A Computer-Based Approach to the Classification of Nilpotent Lie Algebras

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CONTENTS

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We adapt the p-group generation algorithm to classify small-dimensional nilpotent Lie algebras over small fields. Using an implementation of this algorithm, we list the nilpotent Lie algebras of dimension up to 9 over $\mathbb{F}_2$ and those of dimension up to 7 over $\mathbb{F}_3$ and $\mathbb{F}_5$.

1. INTRODUCTION

The classification of n-dimensional nilpotent Lie algebras over a given field $\mathbb{F}$ is a very difficult problem even for relatively small n. The aim of this article is to present a series of computer calculations that imply the following theorem.

Theorem 1.1. *The number of isomorphism types of six-dimensional nilpotent Lie algebras is* 36 *over* $\mathbb{F}_2$*, and* 34 *over* $\mathbb{F}_3$ *and* $\mathbb{F}_5$*. The number of isomorphism types of seven-dimensional nilpotent Lie algebras is* 202 *over* $\mathbb{F}_2$*,* 199 *over* $\mathbb{F}_3$*, and* 211 *over* $\mathbb{F}_5$*. The number of isomorphism types of nilpotent Lie algebras with dimension* 8 *and* 9 *over* $\mathbb{F}_2$ *is* $1,831$ *and* $27,073$*, respectively.*

The classifications in Theorem 1.1 were obtained using a GAP 4 [The GAP Group 04] implementation of a nilpotent Lie algebra generation algorithm. The ideas used in these calculations are the same as those used in the classification of finite 2-groups with order up to 2^9; see [O'Brien 90, Eick and O'Brien 99]. Let $\gamma_i(L)$ denote the ith term of the lower central series of a Lie algebra L, so that $\gamma_1(L) = L$, $\gamma_2(L) = L'$, etc. If L is a finitely generated nilpotent Lie algebra with nilpotency class c, then L is an immediate descendant of $L/\gamma_c(L)$ (see Section 2 for definitions). Further, $L/\gamma_c(L)$ is an immediate descendant of $L/\gamma_{c-1}(L)$. Continuing this way, we can see that every finitely generated nilpotent Lie algebra can be obtained after finitely many steps from a finite-dimensional abelian Lie algebra by computing immediate

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descendants. This suggests that, once we can efficiently compute immediate descendants, a theoretical algorithm to generate all n-dimensional nilpotent Lie algebras can be designed. We will see that every immediate descendant of L is a quotient of another nilpotent Lie algebra, that is referred to as the *cover*. It is shown, in this paper, that, for a finite-dimensional nilpotent $\mathbb{F}_p$-Lie algebra L, it is possible to effectively compute the cover, and then to compute a complete and irredundant list of the isomorphism types of the immediate descendants of L. Repeating the immediate descendant calculation finitely many times, it is, in theory, possible to obtain a complete and irredundant list of all isomorphism types of the nilpotent Lie algebras with a given dimension n over a finite field. In practice, this calculation quickly becomes unfeasible as n grows. Nevertheless, using this approach, it is possible to obtain classifications of Lie algebras that would otherwise be beyond hope; see Theorem 1.1.

The structure of this paper is as follows. In Section 2, we develop the theory of a Lie algebra generation algorithm. An application of the algorithm to prove Theorem 1.1 will be presented in Section 3. Finally, Section 4 will discuss an implementation of the algorithm.

2. A NILPOTENT LIE ALGEBRA GENERATION ALGORITHM

Our nilpotent Lie algebra generation algorithm is an adaptation of O'Brien's p-group generation algorithm, whose details can be found in [O'Brien 90]. In this section we describe our algorithm without proofs. Another variation on this theme is presented in [O'Brien et al. 04] where the authors classified groups and nilpotent Lie rings of order p^6. Recently O'Brien and Vaughan-Lee used the same approach to extend these results to p^7, see [O'Brien and Vaughan-Lee 05].

Throughout this section, L is a finite-dimensional, nilpotent Lie algebra. Let $Z(L)$ denote the center of L. A nilpotent Lie algebra K is said to be a *central extension* of L if K has an ideal I such that $I \leqslant K' \cap Z(K)$ and $K/I \cong L$. Using the terminology of [Batten et al. 96, Batten and Stitzinger 96], (K, I) is said to be a defining pair for L. The algebra K is said to be an *immediate descendant* of L if $L \cong K/\gamma_c(K)$ where c is the nilpotency class of K. Hence an immediate descendant is a special kind of central extension. Suppose that $\dim L/L' = d$. Then L is a d-generator Lie algebra, and so the free Lie algebra F_d with rank d has an ideal I such that $F_d/I \cong L$. The *cover* L^* of L is defined as the Lie algebra $F_d/[I, F_d]$. The *multiplicator* of L^* is the ideal $I/[I, F_d]$. The cover L^* is also a finite-dimensional nilpotent Lie algebra. Moreover, if L has nilpotency class c then the class of L^* is at most $c+1$, and $\gamma_{c+1}(L^*)$ is referred to as the *nucleus* of L^*.

Suppose now, without loss of generality, that $L = F_d/I$ as in the previous paragraph. Let L^* be the cover of L with multiplicator M and nucleus N. Then K is a central extension of L if and only if $K \cong L^*/J$ for some ideal $J \leqslant M$. Further, in this case, K is an immediate descendant of L if and only if $J \neq M$ and $J + N = M$. A proper subspace J of M with $J + N = M$ is said to be *allowable*. Thus it is possible to obtain a complete list of immediate descendants of L by listing all quotients L^*/J where J runs through the allowable subspaces of the multiplicator M. Unfortunately, two different allowable subspaces may lead to isomorphic Lie algebras. This problem can, however, be tackled using the automorphism group of L. If α is an automorphism, then α can be lifted to an automorphism α^* of L^* as follows. Let $\psi : L^* \to L$ denote the natural epimorphism with kernel M. Suppose that $b_1, \ldots, b_d$ is a minimal generating set for L^*; then $b_1\psi, \ldots, b_d\psi$ is a minimal generating set for L. Suppose that $y_1, \ldots, y_d \in L^*$ are chosen so that $b_i\psi\alpha = y_i\psi$ for all $i \in \{1, \ldots, d\}$. Then the map $b_i \mapsto y_i$, for $i = 1, \ldots, d$, can uniquely be extended to an automorphism of L^*. This automorphism is denoted α^*, even though it is not uniquely determined by α. On the other hand the restriction of α^* to $M = I/[I, F_d]$ depends only on α. This defines a linear representation

$$\varrho : \mathsf{Aut}(L) \to \mathsf{GL}(M) \quad \text{given by} \quad \alpha \mapsto \alpha^*|_M. \qquad (2\text{–}1)$$

Using a familiar argument, it is not hard to see that two allowable subspaces J_1 and J_2 give isomorphic Lie algebras L^*/J_1 and L^*/J_2 if and only if J_1 and J_2 are in the same orbit under the action $\mathsf{Aut}(L)\varrho$.

If J is an allowable subspace of the multiplicator then the automorphism group of $K = L^*/J$ can also be computed using $\mathsf{Aut}(L)$. Let S denote the stabilizer in $\mathsf{Aut}(L)$ of J under the representation ϱ. Let X denote a generating set for S. For each $\alpha \in X$ choose $\alpha^* \in \mathsf{Aut}(L^*)$, as in the previous paragraph, and let $X^* = \{\alpha^* \mid \alpha \in X\}$. Suppose that $\{b_1, \ldots, b_d\}$ is a minimal generating set for K and that $\{c_1, \ldots, c_l\}$ is a basis for the last non-trivial term of the lower central series of K. For $i \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, l\}$ let $\psi_{i,j}$ denote the automorphism that maps b_i to $b_i + c_j$ and fixes $b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_d$. Then $X^* \cup \{\psi_{i,j} \mid i = 1, \ldots, d,\ j = 1, \ldots, l\}$ is a generating set for $\mathsf{Aut}(K)$.

A similar approach to compute the automorphism group of a soluble Lie algebra over a finite field is described in [Eick 04]. Our method is, however, more efficient for nilpotent Lie algebras.

The cover of a finite-dimensional nilpotent Lie algebra L can be constructed in a way that is very similar to the construction of the p-covering group of a finite p-group. A good description of this procedure can be found in [Newman et al. 98]. Suppose that L has class c, and hence the lower central series is

$$\begin{aligned} L &= \gamma_1(L) > \gamma_2(L) \\ &= L' > \gamma_3(L) > \cdots > \gamma_c(L) > \gamma_{c+1}(L) \\ &= 0. \end{aligned}$$

We say that a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ for L is *compatible with the lower central series* if there are indices $1 = i_1 < i_2 < \cdots < i_{c-1} < i_c \leqslant n$ such that $\{b_{i_k}, \ldots, b_n\}$ is a basis of $\gamma_k(L)$ for $k \in \{1, \ldots, c\}$.

Suppose that $b_i \in \gamma_j(L) \setminus \gamma_{j+1}(L)$. Then we say that the number j is the *weight* of b_i. We call a basis $\mathcal{B}$ a *nilpotent basis* if the following hold.

(i) The basis $\mathcal{B}$ is compatible with the lower central series.

(ii) For each $b_i \in \mathcal{B}$ with weight $w \geqslant 2$ there are $b_{j_1}, b_{j_2} \in \mathcal{B}$ with weight 1 and $w-1$, respectively, such that $b_i = [b_{j_1}, b_{j_2}]$. The product $[b_{j_1}, b_{j_2}]$ is called the *definition* of b_i.

If $\{b_1, \ldots, b_n\}$ is a nilpotent basis for a Lie algebra L, then there are coefficients $\alpha_{i,j}^k$ for $i < j < k$ such that

$$[b_i, b_j] = \sum_{k=j+1}^{n} \alpha_{i,j}^k b_k. \tag{2–2}$$

It is routine to see that every finitely generated nilpotent Lie algebra has a nilpotent basis.

Suppose that $\mathcal{B} = \{b_1, \ldots, b_n\}$ is a nilpotent basis for a d-generator nilpotent Lie algebra and the $\alpha_{i,j}^k$ are as in (2–2). We build a presentation for the Lie algebra L^* as follows. The set $\{b_{d+1}, \ldots, b_n\}$ is a basis for L'. If, for some $i < j$, the product $[b_i, b_j]$ is not a definition and $w(b_i) + w(b_j) \leqslant c+1$, then we modify the product in (2–2) by introducing a central basis element $b_{i,j}$ and set

$$[b_i, b_j] = \sum_{k=j+1}^{n} \alpha_{i,j}^k b_k + b_{i,j}.$$

We introduce the new basis elements so that different nondefining products $[b_i, b_j]$ are augmented with different basis elements $b_{i,j}$. We also ensure that the newly introduced basis elements $b_{i,j}$ are central. If a product $[b_i, b_j]$ is a definition of b_k, then the product $[b_i, b_j] = b_k$ is not modified. Similarly if $w(b_i) + w(b_j) > c+1$ then $[b_i, b_j]$ is left untouched. This way we obtain an anticommutative algebra $\hat{L}$ with basis $\{b_1, \ldots, b_d\} \cup \{b_{i,j}\}$ where the product of two basis elements is defined using the rules above. We compute the ideal J in $\hat{L}$ generated by the set of elements

$$\{[[b_i, b_j], b_k] + [[b_j, b_k], b_i] + [[b_k, b_i], b_j] \mid i,\ j,\ k \in \{1, \ldots, n\}\}.$$

Then we obtain the cover L^* as $\hat{L}/J$.

It is possible to make this basic algorithm to compute the cover more effective. In practice, we only introduce a new basis element for products of the form $[b_i, b_j]$ where $w(b_i) = 1$ and compute products $[b_i, b_j]$ with $w(b_i) > 1$ using the Jacobi identity. We also use the result in [Havas et al. 90] that J is already generated by the set of element

$$\{[[b_i, b_j], b_k] + [[b_j, b_k], b_i] + [[b_k, b_i], b_j] \mid i \in \{1, \ldots, d\},\ i < j < k \leqslant n\}.$$

The proof that the resulting Lie algebra is isomorphic to L^* is completely analogous to that in the p-group case; see [Newman et al. 98] for details.

3. SOME CLASSIFICATIONS OF SMALL LIE ALGEBRAS

In theory, it is possible to use the procedures described in Section 2 to classify nilpotent $\mathbb{F}_q$-Lie algebras of a given dimension using recursion. It is clear that there is a unique one-dimensional nilpotent Lie algebra over each field $\mathbb{F}_q$; the automorphism group of this algebra is naturally isomorphic to the multiplicative group $\mathbb{F}_q^*$. Suppose that we have a complete and irredundant list of nilpotent $\mathbb{F}_q$-Lie algebras of dimension $1, \ldots, n-1$ for some $n \geqslant 2$ and we are also given the automorphism groups of these algebras. Up to isomorphism, there is exactly one n-dimensional abelian $\mathbb{F}_q$-Lie algebra. Each nonabelian nilpotent Lie algebra with dimension n is an immediate descendant of a smaller-dimensional Lie algebra. Hence, for each algebra L with dimension m in the precomputed list we construct the Lie cover L^*, the multiplicator M, and $\mathsf{Aut}(L)\varrho$ where ϱ is the representation in (2–1). Then, using the fact that M is finite, we construct the orbits of the $(m + \dim M - n)$-dimensional allowable subspaces under the finite linear group $\mathsf{Aut}(L)\varrho$. For each orbit representative U, we construct the quotient L^*/U and the stabiliser of U in $\mathsf{Aut}(L)$ under the

Type	Number of this Type	Type	Number of this Type
[6][6]	1	[3, 2, 1][2]	3
[5, 1][4]	1	[3, 2, 1][1]	3
[5, 1][2]	1	[3, 1, 2][3]	1
[4, 2][3]	1	[3, 1, 2][2]	3
[4, 2][2]	3	[3, 1, 1, 1][2]	2
[4, 1, 1][3]	1	[3, 1, 1, 1][1]	4
[4, 1, 1][2]	1	[2, 1, 2, 1][2]	1
[4, 1, 1][1]	1	[2, 1, 2, 1][1]	2
[3, 3][3]	1	[2, 1, 1, 1, 1][1]	6

TABLE 1. The nilpotent Lie algebras of dimension 6 over $\mathbb{F}_2$.

Type	Number of this Type	Type	Number of this Type
[6][6]	1	[3, 2, 1][2]	3
[5, 1][4]	1	[3, 2, 1][1]	3
[5, 1][2]	1	[3, 1, 2][3]	1
[4, 2][3]	1	[3, 1, 2][2]	3
[4, 2][2]	3	[3, 1, 1, 1][2]	2
[4, 1, 1][3]	1	[3, 1, 1, 1][1]	3
[4, 1, 1][2]	1	[2, 1, 2, 1][2]	1
[4, 1, 1][1]	1	[2, 1, 2, 1][1]	2
[3, 3][3]	1	[2, 1, 1, 1, 1][1]	5

TABLE 2. The nilpotent Lie algebras with dimension 6 over $\mathbb{F}_3$ and $\mathbb{F}_5$.

Type	Number of this Type	Type	Number of this Type	Type	Number of this Type	Type	Number of this Type
[7][7]	1	[5, 1, 1][1]	1	[4, 1, 1, 1][2]	4	[3, 1, 2, 1][2]	11
[6, 1][5]	1	[4, 3][4]	1	[4, 1, 1, 1][1]	5	[3, 1, 2, 1][1]	8
[6, 1][3]	1	[4, 3][3]	5	[3, 3, 1][3]	1	[3, 1, 1, 1, 1][2]	6
[6, 1][1]	1	[4, 2, 1][3]	3	[3, 3, 1][2]	3	[3, 1, 1, 1, 1][1]	21
[5, 2][4]	1	[4, 2, 1][2]	12	[3, 3, 1][1]	2	[2, 1, 2, 2][2]	3
[5, 2][3]	3	[4, 2, 1][1]	9	[3, 2, 2][3]	2	[2, 1, 2, 1, 1][2]	4
[5, 2][2]	2	[4, 1, 2][4]	1	[3, 2, 2][2]	21	[2, 1, 2, 1, 1][1]	14
[5, 1, 1][4]	1	[4, 1, 2][3]	3	[3, 2, 1, 1][2]	9	[2, 1, 1, 1, 2][2]	4
[5, 1, 1][3]	1	[4, 1, 2][2]	5	[3, 2, 1, 1][1]	13	[2, 1, 1, 1, 1, 1][1]	15
[5, 1, 1][2]	1	[4, 1, 1, 1][3]	2	[3, 1, 2, 1][3]	1		

TABLE 3. The nilpotent Lie algebras of dimension 7 over $\mathbb{F}_2$.

representation ϱ. The automorphism group of L^*/U can now be constructed as described in Section 2. The collection of all Lie algebras L^*/U so obtained is a complete and irredundant list of the isomorphism types of the nonabelian nilpotent Lie algebras with dimension n.

Suppose that L is a finite-dimensional nilpotent Lie algebra. Let c denote the class of L. Then the *type* of the Lie algebra L is

$$[\dim L/L', \dim L'/\gamma_3(L), \ldots, \dim \gamma_c(L)][\dim Z(L)].$$

It is well known that, over an arbitrary field, there is just one nilpotent Lie algebra with dimension 1 and 2. There are two nilpotent Lie algebras with dimension 3 (the types are $[3][3]$ and $[2,1][1]$), and three nilpotent Lie algebras with dimension 4 (the types are $[4][4]$, $[3,1][2]$, $[2,1,1][1]$). The number of isomorphism types of five-dimensional nilpotent Lie algebras is nine over all fields; see [Goze and Khakimdjanov 96]. Up to isomorphism, there is exactly one Lie algebra with each of the following types: $[5][5]$, $[4,1][3]$, $[4,1][1]$, $[3,2][2]$, $[3,1,1][2]$, $[3,1,1][1]$, $[2,1,2][2]$; there are two Lie algebras with type $[2,1,1,1][1]$.

The number of six-dimensional nilpotent Lie algebras depends on the underlying field. Using the GAP 4 package described in Section 4, we obtained 36 isomorphism classes of six-dimensional nilpotent Lie algebras over $\mathbb{F}_2$, and 34 such classes over $\mathbb{F}_3$ and $\mathbb{F}_5$. It is mentioned in Wilkinson's paper [Wilkinson 88] that the number of isomorphism classes of finite p-groups with order p^6 and exponent p is 34 whenever $p \geqslant 7$. Though there are several mistakes in the main part of Wilkinson's paper (see discussion after Theorem 1 in [O'Brien and Vaughan-Lee 05]), this particular claim appears to be true, as verified in [O'Brien et al. 04]. Using the Lazard correspondence [O'Brien et al. 04, Section 4] we obtain that, for $p \geqslant 7$, there are 34 pairwise nonisomorphic six-dimensional nilpotent $\mathbb{F}_p$-Lie algebras. In fact, the computation referred to above implies that this claim holds for $p = 3,\ 5$. The number of six-dimensional, nilpotent $\mathbb{F}_2$-Lie algebras for each possible type can be found in Table 1, while Table 2 contains the same information over $\mathbb{F}_3$ and $\mathbb{F}_5$. One can read off, for instance, from these tables that there are six pairwise nonisomorphic nilpotent Lie algebras with type $[2,1,1,1,1][1]$ over $\mathbb{F}_2$ and there are only five such Lie algebras over $\mathbb{F}_3$ and $\mathbb{F}_5$.

Shedler's thesis [Shedler 64] contains a classification of six-dimensional nilpotent Lie algebras over any field. However, this work is unpublished and as [Gong 98]

Type	Number of this Type	Type	Number of this Type	Type	Number of this Type	Type	Number of this Type
[7][7]	1	[5, 1, 1][1]	1	[4, 1, 1, 1][2]	3	[3, 1, 2, 1][2]	10
[6, 1][5]	1	[4, 3][4]	1	[4, 1, 1, 1][1]	5	[3, 1, 2, 1][1]	12
[6, 1][3]	1	[4, 3][3]	5	[3, 3, 1][3]	1	[3, 1, 1, 1, 1][2]	5
[6, 1][1]	1	[4, 2, 1][3]	3	[3, 3, 1][2]	3	[3, 1, 1, 1, 1][1]	17
[5, 2][4]	1	[4, 2, 1][2]	12	[3, 3, 1][1]	5	[2, 1, 2, 2][2]	3
[5, 2][3]	3	[4, 2, 1][1]	9	[3, 2, 2][3]	2	[2, 1, 2, 1, 1][2]	4
[5, 2][2]	2	[4, 1, 2][4]	1	[3, 2, 2][2]	21	[2, 1, 2, 1, 1][1]	16
[5, 1, 1][4]	1	[4, 1, 2][3]	3	[3, 2, 1, 1][2]	8	[2, 1, 1, 1, 2][2]	3
[5, 1, 1][3]	1	[4, 1, 2][2]	5	[3, 2, 1, 1][1]	14	[2, 1, 1, 1, 1, 1][1]	11
[5, 1, 1][2]	1	[4, 1, 1, 1][3]	2	[3, 1, 2, 1][3]	1		

TABLE 4. The nilpotent Lie algebras with dimension 7 over $\mathbb{F}_3$.

Type	Number of this Type	Type	Number of this Type	Type	Number of this Type	Type	Number of this Type
[7][7]	1	[5, 1, 1][1]	1	[4, 1, 1, 1][2]	3	[3, 1, 2, 1][2]	10
[6, 1][5]	1	[4, 3][4]	1	[4, 1, 1, 1][1]	5	[3, 1, 2, 1][1]	16
[6, 1][3]	1	[4, 3][3]	5	[3, 3, 1][3]	1	[3, 1, 1, 1, 1][2]	5
[6, 1][1]	1	[4, 2, 1][3]	3	[3, 3, 1][2]	3	[3, 1, 1, 1, 1][1]	16
[5, 2][4]	1	[4, 2, 1][2]	12	[3, 3, 1][1]	6	[2, 1, 2, 2][2]	3
[5, 2][3]	3	[4, 2, 1][1]	9	[3, 2, 2][3]	2	[2, 1, 2, 1, 1][2]	4
[5, 2][2]	2	[4, 1, 2][4]	1	[3, 2, 2][2]	21	[2, 1, 2, 1, 1][1]	18
[5, 1, 1][4]	1	[4, 1, 2][3]	3	[3, 2, 1, 1][2]	8	[2, 1, 1, 1, 2][2]	3
[5, 1, 1][3]	1	[4, 1, 2][2]	5	[3, 2, 1, 1][1]	18	[2, 1, 1, 1, 1, 1][1]	13
[5, 1, 1][2]	1	[4, 1, 1, 1][3]	2	[3, 1, 2, 1][3]	1		

TABLE 5. The nilpotent Lie algebras with dimension 7 over $\mathbb{F}_5$.

Type	Number of this Type	Type	Number of this Type	Type	Number of this Type	Type	Number of this Type
[8][8]	1	[5, 2, 1][1]	13	[4, 2, 1, 1][1]	54	[3, 2, 1, 1, 1][1]	88
[7, 1][6]	1	[5, 1, 2][5]	1	[4, 1, 2, 1][4]	1	[3, 1, 2, 2][3]	3
[7, 1][4]	1	[5, 1, 2][4]	3	[4, 1, 2, 1][3]	11	[3, 1, 2, 2][2]	37
[7, 1][2]	1	[5, 1, 2][3]	5	[4, 1, 2, 1][2]	48	[3, 1, 2, 1, 1][3]	4
[6, 2][5]	1	[5, 1, 2][2]	14	[4, 1, 2, 1][1]	26	[3, 1, 2, 1, 1][2]	71
[6, 2][4]	3	[5, 1, 1, 1][4]	2	[4, 1, 1, 1, 1][3]	6	[3, 1, 2, 1, 1][1]	82
[6, 2][3]	2	[5, 1, 1, 1][3]	4	[4, 1, 1, 1, 1][2]	21	[3, 1, 1, 1, 2][3]	4
[6, 2][2]	8	[5, 1, 1, 1][2]	5	[4, 1, 1, 1, 1][1]	39	[3, 1, 1, 1, 2][2]	39
[6, 1, 1][5]	1	[5, 1, 1, 1][1]	5	[3, 3, 2][4]	1	[3, 1, 1, 1, 1, 1][2]	15
[6, 1, 1][4]	1	[4, 4][4]	4	[3, 3, 2][3]	15	[3, 1, 1, 1, 1, 1][1]	80
[6, 1, 1][3]	1	[4, 3, 1][4]	1	[3, 3, 2][2]	77	[2, 1, 2, 3][3]	1
[6, 1, 1][2]	1	[4, 3, 1][3]	29	[3, 3, 1, 1][3]	3	[2, 1, 2, 2, 1][2]	26
[6, 1, 1][1]	1	[4, 3, 1][2]	51	[3, 3, 1, 1][2]	13	[2, 1, 2, 2, 1][1]	20
[5, 3][5]	1	[4, 3, 1][1]	25	[3, 3, 1, 1][1]	6	[2, 1, 2, 1, 2][3]	2
[5, 3][4]	5	[4, 2, 2][4]	2	[3, 2, 3][3]	28	[2, 1, 2, 1, 2][2]	24
[5, 3][3]	16	[4, 2, 2][3]	48	[3, 2, 2, 1][3]	11	[2, 1, 2, 1, 1, 1][2]	12
[5, 2, 1][4]	3	[4, 2, 2][2]	209	[3, 2, 2, 1][2]	164	[2, 1, 2, 1, 1, 1][1]	24
[5, 2, 1][3]	12	[4, 2, 1, 1][3]	9	[3, 2, 2, 1][1]	84	[2, 1, 1, 1, 2, 1][2]	11
[5, 2, 1][2]	35	[4, 2, 1, 1][2]	59	[3, 2, 1, 1, 1][2]	49	[2, 1, 1, 1, 1, 1, 1][1]	47

TABLE 6. The nilpotent Lie algebras with dimension 8 over $\mathbb{F}_2$.

points out, contains several mistakes. There exist classifications of six-dimensional nilpotent Lie algebras over infinite fields; see for instance [Goze and Khakimdjanov 96].

A classification of finite p-groups with exponent p and order p^7 was obtained by Wilkinson [Wilkinson 88]. If $p \geqslant 7$ then, by the Lazard correspondence, the number of finite p-groups with exponent p and order p^7 coincides with the number of seven-dimensional nilpotent $\mathbb{F}_p$-Lie algebras. According to Wilkinson this number is $173 + 7p + 2\gcd(p-1, 3)$, but as [O'Brien and Vaughan-

Type	Number of this Type	Type	Number of this Type	Type	Number of this Type	Type	Number of this Type
[9][9]	1	[5, 3, 1][5]	1	[4, 2, 2, 1][4]	11	[3, 2, 2, 1, 1][1]	1282
[8, 1][7]	1	[5, 3, 1][4]	29	[4, 2, 2, 1][3]	402	[3, 2, 1, 1, 2][3]	33
[8, 1][5]	1	[5, 3, 1][3]	327	[4, 2, 2, 1][2]	2859	[3, 2, 1, 1, 2][2]	325
[8, 1][3]	1	[5, 3, 1][2]	318	[4, 2, 2, 1][1]	713	[3, 2, 1, 1, 1, 1][2]	163
[8, 1][1]	1	[5, 3, 1][1]	133	[4, 2, 1, 1, 1][3]	49	[3, 2, 1, 1, 1, 1][1]	435
[7, 2][6]	1	[5, 2, 2][5]	2	[4, 2, 1, 1, 1][2]	487	[3, 1, 2, 3][4]	1
[7, 2][5]	3	[5, 2, 2][4]	48	[4, 2, 1, 1, 1][1]	565	[3, 1, 2, 3][3]	21
[7, 2][4]	2	[5, 2, 2][3]	502	[4, 1, 2, 2][4]	3	[3, 1, 2, 2, 1][3]	26
[7, 2][3]	8	[5, 2, 2][2]	799	[4, 1, 2, 2][3]	37	[3, 1, 2, 2, 1][2]	622
[7, 2][2]	6	[5, 2, 1, 1][4]	9	[4, 1, 2, 2][2]	258	[3, 1, 2, 2, 1][1]	302
[7, 1, 1][6]	1	[5, 2, 1, 1][3]	59	[4, 1, 2, 1, 1][4]	4	[3, 1, 2, 1, 2][4]	2
[7, 1, 1][5]	1	[5, 2, 1, 1][2]	231	[4, 1, 2, 1, 1][3]	71	[3, 1, 2, 1, 2][3]	79
[7, 1, 1][4]	1	[5, 2, 1, 1][1]	129	[4, 1, 2, 1, 1][2]	463	[3, 1, 2, 1, 2][2]	353
[7, 1, 1][3]	1	[5, 1, 2, 1][5]	1	[4, 1, 2, 1, 1][1]	318	[3, 1, 2, 1, 1, 1][3]	12
[7, 1, 1][2]	1	[5, 1, 2, 1][4]	11	[4, 1, 1, 1, 2][4]	4	[3, 1, 2, 1, 1, 1][2]	230
[7, 1, 1][1]	1	[5, 1, 2, 1][3]	48	[4, 1, 1, 1, 2][3]	39	[3, 1, 2, 1, 1, 1][1]	314
[6, 3][6]	1	[5, 1, 2, 1][2]	180	[4, 1, 1, 1, 2][2]	191	[3, 1, 1, 1, 2, 1][3]	11
[6, 3][5]	5	[5, 1, 2, 1][1]	37	[4, 1, 1, 1, 1, 1][3]	15	[3, 1, 1, 1, 2, 1][2]	181
[6, 3][4]	16	[5, 1, 1, 1, 1][4]	6	[4, 1, 1, 1, 1, 1][2]	80	[3, 1, 1, 1, 1, 1, 1][2]	47
[6, 3][3]	122	[5, 1, 1, 1, 1][3]	21	[4, 1, 1, 1, 1, 1][1]	213	[3, 1, 1, 1, 1, 1, 1][1]	423
[6, 2, 1][5]	3	[5, 1, 1, 1, 1][2]	39	[3, 3, 3][4]	16	[2, 1, 2, 3, 1][3]	5
[6, 2, 1][4]	12	[5, 1, 1, 1, 1][1]	47	[3, 3, 3][3]	642	[2, 1, 2, 3, 1][2]	10
[6, 2, 1][3]	35	[4, 5][5]	2	[3, 3, 2, 1][4]	2	[2, 1, 2, 2, 2][3]	19
[6, 2, 1][2]	70	[4, 4, 1][4]	19	[3, 3, 2, 1][3]	104	[2, 1, 2, 2, 2][2]	170
[6, 2, 1][1]	18	[4, 4, 1][3]	77	[3, 3, 2, 1][2]	808	[2, 1, 2, 2, 1, 1][2]	60
[6, 1, 2][6]	1	[4, 4, 1][2]	127	[3, 3, 2, 1][1]	316	[2, 1, 2, 2, 1, 1][1]	98
[6, 1, 2][5]	3	[4, 4, 1][1]	54	[3, 3, 1, 1, 1][3]	16	[2, 1, 2, 1, 2, 1][3]	6
[6, 1, 2][4]	5	[4, 3, 2][5]	1	[3, 3, 1, 1, 1][2]	86	[2, 1, 2, 1, 2, 1][2]	62
[6, 1, 2][3]	14	[4, 3, 2][4]	55	[3, 3, 1, 1, 1][1]	76	[2, 1, 2, 1, 2, 1][1]	16
[6, 1, 2][2]	25	[4, 3, 2][3]	814	[3, 2, 4][4]	12	[2, 1, 2, 1, 1, 1, 1][2]	40
[6, 1, 1, 1][5]	2	[4, 3, 2][2]	2510	[3, 2, 3, 1][3]	258	[2, 1, 2, 1, 1, 1, 1][1]	124
[6, 1, 1, 1][4]	4	[4, 3, 1, 1][4]	3	[3, 2, 3, 1][2]	429	[2, 1, 1, 1, 2, 2][2]	7
[6, 1, 1, 1][3]	5	[4, 3, 1, 1][3]	131	[3, 2, 3, 1][1]	203	[2, 1, 1, 1, 2, 1, 1][2]	45
[6, 1, 1, 1][2]	5	[4, 3, 1, 1][2]	396	[3, 2, 2, 2][3]	44	[2, 1, 1, 1, 2, 1, 1][1]	18
[6, 1, 1, 1][1]	5	[4, 3, 1, 1][1]	296	[3, 2, 2, 2][2]	908	[2, 1, 1, 1, 1, 1, 2][2]	32
[5, 4][5]	4	[4, 2, 3][4]	28	[3, 2, 2, 1, 1][3]	71	[2, 1, 1, 1, 1, 1, 1, 1][1]	124
[5, 4][4]	53	[4, 2, 3][3]	1377	[3, 2, 2, 1, 1][2]	1296		

TABLE 7. The nilpotent Lie algebras with dimension 9 over $\mathbb{F}_2$.

Lee 05] points out there are several mistakes in Wilkinson's calculations and the correct number is

$$174 + 7p + 2\gcd(p-1, 3). \tag{3–1}$$

Computer calculations with the GAP 4 package described in Section 4 show that the number of seven-dimensional nilpotent Lie algebras over $\mathbb{F}_2$, $\mathbb{F}_3$, and $\mathbb{F}_5$ is 202, 199, and 211, respectively; the number of Lie algebras for each possible type is presented in Tables 3–5. This calculation also shows that (3–1) is valid over $\mathbb{F}_5$. Michael Vaughan-Lee independently obtained a classification of nilpotent Lie rings with order p^7, and the numbers above were also confirmed by his computation.

For some classifications of seven-dimensional nilpotent Lie algebras over infinite fields we refer to [Ancochéa-Bermúdez and Goze 89, Romdhani 89, Gong 98, Goze and Remm 04].

The author's GAP 4 program was also used the obtain a classification of nilpotent $\mathbb{F}_2$-Lie algebras with dimension 8 and 9. The total number of such Lie algebras is $1,831$ and $27,073$. More detailed information about the possible types can be found in Tables 6–8.

The classifications of nilpotent Lie algebras in Theorem 1.1 are available in GAP 4 format on the author's web site http://www.sztaki.hu/~schneider/Research/SmallLie/.

4. IMPLEMENTATION OF THE ALGORITHMS

Implementations of all procedures described in Section 2 are available in the GAP 4 computer algebra package Sophus. This program can be freely downloaded from the author's web page http://www.sztaki.hu/~schneider/Research/Sophus/. The current version of Sophus contains

(i) a program to compute the cover of a nilpotent Lie algebra;

(ii) a program to compute the automorphism group of a nilpotent Lie algebra;

(iii) a program to compute the set of immediate descendants of a nilpotent Lie algebra; and

(iv) a program to check if two nilpotent Lie algebras are isomorphic.

The full implementation of these procedures is nearly $4,000$ lines long.

The classifications presented in the previous section were computed on several Pentium 4 computers between 1.7- and 2.5-GHz CPU speed and 1-2 GB memory. The computation of the list of the $\mathbb{F}_2$-Lie algebras with dimension up to 6 takes only a few seconds, while those for dimension 7 take about three minutes.

Determining the remaining classes of nilpotent Lie algebras in Theorem 1.1 is more complicated and requires human intervention. Most of the descendant computations for the eight- and nine-dimensional Lie algebras over $\mathbb{F}_2$ could easily be carried out. However, computing the eight-dimensional descendants of the six-dimensional abelian Lie algebra requires finding representatives of the $\mathsf{GL}(6,2)$-orbits on the set of 178,940,587 allowable subspaces under the action in (2–1). In the computation of the nine-dimensional descendants of the seven-dimensional abelian Lie algebra, the number of allowable subspaces is 733,006,703,275. In such cases we applied the Cauchy-Frobenius Lemma (see [Eick and O'Brien 99, Section 4]) to predict the number of descendants. Then we used either the ideas of O'Brien's extended algorithm presented in [O'Brien 91, Section 2] or the existing classification of 2-groups of order up to 2^9. In the latter case we constructed Lie algebras associated with the 2-central series filtration of the groups, tested them for isomorphism, and eliminated the duplicates.

For computing the seven-dimensional descendants of the five-dimensional abelian Lie algebras over $\mathbb{F}_3$ and $\mathbb{F}_5$, we used the result of the corresponding computation over $\mathbb{F}_2$. The Cauchy-Frobenius Lemma implies that the number of these Lie algebras is the same over $\mathbb{F}_2$, $\mathbb{F}_3$, and $\mathbb{F}_5$. It is possible to interpret the structure constants table of the $\mathbb{F}_2$-Lie algebras over $\mathbb{F}_3$ and $\mathbb{F}_5$ and obtain the required lists. Then the algebras in these lists were tested for nonisomorphism.

The most difficult problem when computing the immediate descendants of a nilpotent Lie algebra is computing the orbits of the allowable subspaces under the representation (2–1). Further, for computing the automorphism group of an immediate descendant, the stabiliser of an allowable subspace must also be calculated; see Section 2. These orbit-stabiliser computations were carried out adopting the procedures described in [Eick et al. 02].

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