Irrational Life

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We prove that in Conway's Game of Life cellular automaton with initial configuration *irrat2*, the linear growth rate of the number of live cells is irrational, as conjectured in 1991 by Dean Hickerson. Our proof uses intertwined substitution sequences.

1. INTRODUCTION

The Game of Life, invented by John Horton Conway over 30 years ago, is probably the best known cellular automaton. An infinite rectangular grid divides the plane into squares called *cells*. Each cell can be *dead* or *alive*, but at any given moment only finitely many cells are alive. Each cell has eight neighbors. The evolution in (discrete) time follows simple rules: if at the moment n a cell is alive and has two or three live neighbors, it stays alive at the next moment $n+1$, otherwise it dies; if at the moment n it is dead and has exactly three live neighbors, then it becomes alive at the moment $n+1$, otherwise it stays dead. Both overcrowding and loneliness are intolerable.

Cellular automata were introduced by von Neumann and Ulam in the late 40s. One of their goals was to study self-reproduction in these spatially discrete dynamical systems with spatially homogeneous, locally determined, discrete time dynamics. Cellular automata have since been studied from many other points of view, including computability and computational complexity, formal language theory, and statistical mechanics, and applied to model widely diverse phenomena in physics, chemistry, biology, and many other fields.

Our primary tool will be substitution sequences. Given a finite alphabet and a suitable *substitution rule*, i.e., a finite word associated to each symbol in the alphabet, one obtains a sequence of symbols by starting with a symbol, replacing it with its associated word, similarly replacing each symbol in this word, and repeating this procedure. (Actually, one has to start with a letter for which the associated word begins with this letter, and this word has to have at least two letters.) The most

FIGURE 1. Initial configuration.

FIGURE 2. After 90 steps.

famous example is the so-called Morse sequence

$$0110100110010110100101100110100 1\ldots,$$

generated by the substitution $0 \to 01$, $1 \to 10$ and used by Morse to study geodesic flows on negatively curved surfaces.

One can get more information about Life in, for instance, [Berlekamp et al. 82] and [Gardner 83]. There is also substantial information on the Internet, including programs for various platforms and online. Two good starting points are (as of April 2005) http://www.radicaleye.com/lifepage/ and http://psoup.math.wisc.edu/Life32.html.

Many initial configurations with very interesting evolution in time have been found. Most of them have been collected and encoded using the `*.lif` format that can be read by standard Life software. One of the most popular collections is `lifep.zip`. Various parts of configurations ("life forms") have their specific names, often referring to spaceships.

2. IRRATIONAL CONFIGURATION

The Life configuration we considered is called *irrat2* (see Figure 1). It was described by Dean Hickerson in 1991 and can be found with his comments at http://www.radicaleye.com/lifepage/patterns/irrat2.html. For related work, see [Griffeath and Hickerson 03].

It consists of two big *mother spaceships* that move apart to the left and right (on the same horizontal level), leaving behind a combination of small *spaceships* and *mines* in two horizontal rows: upper and lower. (In the standard Life terminology, mother spaceships are called puffers, spaceships are called middleweight spaceships, and mines are called boats.) Additional effects, crucial for us, are created when a spaceship hits a mine. Figures 2, 3, and 4 show the evolution of the system. Note that the scale of the figures changes. Although the spaceships in the lower and upper rows may look different, they are simply in different phases.

The mother spaceships are moving apart with speed $c/2$ (one space unit per two time units) each. The left one leaves spaceships in the upper row (they are moving right with speed $c/2$) and stationary mines in the lower row. Both spaceships and mines are 20 units apart. The right mother spaceship leaves spaceships in the lower row (they are moving left with speed $c/2$) and stationary mines in the upper row. Spaceships are 20 units apart, while mines are ten units apart. When a spaceship hits a mine, both of them are destroyed, but a *glider* is created. It moves in the lower right direction if the catastrophe occurred in

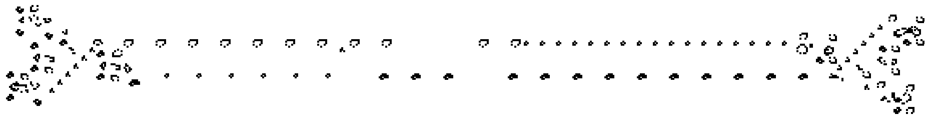

FIGURE 3. After 405 steps.

FIGURE 4. After 1,761 steps (central part).

the upper row and in the upper left direction if it happened in the lower row. In each case, when it reaches the other row, it collides with an incoming spaceship, destroying it (and itself). When a glider created in the lower row hits a spaceship in the upper row, then additionally a secondary mine (of a different shape than the main ones) is created. The next incoming spaceship hits it and is destroyed (the mine is also destroyed) and no more debris remains.

Hickerson's comment to the configuration reads: "Population growth appears to be linear with an irrational multiplier. The probability that a middleweight spaceship will hit a boat seems to be $1/\sqrt{2}$ for the lower (westward) stream, and $\sqrt{2}-1$ for the upper (eastward) stream. If this is true, then the population in generation t is about $(78\sqrt{2}-73)t/40$ for t even, and $(82\sqrt{2}-77)t/40$ for t odd." Our aim is to prove this conjecture about the probabilities.

Noting the position of spaceships and mines, we get two rows of sequences of symbols S (for a spaceship), M (for a mine), and O (for an unoccupied position). It is sufficient for our purposes to record those sequences only every 20 units of time. Each row consists of three blocks. The left-most block of the upper row and the right-most block of the lower row consist of alternating Ss and Os. The central blocks of both rows look similar, except that some Ss are replaced by Os (the spaceships that should occupy those positions have been destroyed). The right-most block of the upper row consists of Ms, while the left-most block of the lower row consists of alternating Ms and Os.

The unit of time we use is equal to 20 elementary units of time. It allows each spaceship to move by one position in its row. Of course, moments at which various events, like destruction of a spaceship, occur are registered approximately, since they take some time to complete. However, this makes no difference for us. Our aim is to identify the sequences (for the central blocks) that are created, so exact times of destruction are irrelevant. It is easier to think of those disappearing elements as remaining there. Then we get longer and longer one-sided sequences whose beginnings remain the same. We want additions from the right, so we have to flip the upper row.

In such a way we get the following model. We start with two short sequences of symbols O and S, upper and lower. Then we read the upper sequence from the left and extend the lower one to the right accordingly. Simultaneously, we read the lower sequence from the left and extend the upper one to the right accordingly.

Both lower and upper sequences consist of alternating Os and Ss, with some Ss replaced by Os. To simplify notation, we remove every second symbol (that is, those that have to be Os). Now, S means a spaceship and O a hole (a destroyed spaceship).

Suppose an S has just been read in the upper sequence. This corresponds to the destruction of a spaceship and a mine in the upper row. The next mine is ten space units farther (this corresponds to one time unit for the movement of a spaceship). Thus, if the next symbols to be read are k Os and then an S, we must wait $2(k+1)+1$ time units for the next spaceship in this row to blow up on a mine. This event will take place ten units to the right compared to the previous one, so the spaceship to be destroyed by the collision with a glider in the lower row will be ten units to the right and $2(k+1)+1$ time units later than the previous one. By this time $k+1$ spaceships in the lower row will travel safely, and only the next one will be destroyed. This means that reading k Os and then an S in the upper row will result in the addition of $k+1$ Ss and then an O in the lower row.

Suppose now that an S has just been read in the lower sequence. This corresponds to the destruction of a spaceship and a mine in the lower row. The next mine is 20 space units farther (this corresponds to two time units for the movement of a spaceship). Thus, if the next symbols to be read are k Os and then an S, we must wait $2(k+1)+2$ time units for the next spaceship in this row to blow up on a mine. This event will take place 20 units to the left compared to the previous one, so the spaceship to be destroyed by the collision with a glider in the upper row will be 20 units to the left and $2(k+1)+2$ time units later than the previous one. By this time $k+1$ spaceships in the upper row will travel safely, and only the next two will be destroyed (one by a glider and the second one by a secondary mine). This means that reading k Os and then an S in the lower row will result in the addition of $k+1$ Ss and then two Os in the upper row.

The above rules can be written in a simpler form. Namely, when we read O in the upper sequence, we append an S to the lower one, and when we read S in the upper sequence, we append SO to the lower one. Similarly, when we read O in the lower sequence, we append an S to the upper one, and when we read S in the lower sequence, we append SOO to the upper one. In other words, we build substitution sequences using the above substitution rules.

When we start with a sufficiently long piece of the upper sequence, reading it results in extending the lower one, and this in turn results in extending the upper one. Thus, we are building a substitution sequence (however, starting with a finite piece that perhaps has a wrong structure). Our substitution rules are compositions of corresponding substitution rules from the preceding paragraph. Concretely, we get $O \to SOO$ and $S \to SOOS$. Similarly, the rules for the lower sequence are $O \to SO$ and $S \to SOSS$.

In order to be able to make precise statements, we have to introduce some notation. Let $u(n,k)$ be the number of Ss in the block of length n starting at place k for the upper row, and $l(n,k)$ a similar number for the lower row. The probability that a spaceship (middleweight spaceship) hits a mine (boat) is equal to the relative density of the Ss in the corresponding row. We will prove a theorem from which the statement about the probabilities from Hickerson's comment will clearly follow.

Theorem 2.1. *For the* irrat2 *initial configuration, the sequences* $(u(n,k)/n)_{n=1}^{\infty}$ *and* $(l(n,k)/n)_{n=1}^{\infty}$ *converge to* $\sqrt{2}-1$ *and* $\sqrt{2}/2$*, respectively, as* $n \to \infty$*, uniformly in* k*. That is,*

$$\lim_{n\to\infty} \sup_k \left| \frac{u(n,k)}{n} - (\sqrt{2}-1) \right| = 0 \qquad \textit{and}$$

$$\lim_{n\to\infty} \sup_k \left| \frac{l(n,k)}{n} - \frac{\sqrt{2}}{2} \right| = 0.$$

Proof: Let us consider the upper sequence. It can be obtained from some initial segment by reading it from the left and appending from the right according to the substitution $O \to SOO$ and $S \to SOOS$. Thus, except perhaps for the initial segment, it is a concatenation of longer and longer blocks which are obtained from one symbol by repetitive substitutions. Let us call such blocks *regular*. Their *generation number* counts how many times we applied a substitution.

We claim that the number of Ss in a regular block of generation m, divided by the length of this block, goes to $\sqrt{2}-1$ as $m \to \infty$. Let A be a regular block of generation m and B the block of generation $m+1$ obtained from it by the substitution. Let the number of Os and Ss in A be a and b, respectively, and the number of Os and Ss in B be c and d, respectively. Since each O in A is replaced by two Os and one S in B, and each S in A is replaced by two Os and two Ss in B, we see that

$$U\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}, \quad \text{where} \quad U = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}.$$

Therefore, in a regular block of generation m the numbers of Os and Ss are equal to the components of the vector $U^m \mathbf{e}_i$, where $i = 1$ or 2 and

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The matrix U has two eigenvalues: $2+\sqrt{2}$, of modulus larger than 1, and $2-\sqrt{2}$, of modulus smaller than 1. The vectors $\mathbf{e}_i$ do not belong to the eigenspace corresponding to the second eigenvalue. Thus, as $m \to \infty$, the vectors $U^m \mathbf{e}_i$ approach the eigenspace corresponding to the first eigenvalue. This means that the number of Ss in a regular block of generation m, divided by the length of this block, goes to b, where $\binom{a}{b}$ is an eigenvector corresponding to the eigenvalue $2+\sqrt{2}$, such that $a+b=1$. An elementary computation shows that $b = \sqrt{2}-1$, thus proving our claim.

Let $\varepsilon > 0$ be given. By the claim that we have just proved, there is m such that the number of Ss in each regular block of generation m, divided by the length of this block, is within $\varepsilon/2$ of $\sqrt{2}-1$. Note than if $m_1 > m$, then each regular block of generation m_1 is a concatenation of regular blocks of generation m. Therefore, there exists k_0 such that starting at this place the upper sequence is a concatenation of regular blocks of generation m. Let k_1 denote the length of the longer of the two regular blocks of generation m. Set $k_2 = \max(k_0, k_1)$. Then any block A of length $n > 2k_2$ is a concatenation of a nonempty block B, which is a concatenation of regular blocks of generation m, and two other blocks (at the beginning and the end) of length at most k_2 each. We admit also a possibility that any of those two blocks may be empty. Let A start at place k, B at place s, and let B have length t. Then we know that

$$\left|\frac{u(t,s)}{t} - (\sqrt{2}-1)\right| \leq \frac{\varepsilon}{2}\,,$$

$|n - t| < 2k_2$, and since B is a subblock of A, also $|u(n,k) - u(t,s)| < 2k_2$. Thus,

$$\begin{aligned}\left|\frac{u(n,k)}{n} - (\sqrt{2}-1)\right| &\\ \leq \left|\frac{u(n,k)}{n} - \frac{u(t,s)}{n}\right| + \left|\frac{u(t,s)}{n} - \frac{u(t,s)}{t}\right| &\\ + \left|\frac{u(t,s)}{t} - (\sqrt{2}-1)\right| &\\ \leq \frac{2k_2}{n} + \left(\sqrt{2}-1+\frac{\varepsilon}{2}\right)\frac{2k_2}{n} + \frac{\varepsilon}{2}\,. &\end{aligned}$$

Thus, there is n_0 such that if $n \geq n_0$, then

$$\left|\frac{u(n,k)}{n} - (\sqrt{2}-1)\right| < \varepsilon$$

independently of k. This proves the theorem for the upper sequence.

For the lower sequence the proof is the same, except that the matrix U has to be replaced by the matrix

$$L = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix},$$

for which we get $b = \sqrt{2}/2$. □

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