

Computing Special Values of Motivic L -Functions

Tim Dokchitser

CONTENTS

1. Introduction
2. Motivic L -Functions
3. Computing $\phi(t)$ and $\frac{\partial^k}{\partial s^k} G_s(t)$ for t Small
4. Computing $\phi(t)$ and $\frac{\partial^k}{\partial s^k} G_s(t)$ for t Large
5. Implementation Notes
6. L -Functions with Unknown Invariants

Acknowledgments

References

We present an algorithm to compute values $L(s)$ and derivatives $L^{(k)}(s)$ of L -functions of motivic origin numerically to required accuracy. Specifically, the method applies to any L -series whose Γ -factor is of the form $A^s \prod_{i=1}^d \Gamma(\frac{s+\lambda_i}{2})$ with d arbitrary and complex λ_j , not necessarily distinct. The algorithm relies on the known (or conjectural) functional equation for $L(s)$.

1. INTRODUCTION

Many L -series in number theory and algebraic geometry can be interpreted as L -series of motives over number fields. For instance, Riemann and Dedekind ζ -function, Dirichlet and Artin L -series, and L -series of elliptic curves are of this kind. They are all of the form $L(X, V, s)$ associated to $V = H^i(X)$ or a “motivic” subspace $V \subset H^i(X)$ of a projective algebraic variety X/K .

Given such series,

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \text{where } \operatorname{Re} s \gg 1, \quad (1-1)$$

standard conjectures state that $L(s)$ extends to a meromorphic function on the whole of \mathbb{C} and satisfies a functional equation of a predicted form. The Riemann hypothesis tells where the zeroes of $L(s)$ are supposed to be located, and numerous conjectures relate values of $L(s)$ at integers to arithmetic invariants of X . The Birch-Swinnerton-Dyer [Birch and Swinnerton-Dyer 63], Zagier [Zagier 91], Deligne-Beilinson-Scholl [Beilinson 86, Scholl 91], and Bloch-Kato [Bloch and Kato 90] conjectures are examples of these.

While the aforementioned conjectures remain unproved in the vast majority of cases, a lot of work has been done to provide numerical evidence for some of them in low-dimensional cases. This applies especially to the Riemann hypothesis for the Riemann ζ -function [van de Lune et al. 86], Dirichlet and Artin L -series [Davies and Haselgrove 61, Keiper 96, Lagarias and Odlyzko 79, Rubinstein 98, Tollis 97], and L -series

2000 AMS Subject Classification: Primary 11M99;
Secondary 14G10, 11G40, 11R42, 11M41, 11F66

Keywords: L -functions, Zeta-functions, motives, Meijer G -function.

$L(E, H^1, s)$ of elliptic curves [Fermigier 92]. Other well-studied cases are the Birch-Swinnerton-Dyer conjecture [Birch and Swinnerton-Dyer 63, Buhler et al. 85] for $L(E, H^1, s)|_{s=1}$ where E/\mathbb{Q} is an elliptic curve as well as various computations for modular forms and their symmetric powers.

To perform this kind of calculations one needs an efficient algorithm to compute numerically to required precision $L(s)$ (or, more precisely, its analytic continuation) for a given complex s . Such algorithms are usually based on writing $L(s)$ as a series in special functions associated to the inverse Mellin transform of the Γ -factor of $L(s)$. In the cases mentioned above these special functions are incomplete Gamma functions for $\dim V = 1$ (Riemann ζ -function, Dirichlet characters) and incomplete Bessel functions for $\dim V = 2$ (modular forms, elliptic curves).

In higher-dimensional cases ($\dim V > 2$) the situation is somewhat complicated by the fact that the special functions in question are rather general Meijer G -functions. It is possible to compute them using expansions at the origin but the resulting scheme is not very efficient due to cancellation problems. See Cohen's exposition in [Cohen 00, Section 10.3], which is based essentially on the work of Lavrik [Lavrik 68] and Tollis [Tollis 97].

The goal of this paper is threefold. First, we deduce analogous formulae to cover derivatives of L -functions. Second, for the special functions in question, we deduce asymptotic expansions at infinity and the form of the associated continued fraction expansions. Using these results, we construct an empirical but efficient algorithm to compute arbitrary motivic L -functions and their derivatives. Finally, we discuss L -functions with partially unknown invariants.

The scheme presented here was implemented as a PARI script [Dokchitser 02]. For an arbitrary motivic L -series for which meromorphic continuation and the functional equation are assumed, the algorithm numerically verifies the functional equation and allows one to compute the values $L(s)$ and derivatives $L^{(k)}(s)$ for complex s to predetermined precision. (The formulae described in the present paper can be used in any other environment that provides arbitrary precision arithmetic, complex numbers, Laurent series and the Taylor series expansion of the Γ -function.) The above PARI implementation also includes examples of computations with Riemann ζ -function, Dirichlet L -functions, Dedekind ζ -function, Shintani's ζ -function, L -series of modular forms, and those associated to curves C/\mathbb{Q} of genus 1, 2, 3, and 4.

The structure of the paper is as follows. In Section 2 we start with generalities on the invariants of L -functions and outline the algorithm. In Section 3 we deduce power series expansions of general Meijer G -functions required in the computations. Our approach here is standard and has been used in most of the algorithms to compute L -functions (e.g., [Lagarias and Odlyzko 79, Rubinstein 98, Tollis 97, van de Lune et al. 86]). These two sections are only included for the sake of completeness and to set up the notation. In Section 4 asymptotic expansions at infinity of the same special functions and associated continued fraction expansions are presented. Then, Section 5 summarises the algorithm and addresses implementation and accuracy issues. Finally, Section 6 contains some remarks on working with L -functions for which not all of the invariants are known.

2. MOTIVIC L -FUNCTIONS

Suppose we are given an L -series,

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \text{with } a_n \in \mathbb{C}.$$

We make the following three assumptions on $L(s)$:

Assumption 2.1. *The coefficients of $L(s)$ grow at most polynomially in n , that is $a_n = O(n^\alpha)$ for some $\alpha > 0$. Equivalently, the defining series for $L(s)$ converges for $\operatorname{Re} s$ sufficiently large.*

Assumption 2.2. *The series $L(s)$ admits a meromorphic continuation to the entire complex plane. There exist weight $w \geq 0$, sign $\epsilon = \pm 1$, real positive exponential factor A , and the Γ -factor*

$$\gamma(s) = \Gamma\left(\frac{s+\lambda_1}{2}\right) \cdots \Gamma\left(\frac{s+\lambda_d}{2}\right)$$

of dimension $d \geq 1$ and with Hodge numbers $\lambda_1, \dots, \lambda_d \in \mathbb{C}$, such that

$$L^*(s) = A^s \gamma(s) L(s)$$

satisfies a functional equation¹

$$L^*(s) = \epsilon L^*(w-s). \quad (2-1)$$

Assumption 2.3. *The function $L^*(s)$ has finitely many simple poles p_j with residues $r_j = \operatorname{res}_{s=p_j} L^*(s)$ and no other singularities.*

¹Functional equation may also involve two different L -functions, see Remark 2.7.

| $L(s)$ | Description | w | d | (λ_j) | N | ϵ | (p_j) |
|------------------------------------|--|-----|------|--|--------------|------------------|------------------------------------|
| $\zeta(s)$ | Riemann ζ -function | 1 | 1 | (0) | 1 | 1 | (0,1) |
| $L(\chi, s)$ $L(\bar{\chi}, s)$ | χ primitive Dirichlet character mod N | 1 | 1 | (0), $\chi(-1)=1$ (1), $\chi(-1)=-1$ | N | $ \epsilon = 1$ | |
| $\zeta(F, s)$ | Dedekind ζ -function $[F : \mathbb{Q}] = d$ | 1 | d | $(0, \dots, 0, 1, \dots, 1)$ $d - \sigma, \sigma$ times | $ \Delta_F $ | 1 | (0,1) |
| $L(f, s)$ | f modular form of weight k on $SL_2(\mathbb{Z})$ | k | 2 | (0, 1) | 1 | $(-1)^k$ | (0,k) |
| $L(f, s)$ | f cusp form of weight k on $SL_2(\mathbb{Z})$ | k | 2 | (0, 1) | 1 | $(-1)^k$ | |
| $L(f, s)$ | f Hecke cusp form of weight k on $\Gamma_0(N)$ | k | 2 | (0, 1) | N | ± 1 | |
| $L(E, s)$ | E/\mathbb{Q} elliptic curve of conductor N | 2 | 2 | (0, 1) | N | ± 1 | |
| $L(C, s)$ | C/\mathbb{Q} genus g curve of conductor N | 2 | $2g$ | $(0, \dots, 0, 1, \dots, 1)$ g, g times | N | ± 1 | |
| $\zeta_{\text{Sh}}(s)$ | Shintani's ζ -function | 1 | 4 | $(0, 1, \frac{1}{6}, -\frac{1}{6})$ | $2^4 3^3$ | 1 | $(0, \frac{1}{6}, \frac{5}{6}, 1)$ |

TABLE 1.

Remark 2.4. Even for motivic L -functions of general kind the parameters can often be restricted further. Usually, a_n lie in the ring of integers of a fixed number field (most often \mathbb{Z}), $A = \sqrt{N}/\pi^{d/2}$ (with conductor $N \in \mathbb{Z}$), and λ_k are integers (or even $\lambda_k \in \{0, 1\}$). Moreover, $L^*(s)$ is usually entire, and there is a product formula for $L(s)$. However, these additional assumptions do not simplify our algorithm. At the same time, there are some L -functions not of motivic origin (e.g., Shintani's ζ -function [Shintani 72]) to which the algorithm still applies, so we do not require more than stated above. The assumption that the poles of $L^*(s)$ are simple is not essential either (see discussion below).

Example 2.5. Table 1 contains some well-known examples of L -series satisfying our assumptions and their basic invariants. For every one of these L -functions, the exponential factor is of the form $A = \sqrt{N}/\pi^{d/2}$ with $N \in \mathbb{Z}$. In the second row, $L(\chi, s)$ satisfies a functional equation that involves the “dual” L -function associated to the complex conjugate character $L(\bar{\chi}, s)$ (see Remark 2.7 below). In the third row, Δ_F is the discriminant of F/\mathbb{Q} and σ is the number of pairs of complex embeddings.

For the latter (non-motivic) example see Shintani's original paper [Shintani 72]. For all the rest (and other motivic examples) see [Manin and Panchishkin 95], Chapter 4 and articles in [Janssen et al. 94] for references and additional information. For actual L -series computations in the above cases, see [Dokchitser 02].

Given an L -function that satisfies Assumptions 2.1–2.3, we would like to

- (a) give a numerical verification of the functional equation for $L(s)$,
- (b) determine the k th derivative $L^{(k)}(s_0)$ to necessary precision for a given $s_0 \in \mathbb{C}$ and an integer $k \geq 0$.

To this end define $\phi(t)$ to be the inverse Mellin transform of $\gamma(s)$, that is

$$\gamma(s) = \int_0^\infty \phi(t) t^s \frac{dt}{t}. \tag{2-2}$$

Henceforth, we let s denote a complex number and t a positive real (and not $\text{Im } s$ as is sometimes customary!). The function $\phi(t)$ exists (for real $t > 0$ that is) and decays exponentially for large t (see Section 3). In particular, the following sum converges exponentially fast:

$$\Theta(t) = \sum_{n=1}^\infty a_n \phi\left(\frac{nt}{A}\right). \tag{2-3}$$

This function is defined so that $L^*(s)$ becomes the Mellin transform of $\Theta(t)$,

$$\begin{aligned} \int_0^\infty \Theta(t) t^s \frac{dt}{t} &= \int_0^\infty \sum_{n=1}^\infty a_n \phi\left(\frac{nt}{A}\right) t^s \frac{dt}{t} \\ &= \sum_{n=1}^\infty a_n \int_0^\infty \phi\left(\frac{nt}{A}\right) t^s \frac{dt}{t} \\ &= \sum_{n=1}^\infty a_n \int_0^\infty \phi(t) \left(\frac{At}{n}\right)^s \frac{dt}{t} \\ &= A^s \sum_{n=1}^\infty \frac{a_n}{n^s} \gamma(s) = L^*(s). \end{aligned} \tag{2-4}$$

By the Mellin inversion formula,

$$\Theta(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L^*(s)t^{-s} ds, \quad \text{with } \operatorname{Re} c \gg 1,$$

if $c \in \mathbb{C}$ is chosen to lie to the right of the poles of $L^*(s)$. By the assumed functional equation (2-1) for $L^*(s)$,

$$\begin{aligned} \Theta(1/t) &= \int_{c-i\infty}^{c+i\infty} L^*(s) t^s ds \\ &= t^w \int_{c-i\infty}^{c+i\infty} \epsilon L^*(w-s) t^{s-w} ds \\ &= t^w \epsilon \int_{w-c-i\infty}^{w-c+i\infty} L^*(s) t^{-s} ds. \end{aligned}$$

This is almost an expression for $\epsilon t^w \Theta(t)$ except that the integration path lies to the *left* of the poles of $L^*(s)$. Shifting this path to the right, we pick up residues of $L^*(s)t^{-s}$ at the poles of $L^*(s)$. Consequently, $\Theta(t)$ enjoys the functional equation

$$\Theta(1/t) = \epsilon t^w \Theta(t) - \sum_j r_j t^{p_j}. \tag{2-5}$$

Note that the assumption that $L^*(s)$ has simple poles is inessential. If the poles are of higher order, the residues of $L^*(s)t^{-s}$ also involve some $\log t$ -terms. Then (2-5) and (2-9) below have extra terms, but this does not affect the reasoning elsewhere.

In Section 3 and Section 4 we describe how to compute $\phi(t)$ for $t > 0$ for a given Γ -factor $\gamma(s)$. Then, $\Theta(t)$ can be also effectively computed numerically since (2-3) converges exponentially fast.

Now we are ready to answer the first question, that of numerical verification of the functional equation for $L^*(s)$. Pick $t > 0$ and check that (2-5) holds numerically for this t . In fact, (2-5) holds for *all* t if and only if the functional equation (2-1) is satisfied. Note that having such a verification is useful when not all of the invariants of $L(s)$ are known (see Section 5).

Example 2.6. Let $L(s) = \zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ be the Riemann ζ -function. Then,

$$a_n \equiv 1, \quad w = 1, \quad \epsilon = 1, \quad A = \frac{1}{\sqrt{\pi}}, \quad d = 1, \quad \text{and } \gamma(s) = \Gamma\left(\frac{s}{2}\right).$$

We have

$$\phi(t) = 2e^{-t^2} \quad \text{and} \quad \Theta(t) = \sum_{n=1}^{\infty} 2e^{-\pi n^2 t^2}.$$

The function $L^*(s)$ has simple poles at $p_1 = 0$ and $p_2 = 1$ with residues $r_1 = 1$ and $r_2 = -1$, so the functional

equation for $\Theta(t)$ reads

$$\Theta(1/t) = t \Theta(t) - 1 + t. \tag{2-6}$$

In fact, applying Poisson's summation formula to $f(x) = e^{-\pi x^2}$ gives (2-6) and this *proves* the functional equation for $\zeta(s)$.

We now proceed to the second problem, that of computing $L(s)$ and $L^{(m)}(s)$. Fix $s \in \mathbb{C}$ and let

$$G_s(t) = t^{-s} \int_t^{\infty} \phi(x) x^s \frac{dx}{x}, \quad \text{for } t > 0. \tag{2-7}$$

Thus, $t^s G_s(t)$ is the incomplete Mellin transform of $\phi(t)$, and $\lim_{t \rightarrow 0} t^s G_s(t) = \gamma(s)$ is the original Γ -factor. As in the case of $\phi(t)$, the function $G_s(t)$ decays exponentially with t and can be effectively computed numerically (Sections 3 and 4).

Consider (2-4), which expresses $L^*(s)$ as the Mellin transform of $\Theta(t)$. Split the integral into two and apply the functional equation (2-5) to the second one:

$$\begin{aligned} L^*(s) &= \int_0^{\infty} \Theta(t) t^s \frac{dt}{t} = \int_1^{\infty} + \int_0^1 \\ &= \int_1^{\infty} \Theta(t) t^s \frac{dt}{t} + \int_1^{\infty} \Theta(1/t) t^{-s} \frac{dt}{t} \\ &= \int_1^{\infty} \Theta(t) t^s \frac{dt}{t} + \int_1^{\infty} \epsilon t^w \Theta(t) t^{-s} \frac{dt}{t} \\ &\quad - \int_1^{\infty} \sum_j r_j t^{p_j} t^{-s} \frac{dt}{t} \\ &= \int_1^{\infty} \Theta(t) t^s \frac{dt}{t} + \epsilon \int_1^{\infty} \Theta(t) t^{w-s} \frac{dt}{t} \\ &\quad + \sum_j \frac{r_j}{p_j - s}. \end{aligned} \tag{2-8}$$

By definition of $\Theta(t)$ and $G_s(x)$, the first integral can be rewritten:

$$\begin{aligned} \int_1^{\infty} \Theta(t) t^s \frac{dt}{t} &= \int_1^{\infty} \sum_{n=1}^{\infty} a_n \phi\left(\frac{nt}{A}\right) t^s \frac{dt}{t} \\ &= \sum_{n=1}^{\infty} a_n \int_1^{\infty} \phi\left(\frac{nt}{A}\right) t^s \frac{dt}{t} \\ &= \sum_{n=1}^{\infty} a_n \int_{n/A}^{\infty} \phi(t) \left(\frac{At}{n}\right)^s \frac{dt}{t} \\ &= \sum_{n=1}^{\infty} a_n G_s\left(\frac{n}{A}\right). \end{aligned}$$

The same applies to the second integral if s is replaced by $w - s$, and (2-8) becomes

$$L^*(s) = \sum_{n=1}^{\infty} a_n G_s\left(\frac{n}{A}\right) + \epsilon \sum_{n=1}^{\infty} a_n G_{w-s}\left(\frac{n}{A}\right) + \sum_j \frac{r_j}{p_j - s}.$$

This formula allows one to determine $L^*(s)$, and hence $L(s) = L^*(s)/\gamma(s)$, for a given $s \in \mathbb{C}$. Differentiating the above equation produces the formula for derivatives,

$$\begin{aligned} \frac{\partial^k}{\partial s^k} L^*(s) &= \sum_{n=1}^{\infty} a_n \frac{\partial^k}{\partial s^k} G_s\left(\frac{n}{A}\right) \\ &+ \epsilon \sum_{n=1}^{\infty} a_n \frac{\partial^k}{\partial s^k} G_{w-s}\left(\frac{n}{A}\right) + \sum_j \frac{k! r_j}{(p_j - s)^{k+1}}. \end{aligned} \quad (2-9)$$

It remains to explain how to compute the functions $\phi(t)$ and $\frac{\partial^k}{\partial s^k} G_s(t)$. This is the content of the next three sections.

Remark 2.7. We assumed that the functional equation (2-1) involves $L^*(s)$ both on the left-hand and on the right-hand side. In fact, for arbitrary motives the functional equation may be of a more general form,

$$L^*(s) = \epsilon \widehat{L}^*(w-s),$$

where

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \widehat{L}(s) = \sum_{n=1}^{\infty} \frac{\widehat{a}_n}{n^s}$$

are L -functions of “dual” motives. For instance, Dirichlet L -series associated to non-quadratic characters are of this nature. The sign ϵ is then an algebraic integer of absolute value 1. Clearly, our arguments go through in this more general case as well. The result is that (2-5) and (2-9) have to be simply replaced by

$$\Theta(1/t) = \epsilon t^w \widehat{\Theta}(t) - \sum_j \widehat{r}_j t^{p_j}$$

and

$$\begin{aligned} \frac{\partial^k}{\partial s^k} L^*(s) &= \sum_{n=1}^{\infty} a_n \frac{\partial^k}{\partial s^k} G_s\left(\frac{n}{A}\right) \\ &+ \epsilon \sum_{n=1}^{\infty} \widehat{a}_n \frac{\partial^k}{\partial s^k} \widehat{G}_{w-s}\left(\frac{n}{A}\right) + \sum_j \frac{k! \widehat{r}_j}{(\widehat{p}_j - s)^{k+1}}. \end{aligned}$$

Here, \widehat{A} , \widehat{p}_j , etc. are associated to $\widehat{L}(s)$ as A , p_j , etc. are to $L(s)$.

3. COMPUTING $\phi(t)$ AND $\frac{\partial^k}{\partial s^k} G_s(t)$ FOR t SMALL

Recall that

$$\gamma(s) = \Gamma\left(\frac{s+\lambda_1}{2}\right) \cdots \Gamma\left(\frac{s+\lambda_d}{2}\right) \quad (3-1)$$

and that $\phi(t)$ is defined as the inverse Mellin transform of $\gamma(s)$. By the Mellin inversion formula (see e.g.,

[Braaksma 64, Section 2]), $\phi(t)$ is given by the residue sum

$$\phi(t) = \sum_{z \in \mathbb{C}} \operatorname{res}_{s=z} \gamma(s) t^{-s}, \quad \text{for } t > 0. \quad (3-2)$$

Since $\Gamma(s)$ has simple poles at zero and negative integers, the function $\gamma(s)$ has a pole at $s \in \mathbb{C}$ if and only if $s = -\lambda_j - 2n$ for some j and an integer n . If $\lambda_j - \lambda_k \notin 2\mathbb{Z}$ for $j \neq k$, then all poles of $\gamma(s)$ are simple and

$$\begin{aligned} \operatorname{res}_{s=-\lambda_j-2n}(\gamma(s)t^{-s}) &= \\ &2 \frac{(-1)^n}{n!} t^{\lambda_j+2n} \prod_{k \neq j} \gamma\left(\frac{-\lambda_j-2n+\lambda_k}{2}\right). \end{aligned}$$

Hence, in this case (3-2) is of the form $\sum_j t^{\lambda_j} p_j(t^2)$ where $p_j(t)$ is a power series in t . The coefficients of $p_j(t)$ satisfy a simple linear recursion coming from the relation $\Gamma(s+1) = s\Gamma(s)$.

Example 3.1. Let $d = 1$ and let λ_1 be arbitrary. Then, $\phi(t)$ is given by

$$\phi(t) = t^{\lambda_1} \sum_{n=0}^{\infty} 2 \frac{(-1)^n}{n!} t^{2n} = 2 t^{\lambda_1} e^{-t^2}.$$

In general, the poles of $\gamma(s)$ are not simple and the residue of $\gamma(s)t^{-s}$ at $s = z$ is t^{-z} times a polynomial in $\ln t$ of the corresponding degree. The reason is that nonconstant terms of the Taylor expansion of t^{-s} at $s = z$,

$$t^{-s} = t^{-z} \sum_{k=0}^{\infty} \frac{(-\ln t)^k}{k!} (s-z)^k,$$

contribute to the residue in the case of a multiple pole. So (3-2) is again of the form $\sum_j t^{\lambda_j} p_j(t^2)$, except now $p_j(t)$ is a power series in t whose coefficients are polynomials in $\ln t$ of a fixed degree depending on j .

Example 3.2. Let $d = 2$ and $\lambda_1 = \lambda_2 = 0$. Then $\phi(t)$ is a Bessel function,

$$\begin{aligned} \phi(t) &= 4K_0(2t) = -4(\ln t + \gamma_e) \\ &- 4(\ln t - 1 + \gamma_e)t^2 - \frac{2\ln t - 3 + 2\gamma_e}{2}t^4 + \dots, \end{aligned}$$

with $\gamma_e = -\Gamma'(1)$ the Euler constant.

Algorithm 3.3. (Expansion of $\phi(t)$ for t small.) The recursions necessary to determine the coefficients of (3-2) for a general Γ -factor $\gamma(s)$ are as follows.

1. Let $\gamma(s)$ and $\phi(t)$ be defined by (3-1) and (3-2), respectively.

2. We say that λ_j and λ_k are equivalent if $\lambda_j - \lambda_k \in 2\mathbb{Z}$. Let $\Lambda_1, \dots, \Lambda_N$ denote the equivalence classes and let $\ell_j = |\Lambda_j|$. Thus, $\sum \ell_j = d$.
3. Let $m_j = -\lambda_{k_j} + 2$, where $\lambda_{k_j} \in \Lambda_j$ is the element with the smallest real part, that is $\inf_{\lambda \in \Lambda_j} \operatorname{Re} \lambda = \operatorname{Re} \lambda_{k_j}$. In other words, $\gamma(s)$ is analytic at $s = m_j$, has a pole of some order at $s = m_j - 2$, and has a pole of order ℓ_j at $s = m_j - 2n$ for $n \gg 1$.
4. Let $c_j^{(0)}(s)$ be the beginning of the Taylor series of $\gamma(s + m_j)$ around $s = 0$ with $O(s^{\ell_j})$ as the last term.
5. For $1 \leq j \leq d$ and $n \geq 1$, define $c_j^{(n)}(s)$ recursively by

$$c_j^{(n)}(s) = c_j^{(n-1)}(s) / \prod_{k=1}^d \left(\frac{s + \lambda_k + m_j}{2} - n \right) \quad (3-3)$$

considered as a quotient of Laurent series in $s = 0$. Note that $c_j^{(n)}(s)$ terminates at $O(1)$ for $n \gg 1$. Let $c_{j,k}^{(n)}$ denote the coefficient of s^{-k} in $c_j^{(n)}(s)$.

6. For t real positive, $\phi(t)$ is given by

$$\phi(t) = \sum_{j=1}^N t^{-m_j} \sum_{n=1}^{\infty} \left(\sum_{k=0}^{\ell_j-1} \frac{(-\ln t)^k}{k!} c_{j,k+1}^{(n)} \right) t^{2n}. \quad (3-4)$$

Remark 3.4. The above series converges exponentially fast since

$$\max_{j \leq N, k \leq \ell_j} |c_{j,-k}^{(n)}| = O((n!)^{-d}) \quad \text{as } n \rightarrow \infty.$$

Nevertheless, this is not an efficient way to compute $\phi(t)$ for large t . Take for instance the series $e^{-t} = \sum_{n=0}^{\infty} (-t)^n/n!$ for $t = 50$. The terms grow up to 3×10^{20} for $n = 50$ before starting to tend to 0. Thus, to determine e^{-50} to 10 decimal digits with this series, one has to require working precision of 30 digits and compute 160 terms until everything happily cancels, producing the answer 0.0000000000. This is clearly not too efficient a procedure. As this is exactly the general behaviour for arbitrary $\gamma(s)$, for large t we use instead a different method based on asymptotic expansions at infinity as described in Section 4 below.

As explained in Section 2, we also need means for computing the incomplete Mellin transform of $\phi(t)$ and its derivatives. Recall that for $s \in \mathbb{C}$ and $t > 0$ we defined $G_s(t)$ to be

$$G_s(t) = t^{-s} \int_t^{\infty} \phi(x) x^s \frac{dx}{x}.$$

Recall also that $\lim_{t \rightarrow 0} t^s G_s(t)$ exists and equals $\gamma(s)$ whenever s is not a pole of $\gamma(s)$. For such s clearly

$$t^s G_s(t) = \gamma(s) - \int_0^t \phi(x) x^s \frac{dx}{x}. \quad (3-5)$$

Since (3-4) expresses $\phi(t)$ as an infinite sum of terms of the form $t^\alpha (\ln t)^\beta$, term by term integration of (3-5) results in a similar expression for $G_s(t)$.

In the points where $\gamma(s)$ does have a pole, the formula (3-5) makes no sense as the right-hand side becomes $\infty - \infty$. However, it is not difficult to locate the terms that contribute to the principal parts of the Laurent series. Ignoring these terms then gives the value of $G_s(t)$ for such s . Note that there could be numerical problems in using (3-5) close to (but not exactly at) a pole of $\gamma(s)$.

Algorithm 3.5. (Expansion of $\frac{\partial^k}{\partial s^k} G_s(t)$ for t small.) All this is summarised in the following formulae which allow us to determine $\frac{\partial^k}{\partial s^k} G_s(t)$ for arbitrary $s \in \mathbb{C}$ and $t > 0$. Here $\alpha \in \mathbb{C}$ and $i, j, k \geq 0$ and $n \geq 1$ are integers.

1. Let $c_{j,i}^{(n)}$ be as in (3-3).
2. Define $L_{\alpha,j,k}(x) \in \mathbb{C}[x]$ by the formula

$$L_{\alpha,j,k}(x) = \begin{cases} k! \sum_{i=0}^{j-1} \binom{i-j}{k} \frac{\alpha^{i-j-k}}{i!} (-x)^i, & \alpha \neq 0, \\ 0, & \alpha = 0. \end{cases}$$

3. Let

$$S_{j,k,s}^{(n)}(x) = \sum_{i=1}^{\ell_j} c_{j,i}^{(n)} L_{2n+s-m_j,i,k}(x) \in \mathbb{C}[x].$$

4. For $t > 0$ consider the infinite sum

$$\tilde{G}_{s,k}(t) = \sum_{j=1}^N t^{2-m_j} \sum_{n=1}^{\infty} S_{j,k,s}^{(n)}(\ln t) t^{2n}. \quad (3-6)$$

5. The formula for $\frac{\partial^k}{\partial s^k} G_s(t)$ reads

$$\frac{\partial^k}{\partial s^k} G_s(t) = \left(\frac{\partial^k}{\partial S^k} \frac{\gamma(S)}{t^S} \right)_{S=s}^* - \tilde{G}_{s,k}(t),$$

where $f(S)_{S=s}^*$ denotes the constant term a_0 of the Laurent expansion $\sum_k a_k (S-s)^k$ of $f(S)$ at $S = s$. Thus $f(S)_{S=s}^* = f(s)$ if $f(S)$ is analytic at $S = s$.

Remark 3.6. The series for $\frac{\partial^k}{\partial s^k} G_s(t)$ converges exponentially fast since the corresponding one for $\phi(t)$ does (see Remark 3.4). Again, however, it is inefficient for large t in which case we use an alternative approach described in the following section.

4. COMPUTING $\phi(t)$ AND $\frac{\partial^k}{\partial s^k} G_s(t)$ FOR t LARGE

To compute $\phi(t)$ and $G_s(t)$ for large t , we begin with the asymptotic expansions of these functions at infinity.

Recall that $\phi(t)$ is defined as the inverse Mellin transform of a product of Γ -functions,

$$\Gamma\left(\frac{s+\lambda_1}{2}\right) \cdots \Gamma\left(\frac{s+\lambda_d}{2}\right) = \int_0^\infty \phi(t) t^s \frac{dt}{t}.$$

In other words, $\phi(t)$ is a special case of Meijer G -function. Given two sequences of complex parameters,

$$a_1, \dots, a_n, a_{n+1}, \dots, a_p \quad \text{and} \quad b_1, \dots, b_m, b_{m+1}, \dots, b_q,$$

a general Meijer G -function $G_{p,q}^{m,n}(t; a_1, \dots, a_p; b_1, \dots, b_q)$ is defined by

$$\int_0^\infty G_{p,q}^{m,n}(t; a_1, \dots, a_p; b_1, \dots, b_q) t^s \frac{dt}{t} = \frac{\prod_{j=1}^m \Gamma(b_j + s) \prod_{j=1}^n \Gamma(1 - a_j - s)}{\prod_{j=m+1}^q \Gamma(1 - b_j - s) \prod_{j=n+1}^p \Gamma(a_j + s)}.$$

We refer to Luke [Luke 69, Sections 5.2–5.11], for basic properties of the G -function.

In our case replacing s by $s/2$ yields an identification

$$\phi(t) = 2 G_{0,d}^{d,0}(t^2; ; \frac{\lambda_j}{2}).$$

As discovered by Meijer (in greater generality), the function $G_{0,d}^{d,0}$ possesses the following asymptotic expansion at infinity ([Luke 69, Theorem 5.7.5]):

$$G_{0,d}^{d,0}(t; ; \frac{\lambda_j}{2}) \sim \frac{(2\pi)^{(d-1)/2}}{\sqrt{d}} e^{-d t^{1/d}} t^{\kappa/d} \sum_{n=0}^\infty M_n t^{-n/d},$$

$$\kappa = (1 - d + \sum_{j=1}^d \lambda_j)/2. \tag{4-1}$$

Here, $M_n = M_n(\lambda_1, \dots, \lambda_d)$ are constants, $M_0 = 1$. As for $\phi(t)$, we have

$$\phi(t^{d/2}) \sim \frac{2(2\pi)^{(d-1)/2}}{\sqrt{d}} e^{-d t} t^\kappa \sum_{n=0}^\infty M_n t^{-n}. \tag{4-2}$$

We would like to note here that the stated asymptotic expansion for large t is much “neater” than the expansion (3–4) for $\phi(t)$ for small t : it involve no logarithmic terms and its shape is independent of whether any of the λ_j are equal modulo $2\mathbb{Z}$.

The coefficients M_n in the asymptotic expansion (4-2) can be found as follows. The defining relation

$$\gamma(s+2) = \gamma(s) \times \prod_{j=1}^d \frac{s+\lambda_j}{2}$$

at the level of inverse Mellin transforms is equivalent to an ordinary differential equation (of degree d) with polynomial coefficients for $\phi(t)$. It follows that the function $t^{-\kappa} e^{dt} \phi(t^{d/2})$ satisfies a different ODE, of degree $d+1$. Formally, substituting $1 + \sum_{n \geq 1} M_n t^{-n}$ as a solution gives a recursion for the M_n with polynomial coefficients. This has been worked out in general by E. M. Wright; for details see Luke [Luke 69, Section 5.11.5], especially formulae (8) and (16).

Algorithm 4.1. (Asymptotic expansion associated to $\phi(t)$.) Here is the answer in our case, rewritten in a slightly different polynomial basis.

1. Let $S_m = S_m(\lambda_1, \dots, \lambda_d)$ denote the m th elementary function of $\lambda_1, \dots, \lambda_d$,

$$S_0 = 1, \quad S_1 = \sum_{j=1}^d \lambda_j, \quad \dots, \quad S_d = \prod_{j=1}^d \lambda_j.$$

2. Define also modified symmetric functions S_m by

$$\tilde{S}_m = \sum_{k=0}^m (-S_1)^k d^{m-1-k} \binom{k+d-m}{k} S_{m-k},$$

$$\text{for } 0 \leq m \leq d, \tilde{S}_{d+1} \equiv 0.$$

3. For $k \geq 0$ define $\Delta_k(x) \in \mathbb{Q}[x]$ by means of the generating function

$$\left(\frac{\sinh t}{t}\right)^x = \sum_{k=0}^\infty \Delta_k(x) t^{2k}.$$

4. For $p \geq 1$ consider the following polynomials:

$$\nu_p(n) = -\frac{d}{(2d)^p} \sum_{m=0}^p \tilde{S}_m \prod_{j=m}^{p-1} (d-j) \sum_{k=0}^{\lfloor \frac{p-m}{2} \rfloor} \frac{(2n-p+1)^{p-m-2k}}{(p-m-2k)!} \Delta_k(d-p).$$

5. The coefficients M_n in the asymptotic expansion (4-2) satisfy a recursion

$$M_n = \begin{cases} 0, & n < 0, \\ 1, & n = 0, \\ \frac{1}{n} \sum_{p=1}^d \nu_p(n) M_{n-p}, & n \geq 1. \end{cases}$$

Applying term-wise integration to (4-2), it is also easy to deduce the asymptotic expansion of $G_s(t)$ for $t \rightarrow \infty$,

$$G_s(t^{d/2}) \sim \frac{(2\pi)^{(d-1)/2}}{\sqrt{d}} e^{-d t} t^{\kappa-1} \sum_{n=0}^\infty \mu_n(s) t^{-n}. \tag{4-3}$$

$$\mu_n = \begin{cases} 0, & n < 0, \\ 1, & n = 0, \\ \frac{1}{n} \sum_{p=1}^d \left(\nu_{p+1}(n) - \frac{S_1+d(s-1)-2(n-p)-1}{2d} \nu_p(n) \right) \mu_{n-p}, & n \geq 1. \end{cases} \tag{4-4}$$

Here, $\kappa = (1 - d + S_1)/2$ as in (4-1), and $\mu_n(s) = \mu_n(\lambda_1, \dots, \lambda_d; s)$ satisfy a recursion (4-4).

By induction one shows that μ_n is a polynomial in s with the leading term $2^{-n} s^n$. So if we differentiate (4-3) k times to s , exactly k terms vanish and we get the following formula for the derivatives $\frac{\partial^k}{\partial s^k} G_s(t)$:

$$\frac{\partial^k}{\partial s^k} G_s(t^{d/2}) \sim \frac{(2\pi)^{(d-1)/2}}{\sqrt{d}} e^{-d t} t^{\kappa-1-k} \sum_{n=0}^{\infty} \frac{\partial^k \mu_{n+k}(s)}{\partial s^k} t^{-n}. \tag{4-5}$$

Equations (4-2), (4-3), and (4-5) provide asymptotic series for the functions $\phi(t)$, $G_s(t)$, and $\frac{\partial^k}{\partial s^k} G_s(t)$ at infinity. For computational purposes, though, it is better to work with continued fraction expansions associated to these series. Consider, for instance, the case of $\phi(t)$, the case of $\frac{\partial^k}{\partial s^k} G_s(t)$ being analogous.

Fix d and $\lambda_1, \dots, \lambda_d$. Letting $x = 1/t$ in (4-2), we get

$$\psi(x) := \frac{\sqrt{d}}{2(2\pi)^{(d-1)/2}} e^{-d x} x^\kappa \phi(x^{-d/2}) \sim \sum_{n=0}^{\infty} M_n x^n \tag{4-6}$$

with M_n constants. As any formal series, the right-hand side can be *formally* written either as a unique infinite continued fraction

$$\sum_{n=0}^{\infty} M_n x^n = \alpha_0 + \frac{x^{k_0}}{\alpha_1 + \frac{x^{k_1}}{\alpha_2 + \frac{x^{k_2}}{\alpha_3 + \dots}}}, \text{ with } \alpha_n \neq 0 \text{ for } n > 0,$$

or as a unique terminating fraction of the same form. To see this start with $p_0(x) = \sum M_n x^n$ and define recursively formal power series $p_{n+1}(x)$ in terms of $p_n(x)$ by

$$p_n(x) = \alpha_n + \frac{x^{k_n}}{p_{n+1}(x)}, \quad \text{for } n \geq 0.$$

Here, k_n is the degree of the first nonzero term in $p_n(x) - p_n(0)$; if $p_n(x) \equiv 0$ for some n , then terminate. This shows the existence of the continued fraction expansion; its uniqueness is not difficult to verify as well. The construction shows how to calculate the α_n for given M_n

with $n \leq N$. There are of course better (computationally more stable) methods to find the α_n , see for instance [Henrici 77, Lorentzen and Waadeland 92].

If the fraction does not terminate, define the partial convergents $C_n(x)$ for all n by

$$C_n(x) = \alpha_0 + \frac{x^{k_0}}{\alpha_1 + \frac{x^{k_1}}{\dots + \frac{x^{k_{n-1}}}{\alpha_n}}}.$$

If the fraction does terminate at C_N , let $C_n = C_N$ for $n > N$.

We can think of $C_n(x)$ as approximants to the original function $\psi(x)$ of (4-6). Indeed, $C_n(x)$ is a rational function whose Taylor expansion at $x = 0$ starts with $M_0 + \dots + M_n x^n$. Hence, $\psi(x)$ and $C_n(x)$ have the same asymptotic expansion at least up to x^n . Therefore, there is a constant $K_n > 0$ such that

$$|\psi(x) - C_n(x)| \leq K_n x^{n+1}.$$

Unfortunately, it seems very difficult to provide explicit bounds for K_n . It appears that $C_n(x)$ converge rapidly to $\psi(x)$, but to prove either “converge” or “rapidly” or “to $\psi(x)$ ” in any generality is hard. So, the last step of the algorithm is based on purely empirical observations concerning the convergence of the continued fractions. The reader uncomfortable with it is referred to Section 5 on how to avoid this. In the implementation [Dokchitser 02] we do use asymptotic expansions with a simple numerical check (Step 7 in Algorithm 4.2 below) to justify the values.

Algorithm 4.2. (Computing $\phi(t)$ for t arbitrary.) The computation of $\phi(t)$ for arbitrary t can be performed as follows.

1. Let $\epsilon > 0$ be the necessary upper bound for the required precision in the computation of $\phi(t)$.
2. Let $\phi_n(t)$ be the n th approximant to $\phi(t)$ defined by (see (4-2))

$$\phi_n(t) = \frac{2(2\pi)^{(d-1)/2}}{\sqrt{d}} e^{-d t^{2/d}} t^{2\kappa/d} C_n(1/t^{2/d}), \quad \text{for } n \geq 0.$$

As we have already mentioned, $\phi(t) - \phi_n(t) = O(t^{-n})$ as $t \rightarrow \infty$. Denote by $\phi_{\text{taylor}}(t)$ the function $\phi(t)$ computed using the power series expansion at the origin as in Section 3.

3. Determine t_0 such that $|\phi_0(t)| < \epsilon/2$ for $t > t_0$.
4. Choose a subdivision of the interval $[0, t_0]$,

$$0 < t_k < t_{k-1} < \dots < t_1 < t_0 < \infty.$$

For every t_i let n_i be an integer for which $|\phi_n(t) - \phi_{n+1}(t)| < \epsilon/2$ and $|\phi_n(t) - \phi_{n+2}(t)| < \epsilon/2$.

5. Determine M_n for $0 \leq n \leq n_k$ using the recursion that they satisfy.
6. The function $\phi(t)$ is then computed as follows:

$$\phi_{\text{general}}(t) = \begin{cases} \phi_{\text{taylor}}(t), & 0 < t \leq t_k, \\ \phi_{n_i}(t), & t_i \leq t < t_{i-1}, \\ 0, & t > t_0. \end{cases}$$

7. As a numerical check, verify that $\phi_{\text{taylor}}(t_k)$ agrees with $\phi_{n_k}(t_k)$.

Example 4.3. Let $d = 2$ and $\lambda_1 = \lambda_2 = 0$ as in Example 3.2. Recall that in this case $\phi(t)$ is a Bessel function $4K_0(2t)$. Asymptotic expansion (4-2) then reads

$$\phi(t) \sim 2\sqrt{\pi} e^{-2t} t^{-1/2} \sum_{n=0}^{\infty} M_n t^{-n},$$

and the coefficients M_n satisfy a recursion

$$16nM_n = -(2n - 1)^2 M_{n-1}.$$

It follows that

$$\begin{aligned} M_0 &= 1, \\ M_1 &= -\frac{1}{16}, \\ M_2 &= \frac{9}{512}, \\ &\vdots \\ M_n &= (-1)^n \frac{(2n-1)!!(2n-1)!!}{16^n n!}, \dots \end{aligned}$$

Choose $\epsilon = \frac{1}{2} \cdot 10^{-10}$ and $t_0 = 12$, $t_1 = 6$, and $t_2 = 2$. Take $n_1 = 6$ and $n_2 = 20$, and compute $\phi(t)$ by

$$\phi_{\text{general}}(t) = \begin{cases} \phi_{\text{taylor}}(t), & 0 < t \leq 2, \\ \phi_{20}(t), & 2 \leq t < 7, \\ \phi_6(t), & 7 \leq t < 12, \\ 0, & t > 12. \end{cases}$$

Numerical check produces $|\phi_{\text{taylor}}(2) - \phi_{20}(2)| \leq 4 \cdot 10^{-14} \leq \epsilon$, as required.

5. IMPLEMENTATION NOTES

Let us begin with a summary of steps of the algorithm presented in the previous sections. We start with an L -function satisfying Assumptions 2.1–2.3 (see also Remark 2.7).

- The formula used for the numerical verification of the functional equation is (2-5) and that for computing $L(s)$ and its derivatives is (2-9) together with $L^*(s) = L(s)/\gamma(s)$. The functions used in these formulae are $\phi(t)$ defined by (2-2) and $G_s(t)$ defined by (2-7).
- To compute $\phi(t)$ numerically we employ Algorithm 4.2. It is based on power series expansions at the origin (Algorithm 3.3), asymptotic formula (4-2), recursion (Algorithm 4.1) and the associated continued fractions.
- The functions $\frac{\partial^k}{\partial s^k} G_s(t)$ are computed in the same manner. The corresponding expansions at the origin are given by Algorithm 3.5, asymptotics by (4-3), and recursions for the coefficients by (4-4).

Now, in order to make a practical algorithm out of these results, we still need to explain how to truncate various infinite sums and to discuss related precision issues.

If one desires to supply our method with rigorous proofs that all of the computations are correct, the following issues have to be addressed. First, one has to keep track of the number of operations used and possible round-off errors, perhaps even using interval arithmetic to justify the results. Second, one needs to have analytic bounds on the size of the functions $\phi(t)$ and $G_s(t)$ for large t rather than just their asymptotic behaviour.

In the PARI implementation [Dokchitser 02] we have chosen to be content with intuitively natural bounds and a few numerical checks to justify the results. A reader wishing to develop a more rigorous approach might consider the following.

Remark 5.1. Let us start with the computations of $\phi(t)$ and $\frac{\partial^k}{\partial s^k} G_s(t)$ by means of series expansions at the origin. Both the defining expressions (3-4) and (3-6) are infinite sums, but it is not difficult to see how to terminate them. The point is that it suffices to give an explicit bound on the coefficients $c_{i,j}^{(n)}$ which goes to zero exponentially with n . Everything else in (3-4) and (3-6) grows at most polynomially in n so that any rough estimate will do. As for an explicit exponential bound on $c_{i,j}^{(n)}$, it can be found

from (3-3) and, say, from the obvious lower bound $\frac{1}{2}n^d$ for $n \gg 1$ on the coefficients in $n^d \prod_{k=1}^d (1 - \frac{s+\lambda_k+m_j}{2n})$ treated as a polynomial in s .

Remark 5.2. The next question is that of working precision. This has already been mentioned in Remarks 3.4 and 3.6. When $\phi(t)$ and $G_s(t)$ are computed from the expansions at the origin, the terms in the defining series can be very large. Then, one needs to work with larger working precision than the desired precision of the answer.

A similar thing occurs when one computes $L(s)$ for large $\text{Im } s$. This is a well-known problem that one has to face when verifying the Riemann hypothesis. The point is that $L(s) = L^*(s)/\gamma(s)$ and both $L^*(s)$ and $\gamma(s)$ decrease exponentially fast on vertical strips. Hence, one needs to compute $L^*(s)$ to more significant digits ($\log_{10} |\gamma(s)|$ more to be precise) to evaluate $L(s)$ to the desired precision.

A solution to this problem has been suggested in Lagarias-Odlyzko [Lagarias and Odlyzko 79] and worked out by Rubinstein [Rubinstein 98]. By modifying $G_s(t)$ with a suitably chosen exponential factor, one obtains a formula for $L(s)$ that does not have the loss-of-precision behaviour. It may be possible to work out the behaviour of the special functions in Rubinstein’s formulae as we did for $\phi(t)$ and $G_s(t)$.

Remark 5.3. The next issue is how to truncate the main formulae used in this paper. Recall that to verify the functional equation numerically we used (2-3),

$$\Theta(t) = \sum_{n=1}^{\infty} a_n \phi\left(\frac{nt}{A}\right).$$

Then to actually compute the L -values, we wrote

$$\begin{aligned} \frac{\partial^k}{\partial s^k} L^*(s) &= \sum_{n=1}^{\infty} a_n \frac{\partial^k}{\partial s^k} G_s\left(\frac{n}{A}\right) \\ &+ \epsilon \sum_{n=1}^{\infty} a_n \frac{\partial^k}{\partial s^k} G_{w-s}\left(\frac{n}{A}\right) + \sum_j \frac{k! r_j}{(p_j - s)^{k+1}}. \end{aligned}$$

(See also Remark 2.7 for the necessary modifications when there are two different L -functions involved in the functional equation.) In any case, one needs analytic estimates on $\phi(t)$ and $\frac{\partial^k}{\partial s^k} G_s(t)$ for large t to carefully estimate the error in truncating these infinite sums.

One possible way to obtain such estimates is to use a method of Tollis [Tollis 97] based on Braaksma’s work [Braaksma 64] on asymptotic behaviour of Meijer

G -functions. By applying the Euler-Maclaurin summation formula to the Mellin-Barnes integral defining $G_s(t)$, Tollis determines an explicit exponential bound for $G_s(t)$ in the case $\lambda = (0, \dots, 0, 1, \dots, 1)$ with $\rho + \sigma$ zeroes and σ ones ($\rho, \sigma \geq 0$). It is likely that his method is general enough to obtain similar estimates for an arbitrary Γ -factor as well.

Remark 5.4. Finally, let us turn to the methods of Section 4, asymptotic expansions, and associated continued fractions.

Unfortunately, there seem to be few cases where one can actually provide explicit estimates for the convergence of the continued fractions of, say, $\phi(t)$. The most general result known to the author in this respect is that of Gargantini and Henrici [Gargantini and Henrici 67]. They study the functions that can be written as a Stieltjes transform of a positive measure,

$$f(x) = \int_0^{\infty} \frac{d\mu(t)}{x+t},$$

and show that such functions admit convergent continued fraction expansions at infinity, with explicit error bounds. This, however, does not seem to apply to our functions in general. See Henrici [Henrici 77, Chapter 12] and Lorentzen-Waadeland [Lorentzen and Waadeland 92] for more information.

The full analysis is available, though, in low-dimensional cases. For instance, for $d = 1$ the function $G_s(x)$ is the incomplete Gamma function for which there are known convergent continued fractions expansions at infinity; see Henrici [Henrici 77, Section 12.13.I]. Also, for $d = 2$ the function $\phi(x)$ reduces to the Whittaker function, which is a Stieltjes transform (basically, of itself). So, in this case, the continued fraction expansion converges; see Henrici [Henrici 77, Section 12.13.II].

Returning to the general case, there always remains a possible way out, which is to compute $\phi(t)$ and $\frac{\partial^k}{\partial s^k} G_s(t)$ only using Taylor expansions at the origin, even for large t . In this case one can give precise estimates for the convergence (see e.g., [Cohen 00, Section 10.3]) that lead to a rigorous algorithm. Then, however, one pays the price of substantial loss of efficiency.

Alternatively, one might try a completely different approach to compute the functions in question at infinity. For instance, one can consider functions related to $\frac{\partial^k}{\partial s^k} G_s(t)$ and their derivatives as Bessel functions $K_n(t)$ are related to $K_0(t)$. They satisfy various relations, and

one may be able to construct an algorithm to compute them using backward recursions, possibly vector-valued, analogous to what one does for Bessel functions. (This idea is due to D. Zagier.)

6. L -FUNCTIONS WITH UNKNOWN INVARIANTS

In a perfect world, one knows all of the invariants associated to one's L -function. In a less perfect world, one may not know the sign ϵ and, perhaps, the residues r_i at the poles of $L^*(s)$. In reality, however, there are plenty of examples where it is even difficult to determine the exponential factor A and some of the coefficients a_n . Fortunately, in some of these cases it is still possible to make computations with L -functions.

To illustrate this hierarchy of missing data, start with an L -function $L(s)$ that is expected to satisfy Assumptions 2.1–2.3, and only the sign ϵ in the functional equation is difficult to determine. As we have already mentioned, the functional equation is equivalent to the statement that, for all $t > 1$,

$$\Theta(1/t) = \epsilon t^w \Theta(t) - \sum_j r_j t^{p_j}. \quad (6-1)$$

Choose $1 < t < \infty$ and evaluate the left-hand and the right-hand side. This gives an equation which can be solved for ϵ . Afterwards it is of course sensible to check that (6-1) holds with the obtained ϵ by verifying it numerically for some other values of t .

The same applies to the case when neither the sign ϵ nor the residues r_i are known. The above equation is linear in them all, so choosing enough t 's produces a linear system of equations from which ϵ and the r_i can be obtained. There might be, of course, precision problems if there are many residues to be determined.

In most cases, actually, $\epsilon = \pm 1$ and $L^*(s)$ has no poles, so simply trying $\epsilon = -1$ and $\epsilon = 1$ for some $t > 1$ immediately yields the right sign.

Next come the dimension d , the Hodge numbers $\lambda_1, \dots, \lambda_d$, and the poles p_j of $L^*(s)$. Fortunately, these can always be determined in practice, at least in all of the cases that the author is aware of.

The next issue is that of the exponential factor A . Consider for instance $L(C, H^1, s)$, the L -function associated to H^1 of a genus g curve C/\mathbb{Q} . Then, $A = \sqrt{N}/\pi^g$ where N is the conductor of C . In practice, to determine N one at least has to be able to find a model of C over \mathbb{Z} that is regular at a given prime of bad reduction. This, in turn, means performing successive blowing-ups over the

unramified closure of \mathbb{Q}_p , an operation which is not without computational difficulties. For curves of genus 1 and 2, there are effective algorithms for doing this, but not for higher genera. So finding N for a given curve might be hard. Also note that (6-1) is absolutely *not* linear in A , so one cannot solve for it directly.

Fortunately, one can usually determine the full set $\Sigma = \{p_1, \dots, p_k\}$ of primes where C has bad reduction. Then, one knows that $N = p_1^{b_1} \cdots p_k^{b_k}$ is composed of those primes and has (hopefully) an upper bound for the b_i , say in terms of the discriminant of C or some similar quantity. This leaves only finitely many choices for N and, as in the case of the sign $\epsilon = \pm 1$, a simple trial-and-error can establish the proper functional equation. It should be noted here that this applies, of course, only to L -functions that have a *unique* A (and ϵ , etc.) for which the functional equation holds.

Finally we turn to the coefficients a_i . Again, take the case of a genus g curve C/\mathbb{Q} with the set $\Sigma = \{p_1, \dots, p_k\}$ of bad primes as above. Then the problem is to determine the local factors at bad primes, that is the coefficients $a_{p_i^j}$ for $1 \leq i \leq k$ and $j \geq 1$. The local factors at good primes can be determined by counting points over finite fields, and the coefficients a_n for composite n can be obtained by the product formula.

Fortunately, again, there are only finitely many choices of possible local factors for a given bad prime p_i . For instance, $|a_p| < 2g\sqrt{p}$ and $|a_{p^j}|$ satisfy similar estimates. Moreover, a_{p^j} with $1 \leq j < 2g$ determine a_{p^j} for all j , as the degree of the local factor is bounded by $2g$. This, however, is not a very practical approach, especially for large primes p_i when there are numerous possibilities for the local factors.

Another approach is to note that the functional equation (6-1) is in fact *linear* in the a_i , since $\Theta(t)$ is linear. If there were only finitely many unknown coefficients a_i , they could have been obtained in the same manner as ϵ and the r_i were.

To illustrate what can be done when infinitely many coefficients are unknown, consider the following typical case.

1. Say, there is just one prime p for which a_p is difficult to determine theoretically;
2. assume that all a_n are integers;
3. assume that there is a product formula for $L(s)$ in question, in particular $a_{mn} = a_m a_n$ for m, n coprime.

Using multiplicativity, write (2–3) as

$$\Theta(t) = \sum_{k=1}^{\infty} a_{p^k} \theta_k(t),$$

where $\theta_k(t)$ are computable functions. Moreover, since we only take finitely many terms when actually computing $\Theta(t)$, we have

$$\Theta(t) \approx \sum_{k=1}^K a_{p^k} \theta_k(t),$$

where “ \approx ” stands for “equal to required precision.” Hence, the functional equation (6–1) for a fixed t becomes simply a linear equation in a_p, \dots, a_{p^K} . After plugging in enough t s, we get a linear system that can be solved for the a_{p^i} . However, the coefficient functions $\theta_k(t)$ decay rapidly with k , so a_{p^i} obtained by solving this system are certainly unreliable for large i . If the first coefficient a_p resembles an integer, we can simply round it off and repeat the same process with a_{p^2}, \dots, a_{p^K} as variables until all the a_{p^i} are determined.

In practice, this works well for a large prime p and even when there are several (large) primes p for which the a_{p^i} are unknown. This does not work for small primes, for instance virtually never for $p = 2$. But then, for *small* primes one may try all possible local factors by trial-and-error and for *large* primes solve for the coefficients as described above.

At this stage the reader is likely to be long horrified by the methods suggested here and might wonder whether the reliability of such an approach is not extremely dubious. In our defence we may say that since there is an effective way to verify the functional equation numerically, *any* method to make an intelligent guess will do, however dubious it might be. When A , ϵ , and the bad local factors are determined (or simply guessed in whatever way), one can make numerous checks that they are of correct shape and that (2–5) holds for various t . Thus, it is hoped that someone who has actually tried to perform blowing-ups on a genus 6 arithmetic surface with 2^{20} in the discriminant will forgive the author for offering desperate tricks to avoid the hard work. After all, the methods of this section do allow us to produce evidence for various conjectures even in the difficult cases where it is hard to determine all of the invariants of the L -function in question-using theoretical arguments.

In conclusion, let us mention that, fortunately, there are better ways to determine the local factors for bad primes, at least for arithmetic surfaces. These were used to make computations with curves of genus $g \leq 8$ and are to appear in [Dokchitser, to appear].

ACKNOWLEDGMENTS

The author is extremely grateful to Don Zagier for suggesting to work on this project to begin with and also for numerous explanations, ideas and suggestions which have finally led to this work. Just saying that the algorithm and this paper would not exist without his influence illustrates only a fraction of the truth. The author would also like to thank Richard Brent and Yuri Dokchitser for their comments, and the stimulating atmosphere of the Max-Planck Institut für Mathematik in Bonn where most of this work was carried out.

REFERENCES

- [Beilinson 86] A. Beilinson. “Higher Regulators of Modular Curves.” In *Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory*, edited by S. J. Bloch, R. K. Dennis, E. M. Friedlander, and M. Stein, pp. 1–34, Contemporary Math. 55. Providence, RI: AMS, 1986.
- [Birch and Swinnerton-Dyer 63] B. J. Birch and H. P. F. Swinnerton-Dyer. “Notes on Elliptic Curves, I.” *J. Reine Angew. Math.* 212 (1963), 7–25.
- [Bloch and Kato 90] S. Bloch and K. Kato. “ L -Functions and Tamagawa Numbers of Motives.” In *The Grothendieck Festschrift*, Vol. I, edited by A. Johnson, pp. 333–400, Progress in Mathematics 86. Boston: Birkhäuser, 1990.
- [Braaksma 64] B. L. J. Braaksma. “Asymptotic Expansions and Analytic Continuations for a Class of Barnes-Integrals.” *Compositio Math.* 15 (1964), 239–241.
- [Buhler et al. 85] J. P. Buhler, B. H. Gross, and D. B. Zagier. “On the Conjecture of Birch and Swinnerton-Dyer for an Elliptic Curve of Rank 3.” *Math. Comp.* 44 (1985), 473–481.
- [Cohen 00] H. Cohen. *Advanced Topics in Computational Number Theory*. New York: Springer-Verlag, 2000.
- [Davies and Haselgrove 61] D. Davies and C. B. Haselgrove. “The Evaluation of Dirichlet L -Functions.” *Proc. Royal Soc. Ser. A* 264 (1961), 122–132.
- [Dokchitser 02] T. Dokchitser. “ComputeL: Pari Package to Compute Motivic L -Functions.” Available from World Wide Web (<http://maths.dur.ac.uk/~dma0td/computel/>), 2002.
- [Dokchitser, to appear] T. Dokchitser. “Computations with L -functions of Higher Genus Curves.” In preparation.
- [Fermigier 92] Stéphane Fermigier. “Zéros des fonctions L de courbes elliptiques.” *Experimental Math.* 1:2 (1992), 167–173.
- [Gargantini and Henriци 67] I. Gargantini and P. Henriци. “A Continued Fraction Algorithm for the Computation of Higher Transcendental Functions in the Complex Plane.” *Math. Comp.* 21 (1967), 18–29.
- [Henrici 77] P. Henriци. *Applied and Computational Complex Analysis*, Vol. II. New York: Wiley-Interscience, 1977.

- [Janssen et al. 94] U. Janssen, S. Kleiman, and J. -P. Serre, editors. *Motives*. Proc. Symp. Pure Math., 55. Providence: AMS, 1994.
- [Keiper 96] J. B. Keiper. “On the Zeros of the Ramanujan τ -Dirichlet Series in the Critical Strip.” *Math. Comp.* 65:216 (1996), 1613–1619.
- [Kontsevich and Zagier 01] M. Kontsevich and D. Zagier. “Periods.” In *Mathematics Unlimited—2001 and Beyond*, edited by B. Engquist and W. Schmid, pp. 771–808. Berlin: Springer, 2001.
- [Lagarias and Odlyzko 79] J. C. Lagarias and A. M. Odlyzko. “On Computing Artin L -Functions in the Critical Strip.” *Math. Comp.*, 33:147 (1979), 1081–1095.
- [Lavrik 68] A. F. Lavrik. “Approximate Functional Equation for Dirichlet Functions.” *Izv. Akad. Nauk SSSR* 32 (1968), 134–185.
- [Lorentzen and Waadeland 92] L. Lorentzen and H. Waadeland. *Continued Fractions with Applications*. Studies in Computational Mathematics, 3. Amsterdam: North-Holland, 1992.
- [Luke 69] Y. L. Luke. *The Special Functions and Their Approximations*, Vol. 1. New York: Academic Press, 1969.
- [Manin and Panchishkin 95] Yu. I. Manin and A. A. Panchishkin. “Number Theory I.” In *Encyclopedia of Math. Sciences*, 49, edited by A. N. Parshin and I. R. Shafarevich. Berlin: Springer-Verlag, 1995.
- [Rubinstein 98] M. O. Rubinstein. “Evidence for a Spectral Interpretation of the Zeros of L -Functions.” PhD diss., Princeton University, 1998.
- [Scholl 91] A. Scholl. “Remarks on Special Values of L -Functions.” In *L -Functions and Arithmetic*, edited by J. Coates and M. J. Taylor, pp. 373–392, London Mathematical Society Lecture Notes 153. Cambridge: Cambridge University Press, 1991.
- [Serre 65] J. -P. Serre. “Zeta and L -Functions.” In *Arithmetical Algebraic Geometry, Proc. Conf. Purdue 1963*, edited by O. F. G. Schilling, pp. 82–92. New York: Harper & Row, 1965.
- [Shintani 72] T. Shintani. “On Dirichlet Series Whose Coefficients Are Class Numbers of Integral Binary Cubic Forms.” *J. Math. Soc. Japan* 24 (1972), 132–188.
- [Tollis 97] E. Tollis. “Zeros of Dedekind Zeta Functions in the Critical Strip.” *Math. Comp.* 66:219 (1997), 1295–1321.
- [van de Lune et al. 86] J. van de Lune, H. J. J. te Riele, and D. T. Winter. “On the Zeros of the Riemann Zeta Function in the Critical Strip IV.” *Math. Comp.* 46:174 (1986), 667–681.
- [Yoshida 95] H. Yoshida. “On Calculations of Various L -Functions.” *J. Math. Kyoto Univ.* 35:4 (1995), 663–696.
- [Zagier 91] D. Zagier. “Polylogarithms, Dedekind Zeta Functions and the Algebraic K -Theory of Fields.” In *Arithmetical Algebraic Geometry*, edited by G. van der Greer, F. Oort, and J. Steenbrink, pp. 391–430, Progress in Mathematics 89. Boston: Birkhäuser, 1991.

Tim Dokchitser, Department of Math. Sciences, University of Durham, Durham DH1 3LE, United Kingdom
(tim.dokchitser@durham.ac.uk)

Received June 9, 2003; accepted October 30, 2003.

