# The Diophantine Equation $xy + yz + zx = n$ and Indecomposable Binary Quadratic Forms

Meinhard Peters

There are 18 (and possibly 19) integers that are not of the form $xy+yz+xz$ with positive integers $x, y, z$. The same 18 integers appear as exceptional discriminants for which no indecomposable positive definite binary quadratic form exists. We show that the two problems are equivalent.

Recently Borwein and Choi [Borwein and Choi 00], and independently Le [Le 98], have shown that the Diophantine equation $xy + yz + zx = n$ has solutions $x, y, z$ with $x, y, z \geq 1$ for all natural numbers $n$ with the exception of 1, 2, 4, 6, 10, 18, 22, 30, 42, 58, 70, 78, 102, 130, 190, 210, 330, 462 and possibly one further number $> 2 \cdot 10^{11}$. The same numbers appear as exceptional discriminants for which no indecomposable positive definite binary quadratic form exists, as shown in [Zhu and Shao 88] and [Peters 91]. We show the equivalence of the two problems.

An indecomposable binary positive definite quadratic form with discrimant $d$ (in the terminology of O'Meara [O'Meara 63]) exists iff $d = ac - b^2$ with positive integers $a, b, c$ with the reduction conditions $2b \leq a \leq c$. In other words: $d$ is represented by the ternary quadratic form $xy - z^2$ with positive integers $x, y, z$ with $2z \leq x \leq y$. We show that this is equivalent to a representation of $d$ by $xy + yz + zx$ with positive integers. The matrices of the ternary forms

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

are equivalent by means of the transformation matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Explicitly we have the following: if $xy + yz + xz = d$ with $1 \leq z \leq x \leq y$, then $(x+z)(y+z) - z^2 = d$ with $1 \leq$

2000 AMS Subject Classification: Primary 11E12, 11E96; Secondary 11D09

Keywords: Binary quadratic forms, Diophantine equations

$2z \leq x+z \leq y+z$. On the other hand: if $xy - z^2 = d$ with $1 \leq 2z \leq x \leq y$, then $(x-z)(y-z)+(y-z)z+(x-z)z = d$ with $x - z \geq 1, y - z \geq 1$.

Thus, we have seen the equivalence of both problems and it remains the open question of the possible further exception $> 2 \cdot 10^{11}$. The numbers in question are—if we exclude 1, 4, and 18—the disjoint discriminants of the second type; see [Borwein and Borwein 87] and N. J. A. Sloane's *On-Line Encyclopedia of Integer Sequences*: www.research.att.com/~njas/sequences/index.html, sequence A034168.

## REFERENCES

[Borwein and Borwein 87] J. M. Borwein and P. B. Borwein. *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity*, Canadian Math. Soc. Series of Monographs and Advanced Texts, 4. New York: Wiley, 1987.

[Borwein and Choi 00] J. Borwein and K. K. S. Choi. "On the Representations of $xy + yz + zx$." *Exp. Math.* 9:1 (2000), 153–158.

[Le 98] Maohua Le. "A Note on Positive Integer Solutions of the Equation $xy + yz + zx = n$." *Publ. Math. Debrecen* 52 (1998), 159–165.

[O'Meara 63] O. T. O'Meara. *Introduction to Quadratic Forms*. Berlin-Heidelberg-New York: Springer-Verlag, 1963.

[Peters 91] M. Peters. "Indecomposable Binary Quadratic Forms." *Arch. Math.* 57 (1991), 467–468.

[Zhu and Shao 88] F. Z. Zhu and Y. Y. Shao. "On the Construction of Indecomposable Positive Definite Quadratic Forms Over $\mathbb{Z}$." *Chinese Ann. Math. Ser. 13(1)* 9 (1988), 79–94.

Meinhard Peters, Mathematisches Institut, Universität Münster, Einsteinstr. 62, 48149 Münster, Germany (petersm@math.uni-muenster.de)

Received April 20, 2004; accepted April 22, 2004.