Computing Varieties of Representations of Hyperbolic 3-Manifolds into $\mathrm{SL}(4, \mathbb{R})$

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CONTENTS

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The geometric structure on a closed orientable hyperbolic 3-manifold determines a discrete faithful representation ρ of its fundamental group into $\mathrm{SO}^+(3,1)$, unique up to conjugacy. Although Mostow rigidity prohibits us from deforming ρ, we can try to deform the composition of ρ with inclusion of $\mathrm{SO}^+(3,1)$ into a larger group. In this sense, we have found by exact computation a small number of closed manifolds in the Hodgson-Weeks census for which ρ deforms into $\mathrm{SL}(4,\mathbb{R})$, thus showing that the hyperbolic structure can be deformed in these cases to a real projective structure. In this paper we describe the method for computing these deformations, particular attention being given to the manifold Vol3.

1. INTRODUCTION

Following the seminal work of M. Culler and P. Shalen [Culler and Shalen 83], and that of A. Casson [Akbulut and McCarthy 90], the theory of representation and character varieties of 3-manifolds has come to be recognized as a powerful tool, and has duly assumed an important place in low-dimensional topology. Among the many papers that have appeared in this context, we mention [Culler et al. 87, Cooper et al. 94, Boyer and Zhang 98]. Most of the work carried out to date is concerned with representations into Lie groups of 2×2 matrices, owing mainly to connections with actions on trees and the isometry groups of hyperbolic space in dimensions 2 and 3, but also owing to the extreme difficulty of computations beyond the realm of such matrices.

This paper was originally motivated by the following question: under what circumstances can one take the hyperbolic structure on a closed hyperbolic 3-manifold and deform it to a real projective structure? In the language of representations, this amounts to beginning with an $\mathrm{SO}^+(3,1)$-representation ϕ_0 of the fundamental group of the manifold, given by the hyperbolic structure, and then endeavoring to compute the component of the $\mathrm{SL}(4,\mathbb{R})$-representation variety containing ϕ_0. Using a computer, it is relatively easy to see that for many closed hyperbolic 3-manifolds there are linear obstructions to deforming, but when these obstructions vanish it is of considerable interest to see whether genuine deformations exist. The purpose of this article, then, is to describe a method for the exact computation of the representation varieties of closed hyperbolic 3-manifolds into $\mathrm{SL}(4,\mathbb{R})$. For simplicity we restrict our attention to orientable manifolds with 2-generator fundamental groups.

The technique is sufficiently practical that we have used it to compute 21 varieties of this type exactly, and have investigated numerically the first 4500 closed orientable manifolds with 2-generator groups in the Hodgson–Weeks census [Hodgson and Weeks 00]. Numerical evidence strongly suggests that only 52 of these 4500 manifolds admit nontrivial deformations of ϕ_0 into $\mathrm{SL}(4,\mathbb{R})$. When these deformations do occur, they lead to a number of interesting constructions, including families of real projective structures on the manifold and families of discrete faithful representations into $\mathrm{PU}(3,1)$, the group of orientation-preserving isometries of 3-dimensional complex hyperbolic space. These and other theoretical aspects are considered in more depth in [Cooper et al. 05]; in this paper we concentrate on the computational aspects of the investigation.

The presence of an embedded totally geodesic surface in the manifold guarantees the existence of a well-established type of deformation known as *bending*, but owing to their small volume, the census manifolds cannot contain such surfaces [Kojima and Miyamoto 91]. The underlying reason for the sporadic occurrence of these deformations is still a mystery.

In order to make the paper reasonably self-contained, there now follows a section summarizing the necessary background information.

2. BACKGROUND

2.1 The Minkowski Model for Hyperbolic Space

A hyperbolic structure on a 3-manifold M corresponds to a discrete faithful representation of the fundamental group of M into the group of isometries of 3-dimensional hyperbolic space $\mathbb{H}^3$. In the case that M is closed, by Mostow rigidity the structure is unique up to isometry (if M is noncompact and of finite volume, we also have uniqueness if we add the requirement that the geometric structure be complete).

In this article we shall focus our attention on closed orientable hyperbolic 3-manifolds M. Thus the geometric structure on M corresponds to a discrete faithful representation $\phi_0 : \pi_1(M) \to \mathrm{Isom}^+ \mathbb{H}^3$, unique up to conjugacy; here $\mathrm{Isom}^+ \mathbb{H}^3$ denotes the group of orientation-preserving isometries of $\mathbb{H}^3$. In the upper-half-space model for $\mathbb{H}^3$, the group $\mathrm{Isom}^+ \mathbb{H}^3$ is naturally identified with $\mathrm{PSL}(2,\mathbb{C})$, the group of Möbius transformations of the boundary, but for us it will be more propitious to work in the *Minkowski model* for $\mathbb{H}^3$, since our intention is to consider $\mathrm{Isom}^+ \mathbb{H}^3$ as a subgroup of $\mathrm{SL}(4,\mathbb{R})$ and search for deformations of the composite homomorphism $\pi_1(M) \xrightarrow{\phi_0} \mathrm{Isom}^+ \mathbb{H}^3 \hookrightarrow \mathrm{SL}(4,\mathbb{R})$.

Minkowski space of dimension $n+1$, denoted by $\mathbb{M}^{n+1}$, is the real vector space $\mathbb{R}^{n+1}$ endowed with a quadratic form Q of signature $(n,1)$, which we take without loss of generality to be

$$Q(x_0, x_1, \ldots, x_n) = -x_0^2 + x_1^2 + \cdots + x_n^2.$$

The group of isometries of $\mathbb{M}^{n+1}$ is the *Lorentz group* $\mathrm{O}(n,1)$; it consists of those $(n+1)\times(n+1)$ matrices A satisfying $A^{-1} = FA^tF$, where the superscript ^{t} denotes transpose, and where F is the diagonal $(n+1)\times(n+1)$ matrix with entries $-1, 1, \ldots, 1$. An excellent reference is [Epstein and Penner 88].

Let $\mathbf{x}$ denote $(x_0, x_1, \ldots, x_n) \in \mathbb{M}^{n+1}$. In the Minkowski model, *real hyperbolic n-space* is the "upper sheet" of the hyperboloid $\mathbb{H}^n = \{\mathbf{x} : Q(\mathbf{x}) = -1\}$, namely the set of points of this hyperboloid for which $x_0 > 0$. We note that $\mathbb{H}^n$ is asymptotic to the *light cone* $C = \{\mathbf{x} : Q(\mathbf{x}) = 0\}$, and that $Q(\mathbf{x}) > 0$ if and only if the vector $\mathbf{x}$ points in a direction outside C. It is not hard to see that the tangent vectors at any point of $\mathbb{H}^n$ satisfy this condition, whence the $(3,1)$-form of $\mathbb{M}^{n+1}$ is positive definite on the tangent bundle of $\mathbb{H}^n$; indeed, it induces the expected Riemannian metric of constant negative curvature. The *boundary* of $\mathbb{H}^n$ in the Minkowski model is the space whose points are rays of the light cone C. The group $\mathrm{Isom}^+ \mathbb{H}^n$ is the subgroup $\mathrm{SO}^+(n,1)$ of $\mathrm{O}(n,1)$ consisting of linear transformations that (i) have determinant 1, and (ii) preserve the sheets of the hyperboloid $\{\mathbf{x} : Q(\mathbf{x}) = -1\}$. The group $\mathrm{SO}^+(n,1)$ has index 4 in $\mathrm{O}(n,1)$, and is the component subgroup of the identity.

Let us specialize to the case $n = 3$. The assignment

$$(t, x, y, z) \mapsto \begin{bmatrix} t+z & x+iy \\ x-iy & t-z \end{bmatrix}$$

defines an $\mathbb{R}$-vector space isomorphism from $\mathbb{R}^4$ to the space $\mathcal{H}^2$ of 2×2 Hermitian matrices over the complex numbers; the quadratic form in $\mathcal{H}^2$ corresponding to Q under this isomorphism is simply the negative of the determinant. We therefore have an isomorphism $\mathbb{M}^4 = (\mathbb{R}^4, Q) \approx (\mathcal{H}^2, -\det)$. It is then easy to construct an isomorphism from PSL$(2,\mathbb{C})$ to the group of isometries of $\mathbb{M}^4$: we simply assign to each matrix $A \in \mathrm{SL}(2,\mathbb{C})$ the map $H \mapsto A^*HA (H \in \mathcal{H}^2)$, where * denotes Hermitian transpose. We may construct an explicit isomorphism from PSL$(2,\mathbb{C})$ to SO$^+(3,1)$ by choosing a specific basis for $\mathcal{H}^2$, for example

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \right\},$$

and then computing the matrix of the linear transformation $H \mapsto A^*HA$ with respect to this basis.

2.2 Eigensystems of Isometries of Hyperbolic Space

The isomorphism given above may be used to determine the eigenvectors and corresponding eigenvalues of the various kinds of isometry of $\mathbb{H}^3$ in the Minkowski model. The situation may be summarized as follows.

(i) The eigensystem of a loxodromic. Let $g \in \mathrm{Isom}^+ \mathbb{H}^3$ be a loxodromic; then g acts freely on $\mathbb{H}^3$ and fixes two points on the boundary of $\mathbb{H}^3$. In the upper-half-space model, the boundary is identified with the extended complex plane, and with suitable choice of coordinates we may assume that the fixed points of g are $0, \infty$. Then g corresponds to a dilation $z \mapsto az$ on the boundary, where the complex number a is the *dilation factor* of g (see [Maskit 80]). We may think of g as the composition of a "pure translation" with axis the geodesic joining 0 and ∞, together with a rotation (elliptic) about that axis through an angle $\arg(a)$. The isometry g corresponds to the element of PSL$(2,\mathbb{C})$ represented by the matrix $\left[\begin{smallmatrix} a^{1/2} & 0 \\ 0 & 1/a^{1/2} \end{smallmatrix}\right]$, where $a^{1/2}$ is either of the square roots of a.

Application of the above isomorphism yields the following matrix in SO$^+(3,1)$:

$$\begin{bmatrix} p & q & 0 & 0 \\ q & p & 0 & 0 \\ 0 & 0 & r & s \\ 0 & 0 & -s & r \end{bmatrix},$$

where

$$p = \frac{1}{2}\left(|a| + \frac{1}{|a|}\right), \quad q = \frac{1}{2}\left(|a| - \frac{1}{|a|}\right),$$
$$r = \Re\mathrm{e}\left(\frac{a}{|a|}\right), \quad s = \Im\mathrm{m}\left(\frac{a}{|a|}\right).$$

The eigenvalues of this matrix are $|a|$, $\frac{1}{|a|}$, $e^{i\theta}$, $e^{-i\theta}$, where $\theta = \arg(a)$. The eigenspaces of the real eigenvalues $|a|$, $\frac{1}{|a|}$ are the two rays of the light cone identified with the fixed points of g. Indeed, the subspace spanned by these two rays meets $\mathbb{H}^3$ precisely in the axis of g. The pairing of each eigenvalue with its inverse corresponds to the fact that the axis of g admits two orientations. It can happen that $\theta = 0$, in which case the isometry is a pure translation.

(ii) The eigensystem of an elliptic. Let g be an elliptic isometry. Since we may regard g as a degenerate loxodromic, with dilation factor on the unit circle, the corresponding matrix in SO$^+(3,1)$ will have eigenvalues 1, $e^{i\theta}$, $e^{-i\theta}$, where θ is the angle of rotation, as before. The eigenspace of the eigenvalue 1 is the 2-dimensional subspace of $\mathbb{M}^4$ containing the two rays on the light cone corresponding to the ends of the axis of g.

(iii) The eigensystem of a parabolic. Let g be parabolic; then g has one fixed point on the boundary, and g effects a Euclidean translation along each horosphere centered at this fixed point. If we choose coordinates in the upper-half-space model such that the fixed point is ∞, then g corresponds to the element of PSL$(2,\mathbb{C})$ represented by $\left[\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right]$, where the complex number a describes the Euclidean translation. The corresponding matrix in SO$^+(3,1)$ is

$$\begin{bmatrix} 1+p & p & r & s \\ -p & 1-p & -r & -s \\ r & r & 1 & 0 \\ s & s & 0 & 1 \end{bmatrix},$$

where

$$p = \frac{1}{2}|a|^2, \quad r = \Re\mathrm{e}\, a, \quad s = \Im\mathrm{m}\, a.$$

This matrix has a single eigenvalue 1 with algebraic multiplicity 4 and geometric multiplicity 2. The eigenspace is spanned by a vector (in this case $(-1,1,0,0)$) pointing along the ray on the light cone corresponding to the single fixed point of g, and an orthogonal vector (in this case $(0,0,-\Im\mathrm{m}\, a, \Re\mathrm{e}\, a)$), whose direction encodes the direction of the Euclidean translation.

3. TRACE CALCULUS

The ultimate aim of this paper is to compute representation varieties of $\pi_1(M)$ into SL$(4,\mathbb{R})$. Since any representation can be composed with an inner automorphism of the target group to produce another, we are content to find just one representation in each such equivalence class. In essence we are computing a variety $\mathcal{V}$ that is an embedded copy of the character variety in the full representation variety.

If the variety $\mathcal{V}$ has dimension n, then the image of each generator of $\pi_1(M)$ is a matrix whose entries are algebraic functions of n independent parameters $u_1, \dots, u_n$. The matrix entries at a generic point of the variety can therefore be considered to lie in a field $\mathbb{F}$ of transcendence degree n over $\mathbb{R}$, and $\mathcal{V}$ is then specified by a single "tautological" representation Ψ into $\mathrm{SL}(4, \mathbb{F})$. Individual representations are obtained from Ψ by evaluating at specific points $(u_1, \dots, u_n)$ in parameter space. Clearly Ψ depends on a choice of parameterization of $\mathcal{V}$.

Two important fields in this context are (i) the field K generated by the entries of image matrices, and (ii) the subfield T of K generated by the traces of image matrices. By conjugating judiciously, we can guarantee that the field K (hence also T) is algebraic of finite degree over the purely transcendental extension $\mathbb{Q}(u_1, \dots, u_n)$ of the rationals. The trace field T is independent of the choice of conjugation, and in practice it is often easy to guess T. However, in order to compute the variety $\mathcal{V}$, we shall need to specify generators for K over the base field $\mathbb{Q}(u_1, \dots, u_n)$; the elementary proposition given below will be helpful in that regard.

In order to state Proposition 3.1 it will be convenient to introduce some notation. Let $a = (a_{ij})$ be any $n \times n$ matrix over a commutative ring R, and let $\sigma = (n_1, \dots, n_k)$ be any cyclically ordered sequence of distinct numbers from $\{1, 2, \dots, n\}$ (thus we regard the sequences $(n_1, n_2, n_3, \dots, n_k)$ and $(n_2, n_3, \dots, n_k, n_1)$ as being identical). Then we define a_σ to be the following element of R: $a_\sigma = a_{n_1 n_2} a_{n_2 n_3} \cdots a_{n_k n_1}$.

The proof of Proposition 3.1 is greatly facilitated by having a computer algebra system to hand, for example Mathematica or Maple.

Proposition 3.1. *Let G be a subgroup of $\mathrm{SL}(4, \mathbb{C})$ generated by matrices a, b, where b is a diagonal matrix*

$$b = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}.$$

Suppose further that

(i) *the λ_i are all distinct;*

(ii) $\mathrm{tr}(b) \neq 0$*;*

(iii) $(\mathrm{tr}(b))^3 \neq \mathrm{tr}(b^3)$.

Let T be the trace field of G, and let K be the field obtained by adjoining $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ to T. Then for each cycle σ of length 1, 2, or 3 in $\{1, 2, 3, 4\}$, we have $a_\sigma \in K$.

Proof: First we deal with the case in which σ has length 1, i.e., the diagonal entries a_{ii} $(1 \leq i \leq 4)$. It is readily checked that the four traces $\mathrm{tr}(a)$, $\mathrm{tr}(ab)$, $\mathrm{tr}(ab^2)$, $\mathrm{tr}(ab^3)$ are all linear expressions in the entries a_{ii} over the field K. We therefore have a system of linear equations for the a_{ii} over K, and it is easily verified (using, for example, Mathematica) that the determinant of the matrix of coefficients is $\prod_{i<j}(\lambda_i - \lambda_j)$. Therefore, from the hypothesis that the λ_i are distinct, the system has a (unique) solution and the a_{ii} have been shown to lie in the field K.

Next, consider the six products corresponding to cycles of length 2, namely

$$a_{12}a_{21},\ a_{13}a_{31},\ a_{14}a_{41},\ a_{23}a_{32},\ a_{24}a_{42},\ a_{34}a_{43}.$$

Let B denote the inverse of b. The trace of any word in the generators a, b involving precisely two occurrences of a (and no occurrences of a^{-1}) is a linear expression in these six products with coefficients in the field $T(\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ a_{11},\ a_{22},\ a_{33},\ a_{44})$, which we now know to be equal to K. The six traces

$$\mathrm{tr}(aa),\ \mathrm{tr}(aab),\ \mathrm{tr}(aaB),\ \mathrm{tr}(aabb),\ \mathrm{tr}(abab),\ \mathrm{tr}(aBab)$$

therefore give rise to a linear system in the six products over this field, and one can verify that the determinant of the matrix of coefficients is $4\prod_{i<j}(\lambda_i - \lambda_j)^2$. Again, this determinant is nonzero, and it follows that the six products corresponding to cycles of length 2 are in the field K.

The eight products corresponding to cycles of length 3 are dealt with similarly, using the traces of the words

$$aaa,\ aaab,\ aaabb,\ aaBab,\ aabaB,$$
$$aaBabb,\ abaBabb,\ abaBBabbb.$$

This time the determinant of the 8×8 matrix of coefficients is

$$\left(\prod_{i<j}(\lambda_i - \lambda_j)^4\right) \mathrm{tr}(b) \left((\mathrm{tr}(b))^3 - \mathrm{tr}(b^3)\right),$$

and the result follows. □

Remark 3.2. The choice of words at each stage of the proof of Proposition 3.1 is certainly not unique, but the eight words used for the last stage need to be chosen quite carefully in order that the matrix of coefficients should have full rank.

Remark 3.3. If M is a closed, orientable, hyperbolic 3-manifold, then each nontrivial element $g \in \pi_1(M)$ is loxodromic. Let $\phi_0 : \pi_1(M) \to \mathrm{SO}^+(3,1)$ be the representation given by the geometric structure; then the eigenvalues of $\phi_0(g)$ are distinct if and only if g is not a pure translation. Therefore, if b is the diagonalization of $\phi_0(g)$ and g has a nontrivial rotational component, then at least condition (i) of Proposition 3.1 is met. Condition (ii) is automatically met because the trace of $\phi_0(g)$ cannot equal zero; this follows directly from the fact, explained in Section 2.2, that $\phi_0(g)$ has eigenvalues $|a|$, $\frac{1}{|a|}$, $e^{i\theta}$, $e^{-i\theta}$ with $|a| \neq 1$. Condition (iii) has also been met for all manifolds we have investigated. Furthermore, these three properties are "open conditions," so if they are satisfied for $\phi_0(g)$, they are also satisfied for $\phi_1(g)$, given that ϕ_1 is sufficiently close to ϕ_0.

Remark 3.4. It often happens in practice that the characteristic polynomial of the image of $g \in \pi_1(M)$ under the tautological representation Ψ is *reciprocal*, meaning that its roots are in reciprocal pairs; equivalently, the coefficients of the polynomial form a palindromic sequence. If $\phi_0(g)$ is not a pure translation, and if ϕ is an evaluation of Ψ close to the $\mathrm{SO}^+(3,1)$ representation ϕ_0, then the two "rotational" eigenvalues λ_3, λ_4 of $\phi(g)$ are nonreal. If $\lambda_2 = 1/\lambda_1$ and $\lambda_4 = 1/\lambda_3$, then we have $\lambda_3 + 1/\lambda_3 \in T(\lambda_1)$, whence $\lambda_3, 1/\lambda_3$ are roots of a quadratic over the field $T(\lambda_1)$. Therefore K has degree 2 over $T(\lambda_1)$, and $T(\lambda_1) = K \cap \mathbb{R}$.

The hypotheses of the following corollary are obviously not best possible, but the corollary suits our purpose in obtaining representations for which the matrix entry field is algebraic over $\mathbb{Q}(u_1, \ldots, u_n)$.

Corollary 3.5. *Let G, K be as in Proposition 3.1, and let us impose the extra hypothesis that all off-diagonal entries of a are nonzero. Let c be the matrix*

$$c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{12} & 0 & 0 \\ 0 & 0 & a_{12}a_{23} & 0 \\ 0 & 0 & 0 & a_{12}a_{23}a_{34} \end{bmatrix}.$$

Then the field generated by the matrix entries of cGc^{-1} is precisely K.

Proof: The group cGc^{-1} is generated by $cac^{-1} = a'$, say, and $cbc^{-1} = b$. Let K' be the field generated by the matrix entries of cGc^{-1}. Since conjugation by c has not affected traces or the diagonal matrix b, K is contained in K'. On the other hand, the $(1,2)$, $(2,3)$, $(3,4)$ entries of the matrix a' are all equal to 1, whence the matrix entries of a' are all expressible as products of the a'_σ and their inverses. We now apply Proposition 3.1 to the group $\langle a', b\rangle$, deducing that each entry of a' is in K; it then follows at once that $K' \subset K$. □

4. CANONICAL FORM FOR REPRESENTATIONS

In this section we explain how to conjugate a representation $\phi : \pi_1(M) \to \mathrm{SL}(4,\mathbb{R})$ into a convenient "standard" form. As always, we are assuming that M is a closed, orientable, hyperbolic 3-manifold and that its fundamental group is generated by a pair of elements α, β. Let a, b denote the images of α, β respectively under ϕ, and let a_0, b_0 denote the corresponding images under the "base" representation $\phi_0 : \pi_1(M) \to \mathrm{SO}^+(3,1)$. Since all nontrivial elements of $\pi_1(M)$ are loxodromic, the matrix b_0 is diagonalizable; also, as explained in Section 2.2, b_0 has two distinct real eigenvalues and two other mutually conjugate eigenvalues. If ϕ is sufficiently close to ϕ_0, continuity together with the fact that the characteristic polynomial of b has real coefficients ensures that the matrix b enjoys the same properties. We make the further assumption that the diagonalization of b satisfies conditions (i), (ii), (iii) of Proposition 3.1. As explained in Remark 3.3 above, condition (ii) is automatically met for representations ϕ close to ϕ_0, and in all observed cases the other two conditions are also met; should this happen not to be so, one could resort to changing the generating set for $\pi_1(M)$.

Let the two real eigenvalues of b be λ_1, λ_2, and let the other two eigenvalues be $\lambda_3, \overline{\lambda}_3$. Since b is an automorphism of $\mathbb{R}^4$, we may choose real eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ for λ_1, λ_2 respectively, and eigenvectors $\mathbf{v}_3, \overline{\mathbf{v}}_3$ for $\lambda_3, \overline{\lambda}_3$. Since these four eigenvectors form a linearly independent set over $\mathbb{C}$, the set of real vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 + \overline{\mathbf{v}}_3, i(\mathbf{v}_3 - \overline{\mathbf{v}}_3)\}$ is also linearly independent over $\mathbb{C}$, hence also over $\mathbb{R}$. Therefore each vector in $\mathbb{R}^4$ is uniquely expressible as a real linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 + \overline{\mathbf{v}}_3, i(\mathbf{v}_3 - \overline{\mathbf{v}}_3)$, or alternatively as a linear combination $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \overline{k_3\mathbf{v}_3}$, where k_1, k_2 are real.

Let us now consider the matrices $a_1, b_1 \in \mathrm{SL}(4,\mathbb{C})$ representing the same linear transformations as a, b respectively, but with respect to the basis of eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \overline{\mathbf{v}}_3\}$. Then b_1 is diagonal, and from the discussion in the previous paragraph, the entries of the upper left 2×2 submatrix of a_1 are real. Assuming that the off-diagonal entries of a_1 are nonzero, we see that we may adjust $\mathbf{v}_2$ by a real scalar so that the $(1,2)$ entry of a_1 is 1. We may independently adjust the last two eigenvectors by scalars so that the $(2,3)$ and $(3,4)$ entries are also 1. Then, from Corollary 3.5, all entries of a_1 lie in the field K obtained by adjoining the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ to the trace field T.

We can improve matters slightly by conjugating a_1, b_1 to new matrices a_2, b_2 by means of a further change of basis, namely to $\{\mathbf{v}_1, \mathbf{v}_2, a(\mathbf{v}_1), a(\mathbf{v}_2)\}$.[1] Note that each of these basis vectors is real, so the resulting matrices have real entries. Indeed, they have the convenient form

$$a_2 = \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}, \quad b_2 = \begin{bmatrix} \lambda_1 & 0 & * & * \\ 0 & \lambda_2 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}.$$

Since the transition matrix corresponding to this latest change of basis has columns

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad a_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad a_1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

its entries are in K, and we infer that the entries of a_2, b_2 are in $K \cap \mathbb{R}$. In particular, if the characteristic polynomial of b is reciprocal, by Remark 3.4, a_2, b_2 are matrices over the field $T(\lambda_1)$.

Remark 4.1. Matrices with reciprocal characteristic polynomials are particularly desirable, because their eigenvalues admit relatively simple expressions in terms of the coefficients. Indeed, the roots of $1 + px + qx^2 + px^3 + x^4$ are

$$\frac{1}{4}\left(-p - \sqrt{p^2 - 4q + 8} \pm \sqrt{2\left(p^2 - 2q - 4 + p\sqrt{p^2 - 4q + 8}\right)}\right)$$

and

$$\frac{1}{4}\left(-p + \sqrt{p^2 - 4q + 8} \pm \sqrt{2\left(p^2 - 2q - 4 - p\sqrt{p^2 - 4q + 8}\right)}\right),$$

as is easily verified by substituting $y = x + 1/x$.

[1] In practice, this set of vectors has always been found to be linearly independent within a neighborhood of ϕ_0.

5. INFINITESIMAL AND ACTUAL DEFORMATIONS

Since a representation of $\pi_1(M)$ into a linear group G is determined by its effect on the generators of $\pi_1(M)$, we may consider the representation variety $\mathrm{Hom}(\pi_1(M), G)$ as being a subspace of the k-fold product of G, where k is the number of generators. In particular, if $k = 2$ and $G = \mathrm{SL}(4,\mathbb{R})$, we consider the variety to be a subspace of $\mathrm{SL}(4,\mathbb{R}) \times \mathrm{SL}(4,\mathbb{R})$, which in turn embeds naturally into $\mathbb{R}^{32}$ if we use matrix entries as coordinates. Since M is closed, $\pi_1(M)$ has deficiency zero (recall that the *deficiency* of a finitely presented group is the maximum over all finite presentations of the number of generators minus the number of relators).

Given a closed orientable hyperbolic 3-manifold M, the first step in deciding whether $\phi_0 : \pi_1(M) \to \mathrm{SO}^+(3,1)$ deforms into $\mathrm{SL}(4,\mathbb{R})$ is to linearize the problem and see whether there exist perturbations $\phi : \pi_1(M) \to \mathrm{GL}(4,\mathbb{R})$ of ϕ_0 that preserve the group relations to first order. Such perturbations are called *infinitesimal deformations* or *deformations to first order*, and form the *Zariski tangent space* at ϕ_0.

There now follows a brief review of the Zariski tangent space. Our discussion is specific to the current context, but is easy to generalize. Suppose that we have a group $G = \langle x, y \mid R_1, R_2 \rangle$ and that we wish to investigate the possibility that a representation $\rho : G \to \mathrm{SL}(4,\mathbb{R})$ can be smoothly deformed. If such a deformation were possible, there would exist a smooth path of representations (x_t, y_t) for which (x_0, y_0) was the given initial representation $(\rho(x), \rho(y))$. Clearly, we must have $R_1(x_t, y_t) = I$, $R_2(x_t, y_t) = I$ all along the path, but this condition is usually hard to work with; indeed, it is the main concern of this article. It is much easier to focus on the corresponding linearized condition at $t = 0$, thereby obtaining a necessary (but not sufficient) condition for the existence of deformations. Writing $x_t = v_t x_0,\ y_t = w_t y_0$, we see that a smooth path (x_t, y_t) corresponds to pair of paths v_t, w_t in the Lie group $\mathrm{SL}(4,\mathbb{R})$ through the identity. Let $\mathfrak{v}, \mathfrak{w}$ be the tangent vectors at I to the paths v_t, w_t, respectively. The vectors $\mathfrak{v}, \mathfrak{w}$ are elements of the Lie algebra $\mathfrak{sl}(4,\mathbb{R})$ and satisfy a certain linear condition easily obtainable from the group relators R_1, R_2 by taking Fox derivatives. The *Zariski tangent space* for the deformation problem is the subspace of $\mathfrak{sl}(4,\mathbb{R}) \oplus \mathfrak{sl}(4,\mathbb{R})$ consisting of all ordered pairs $(\mathfrak{v}, \mathfrak{w})$ satisfying this linear condition. Intuitively, these ordered pairs can be considered as "infinitesimal deformations." There is of course no guarantee that a given vector in the Zariski tangent space is integrable, i.e., that it corresponds to an actual deformed representation; indeed, there is in general an infinite sequence of obstructions that have to be surmounted (see [Kapovich 00, p. 71] for a discussion of this issue).

The Zariski tangent space is relatively easy to compute in Mathematica or Maple as the kernel of a Jacobian matrix J; since usually we are interested only in its dimension, we merely compute the nullity of J. The method is illustrated in Section 7 for the fundamental group of the manifold Vol3.

At this point it is necessary to discuss a fundamental issue regarding the practicalities of computation. A computer has two distinct modes of computation, namely integer and floating-point, each with its advantages and drawbacks. Assuming freedom from programming error, integer computations are exact, and their output may be used directly in a mathematical proof. However, in many situations, for example if one wishes to solve a nonlinear equation by means of an iterative method based on analytical principles, integer computations are not appropriate. Floating-point computations, on the other hand, are inherently inexact, however many decimal places of accuracy are used, since the result of any arithmetic operation is rounded off before being stored for further processing. With floating-point calculations it is possible to assert that a result is accurate within certain specified bounds if one keeps track of the propagation of roundoff errors. Therefore, for example, with sufficient working accuracy and with due cause, one can assert that the result of a computation is nonzero; however, one can never infer, solely on the basis of a floating-point computation, that a number is exactly zero.

The computation of the dimension of the Zariski tangent space takes place in floating-point mode. Therefore, strictly speaking, by computing the nullity of the Jacobian J we are computing only an upper bound for the dimension. However, for theoretical reasons we do have a lower bound. There are two kinds of "inessential" infinitesimal deformations that we must exclude from our count. First, composition with inner automorphisms of $\mathrm{GL}(4,\mathbb{R})$ gives rise to $16-1=15$ dimensions of inessential infinitesimal deformations (we subtract 1 from 16 since the center of $\mathrm{GL}(4,\mathbb{R})$ has dimension 1, and absolute irreducibility of the representation guarantees that only scalar matrices can commute with the generators). Second, suppose that $H_1(M)$ has a free summand of rank r. Then there exists an epimorphism η of $\pi_1(M)$ to the direct sum of r copies of $\mathbb{Z}$, and we have r independent 1-parameter families of inessential infinitesimal deformations $\phi_{i,\lambda} : g \mapsto \lambda^{w_i(g)} \cdot \phi_0(g)$, where the parameter λ is a nonzero real number, and the "weight function" w_i is the composition of η with projection to the ith summand of $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ $(1 \leq i \leq r)$.[2] Manifolds for which $\nu(J)$ is found to equal $15+r$ are *rigid*, in the sense that $\chi(\phi_0)$ is an isolated point in the character variety. This assertion is justified by the fact that $15+r$ is both an upper bound (as a result of the computation) and a lower bound (from the theory). The excess of $\nu(J)$ over $15+r$ is the potential dimension of the character variety of deformations of ϕ_0 into $\mathrm{SL}(4,\mathbb{R})$; as already mentioned, quite apart from the uncertainty due to roundoff error, there is no guarantee that an infinitesimal deformation is integrable to an actual deformation.

It is possible in principle to calculate $\nu(J)$ exactly, since J depends only on the base representation $\phi_0 : \pi_1(M) \to \mathrm{SO}^+(3,1)$. However, in order to be able to paint a broad picture relatively quickly, we have chosen to compute $\nu(J)$ in floating-point mode, using an approximation of J to 1000 decimal places. To obtain this accuracy, we start with the representation $\rho : \pi_1(M) \to \mathrm{SL}(2,\mathbb{C})$ given by SnapPea [Weeks 90] to machine accuracy, and then increase the accuracy of ρ to the required level by means of a few iterations of Newton's method, using the group relations, before converting to a very accurate approximation of $\phi_0 : \pi_1(M) \to \mathrm{SO}^+(3,1)$. In cases in which the computed dimension of the Zariski tangent space exceeds the known dimension of the representation variety, the stated dimension of the Zariski tangent space is therefore not rigorous, albeit almost certainly correct.

This computation was carried out for the first 4500 2-generator closed orientable manifolds in the Hodgson–Weeks census [Hodgson and Weeks 00], and it was found that only 61 of these manifolds admit nontrivial infinitesimal deformations. Of these, 21 have been shown rigorously to admit actual deformations, and there is compelling numerical evidence that a further 31 do. Of the remaining 9 manifolds, 3 have been proved to be rigid using a certain third-order obstruction explained in [Cooper et al. 05], and numerical evidence strongly suggests that the remaining 6 are rigid. These results are set out in Table 1. A check mark in the last column indicates that the variety has been computed exactly and has been shown rigorously to have the stated dimension; absence of a check mark should be interpreted as "compelling numerical evidence only." Details of the computation for the manifold m007(3, 1) are given in Section 7 of this paper.

[2]Of course, these are not infinitesimal deformations into $\mathrm{SL}(4,\mathbb{R})$, but for computational expediency we first consider all infinitesimal deformations into $\mathrm{GL}(4,\mathbb{R})$ and then take an appropriate subspace.

The outcome for v2678(2, 1) is of interest. This manifold apparently has a 5-dimensional space of essential infinitesimal deformations, whereas the $SL(4,\mathbb{R})$ character variety apparently has two 3-dimensional branches meeting in a 1-dimensional subvariety containing $\chi(\phi_0)$. It would follow that $\chi(\phi_0)$ is not a smooth point of the variety. It is also interesting that from the first 2000 manifolds in the census, v2678(2, 1) appears from numerical evidence to be the only example admitting deformations into $SO(4,1)$. K. Scannell [Scannell 00] has proved that m036(−3, 2), the double cover of Vol3, admits infinitesimal deformations into $SO(4,1)$; however, numerical evidence suggests that these are not integrable.

6. SUMMARY OF THE COMPUTATIONAL PROCEDURE

We are now ready to give an outline of the entire computational procedure. A feature of the method is that the trace field, matrix entry field, and exact matrix entries are derived by informed guesswork based on numerical data; it is only at the very last step that the existence of the representation variety is proved, by checking formally that the proposed matrices $\Psi(\alpha), \Psi(\beta)$ satisfy the two relations of $\pi_1(M)$. Initial data needed to get started, i.e., generators and relations for $\pi_1(M)$ and ϕ_0 accurate to a few decimal places, can easily be obtained using SnapPea [Weeks 90] or Snap [Coulson et al. 00].

Step 1. Compute the dimension of the Zariski tangent space at the $SO^+(3,1)$ representation ϕ_0, using a highly accurate approximation to the Jacobian matrix; the computational details are best understood by reading the account of this step in Section 7. If the manifold is found to be rigid, then there are no deformations, and we quit.

Step 2. Apply a small random perturbation to ϕ_0 by slightly modifying the matrices $\phi_0(\alpha)$, $\phi_0(\beta)$, and then perform Newton's method to try to converge to a representation $\phi_1 : \pi_1(M) \to SL(4,\mathbb{R})$ not conjugate to ϕ_0 (the test is to compare characteristic polynomials of matrices $\phi_0(g), \phi_1(g)$ for various elements $g \in \pi_1(M)$). The 32 unknowns at each stage of the Newton process are the adjustments to the 32 matrix entries needed to cancel out the residuals to first order. The two defining relations of the group provide 32 constraints for these unknowns, but in the case that $H_1(M)$ has a free direct summand it will be necessary to add constraints $\det(\phi_1(\alpha)) = 1$, $\det(\phi_1(\beta)) = 1$. Since there are now more equations than unknowns, we use the QR decomposition of the matrix of coefficients (i.e., the Jacobian matrix) to find a least-squares solution to the linear system.

It can happen that the manifold is rigid despite the existence of a nontrivial Zariski tangent; this phenomenon manifests itself here by the Newton process refusing to converge.[3] To prove rigidity one then has to compute a higher-order obstruction, for example the third-order obstruction described in [Cooper et al. 05]. Referring to Table 1, in this way it was proved that m149(−4, 1), m159(2, 3), m293(4, 1) are all rigid. The six outstanding cases listed in Table 1 have not been checked rigorously, but there is strong evidence that they are rigid.

If the Newton process converges satisfactorily to a representation ϕ_1 distinguished from ϕ_0 by examination of characteristic polynomials, we move on to Step 3. We note that because we are using floating-point arithmetic, we cannot yet assert definitely that we have found a genuine representation ϕ_1.

Step 3. Find a suitable parameterization of the character variety. The character variety can always be parameterized by means of coefficients of characteristic polynomials, but the aim is to find parameters $u_1, \ldots, u_n$ for which the trace field has small degree over the field $\mathbb{Q}(u_1, \ldots, u_n)$. This usually involves a modest amount of experimentation. From Step 1 we have an upper bound on the number of parameters n.

From Step 2 we already know of at least one trace that varies as one moves away from ϕ_0, say $\mathrm{tr}(\phi(g))$, where $g \in \pi_1(M)$. Add a constraint to the Newton process of Step 2, declaring that this trace is some rational number reasonably close to the value of this trace at ϕ_0. This might make the convergence of the Newton process less robust, in which case it will be necessary to control the step length, by multiplying the adjustments at each stage by some dynamically controlled scale factor. Once one has achieved convergence, determine by means of LLL [Lenstra et al. 82] (or an alternative, e.g., PSLQ [Bailey and Ferguson 91]) whether all other traces now appear to be algebraic numbers, and if so, note the degree and complexity of their minimal polynomials. If some trace appears not to be algebraic, select it as an additional parameter, add an extra constraint declaring it to equal an appropriate rational number, and repeat the process. We note that the proof of Proposition 3.1 provides a list of traces that is sufficient for this purpose.

Eventually, we should achieve a numerical approximation to a representation where LLL declares that all traces are algebraic. In practice, one would not wish to compute a variety for which $n > 2$, although we have used this method to work out the deformation variety for one 3-dimensional example, the cusped manifold m007. In the last section of this paper, we take the reader in some detail through a 1-dimensional example, namely the closed manifold m007(3, 1) known as Vol3.

[3]This is probably because the vanishing of the first-order obstruction to the existence of a variety implies the vanishing of the second-order obstruction also; see, for example, [Cooper et al. 05].

manifold	volume	inf.	actual	manifold	volume	inf.	actual
m007(3, 1)	1.014941	1	1 ✓	s912(0, 1)	4.059766	2	2
m036(−3, 2)	2.029883	1	1 ✓	m401(−2, 3)	4.059766	2	2
m034(−4, 1)	2.195964	1	1 ✓	v825(4, 1)	4.059766	1	1
m160(−3, 2)	2.595387	1	1 ✓	m358(1, 3)	4.059766	1	1
m082(1, 3)	2.786804	1	1 ✓	m368(−4, 1)	4.059766	1	0
m078(5, 1)	2.816179	1	1 ✓	s778(−3, 2)	4.059766	2	2 ✓
m100(2, 3)	2.882494	1	1 ✓	s779(1, 2)	4.059766	2	2
m149(−4, 1)	3.044824	1	0 ✓	m395(−2, 3)	4.059766	2	2
m188(2, 3)	3.044824	1	1 ✓	s440(−1, 3)	4.059766	1	1
m247(−1, 3)	3.044824	1	1 ✓	v2678(2, 1)	4.116968	5	3
m159(2, 3)	3.044824	1	0 ✓	s500(4, 1)	4.116968	1	1
m115(5, 2)	3.060334	1	1 ✓	v2334(−1, 2)	4.116968	2	0
m121(−4, 3)	3.195780	1	1 ✓	s490(−4, 1)	4.116968	1	1
m336(−1, 3)	3.663862	2	2 ✓	s668(4, 1)	4.221804	1	1
m303(−1, 3)	3.663862	1	1 ✓	s518(−1, 4)	4.400901	1	1 ✓
s572(1, 2)	3.663862	1	1 ✓	v2817(−3, 1)	4.407345	1	1
m293(4, 1)	3.663862	1	0 ✓	s533(1, 4)	4.422687	1	1
s645(−1, 2)	3.663862	1	1 ✓	m402(2, 3)	4.436783	1	1
m312(−1, 3)	3.663862	2	2 ✓	v1461(1, 3)	4.598034	1	1
s778(−3, 1)	3.663862	1	1 ✓	s636(−1, 4)	4.598853	1	1 ✓
m304(5, 1)	3.663862	1	1	s618(1, 4)	4.598853	1	1 ✓
s682(−3, 1)	3.663862	3	2	v1222(−5, 1)	4.626243	1	1
s350(−4, 1)	3.663862	1	0	v1251(4, 3)	4.686034	1	1
m294(4, 1)	3.663862	1	0	s666(−4, 3)	4.686034	1	1
s495(1, 2)	3.663862	1	0	v2413(−3, 2)	4.686034	2	0
s235(−3, 4)	3.794090	1	1	v1695(−5, 1)	4.834441	1	1
m290(−3, 4)	3.818259	1	1	v1860(2, 3)	4.974542	1	1
m350(−1, 3)	3.861814	1	1	v1847(−4, 3)	5.016110	1	1
m360(−2, 3)	3.861814	1	1	v1845(−5, 2)	5.017640	1	1
s287(3, 4)	3.896345	1	1	v3283(−3, 1)	5.171469	1	1
m346(2, 3)	3.933297	1	1				

TABLE 1. Infinitesimal and actual deformations of closed manifolds.

Step 4. Compute the trace field T, as an extension of finite degree over the field $\mathbb{Q}(u_1, \ldots, u_n)$. This is best explained by means of a worked example, but broadly speaking, the method is to compute generators for the trace field evaluated at each point of a cubic lattice in parameter space, using LLL, and then obtain generators for T over $\mathbb{Q}(u_1, \ldots, u_n)$ using polynomial interpolation.[4] The points of the lattice should be chosen to have rational coordinates whose denominators are not too large; moreover, the lattice points should be reasonably close together. If $n > 1$ it will be necessary to interpolate in each of the n coordinate directions, so as to obtain a polynomial in the n variables $u_1, \ldots, u_n$. In practice, a row of data points $(x_1, y_1), \ldots, (x_k, y_k)$ for the interpolation may not lie on the graph of the desired polynomial, but $(x_1, \lambda_1 y_1), \ldots, (x_k, \lambda_k y_k)$ will lie on the graph for integers λ_i that are small relative to the y_i. Determining these "multipliers" λ_i is perhaps the trickiest part of the entire process, but skill comes with practice! It is usually self-evident when the correct λ_i have been found, since the degree of the interpolating polynomial is then much smaller than the number of data points.

Once the trace field T has been computed, choose once and for all a basis $\tau_1, \ldots, \tau_m$ for T as a vector space over $\mathbb{Q}(u_1, \ldots, u_n)$.

[4]Here we use polynomial interpolation of the most basic kind, namely fitting a polynomial of least degree to finitely many data points. Functions for this purpose are provided in all standard computer algebra systems.

Step 5. Choose an element of $\pi_1(M)$ whose image is to fulfill the role of the matrix b in Section 3. If at all possible, to avoid field extensions of uncomfortably large degree, b should have a reciprocal characteristic polynomial. Fortunately, such elements have proved to be available for all varieties that we have computed. Once this matrix has been chosen, decide on a basis for the field K generated by the matrix entries over $\mathbb{Q}(u_1, \ldots, u_n)$. This basis will of course be a function of the parameters.

Step 6. Express each trace used in the proof of Proposition 3.1 as a linear combination $\sum_{i=1}^{n} f_i \tau_i$, where the f_i are rational functions of the parameters $u_1, \ldots, u_n$. Again one uses polynomial interpolation, but for determining the coefficients f_i at each lattice point the required tool is a facility for detecting integer relations. Pari's *lindep* is such a function [Batut et al. 90]. Neither Mathematica nor Maple has a built-in integer-relation facility, but notebooks (for Mathematica) and worksheets (for Maple) are available that will perform this task.

Step 7. Write a program that uses the method of Section 4 to compute generating matrices a, b in standard form, to a large number of decimal places, for each point of a lattice in parameter space. The "large number" just referred to depends on the complexity of the situation, but typically 2000 decimal places are appropriate, so that the integer-relation detector *lindep* can produce results that are not spurious. The number of points in the lattice depends also on the complexity, but usually we have found that an n-cube of edge length 50 is sufficient. Fortunately, this method of computing numerical approximations to representations is remarkably fast, and even the computation of the 2500 representations for a 50×50 lattice can usually be accomplished in an hour or two.

Step 8. Determine an exact expression for each matrix entry in terms of the parameters, using *lindep* and polynomial interpolation, as in Step 4.

Step 9. Use the formal algebra capabilities of Maple or Mathematica to verify that the exact matrices obtained in the previous step satisfy the group relations.

7. THE MANIFOLD VOL3

The first manifold in the census to admit $\mathrm{SL}(4,\mathbb{R})$ deformations is the third manifold listed in the census, known as "Vol3." Its denotation in the census is m007(3, 1), meaning that it is obtained by (3, 1)-surgery on the cusped manifold m007. The manifold m007, in turn, is the seventh in the census of cusped manifolds obtained by gluing together up to five ideal tetrahedra (for historical reasons, the prefixes "s" and "v" are used for 6 and 7 ideal tetrahedra, respectively). Surgery coefficients are given relative to the basis {[shortest curve], [second-shortest curve]} for the first homology group of the cusp cross section.

SnapPea gives the following presentation for the fundamental group of Vol3:

$$\pi_1(\mathrm{Vol3}) = \langle a, b | aabbABAbb, aBaBabaaab\rangle,$$

and the following numerical approximation to a lift to $\mathrm{SL}(2,\mathbb{C})$ of the discrete faithful representation into $\mathrm{PSL}(2,\mathbb{C})$:

$$a \mapsto \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix},$$

where

$$\begin{aligned}
\alpha_1 &= 0.853230696696 - 1.252448658070i,\\
\alpha_2 &= -0.500000000000 + 0.866025403784i,\\
\alpha_3 &= 0.159374980683 - 2.137255282203i,\\
\alpha_4 &= 0.371514174695 + 1.959555439256i,
\end{aligned}$$

and

$$b \mapsto \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix},$$

where

$$\begin{aligned}
\beta_1 &= 0.500000000000 + 0.866025403784i,\\
\beta_2 &= -0.286522831781 + 0.594411651593i,\\
\beta_3 &= 0.658037006476 + 1.365143787662i,\\
\beta_4 &= 0.000000000000 + 0.000000000000i,
\end{aligned}$$

Here, for notational convenience, we are using the uppercase letters A, B to denote a^{-1}, b^{-1} respectively. Since we shall be considering only representations of $\pi_1(\mathrm{Vol3})$ into linear groups within matrix algebras, we shall take the liberty of considering a, b as matrices and write the group relations as

$$\begin{aligned}
r_1(a,b) &:= aabb - BBaba = 0,\\
r_2(a,b) &:= aBaBa - BAAAB = 0.
\end{aligned}$$

We begin by using Newton's method to improve the accuracy of the $\mathrm{SL}(2,\mathbb{C})$ representation. The procedure is to compute the residuals y_1, y_2 of r_1, r_2, namely their actual starting values (which are already close to the zero 2×2 matrix), and then solve a linear system to find changes da, db in the matrices a, b that cancel out these residuals to first order. For this we compute formal expressions for the changes dr_1, dr_2 in r_1, r_2 effected by changing a, b to $a + da, b + db$; we then solve the system $dr_1 = -y_1,\ dr_2 = -y_2$.

"Differentiating" $mM = I$, we obtain $(dm)M + m(dM) = 0$, whence $dM = -M(dm)M$. Therefore

$$\begin{aligned}
dr_1 = {}& \big((da)abb + a(da)bb + aa(db)b + aab(db)\big)\\
&- \big((-B(db)B)Baba + B(-B(db)B)aba\\
&\quad + BB(da)ba + BBa(db)a + BBab(da)\big),\\
dr_2 = {}& \big((da)BaBa + a(-B(db)B)aBa + aB(da)Ba\\
&\quad + aBa(-B(db)B)a + aBaB(da)\big)\\
&- \big((-B(db)B)AAAB + B(-A(da)A)AAB\\
&\quad + BA(-A(da)A)AB + BAA(-A(da)A)B\\
&\quad + BAAA(-B(db)B)\big).
\end{aligned}$$

The eight unknowns of the linear system are the entries of the matrices da, db, and entry-by-entry comparison of dr_i with $-y_i$ $(i = 1, 2)$ provides eight equations in these unknowns. All this is easy to program in Mathematica or Maple, and after a small number of iterations we arrive at an $\mathrm{SL}(2,\mathbb{C})$ representation accurate to 1000 decimal places. We note in the case that $H_1(M)$ has a free summand, it is necessary to add constraints $\det(a) = 1, \det(b) = 1$ and then use a QR decomposition on the resulting rectangular matrix of coefficients.

The accurate $\mathrm{SL}(2,\mathbb{C})$ representation that we have just obtained is now converted to an $\mathrm{SO}^+(3,1)$ representation, using the method explained in Section 2.1. Let a_0, b_0 denote the images in $\mathrm{SO}^+(3,1)$ of the group generators. The Jacobian matrix J, whose nullity we need to compute for Step 1, is obtained using the above expressions for dr_1, dr_2, with a_0, b_0 in place of a, b. Since a_0, b_0 have in total 32 entries, J has size 32×32. We then find that the rank of J is 16, whence the nullity of J is also

16 and from the discussion of Section 5 there is one dimension's worth of essential infinitesimal deformations.[5]

We now proceed to Step 2. We perturb a_0, b_0 very slightly to matrices a_1, b_1, and then try to converge to a representation using Newton's method. The linear system to be solved at each iteration uses the matrix J of the previous step, but with the current matrices a_1, b_1 in place of a_0, b_0. As an example, one can obtain matrices a_1, b_1 giving a representation to 500 decimal places of accuracy, with characteristic polynomials as follows:

$$\begin{aligned}
&\mathrm{charpoly}(a_1)\\
&\quad = 1.0000000000000000000000000000000000000000000000\\
&\qquad - 2.0000000000000886784402728253059992711608536655x\\
&\quad - 2.0000000000000886784402728253059992711608536655x^3 + x^4
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{charpoly}(b_1)\\
&\quad = 1.0000000000000000000000000000000000000000000000\\
&\qquad - 1.0000000000000000000000000000000000000000000000x\\
&\qquad - 3.0000000000000354713761091309087862853864459793 5x^2\\
&\qquad - 1.0000000000000000000000000000000000000000000000x^3\\
&\qquad + x^4.
\end{aligned}$$

On the other hand, the characteristic polynomials of a_0, b_0 are

$$\mathrm{charpoly}(a_0) = 1 - 2x - 2x^3 + x^4,$$
$$\mathrm{charpoly}(b_0) = 1 - x - 3x^2 - x^3 + x^4,$$

and we are encouraged to try taking the trace of a_1 as parameter, v say. Note the symmetric nature of the characteristic polynomials of a_1, b_1; also note that the trace of b_1 appears to be constant.

Running the Newton program again with the extra constraint $\mathrm{tr}(A_1) = 2.001$, we obtain a representation (to the same accuracy), with

$$\begin{aligned}
&\mathrm{charpoly}(a_1) =\\
&\quad 1.0000000000000000000000000000000000000000000000\\
&\qquad - 2.0010000000000000000000000000000000000000000000x\\
&\qquad - 2.0010000000000000000000000000000000000000000000x^3\\
&\qquad + x^4
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{charpoly}(b_1) =\\
&\quad 1.0000000000000000000000000000000000000000000000\\
&\qquad - 1.0000000000000000000000000000000000000000000000x\\
&\qquad - 3.0040010000000000000000000000000000000000000000x^2\\
&\qquad - 1.0000000000000000000000000000000000000000000000x^3\\
&\qquad + x^4,
\end{aligned}$$

[5] Strictly speaking, because we are working in floating-point mode, we may assert at present only that the space of essential infinitesimal deformations has dimension at most 1.

i	$p_i(x)$
1	$5181915799 - 1490060879x + 104060401x^2$
2	$5389356866 - 1549614193x + 108243216x^2$
3	$5602964647 - 1610935039x + 112550881x^2$
4	$5822860162 - 1674058049x + 116985856x^2$
5	$6049165607 - 1739018191x + 121550625x^2$
6	$6282004354 - 1805850769x + 126247696x^2$
7	$6521500951 - 1874591423x + 131079601x^2$
8	$6767781122 - 1945276129x + 136048896x^2$
9	$7020971767 - 2017941199x + 141158161x^2$
10	$83041 - 10000x$

TABLE 2. Minimal polynomials of $\mathrm{tr}(abAB)$ at selected points of $\mathcal{V}$.

and we suspect strongly that the middle term of the characteristic polynomial of b_1 is $-(v^2-1)$. (A keen observer might have noticed this earlier.)

We now try to identify the trace field. A quick search reveals that the trace of the commutator $a_1b_1A_1B_1$ appears to be irrational, and that according to LLL, for rational v it appears to be a root of a quadratic over $\mathbb{Q}$ (see the Appendix, Section 8, for a brief description of LLL). We run the Newton program again for $v = 2 + \frac{1}{100+i}$ $(1 \le i \le 10)$, and print out the minimal polynomials of this trace, as given by LLL. The results are displayed in Table 2.

We proceed to interpolate the first nine polynomials, assuming that the anomalous degree of the polynomial for $i = 10$ is caused by accidental rationality of $\mathrm{tr}(a_1b_1A_1B_1)$ at $v = 2+\frac{1}{110}$. The polynomials are defined only up to integer multiples; however, the coefficients appear to lie on a "nice" curve, so it probably will not be necessary to find "multipliers" λ_i.

Indeed, interpolation reveals that the minimal polynomial for this trace is

$$1 - (v^4 - 2)x + (2v^4 + 4v^2 + 1)x^2,$$

with discriminant $v^2(v^2+2)^2(v^2-4)$. We are led to surmise that the trace field is

$$T = \mathbb{Q}(v)(\alpha), \quad \text{where } \alpha = \sqrt{v^2-4};$$

some support for this conjecture is provided by computation of several other traces. Incidentally, we note that for $v = 2+\frac{1}{110}$, the quantity $\sqrt{v^2-4}$ is the rational number $\frac{21}{110}$, explaining the anomalous polynomial of degree 1 for $i = 10$.

We now proceed to Step 5, where we choose the matrix to be diagonalized. Since the characteristic polynomial of b_1 is reciprocal, satisfies the conditions of Proposition 3.1, and has roots that are relatively simple expressions in v, we select b_1 for this purpose (in fact, a_1 would have done equally well). Two of the eigenvalues of b_1 are real for $v > 2$; they are

$$\lambda_1 = \frac{1}{4}\left(1 + \sqrt{5+4v^2} + \sqrt{2\left(-5+2v^2+\sqrt{5+4v^2}\right)}\right)$$

and

$$\lambda_2 = \frac{1}{4}\left(1 + \sqrt{5+4v^2} - \sqrt{2\left(-5+2v^2+\sqrt{5+4v^2}\right)}\right).$$

If it should transpire that our guess for the trace field is correct, we can already predict from the discussion of Section 4 that the field generated by the matrix entries will be $K = \mathbb{Q}(v)(\alpha, \lambda_1)$, and that a vector space basis for K over $\mathbb{Q}(v)$ will be $\{1, \gamma, \beta, \beta\gamma, \alpha, \alpha\gamma, \alpha\beta, \alpha\beta\gamma\}$, where

$$\alpha = \sqrt{v^2-4}, \;\; \beta = \sqrt{4v^2+5}, \;\; \gamma = \sqrt{2(-5+2v^2+\beta)}.$$

We note that the field K is not a Galois extension of $\mathbb{Q}(v)$, since for $v > 2$ it does not contain the two nonreal roots of the minimal polynomial of λ_1. It has an automorphism σ_1 negating α and fixing γ, and an automorphism σ_2 fixing α and negating γ; these automorphisms commute and generate the automorphism group of $K : \mathbb{Q}(v)$, which is therefore a Klein group of order 4.

Moving on to Step 6, we would now like to produce representations of high numerical accuracy (2000 decimal places to be safe) for a sequence of values of the parameter v, say $v = 2 + \frac{1}{110+i}$ $(1 \le i \le 20)$ (recall that we wish to avoid $v = 2 + \frac{1}{110}$). In the case of Vol3 it is feasible to obtain these directly from the Newton program; however, typically this is too slow, and a much better approach is to use the method of Proposition 3.1.

Thus our immediate task is to identify each of the 18 traces used in that proposition as elements of the trace field $T = \mathbb{Q}(v)(\alpha)$. For this we run the Newton program as we did earlier for the trace of $a_1b_1A_1B_1$, but this time, for each trace t, we run the integer-relation detector *lindep* on the vector $(1, \alpha, t)$ and interpolate the resulting coefficients over the data points. In this way we obtain a generic relation $p_0(v) + p_1(v)\alpha + p_2(v)t = 0$, where each p_i is a polynomial in v with integer coefficients, and we record that $t = -\frac{p_0(v)}{p_2(v)} - \frac{p_1(v)}{p_2(v)}\alpha$. The results for the 18 traces are set out in Table 3.

These are incorporated into a short program that produces (numerical approximations to) the canonical forms a_2, b_2 defined in Section 4 for the generating matrices, by solving the linear systems given in the proof of Proposition 3.1 and then conjugating as described in Section 4.

The penultimate stage of the process is to apply *lindep* and polynomial interpolation to our data, so as to obtain exact expressions for the matrix entries. The $(2,3)$ entry of the matrix a_2 gives a good idea as to what is involved here. Let us denote this entry by x. For each $v = 2 + \frac{1}{110+i}$ $(1 \le i \le 20)$, we apply *lindep* to the nine-component vector $(1, \gamma, \beta, \beta\gamma, \alpha, \alpha\gamma, \alpha\beta, \alpha\beta\gamma, x)$. The output from Pari is a sequence of 20 nine-component vectors $\mathbf{w}_i$ $(1 \le i \le 20)$, where each component is an integer of approximately 17 digits. The first component of each vector is the integer y_i in Table 4, from which the prospect of fitting a reasonable polynomial admittedly looks bleak.

But we have to bear in mind that the vectors output by *lindep* are homogeneous, and that Pari will reduce each vector so that the greatest common divisor of its components is 1. Note that $y_1, y_5, y_{11}, y_{13}, y_{19}$ look (up to sign) as if they stand a fair chance of lying on the correct polynomial curve, so we begin by fitting a polynomial $p(x)$ to these five data points. Our desired polynomial will probably have higher degree than $p(x)$, but we hope that $p(x)$ will be a close approximation. This is given support by the fact that $p(i)/y_i$ is very close to being an integer for all data points, so we take these integers as

$\mathrm{tr}(a) = v$	$\mathrm{tr}(aBab) = 1$
$\mathrm{tr}(ab) = v$	$\mathrm{tr}(aaa) = 3v + v^3$
$\mathrm{tr}(abb) = v$	$\mathrm{tr}(aaab) = v$
$\mathrm{tr}(abbb) = v^3$	$\mathrm{tr}(aaabb) = v$
$\mathrm{tr}(aa) = v^2$	$\mathrm{tr}(aaBab) = \frac{1}{2}\left(-2v + v^3 + (2+v^2)\alpha\right)$
$\mathrm{tr}(aab) = 1$	$\mathrm{tr}(aabaB) = \frac{1}{2}\left(-2v + v^3 - (2+v^2)\alpha\right)$
$\mathrm{tr}(aaB) = -1 + 2v^2$	$\mathrm{tr}(aaBabb) = \frac{1}{2}\left(2v - v^3 + v^5 + (2 - v^2 - v^4)\alpha\right)$
$\mathrm{tr}(aabb) = 1$	$\mathrm{tr}(abaBabb) = \frac{1}{2}\left(2v - v^3 + v^5 + (2 - v^2 - v^4)\alpha\right)$
$\mathrm{tr}(abab) = v^2$	$\mathrm{tr}(abaBBabbb) = -3v + 2v^3 + 2v^5$

TABLE 3. Traces of selected words in the fundamental group of Vol3.

i	y_i	λ_i
1	9671015960804800	1
2	−510168310445250	−20
3	−1195546153052400	−9
4	2835383895633100	4
5	11949122303635440	1
6	−349544299928850	−36
7	−2649179302985600	−5
8	−3484248880566300	−4
9	−1628653880584800	−9
10	3852395824372810	4
11	16193135112008400	1
12	−94497839870700	−180
13	−17860109073398800	−1
14	4686438400061250	4
15	2185299690910560	9
16	−5156780125192300	−4
17	−4325046517516800	−5
18	−629535323374650	−36
19	23742505309322800	1
20	6216057796580460	4

TABLE 4. Interpolation data for the $(2,3)$ entry of the generator a.

our multipliers λ_i. The list of the 20 multipliers λ_i is given in Table 4, alongside the y_i.

We now interpolate the components of the vectors $\lambda_i \mathbf{w}_i$ at all 20 data points, obtaining the following 9-component vector of polynomials in v:

$$\begin{aligned}
p_1 &= 10v(v^2-1)(v^2-4)(4v^2+5),\\
p_2 &= -v(v^2+2)(v^2-4)(4v^2+5),\\
p_3 &= -2v(v^2-1)(v^2-4)(2v^2+15),\\
p_4 &= -v(v^2+2)(v^2-10),\\
p_5 &= 2v^2(v^2-1)(4v^2+5),\\
p_6 &= -v^2(v^2+2)(4v^2+5),\\
p_7 &= -2v^2(v^2-1)(2v^2-5),\\
p_8 &= (v^2+2)(7v^2-10),\\
p_9 &= -32(v^2-1)(v^2-5)(4v^2+5).
\end{aligned}$$

Since the degree of each of these polynomials is much smaller than the degree of 19 that one would obtain generically, we are confident that we have the right answer, although this will not be proved until the final formal check of the two group relations.

Therefore we record $x = -\frac{1}{p_9}\sum_{i=1}^{8} p_i \nu_i$, where $\{\nu_1, \ldots, \nu_8\}$ is the basis we have chosen for the matrix entry field over $\mathbb{Q}(v)$, and we are done. The other entries of a_2, b_2 are computed similarly.

Once exact expressions for all entries of the generators a_2, b_2 have been computed, it remains for Maple or Mathematica to check that they are correct by verifying formally that the group relations are satisfied, i.e., that $r_1(a_2, b_2) = r_2(a_2, b_2) = 0$. This can take a moderate amount of time; the key is to simplify aggressively at each stage of the computation.

Remark 7.1. The real hyperbolic representation occurs at $v = 2$. There are in fact two nonconjugate $\mathrm{SL}(4,\mathbb{C})$ representations for each $v \in \mathbb{C}$, $v \neq 2$, corresponding to the two square roots of $\alpha^2 = v^2 - 4$. There is a symmetry of the variety interchanging these two representations, induced by the *contragredient* automorphism of $\mathrm{GL}(4,\mathbb{C})$ that assigns to each matrix the transpose of its inverse.

Remark 7.2. Further experimentation can sometimes result in a conjugate family of representations with simpler expressions for the matrix entries. In the case of Vol3, a significant improvement may be obtained by exploiting symmetries of the manifold. The symmetry group of Vol3 is semihedral of order 16 [Hodgson and Weeks 94], and in particular, there is a rotational symmetry ρ of order 4, fixing the axis of $aBaba$ and inducing an automorphism ρ_* of $\pi_1(\mathrm{Vol3})$ sending a to BA and b to aba. Thus we may form the orbifold fundamental group $\pi_1(\mathrm{Vol3}/\langle\rho\rangle)$ as a supergroup of $\pi_1(\mathrm{Vol3})$ by adding a new generator u together with relations $u^4 = 1$, $Uau = BA$, $Ubu = aba$. It follows easily that u^2a has order 2, and that $\pi_1(\mathrm{Vol3}/\langle\rho\rangle)$ is generated by u, u^2a. Starting from the above family of representations and considering $\pi_1(\mathrm{Vol3}/\langle\rho\rangle)$ as a carefully chosen group of isometries of 3-dimensional complex hyperbolic space, for a suitable range of values of the parameter v (see [Cooper et al. 05]), the following much simpler curve of representations was found:

$$\Psi(u) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{v^2-4}{v^2+8}} & \sqrt{\frac{-2v^2-4}{v^2+8}} \\ 0 & 0 & \sqrt{\frac{-2v^2-4}{v^2+8}} & -\sqrt{\frac{v^2-4}{v^2+8}} \end{bmatrix},$$

$$\Psi(u^2a) = \begin{bmatrix} p & 0 & q & 0 \\ 0 & p^* & 0 & q^* \\ q & 0 & -p & 0 \\ 0 & q^* & 0 & -p^* \end{bmatrix},$$

where

$$\begin{aligned}
p &= \frac{1}{4}\left(v + \sqrt{v^2+8}\right),\\
q &= -\frac{1}{2\sqrt{2}}\sqrt{4 - v^2 - v\sqrt{v^2+8}},\\
p^* &= \frac{1}{4}\left(v - \sqrt{v^2+8}\right),\\
q^* &= -\frac{1}{2\sqrt{2}}\sqrt{4 - v^2 + v\sqrt{v^2+8}}.
\end{aligned}$$

Remark 7.3. Earlier, we showed that the trace field contained $\mathbb{Q}(v)(\alpha)$, but did not prove equality. However, we can show equality as follows. Let K be the matrix entry field used in the main part of this section, and let K' be the matrix entry field of Remark 7.2. Since the trace field is invariant under conjugation, it must lie in $K \cap K' = \mathbb{Q}(v)(\alpha)$.

8. APPENDIX: USING LLL

For readers not familiar with this wonderful tool, here is a very brief description. LLL is available in Pari as the function **algdep( )**, in Mathematica as **Recognize**[ ], and in Maple as **MinimalPolynomial( )**. Here we describe the Pari version.

For **algdep( )**, one enters as input a complex number t in floating-point format, a maximum degree $d \geq 1$, and a working precision p. In return, one receives a polynomial in $\mathbb{Z}[x]$ of degree at most d, of which t is a root to within the working precision. For example, the command

$$\mathrm{algdep}(1.41421356, 2, 6)$$

yields the answer $x^2 - 2$.

Of course, the input t is always rational if interpreted literally, and one should regard as spurious any polynomial whose coefficients are large relative to the declared working precision p. A good, conservative rule of thumb is that an answer is trustworthy if the total number of digits over all coefficients is at most $\frac{2p}{3}$, and that the answer is suspect otherwise. For example, one should reject

$$\text{algdep}(3.1415926535, 10, 10) \longrightarrow$$
$$3x^{10} - 9x^9 - 5x^7 + 9x^5 - 5x^4 + 3x^3 + 6x^2 + 6x + 1.$$

The time taken for **algdep( )** to complete its task appears to depend exponentially on the degree d, and also on the precision p; however, it works well for, say, $d = 20$ and $p = 1000$.

The related **lindep( )** function of Pari accepts as input a vector $[t_1, \ldots, t_n]$ of floating-point numbers and a working precision p, and returns a vector $[\lambda_1, \ldots, \lambda_n]$ of integers for which $\sum \lambda_i t_i = 0$ to within the precision p. For example, if we set $t = 2^{1/2} + 3^{1/3}$, the command

$$\text{lindep}([1, t, t^2, t^3, t^4, t^5, 2^{1/2}], 100)$$

instantly yields the answer

$$[-1092,\ 879,\ -468,\ -320,\ 27,\ 48,\ -755],$$

from which one deduces that

$$\sqrt{2} = \frac{1}{755}\left(-1092 + 879t - 468t^2 - 320t^3 + 27t^4 + 48t^5\right).$$

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