have $L = D$, $R = 0$. For X_1, X_2, X_3, X_4 we have $R = L/2$.*

D. Gabai, R. Meyerhoff, and N. Thurston [Gabai et al. 03] proved that Vol3, the closed hyperbolic 3-manifold with conjecturally the third-smallest volume, is the unique exceptional manifold associated with the region X_0. Jones and Reid [Jones and Reid 01] proved that Vol3 does not nontrivially cover any manifold and that the exceptional manifolds associated with the regions X_5 and X_6 are isometric.

In this paper we investigate the seven exceptional regions and the associated exceptional hyperbolic 3-manifolds. In Section 3, using Newton's method for finding roots of polynomials in several variables, we solve the equations obtained from the entries of the quasi-relators to very high precision. Then, using the program PARI-GP [PARI 02], we find entries of the generating matrices as algebraic numbers, find the group coefficient fields, and verify with exact arithmetic that the quasi-relators are relations for all the regions. We also find the invariant trace fields for all the groups, verifying and extending the data given in [Jones and Reid 01].

In Section 4 we show that the manifolds $v2678(2,1)$, $s778(-3,1)$, and $s479(-3,1)$ from SnapPea's census of closed hyperbolic 3-manifolds [Weeks 93] are the exceptional manifolds associated with the regions X_1, X_2, X_5, and X_6. We also show that their fundamental groups are isomorphic to the marked groups G_i for $i = 1,2,5,6$. In Section 5 we find an exceptional manifold associated with the region X_4 and show that its fundamental group is isomorphic to the marked group G_4. This manifold, which we denote by N_4, is commensurable with the SnapPea census manifold $m369(-1,3)$. Lipyanskiy has described a sixth exceptional manifold in [Lipyanskiy 02].

In Section 6, using Gröbner bases we show that the quasi-relators have a unique solution in every region. Let $N_0 = $ Vol3, $N_1 = v2678(2,1)$, $N_2 = s778(-3,1)$, N_3 the exceptional manifold associated with X_3 found in [Lipyanskiy 02], N_4 the exceptional manifold associated with X_4 found in Section 5, and $N_5 = s479(-3,1)$. We shall prove the following theorem.

Region	Quasi-relators
X_0	$r_1 = fwf^{-1}w^2f^{-1}wfw^2$ $r_2 = f^{-1}wfwfw^{-1}fwfw$
X_1	$r_1 = f^{-2}wf^{-1}w^{-1}f^{-1}w^{-1}fw^{-1}f^{-1}w^{-1}f^{-1}wf^{-2}w^2$ $r_2 = f^{-2}w^2f^{-1}wfwfw^{-1}fwfwf^{-1}w^2$
X_2	$r_1 = f^{-1}wfwfw^{-1}f^2w^{-1}fwfwf^{-1}w^2$ $r_2 = f^{-2}wf^{-2}w^2f^{-1}wfwfwf^{-1}w^2$
X_3	$r_1 = f^{-2}wfwf^{-2}w^2f^{-1}w^{-1}f^{-1}wf^{-1}w^{-1}(fw^{-1}f^{-1}w^{-1}f)^2w^{-1}f^{-1}wf^{-1}w^{-1}f^{-1}w^2$ $r_2 = f^{-2}wfwf^{-1}wf(w^{-1}fwfw^{-1})^2fwf^{-1}wfwf^{-2}w^2f^{-1}w^{-1}f^{-1}w^2$
X_4	$r_1 = f^{-2}wfwf^{-1}(wfw^{-1}f)^2wf^{-1}wfwf^{-2}w^2(f^{-1}w^{-1}f^{-1}w)^2w$ $r_2 = f^{-1}(f^{-1}wfw)^2f^{-2}w^2f^{-1}w^{-1}f^{-1}w(f^{-1}w^{-1}fw^{-1})^2f^{-1}wf^{-1}w^{-1}f^{-1}w^2$
X_5	$r_1 = f^{-1}wf^{-1}w^{-1}f^{-1}wf^{-1}wfwfw^{-1}fwfw$ $r_2 = f^{-1}wfwfw^{-1}fw^{-1}f^{-1}w^{-1}fw^{-1}fwfw$
X_6	$r_1 = f^{-1}w^{-1}f^{-1}wf^{-1}w^{-1}f^{-1}w^{-1}fw^{-1}fwfw^{-1}fw^{-1}$ $r_2 = f^{-1}w^{-1}fw^{-1}fwfwf^{-1}wfwfw^{-1}fw^{-1}$

TABLE 2. Quasi-relators for all the regions.

Theorem 1.2. *Let N be an exceptional manifold. Then N is covered by N_i for some $i = 0, 1, 2, 3, 4, 5$.*

2. INVARIANT TRACE FIELDS AND 2-GENERATOR SUBGROUPS

Two finite-volume orientable hyperbolic 3-manifolds are said to be *commensurable* if they have a common finite-sheeted cover. Subgroups $G, G' \subset \mathrm{PSL}(2,\mathbb{C})$ are *commensurable* if there exists $g \in \mathrm{PSL}(2,\mathbb{C})$ such that $g^{-1}Gg \cap G'$ is a finite-index subgroup of both $g^{-1}Gg$ and G'. It follows by Mostow rigidity that finite-volume orientable hyperbolic 3-manifolds are commensurable if and only if their fundamental groups are commensurable as subgroups of $\mathrm{PSL}(2,\mathbb{C})$.

Let G be a group of covering transformations and let $\tilde{G}$ be its preimage in $\mathrm{SL}(2,\mathbb{C})$. It is shown in [Macbeath 83] that the traces of elements of $\tilde{G}$ generate a number field $\mathbb{Q}(\operatorname{tr} G)$ called the *trace field* of G. The *invariant trace field* $k(G)$ of G is defined as the intersection of all the fields $\mathbb{Q}(\operatorname{tr} H)$, where H ranges over all finite-index subgroups of G. The definition already makes clear that $k(G)$ is a commensurability invariant. In [Reid 90], Alan Reid proved the following result.

Theorem 2.1. [Reid 90] *The invariant trace field $k(G)$ is equal to*

$$\mathbb{Q}(\{\operatorname{tr}^2(g) : g \in G\}) = \mathbb{Q}(\operatorname{tr} G^{(2)}),$$

where $G^{(2)}$ is the finite-index subgroup of G generated by squares $\{g^2 : g \in G\}$.

From [Hilden et al. 92, Corollary 3.2], the invariant trace field of a 2-generator group $\langle f, w \mid r_1, r_2 \rangle$ is generated by $\operatorname{tr}(f^2)$, $\operatorname{tr}(w^2)$, and $\operatorname{tr}(f^2w^2)$.

Using trace relations (see [Coulson et al. 00, Theorem 4.2]), as well as [Hilden et al. 92, Corollary 3.2], which says that $\mathbb{Q}(\operatorname{tr} G^{(2)}) = \mathbb{Q}(\operatorname{tr} G^{SQ})$, where $G^{SQ} = \langle g_1^2, \dots, g_n^2 \rangle$ with g_i's generators of G such that $\operatorname{tr}(g_i) \neq 0$, we see that the invariant trace field of a 2-generator group $\langle f, w \mid r_1, r_2 \rangle$ is generated by $\operatorname{tr}(f^2)$, $\operatorname{tr}(w^2)$, and $\operatorname{tr}(f^2w^2)$.

As described in Section 1, a marked group G is generated by f and w, where f and w are seen as covering transformations such that f represents the shortest geodesic and w takes the axis of f to its nearest translate. Let $\mathbb{H}^3$ denote the upper-half-space model of hyperbolic 3-space and let the sphere at infinity be the xy-plane. Conjugate G so that the axis of f is the geodesic line $B_{(0,\infty)}$ in $\mathbb{H}^3$ with endpoints 0 and ∞ on the sphere at infinity, and the geodesic line perpendicular to $w^{-1}(B_{(0,\infty)})$ and $B_{(0,\infty)}$ (orthocurve) lies on the geodesic line $B_{(-1,1)}$ in $\mathbb{H}^3$ with endpoints -1 and 1 on the sphere at infinity.

We can parameterize such a marked group with three complex numbers L, D, and R, where f is an L-translation of $B_{(0,\infty)}$ and w is a D-translation of $B_{(-1,1)}$ followed by an R-translation of $B_{(0,\infty)}$. We can write matrix representatives for f and w using the exponentials L', D', and R' of L, D, and R, respectively (see [Gabai et al. 03, Chapter 1]). We have

$$f = \begin{pmatrix} \sqrt{L'} & 0 \\ 0 & 1/\sqrt{L'} \end{pmatrix}, \quad w = \begin{pmatrix} \sqrt{R'} * ch & \sqrt{R'} * sh \\ sh/\sqrt{R'} & ch/\sqrt{R'} \end{pmatrix}, \tag{2–1}$$

where

$$ch = (\sqrt{D'} + 1/\sqrt{D'})/2$$

and

$$sh = (\sqrt{D'} - 1/\sqrt{D'})/2.$$

We can write down the generators of the invariant trace field of G in terms of L', D', and R' as follows:

$$\begin{aligned} \operatorname{tr}(f^2) &= L' + \frac{1}{L'}, \\ \operatorname{tr}(w^2) &= \frac{1}{4}\left[\left(R' + \frac{1}{R'} + 2\right)\left(D' + \frac{1}{D'} + 2\right) - 8\right], \\ \operatorname{tr}(f^2w^2) &= \frac{1}{4}\left[\left(D' + \frac{1}{D'} + 2\right)\left(R'L' + \frac{1}{R'L'}\right)\right. \\ &\qquad \left. + \left(D' + \frac{1}{D'} - 2\right)\left(L' + \frac{1}{L'}\right)\right]. \end{aligned}$$

3. GUESSING THE ALGEBRAIC NUMBERS AND EXACT ARITHMETIC

In this section we find the marked groups for the regions as subgroups of $\mathrm{PSL}(2,\mathbb{C})$ with algebraic entries and find their invariant trace fields. In [Gabai et al. 03], parameter ranges for the seven regions are specified. For example, for the parameter range for region X_3, see Table 1.

We solve for L', D', and R' such that the quasi-relators are relations in the group. We obtain eight equations in three complex variables out of which three are independent, and we use Newton's method with the parameter range as approximate solutions to find high-precision solutions, e.g., one hundred significant digits, for the parameters satisfying the equations.

This allows us to compute $a = \sqrt{L'}$, $b = \sqrt{R'}$, $c = \sqrt{D'}$, and $\operatorname{tr}(f^2)$, $\operatorname{tr}(w^2)$, $\operatorname{tr}(f^2w^2)$ to high precision. Once the numbers are obtained to high precision, we use the `algdep()` function of the PARI-GP package [PARI 02] to guess a polynomial over the integers that has the desired number as a root.

Although the `algdep()` function cannot prove that the guess is in fact correct, we prove this using the guessed values to perform exact arithmetic and verify the relations. Once we obtain $a = \sqrt{L'}$, $b = \sqrt{R'}$, and $c = \sqrt{D'}$ as roots of polynomials, we find a primitive element that generates the field that contains all three numbers.

For the regions X_0, X_5, and X_6, a, b, and c are all contained in $\mathbb{Q}(a)$. By expressing the matrix entries as algebraic numbers one can verify the relations directly. For example, for X_0, the minimal polynomial for a and c is $x^8 + 2x^6 + 6x^4 + 2x^2 + 1$, and $b = 1$, so we can express a, b, and c as follows:

```
a = Mod(x, x^8 + 2*x^6 + 6*x^4 + 2*x^2 + 1),
b = Mod(1, x^8 + 2*x^6 + 6*x^4 + 2*x^2 + 1),
c = Mod(x, x^8 + 2*x^6 + 6*x^4 + 2*x^2 + 1).
```

Then, using the formulas of Section 2, PARI-GP calculates the quasi-relators exactly as follows:

```
[Mod(1, x^8 + 2*x^6 + 6*x^4 + 2*x^2 + 1) 0],
[0 Mod(1, x^8 + 2*x^6 + 6*x^4 + 2*x^2 + 1)].
```

Thus, exact arithmetic verifies rigorously that the L', D', and R' calculated for X_0 using Newton's method are correct and that the quasi-relators are in fact relations.

In general, the group coefficient field can have arbitrary index over the trace field. In order to keep the degree of the group coefficient field low, we follow the method described in [Lipyanskiy 02]. Given that f, w are generic ($fw - wf$ is nonsingular), if f_2, w_2 are any matrices in $\mathrm{SL}(2,\mathbb{C})$ such that $\operatorname{tr}(f_2) = \operatorname{tr}(f)$, $tr(w_2) = \operatorname{tr}(w)$, and $\operatorname{tr}(f_2^{-1}w_2) = \operatorname{tr}(f^{-1}w)$, then the two pairs are conjugate.

Let $\mathrm{tr}_1 = \operatorname{tr}(f)$, $\mathrm{tr}_2 = \operatorname{tr}(w)$, $\mathrm{tr}_3 = \operatorname{tr}(f^{-1}w)$. Furthermore let

$$f_2 = \begin{pmatrix} 0 & 1 \\ -1 & \mathrm{tr}_1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} z & 0 \\ \mathrm{tr}_1 * z - \mathrm{tr}_3 & \mathrm{tr}_2 - z \end{pmatrix}, \tag{3–1}$$

where $(\mathrm{tr}_2 - z) * z = 1$. Then the pair (f_2, w_2) is conjugate to (f, w). The coefficients of the original f and w may have arbitrary index over the trace field, but in this form the entries of the matrices are in an extension of the trace field of degree at most two. Tables 3 and 4 display the computation of z, tr_1, tr_2, and tr_3 for all regions. In all cases, z is the primitive element and $\mathrm{tr}_i \in \mathbb{Q}(z)$. One easily verifies the relations using the tables. We have the following theorem.

Theorem 3.1. *The marked groups G_i are 2-generator subgroups of $\mathrm{PSL}(2,\mathbb{C})$ with entries in the number fields as given in Tables 3 and 4. Furthermore, the quasi-relators are relations in these groups.*

Remark 3.2. It is proved in [Lipyanskiy 02] that the quasi-relators generate all the relations for these groups and that the groups G_i are discrete cocompact subgroups of $\mathrm{PSL}(2,\mathbb{C})$.

Region	Minimal Polynomial	Numerical Value
X_0	$\tau^8+2\tau^6+6\tau^4+2\tau^2+1$	$0.853230697-1.252448658i$
X_1	$\tau^8-2\tau^7+5\tau^6-4\tau^5+7\tau^4-4\tau^3+5\tau^2-2\tau+1$	$0.904047196-1.471654224i$
X_2	$\tau^4-2\tau^3+4\tau^2-2\tau+1$	$0.742934136-1.529085514i$
X_3	$\tau^{24}-8\tau^{23}+35\tau^{22}-107\tau^{21}+261\tau^{20}-538\tau^{19}$ $+972\tau^{18}-1565\tau^{17}+2282\tau^{16}-3034\tau^{15}+3706\tau^{14}$ $-4171\tau^{13}+4339\tau^{12}-4171\tau^{11}+3706\tau^{10}-3034\tau^{9}$ $+2282\tau^{8}-1565\tau^{7}+972\tau^{6}-538\tau^{5}+261\tau^{4}-107\tau^{3}$ $+35\tau^{2}-8\tau+1$	$1.404292212-1.179267298i$
X_4	$\tau^6-3\tau^5+5\tau^4-4\tau^3+5\tau^2-3\tau+1$	$1.354619901-1.225125454i$
X_5	$\tau^{12}+2\tau^{10}+7\tau^8-4\tau^6+7\tau^4+2\tau^2+1$	$0.868063287-1.460023666i$
X_6	$\tau^{12}-2\tau^{10}+7\tau^8+4\tau^6+7\tau^4-2\tau^2+1$	$1.460023666-0.868063287i$

TABLE 3. Field containing z for all regions.

Region	tr_1	tr_2	tr_3
X_0	$-z-6z^3-2z^5-z^7$	tr_1	$(-5z^2-2z^4-z^6)/2$
X_1	$2-4z+4z^2-7z^3+4z^4-5z^5+2z^6-z^7$	tr_1	tr_1
X_2	$2-3z+2z^2-z^3$	tr_1	tr_1
X_3	$8-34z+107z^2-261z^3+538z^4-972z^5+1565z^6$ $-2282z^7+3034z^8-3706z^9+4171z^{10}-4339z^{11}$ $+4171z^{12}-3706z^{13}+3034z^{14}-2282z^{15}+1565z^{16}$ $-972z^{17}+538z^{18}-261z^{19}+107z^{20}-35z^{21}+8z^{22}-z^{23}$	tr_1	tr_1
X_4	$3-4z+4z^2-5z^3+3z^4-z^5$	tr_1	tr_1
X_5	$-z-7z^3+4z^5-7z^7-2z^9-z^{11}$	tr_1	$(-6z^2+4z^4-7z^6-2z^8-z^{10})/2$
X_6	$3z-7z^3-4z^5-7z^7+2z^9-z^{11}$	tr_1	$(4-6z^2-4z^4-7z^6+2z^8-z^{10})/2$

TABLE 4. Group coefficients as polynomials in z in respective field.

Region	Minimal Polynomial	Numerical Value
X_0	τ^2+3	$1.732050808i$
X_1	$\tau^4-2\tau^3+\tau^2-2\tau+1$	$-0.207106781+0.978318343i$
X_2	τ^2+1	i
X_3	$\tau^{12}+6\tau^{11}+23\tau^{10}+91\tau^9+257\tau^8+489\tau^7+823\tau^6$ $+1054\tau^5-13\tau^4-2445\tau^3-3405\tau^2-1847\tau-337$	$0.632778000-3.019170376i$
X_4	$\tau^3-\tau-2$	$-0.760689853+0.857873626i$
X_5	$\tau^3-\tau^2+\tau+1$	$0.771844506+1.11514250i$
X_6	$\tau^3-\tau^2+\tau+1$	$0.771844506-1.11514250i$

TABLE 5. Invariant trace fields for all the regions.

Remark 3.3. f_2 and w_2 give an efficient way to solve the word problem in these groups.

In this way we also obtain $\mathrm{tr}(f^2)$, $\mathrm{tr}(w^2)$, and $\mathrm{tr}(f^2w^2)$ as roots of polynomials and find a primitive element that generates the field that contains all three traces. We have the following result.

Theorem 3.4. *The invariant trace fields for all the regions are as given in Table 5.*

Remark 3.5. The invariant trace field descriptions in Table 5 for X_i, $i \neq 3$, are the canonical field descriptions given by Snap [Coulson et al. 00].

Region	V	H_1	$l_{\min}$	Manifolds
X_1	4.11696874	$\mathbb{Z}_7 \oplus \mathbb{Z}_7$	1.0930	$v2678(2,1), v2796(1,2)$
X_2	3.66386238	$\mathbb{Z}_4 \oplus \mathbb{Z}_{12}$	1.061	$s778(-3,1), v2018(2,1)$
X_4	7.517689	$\mathbb{Z}_4 \oplus \mathbb{Z}_{12}$	1.2046	NA
X_5 or X_6	3.17729328	$\mathbb{Z}_4 \oplus \mathbb{Z}_4$	1.0595	$s479(-3,1), s480(-3,1),$ $s645(1,2), s781(-1,2),$ $v2018(-2,1)$

TABLE 6. Data for regions X_1, X_2, X_4, X_5, X_6.

4. THE MANIFOLDS FOR THE REGIONS X_1, X_2, X_5, AND X_6

In this section we find manifolds from the Hodgson and Weeks census of closed hyperbolic 3-manifolds whose fundamental groups are isomorphic to the groups G_i, $i = 1, 2, 5, 6$. This census is included in Jeff Weeks's program SnapPea [Weeks 93] and is referred to as SnapPea's census of closed hyperbolic 3-manifolds.

These manifolds are described as Dehn surgeries on cusped hyperbolic 3-manifolds from SnapPea's census of cusped manifolds [Callahan et al. 99], [Hildebrand and Weeks 89]. We use the invariant trace fields and volume estimates for the regions given in [Jones and Reid 01] and [Lipyanskiy 02] to search through the roughly 11,000 manifolds in the closed census.

The package Snap [Goodman et al. 98] includes a text file called `closed.fields`, which lists the invariant trace fields for the manifolds in the closed census. Using this file to compare the invariant trace fields, we narrowed our search to fewer than 50 manifolds for each region. Then using the homology, volume estimates, and length of shortest geodesic, we further narrowed the search to fewer than five manifolds. Table 6 gives the approximate volume (V) as given in [Jones and Reid 01], first homology (H_1), approximate length of shortest geodesic, which is the value of the parameter L ($l_{\min}$), and the manifold description as given in SnapPea.

The above manifolds include those mentioned in [Gabai et al. 03] for the regions X_1, X_2, and X_5. All the SnapPea manifolds associated with a region in Table 6 are isometric. It is shown in [Jones and Reid 01] that the manifolds associated with the regions X_5 and X_6 are isometric with an orientation-reversing isometry. The manifold associated with X_4 is discussed in the next section and that for X_3 is discussed in [Lipyanskiy 02].

The fundamental groups of the above manifolds have two generators and two relations. Tables 2 and 7 give the relators for the marked groups and those for the fundamental groups of the corresponding manifolds. One can verify the isomorphisms between the groups given in Table 8. We have the following theorem.

Theorem 4.1. *The manifolds $v2678(2,1)$, $s778(-3,1)$, and $s479(-3,1)$ in SnapPea's census of closed manifolds are exceptional manifolds associated with the respective regions X_1, X_2, and X_5.*

5. THE MANIFOLD ASSOCIATED WITH THE REGION X_4

In this section we give a description of the manifold associated with the region X_4 as a double cover of an orbifold commensurable with the manifold $m369(-1,3)$ in SnapPea's census of closed manifolds.

In Section 4, using the approximate volumes and other data given in [Jones and Reid 01] and [Lipyanskiy 02], we found manifolds from SnapPea's census of closed manifolds with fundamental groups isomorphic to the groups for the regions X_1, X_2, X_5, and X_6. The regions X_3 and X_4 could not be handled because of their large volumes. However, for the region X_4, a list of manifolds was found in the closed census having approximately half the volume of X_4 and the same commensurability invariants.

In hope of obtaining the manifold for X_4 as a double cover of one of these manifolds, we compared index-two subgroups of the fundamental groups of each of these manifolds to G_4, the marked group for X_4. Most of the subgroups were eliminated on the basis of homology. However, one index-two subgroup of the census manifold $m369(-1,3)$ had the correct homology, and the same lengths for its elements as for X_4. Using the program *testisom* [Holt and Rees 97], it was checked that this subgroup is not isomorphic to G_4.

Theorem 5.1. *The manifold N_4 associated with the region X_4 is commensurable with the manifold $m369(-1,3)$ in SnapPea's census of closed manifolds. This manifold is obtained as a double cover of an orbifold that is double covered by a double cover of $m369(-1,3)$.*

Manifold	π_1 Relators
$v2678(2,1)$	$q_1 = a^2b^2aba^{-1}ba^{-1}b^{-1}a^{-1}ba^{-1}bab^2$ $q_2 = ab^{-1}ab^{-1}a^{-1}b^{-1}ab^{-1}aba^2b^2a^2b$
$s778(-3,1)$	$q_1 = ab^{-1}aba^2b^2ab^2a^2bab^{-1}$ $q_2 = ab^2a^2ba^2b^2aba^{-1}ba^{-1}b$
$s479(-3,1)$	$q_1 = aba^2b^2a^2bab^{-2}a^{-2}b^{-2}$ $q_2 = a^2b^2ab^2a^2bab^{-1}ab^{-1}ab$

TABLE 7. Relators for manifolds.

Region	Manifolds	Isomorphism	Inverse
X_1	$v2678(2,1)$	$f \longrightarrow a^{-1}$, $w \longrightarrow b$	$a \longrightarrow f^{-1}$, $b \longrightarrow w$
X_2	$s778(-3,1)$	$f \longrightarrow a$, $w \longrightarrow b^{-1}$	$a \longrightarrow f$, $b \longrightarrow w^{-1}$
X_5	$s479(-3,1)$	$f \longrightarrow ab$, $w \longrightarrow b$	$a \longrightarrow fw^{-1}$, $b \longrightarrow w$

TABLE 8. Isomorphisms.

Proof: Let $M = m369(-1,3)$. We will construct the following diagram of 2 : 1 covers:

$$\begin{array}{ccc} N & & N_4 \\ \downarrow & \searrow & \downarrow \\ M & & O \end{array}$$

We obtain a presentation of $\pi_1(M)$ from SnapPea:

$$\pi_1(M) = \langle a, b, c \mid ab^{-1}a^{-1}c^2bc, abcb^3a^{-1}c^{-1}, acbc^{-1}b^{-1}cbacb\rangle.$$

Let $\phi : \pi_1(M) \to \mathbb{Z}_2$ be defined by $\phi(a) = 1, \phi(b) = \phi(c) = 0$. Then ϕ is a homomorphism and $\ker(\phi)$ is an index-two subgroup of $\pi_1(M)$ generated by b and c. Let N denote the double cover of M corresponding to this subgroup, so that $\pi_1(N) = \ker(\phi)$. A presentation of $\pi_1(N)$ is

$$\pi_1(N) = \langle b, c \mid r_1, r_2\rangle,$$

where

$$r_1 = bcb^3cbc^{-1}b^{-1}cbc^{-1}b^{-1}cbc^2(bc^3)^2bc^2bcb^{-1}c^{-1} \times bcb^{-1}c^{-1}$$

and

$$r_2 = cbc^{-1}b^{-1}cbc^{-1}b^{-1}cbc^2(bc^3)^2(bc^2bc^3bc^3bc^3)^2 \times bcb^{-1}c^{-1}bcb^{-1}c^{-1}b.$$

Let $\psi : \pi_1(N) \to \pi_1(N)$ be defined by $\psi(c) = c^{-1}$ and $\psi(b) = c^3b$. Then ψ is an automorphism of $\pi_1(N)$ of order two. Extending the group $\pi_1(N)$ by this automorphism, we obtain a group H whose presentation is

$$H = \langle b, c, t \mid r_1, r_2, tct^{-1}c,\ tbt^{-1}b^{-1}c^{-3},\ t^2\rangle.$$

Moreover, $\pi_1(N)$ is a subgroup of H of index two, and the quotient of $\mathbb{H}^3$ by H is an orbifold O (due to the torsion element t) that is double covered by N. Let $\mu : H \to \mathbb{Z}_2$ be defined by $\mu(c) = 0$, $\mu(b) = \mu(t) = 1$. Then μ is a homomorphism and $\ker(\mu)$ is an index-two subgroup of H generated by elements c and $b * t$. Let $x = c$ and $y = b * t$. Then a presentation of $\ker(\mu) = G$ is

$$G = \ker(\mu) = \langle x,\ y \mid s_1,\ s_2,\ s_3\rangle,$$

where

$$\begin{aligned} s_1 &= (yx^{-1}y^{-1}x^{-1})^2yx^2y^2x^3yxyxy^{-1}xyxy^{-1}xyxy \\ &\quad \times x^3y^2x^2, \\ s_2 &= (yxy^{-1}x)^2yxyx^3(y^2x^2y^2x^3)^2yxyxy^{-1}xyxy^{-1}xyx, \\ s_3 &= y^{-1}x^{-3}(y^{-1}x^{-1})^2yx^{-1}y^{-1}x^{-1}yx^2y^2x^3yxyx^3y^2x^2 \\ &\quad \times (yx^{-1}y^{-1}x^{-1})^2. \end{aligned}$$

The presentation for the marked group G_4 as given in [Gabai et al. 03] is

$$G_4 = \langle f,\ w \mid r_1(X_4),\ r_2(X_4)\rangle,$$

where

$$\begin{aligned} r_1(X_4) &= f^{-2}wfwf^{-1}(wfw^{-1}f)^2wf^{-1}wfwf^{-2}w^2 \\ &\quad \times (f^{-1}w^{-1}f^{-1}w)^2w, \\ r_2(X_4) &= f^{-1}(f^{-1}wfw)^2f^{-2}w^2f^{-1}w^{-1}f^{-1}w \\ &\quad \times (f^{-1}w^{-1}fw^{-1})^2f^{-1}wf^{-1}w^{-1}f^{-1}w^2. \end{aligned}$$

One easily verifies that the map $\nu : G_4 \to G$ given by $\nu(x) = f$ and $\nu(y) = f^{-1}w^{-1}$ is an isomorphism. The inverse of ν is given by $\nu^{-1}(f) = y$ and $\nu^{-1}(w) = y^{-1}x^{-1}$.

Lipyanskiy [Lipyanskiy 02] constructed a Dirichlet domain for all the regions whose groups are isomorphic to the marked groups. It follows that G_4 is torsion-free, and hence G is a torsion-free subgroup of H of index 2. Hence it gives the manifold N_4, which double covers the orbifold O. □

Remark 5.2. The symmetries of the configuration of lines in $\mathbb{H}^3$ consisting of the axis of f, w, their translations, and the orthocurves between them led us to study the above-mentioned subgroups and automorphisms. The manifold N has a geodesic of the same length and the same translation length as N_4, but it is not the shortest geodesic in N.

6. UNIQUENESS

In this section we address the issue of uniqueness of the solutions in the given region. In [Gabai et al. 03, Chapter 3], the authors showed the existence and uniqueness of the solution for the region X_0 using a geometric argument to establish $R' = 1$ and then using the symmetry of the region X_0 to reduce the number of variables and obtain a one-variable equation that has only one solution in the region X_0. Using a Gröbner basis we show that there is a unique point in every region X_i for which the quasi-relators equal the identity.

Let I be the ideal generated by the equations formed by the entries of the quasi-relators of a region subtracted from the identity matrix. We compute a Gröbner basis for I and verify that there is a unique solution to equations in the Gröbner basis in that region. For computational convenience we split the relations in half, which reduces the degree of the polynomials generating the ideal. Let $p = \mathrm{tr}_1 = \mathrm{tr}(f)$, $q = \mathrm{tr}_2 = \mathrm{tr}(w)$, and $r = \mathrm{tr}_3 = \mathrm{tr}(f^{-1}w)$ as in Section 3. Using (2–1), L', D', and R' can be expressed in terms of p, q, and r as follows:

$$L' = \left(\frac{p \pm \sqrt{p^2-4}}{2}\right)^2, \tag{6–1}$$

$$D' = \left(\frac{2q\sqrt{R'} \pm \sqrt{4q^2R' - 4(1+R')^2}}{2(1+R')}\right)^2, \tag{6–2}$$

$$R' = \frac{qL' - r\sqrt{L'}}{r\sqrt{L'} - q}. \tag{6–3}$$

Using (3–1), we can write the entries of conjugates of f and w in terms of p, q, r, and z, where $qz - z^2 = 1$. The equations for quasi-relators using (3–1) are simpler for computing Gröbner bases.

For example, for the region X_0, using the ordering z, r, q, p on the variables, the last entry of the Gröbner basis is $(p-1)(p+1)(p^4-2p^2+4)$. Using (6–1) it can be easily checked that only one root of the above equation gives the value of L' lying in the region X_0. Similarly, the last entries of the Gröbner bases in orders z, r, p, q and z, p, q, r are $(q-2)(q+2)(q^4-2q^2+4)$ and $(r+1)(r^2-r+1)$. Using (6–2) and (6–3), it can be easily checked that only one root of $q^4 - 2q^2 + 4$ and one of $r^2 - r + 1$ give the values of D' and R' lying in the region X_0. This shows that there is a unique solution for the quasi-relators in the region X_0.

Similarly, for the regions X_2, X_4, X_5, X_6, the last entry of the Gröbner basis is a polynomial in either p, q, or r, depending on the ordering of the variables. Using (6–1), (6–2), (6–3), we check that there is a unique solution in the respective region.

For the regions X_1 and X_3 we obtain a multivariable polynomial in p, q, and r as a factor of the last entry of the Gröbner basis, along with a single-variable polynomial. We eliminate this factor using the mean value theorem.

For example, for the region X_3 the last entry of the Gröbner basis with the ordering $z > r > q > p$ on the variables factors as

$$\begin{aligned}
&(p^3 + p^2 - 2p - 1)\\
&\quad \times (p^{10} - 7p^9 + 15p^8 + 4p^7 - 49p^6 + 11p^5 + 88p^4\\
&\qquad + 87p^3 - 501p^2 + 543p - 193)\\
&\quad \times (p^{10} + 5p^9 + 6p^8 - 6p^7 - 10p^6 + 12p^5 + 13p^4 - 11p^3\\
&\qquad - 6p^2 + 4p + 1)(p^{12} - 8p^{11} + 23p^{10} - 19p^9 - 35p^8\\
&\qquad + 73p^7 - 3p^6 - 72p^5 + 25p^4 + 29p^3 - 11p^2 - 3p\\
&\qquad + 1)\\
&\quad \times (rq + rp - r + qp - q - p + 1).
\end{aligned}$$

It can be checked that only one root of the polynomial in p above gives the value of L' lying in the region X_3. We will show that the polynomial $rq+rp-r+qp-q-p+1$ has no root in the region X_3.

From Table 1, we see that

$$\begin{aligned}
L' &= 0.581385 - 3.312055\, i,\\
D' &= 1.15663 - 2.756005\, i,\\
R' &= 1.40437 - 1.179435\, i
\end{aligned}$$

is the midpoint of region X_3. Then using (2–1), we see that the p, q, r values corresponding to this point are

$$\begin{aligned} p_0 &= 1.8219 - 0.828571\,i, \\ q_0 &= 1.82191 - 0.828633\,i, \\ r_0 &= 1.82192 - 0.828537\,i. \end{aligned}$$

If the polynomial $f(p,q,r) = rq+rp-r+qp-q-p+1$ has a root, say (p_1, q_1, r_1), in the region X_3, then by the mean value theorem, for some point (p,q,r) in X_3 we obtain

$$\begin{aligned} |f(p_0,q_0,r_0)| &= |\nabla f(p,q,r) \cdot (p_1 - p_0, q_1 - q_0, r_1 - r_0)| \\ &\leq \|\nabla f(p,q,r)\| \, \|(p_1 - p_0, q_1 - q_0, r_1 - r_0)\|. \end{aligned}$$

From the parameter ranges of region X_3 from Table 1, we know that $\|(p_1 - p_0, q_1 - q_0, r_1 - r_0)\| < 0.002$. Hence $\|\nabla f(p,q,r)\| \geq |f(p_0,q_0,r_0)|/0.002$ at some point in the region X_3. It can be checked that $|f(p_0,q_0,r_0)|/0.002 \approx 3000$ and that

$$\begin{aligned} &\|\nabla f(p,q,r)\| \\ &\quad \leq \big((|p|+|q|+1)^2 + (|q|+|r|+1)^2 \\ &\qquad + (|r|+|p|+1)^2\big)^{1/2} \\ &\quad \leq \big((|p_0|+|q_0|+2)^2 + (|q_0|+|r_0|+2)^2 \\ &\qquad + (|r_0|+|p_0|+2)^2\big)^{1/2} \\ &\quad < 11 \end{aligned}$$

in the region X_3, and hence $f(p,q,r)$ does not have a root in X_3.

We can similarly check for the other variables by changing the order of the variables. The last entry of the Gröbner basis with the ordering $z > r > p > q$ on the variables factors as

$$\begin{aligned} &(q-2) \\ &\quad \times \big(q^{10} - 7q^9 + 15q^8 + 4q^7 - 49q^6 + 11q^5 + 88q^4 \\ &\qquad + 87q^3 - 501q^2 + 543q - 193\big) \\ &\quad \times \big(q^{10} + 5q^9 + 6q^8 - 6q^7 - 10q^6 + 12q^5 + 13q^4 \\ &\qquad - 11q^3 - 6q^2 + 4q + 1\big) \\ &\quad \times \big(q^{12} - 8q^{11} + 23q^{10} - 19q^9 - 35q^8 + 73q^7 - 3q^6 \\ &\qquad - 72q^5 + 25q^4 + 29q^3 - 11q^2 - 3q + 1\big) \\ &\quad \times (rp + rq - r + pq - p - q + 1), \end{aligned}$$

and the last entry of the Gröbner basis with the ordering $z > p > q > r$ on the variables factors as

$$\begin{aligned} &(r^3 + r^2 - 2r - 1) \\ &\quad \times (r^{10} - 7r^9 + 15r^8 + 4r^7 - 49r^6 + 11r^5 + 88r^4 \\ &\qquad + 87r^3 - 501r^2 + 543r - 193) \\ &\quad \times (r^{10} + 5r^9 + 6r^8 - 6r^7 - 10r^6 + 12r^5 + 13r^4 \\ &\qquad - 11r^3 - 6r^2 + 4r + 1) \\ &\quad \times (r^{12} - 8r^{11} + 23r^{10} - 19r^9 - 35r^8 + 73r^7 - 3r^6 \\ &\qquad - 72r^5 + 25r^4 + 29r^3 - 11r^2 - 3r + 1) \\ &\quad \times (pq + pr - p + qr - q - r + 1). \end{aligned}$$

Using (6–1), (6–2), (6–3) above, it can be checked that there is only one root of the above polynomial in q and one root of the polynomial in r that give values for D' and R' lying in the region X_3. The polynomial in p, q, and r is the same as above. Hence the quasi-relators for the region X_3 have a unique solution in the region X_3. The uniqueness of solutions is proved similarly for the region X_1.

Proposition 6.1. *Let f and w be as in* (2–1) *and let $r_1(X_i)$, $r_2(X_i)$ be quasi-relators for the region X_i. Then there is a unique triple (L', D', R') in the region X_i for which the quasi-relators equal the identity matrix.*

Proof: It follows from Theorem 3.1 that there is a triple (L', D', R') in the region X_i for which the quasi-relators equal the identity matrix. The uniqueness follows from the Gröbner-basis computation for every region. □

We now give the proof of our main theorem.

Proof of Theorem 1.2: Let N be an exceptional manifold and let δ be the shortest geodesic in N. Let $f \in \pi_1(N)$ be a primitive hyperbolic isometry whose fixed axis $\delta_0 \in \mathbb{H}^3$ projects to δ and let $w \in \pi_1(N)$ be a hyperbolic isometry that takes δ_0 to its nearest translate. Let G be the subgroup of $\pi_1(N)$ generated by f and w. Then the manifold $N' = \mathbb{H}^3/G$ is exceptional and δ_0 projects to the shortest geodesic in N'.

It follows from [Gabai et al. 03, Corollary 1.29] that the (L', D', R') parameter for G lies in the region X_i for some $i = 0, 1, \ldots, 6$. By definition of the quasi-relators [Gabai et al. 03], Relength(r_1) and Relength(r_2) are less than Relength(f). Since f is the shortest element in G, it follows that r_1 and r_2 equal the identity in G, that is, they are relations in G. It is proved in [Lipyanskiy 02] that the quasi-relators generate all the relations in the groups $G_i = \langle f, w \mid r_1(X_i), r_2(X_i) \rangle$ for $i = 0, \ldots, 6$.

Hence $G = G_i$, and $\pi_1(N)$ contains the marked group G_i for some $i = 0, 1, \ldots, 6$.

It follows from Proposition 6.1 that the quasi-relators for a given region equal the identity at a unique point in that region. Hence N is covered by N_i for some $i = 0, 1, 2, 3, 4, 5$, where N_i are the manifolds described in Section 1. □

7. CONCLUSIONS

The first part of Conjecture 1.1 follows from Proposition 6.1. The results in Sections 4 and 5 prove part (i) of Conjecture 1.1 for regions X_i, $i = 1, 2, 4, 5, 6$, and the exact arithmetic from Section 3 proves part (v) of Conjecture 1.1 for all the regions. The question about the uniqueness of the manifolds remains open for all regions except X_0. It is reasonable to make the following conjecture:

Conjecture 7.1. *The manifolds N_i for $i = 1, 2, 3, 4, 5, 6$ do not nontrivially cover any manifold.*

Alan Reid proves the conjecture for N_1 and N_5 in the following appendix.

8. APPENDIX: THE MANIFOLDS N_1 AND N_5 (written by Alan W. Reid)

In this appendix we prove the following theorem.

Theorem 8.1. *The manifolds N_1 and N_5 do not properly cover any closed hyperbolic 3-manifold.*

Proof: We give the proof in the case N_1. The case N_5 is similar. Both arguments follow that given for Vol3 in [Jones and Reid 01]. We refer the reader to [Maclachlan and Reid 03] for details about arithmetic Kleinian groups and quaternion algebras.

Thus, suppose that $N_1 = \mathbb{H}^3/\Gamma_1$ nontrivially covers a closed hyperbolic 3-manifold $N = \mathbb{H}^3/\Gamma$, say, with covering degree d. Using the identification of N_1 as $v2678(2, 1)$ given in this paper, it follows that the volume of N_1 is approximately $4.116968736384613\ldots$ and that $H_1(N_1; \mathbb{Z}) = \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z}$ is finite of odd order. Note that since $H_1(N_1; \mathbb{Z})$ is finite, the closed hyperbolic 3-manifold N is orientable. Since the volume of the smallest arithmetic manifold is approximately 0.94 [Chinburg et al. 01], it follows that $d \leq 4$.

Using Snap (or from computations in this paper), the Kleinian group Γ_1, and hence Γ, is arithmetic with invariant trace field k, say. This has degree 4 and discriminant -448, and the invariant quaternion algebra B/k is unramified at all finite places. We remark that there is a unique such field.

Since $|H_1(N_1; \mathbb{Z})|$ is odd, Γ_1 is derived from a quaternion algebra. Furthermore, B has type number 1, and so Γ_1 is conjugate to the group of elements of norm 1 of a maximal order $\mathcal{O}$ of B. The image of the elements of norm 1 of $\mathcal{O}$ in $\mathrm{PSL}(2, \mathbb{C})$ can be shown (see [Maclachlan and Reid 89]) to coincide with the orientation-preserving subgroup of index 2 in the Coxeter simplex group $T[2, 3, 3; 2, 3, 4]$. The notation for the Coxeter group is that of [Maclachlan and Reid 89].

Denote this group by C. The minimal index of a torsion-free subgroup of C is at least 24, since by inspection of the Coxeter diagram, C contains a subgroup isomorphic to S_4. Therefore the volume calculations of [Maclachlan and Reid 89] show that Γ_1 is a minimal-index torsion-free subgroup of this group.

The analysis in [Jones and Reid 01, Section 4] shows that the possible maximal group in the commensurability class of Γ_1 that contains Γ is either the group $\Gamma_{\mathcal{O}}$ (in the notation of [Jones and Reid 01]), where $\mathcal{O}$ is the maximal order above, or $\Gamma_{\{\mathcal{P}_7\}, \mathcal{O}}$ (in the notation of [Jones and Reid 01]).

Suppose first that $\Gamma < \Gamma_{\mathcal{O}}$. By the remarks above, Γ is not a subgroup of C. Now, [Maclachlan and Reid 89] shows that $\Gamma_{\mathcal{O}}$ contains C as a subgroup of index 2. It follows that Γ must contain Γ_1 as a subgroup of index 2, and that Γ is a torsion-free subgroup of index 24 in $\Gamma_{\mathcal{O}}$. We claim that this is impossible.

First, we can obtain a presentation of the group $\Gamma_{\mathcal{O}}$ using the geometric description of C above; namely, the group $\Gamma_{\mathcal{O}}$ is obtained from C by adjoining an orientation-preserving involution t that is visible in the Coxeter diagram. On checking the action of this involution, one gets that a presentation for $\Gamma_{\mathcal{O}}$ is given by

$$\langle t, a, b, c \mid t^2, a^2, b^3, c^3, (b * c)^2, (c * a)^3, (a * b)^4, \\ t * a * t^{-1} * c * b, t * b * t^{-1} * a * c^{-1}, t * c * t^{-1} * c \rangle.$$

A check with Magma (for example) shows that there are 24 subgroups of index 24 and all but two are easily seen to have elements of finite order by inspection of presentations. The remaining two have abelianizations $\mathbb{Z}/22\mathbb{Z}$. The index-2 subgroups in these groups all have $\mathbb{Z}/11\mathbb{Z}$ in their abelianizations by a standard cohomology-of-groups argument (or further checking with Magma). In particular, these index-2 subgroups cannot coincide with Γ_1, which completes the analysis in this case.

For the second case, the covolume of $G = \Gamma_{\{\mathcal{P}_7\},\mathcal{O}}$ can be computed (using the formula in [Jones and Reid 01, Section 2]) to be 12 times that of Γ_1. An alternative, equivalent, description of this maximal group is as the normalizer of an Eichler order $\mathcal{E}$ of level $\mathcal{P}_7$ in $\mathcal{O}$ (see [Maclachlan and Reid 03, Chapter 11], for example).

The results of [Chinburg and Friedman 00] (see in particular Theorems 3.3 and 3.6) show that G contains elements of orders 2 and 3, and so the minimal index of a torsion-free subgroup in G is at least 6.

Now, $\Gamma_1 \subset G \cap C$. Furthermore, if we denote the image of the group of elements of norm 1 in $\mathcal{E}$ in $\mathrm{PSL}(2,\mathbb{C})$ by $\Gamma^1_{\mathcal{E}}$, then since the level is $\mathcal{P}_7$, it follows that the index $[C : \Gamma^1_{\mathcal{E}}]$ is 8 (see [Maclachlan and Reid 03, Chapters 6, 11]). It is not difficult to see that $G \cap C = \Gamma^1_{\mathcal{E}}$. One inclusion is clear, and the other follows because from above, we have $[C : \Gamma^1_{\mathcal{E}}] = 8$, so that the only possible indices for $[C : G \cap C]$ are 2 and 4 (it cannot be 1, since G is a different maximal group from $\Gamma_{\mathcal{O}}$ above). Now, C is perfect, and so has no solvable quotients. Hence this excludes C from having subgroups of index 2 or 4.

We deduce from the above that Γ_1 is a subgroup of $\Gamma^1_{\mathcal{E}}$. Using the presentation of C, and on checking with Magma, for instance, we find that there are five subgroups of index 8, and some further low-index computations on these subgroups using Magma shows that only one such subgroup can contain Γ_1. This subgroup (denoted by H in what follows) is generated by two elements of order 3 (b and $c * a$ in the generators above).

As in the first case, we can use the geometry associated with H to construct a presentation for the group G. The subgroup H has two generators with both generators of order 3, so we can adjoin involutions s and t, so that a presentation for G is

$$\begin{aligned}&\langle x, y, s, t \mid s^2, t^2, (s*t)^2, s*a*s*y^{-1}, s*y*s^{-1}*x^{-1},\\ &\quad t*x*t*x, t*y*t^{-1}*y, x^3, y^3,\\ &\quad x*y^{-1}*x^{-1}*y*x*y*x^{-1}*y^{-1}*x*y*x*y^{-1}\\ &\qquad *x^{-1}*y^{-1}*x*y*x^{-1}*y^{-1}*x^{-1}*y*x\\ &\qquad *y^{-1}*x^{-1}*y^{-1}*x*y\rangle.\end{aligned}$$

From our remarks above, we need only check for torsion-free subgroups of index 6. However, an easy inspection using Magma shows that all the subgroups of index 6 (of which there are four) have elements of finite order. This completes the proof. □

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