The Hypermetric Cone on Seven Vertices

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CONTENTS

1. Introduction
2. Hypermetrics and Delaunay Polytopes
3. Computing the Extreme Rays of HYP_7
4. Adjacency Properties of the Skeleton of HYP_7 and of Its Dual
5. Final Remarks
References

The hypermetric cone HYP_n is the set of vectors $(d_{ij})_{1\le i<j\le n}$ satisfying the inequalities

$$\sum_{1\le i<j\le n} b_i b_j d_{ij} \le 0 \text{ with } b_i \in \mathbb{Z} \text{ and } \sum_{i=1}^{n} b_i = 1\,.$$

A Delaunay polytope of a lattice is called *extreme* if the only affine bijective transformations of it into a Delaunay polytope, are the homotheties; there is correspondence between such Delaunay polytopes and extreme rays of HYP_n. We show that unique Delaunay polytopes of root lattices A_1 and E_6 are the only extreme Delaunay polytopes of dimension at most 6. We describe also the skeletons and adjacency properties of HYP_7 and of its dual.

The computational technique used is polyhedral, i.e., enumeration of extreme rays, using the program cdd [Fukuda 03], and groups to reduce the size of the computations.

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1. INTRODUCTION

A vector $(d_{ij})_{1\le i<j\le n} \in \mathbb{R}^N$ with $N = \binom{n}{2}$ is called an *n-hypermetric*, if it satisfies the following *hypermetric inequalities*:

$$\sum_{1\le i<j\le n} b_i b_j d_{ij} \le 0 \text{ with } b = (b_i) \in \mathbb{Z}^n \text{ and } \sum_{i=1}^{n} b_i = 1\ . \tag{1–1}$$

The set of vectors satisfying (1–1) is called the *hypermetric cone* and denoted by HYP_n.

We have the inclusions $CUT_n \subset HYP_n \subset MET_n$, where MET_n denotes the cone of all semimetrics on n points and CUT_n (see Section 3 below and Chapter 4 of [Deza and Laurent 97]) is the cone of all semimetrics on n points, which are isometrically embeddable into some space l_1^m. In fact, the triangle inequality $d_{ij} \le d_{ik} + d_{jk}$ is the hypermetric inequality with vector b, such that $b_i = b_j = 1$, $b_k = -1$ and $b_l = 0$, otherwise.

1058-6458/2003 $0.50 per page

For $n \leq 4$, all three cones coincide and $HYP_n = CUT_n$ for $n \leq 6$; so, the cone HYP_7 is the first proper hypermetric cone.[1] See [Deza and Laurent 97] for a detailed study of those cones and their numerous applications in combinatorial optimization, analysis, and other areas of mathematics. In particular, the hypermetric cone had direct applications in the geometry of quadratic forms; see Section 2.

In fact, HYP_n is a polyhedral cone (see [Deza et al. 93]). Lovasz (see [Deza and Laurent 97, pages 201–205]) gave another proof of it and the bound $max|b_i| \leq n!2^n\binom{2n}{n}^{-1}$ for any vector $b = (b_i)$, defining a facet of HYP_n.

The group of all permutations on n vertices induces a partition of the set of k-dimensional faces of HYP_n into orbits. Baranovskii, using his method presented in [Baranovskii 70], found in [Baranovskii 99] the list of all facets of HYP_7: 3,773 facets, divided into 14 orbits. On the other hand, in [Deza et al. 92] 29 orbits of extreme rays of HYP_7 were found by classifying the basic simplexes of the Schläfli polytope of the root lattice E_6.

In Section 3, we show that the 37,170 extreme rays contained in those 29 orbits are, in fact, the complete list. It also implies that the Schläfli polytope (unique Delaunay polytope of E_6) and the segment α_1 (the Delaunay polytope of A_1) are only extreme Delaunay polytopes of dimension at most six. In Section 4, we give adjacency properties of the skeletons of HYP_7 and of its dual.

The computations were done using the programs `cdd` with rational exact arithmetic (see [Fukuda 03]) and `nauty` (see [McKay 03]). Certain errors can arise from any of those programs and Dutour's programs (see [Dutour 03b]).

2. HYPERMETRICS AND DELAUNAY POLYTOPES

For more details on the material of this section, see Chapters 13–16 of [Deza and Laurent 97]. Let $L \subset \mathbb{R}^k$ be a k-dimensional lattice and let $S = S(c, r)$ be a sphere in $\mathbb{R}^k$ with center c and radius r. Then, S is said to be an *empty sphere* in L if the following two conditions hold:

$\|v - c\| \geq r$ for all $v \in L$;

the set $S \cap L$ has affine rank $k + 1$.

[1] Apropos, MET_7 has 46 orbits of extreme rays and not 41 as, by a technical mistake, was given in [Grishushin 92] and [Deza and Laurent 97].

Then, the center c of S is called a *hole* in [Conway and Sloane 99]. The polytope P, which is defined as the convex hull of the set $P = S \cap L$, is called a *Delaunay polytope*, or (in the original terms of Voronoi, who introduced them in [Voronoi 08]), an *L-polytope*.

On every set $A = \{v_1, \ldots, v_m\}$ of vertices of a Delaunay polytope P, one can define a distance function $d_{ij} = \|v_i - v_j\|^2$. The function d turns out to be a metric and, moreover, a hypermetric. It follows from the following formula (see [Assouad 82] and [Deza and Laurent 97, page 195]):

$$\sum_{1 \leq i,j \leq m} b_i b_j d_{ij} = 2(r^2 - \|\sum_{i=1}^{m} b_i v_i - c\|^2) \leq 0\,.$$

On the other hand, Assouad has shown in [Assouad 82] that a distance in *every finite* hypermetric space is the square of Euclidean distance on a generating set of vertices of a Delaunay polytope of a lattice.

For example, in dimension two, there are two combinatorial types of Delaunay polytopes: triangle and rectangle. Since $HYP_3 = MET_3$, we see that a triangle is a Delaunay polytope if and only if it has no obtuse angles.

A Delaunay polytope P is said to be *extreme* if the only (up to orthogonal transformations and translations) affine bijective transformations T of $\mathbb{R}^k$, for which $T(P)$ is again a Delaunay polytope, are the homotheties. In [Deza et al. 92], the authors show that the hypermetric on any generating subset of an extreme Delaunay polytope (see above) lies on an extreme ray of HYP_n and that a hypermetric, lying on an extreme ray of HYP_n, is the square of Euclidean distance on a generating subset of extreme Delaunay polytope of dimension at most $n - 1$.

In [Deza and Laurent 97, page 228], there is a more complete dictionary translating the properties of Delaunay polytopes into those of the corresponding hypermetrics.

Recall that E_6, E_7, and E_8 are *root* lattices defined by

$$\begin{aligned} E_6 &= \{x \in E_8 : x_1 + x_2 = x_3 + \cdots + x_8 = 0\}, \\ E_7 &= \{x \in E_8 : x_1 + x_2 + x_3 + \cdots + x_8 = 0\}, \\ E_8 &= \{x \in \mathbb{R}^8 : x \in \mathbb{Z}^8 \cup (\frac{1}{2} + \mathbb{Z})^8 \text{ and } \sum_i x_i \in 2\mathbb{Z}\}\ . \end{aligned}$$

The skeleton of the unique Delaunay polytope of E_6 is a 27-vertex strongly regular graph, called the *Schläfli graph*. In fact, the 29 orbits of extreme rays of HYP_7, found in [Deza et al. 92], were three orbits of extreme rays of CUT_7 (*cuts*) and 26 orbits corresponding to all sets of seven vertices of the Schläfli graph, which are affine bases (over $\mathbb{Z}$) of E_6. The root lattice E_7 has two Delaunay polytopes: a 7-simplex and a 56-vertex polytope, called the *Gosset polytope*, which is extreme. In [Dutour 03a], all 374 orbits of affine bases for the Gosset polytope were found.

3. COMPUTING THE EXTREME RAYS OF HYP_7

We recall some terminology. Let C be a polyhedral cone in $\mathbb{R}^n$. Given $v \in \mathbb{R}^n$, the inequality $\sum_{i=1}^n v_i x_i \geq 0$ is said to be *valid* for C, if it holds for all $x \in C$. Then, the set $\{x \in C | \sum_{i=1}^n v_i x_i = 0\}$ is called the *face of C, induced by the valid inequality* $\sum_{i=1}^n v_i x_i \geq 0$. A face of dimension $\dim(C) - 1$ is called a *facet* of C; a face of dimension 1 is called an *extreme ray* of C.

An extreme ray e is said to be *incident* to a facet F if $e \subset F$. A facet F is said to be *incident* to an extreme ray e if $e \subset F$. Two extreme rays of C are said to be *adjacent* if they span a two-dimensional face of C. Two facets of C are said to be *adjacent* if their intersection has dimension $\dim(C) - 2$.

All 14 orbits F_m, $1 \leq m \leq 14$, of facets of HYP_7, found by Baranovskii, are represented below by the corresponding vector b^m:

$$
\begin{aligned}
b^1 &= (1, 1, -1, 0, 0, 0, 0);\\
b^2 &= (1, 1, 1, -1, -1, 0, 0);\\
b^3 &= (1, 1, 1, 1, -1, -2, 0);\\
b^4 &= (2, 1, 1, -1, -1, -1, 0);\\
b^5 &= (1, 1, 1, 1, -1, -1, -1);\\
b^6 &= (2, 2, 1, -1, -1, -1, -1);\\
b^7 &= (1, 1, 1, 1, 1, -2, -2);\\
b^8 &= (2, 1, 1, 1, -1, -1, -2);\\
b^9 &= (3, 1, 1, -1, -1, -1, -1);\\
b^{10} &= (1, 1, 1, 1, 1, -1, -3);\\
b^{11} &= (2, 2, 1, 1, -1, -1, -3);\\
b^{12} &= (3, 1, 1, 1, -1, -2, -2);\\
b^{13} &= (3, 2, 1, -1, -1, -1, -2);\\
b^{14} &= (2, 1, 1, 1, 1, -2, -3).
\end{aligned}
$$

It gives a total of 3,773 inequalities. The first ten orbits are the orbits of hypermetric facets of the cut cone CUT_7; the first four of them come as a *0-extension* of facets of the cone HYP_6 (see [Deza and Laurent 97, Chapter 7]). The orbits F_{11}–F_{14} consist of some 19-dimensional simplex faces of CUT_7, becoming simplex facets in HYP_7.

The proof (see [Baranovskii 70] and [Ryshkov and Baranovskii 98]) was in terms of volume of simplexes; this result implies that for any facet of HYP_7 the bound $|b_i| \leq 3$ holds (compare with the bound in the Introduction).

Because of the large number of facets of HYP_7, it is difficult to find extreme rays just by application of existing programs (see [Fukuda 03]). So, let us consider in more detail the cut cone CUT_7.

Denote by CUT_n (and call it the *cut cone*)), the cone generated by all *cuts* δ_S defined by

$$(\delta_S)_{ij} = 1 \text{ if } |S \cap \{i, j\}| = 1 \text{ and } (\delta_S)_{ij} = 0, \text{ otherwise},$$

where S is any subset of $\{1, \ldots, n\}$. The cone CUT_n has dimension $\binom{n}{2}$ and $2^{n-1} - 1$ nonzero cuts as generators of extreme rays. There are $\lfloor \frac{n}{2} \rfloor$ orbits of those cuts, corresponding to all nonzero values of $min(|S|, n - |S|)$. The skeleton of CUT_n is the complete graph $K_{2^{n-1}-1}$. See Part V of [Deza and Laurent 97] for a survey on facets of CUT_n.

The 38,780 facets of the cut cone CUT_7 are partitioned in 36 orbits (see [Grishukhin 90], [De Simone et al. 94], and Chapter 30 of [Deza and Laurent 97] for details). Of these 36 orbits, 10 are orbits of hypermetric facets. We computed the diameter of the skeleton of the dual CUT_7: It is exactly 3 (apropos, the diameter of the skeleton of MET_n, $n \geq 4$, is 2; see [Deza and Deza 94]). So, we have $CUT_n \subset HYP_n$ and the cones CUT_7, HYP_7 have 10 common (hypermetric) facets: F_1–F_{10}.

Each of the 26 orbits of nonhypermetric facets of CUT_7 consists of simplex cones, i.e., those facets are incident exactly to 20 cuts or, in other words, adjacent to 20 other facets. It turns out that the 26 orbits of nonhypermetric facets of CUT_7 correspond exactly to 26 orbits of noncut extreme rays of HYP_7.

In fact, if d is a point of an extreme ray of HYP_7, which is not a cut, then it violates one of the nonhypermetric facet inequalities of CUT_7. More precisely, our computation consists of the following steps:

1. If d belongs to a noncut extreme ray of HYP_7, then $d \notin CUT_7$.
2. So, there is at least one nonhypermetric facet F of CUT_7 with $F(d) < 0$.
3. Select a facet F_i for each nonhypermetric orbit O_i with $1 \leq i \leq 26$ and define 26 subcones C_i, $1 \leq i \leq 26$, by $C_i = \{d \in HYP_7 : F_i(d) \leq 0\}$.
4. The initial set of 3,773 hypermetric inequalities is nonredundant, but adding the inequality $F_i(d) \leq 0$

	12	13	14	15	16	17	23	24	25	26	27	34	35	36	37	45	46	47	56	57	67
O_1	-1	-1	0	0	1	1	-1	0	1	0	1	1	0	1	0	1	-1	1	1	-1	0
$R_4; \overline{G}_{24}$	2	2	2	2	1	1	2	1	1	2	1	1	1	1	2	1	2	1	1	2	2
$O_2; \delta_{\{\underline{3},5,6\}}$	-1	1	0	0	-1	1	1	0	-1	0	1	-1	0	1	0	-1	1	1	1	1	0
$R_5; \overline{G}_4$	2	1	2	1	2	1	1	1	2	1	1	2	1	1	1	2	1	1	1	1	1
$O_3; \delta_{\{\underline{3},5,4\}}$	-1	1	0	0	1	1	1	0	-1	0	1	1	0	-1	0	1	1	-1	-1	1	0
$R_6; \overline{G}_{23}$	2	1	1	1	1	1	1	2	2	2	1	1	1	2	1	1	1	2	2	1	2
O_4	-1	-1	-1	1	1	1	-1	0	0	1	1	0	1	0	1	1	1	1	0	-1	-1
$R_7; \overline{G}_{25}$	2	2	2	1	1	2	2	1	2	1	1	1	1	2	1	1	1	1	1	2	2
$O_5; \delta_{\{\underline{3},7\}}$	-1	1	-1	1	1	-1	1	0	0	1	-1	0	-1	0	1	1	1	-1	0	1	1
$R_8; \overline{G}_5$	2	1	2	1	1	1	1	1	2	1	2	2	2	1	1	1	1	2	1	1	1
$O_6; \delta_{\{\underline{2},3,7\}}$	1	1	-1	1	1	-1	-1	0	0	-1	1	0	-1	0	1	1	1	-1	0	1	1
$R_9; \overline{G}_{26}$	1	1	2	1	1	1	2	2	1	2	1	2	2	1	1	1	1	2	1	1	1
$O_7; \delta_{\{1,\underline{5},6\}}$	1	1	1	1	1	-1	-1	0	0	-1	1	0	-1	0	1	-1	-1	1	0	1	1
$R_{10}; \overline{G}_1$	1	1	1	1	1	1	2	1	1	2	1	1	2	1	1	2	2	1	1	1	1
O_8	-1	-1	-1	0	1	2	-1	0	1	1	2	0	1	1	2	-1	1	1	0	-1	-2
$R_{11}; \overline{G}_{22}$	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	2	1	1	1	2	2
$O_9; \delta_{\{\underline{1},4,6\}}$	1	1	-1	0	1	-2	-1	0	1	-1	2	0	1	-1	2	1	1	-1	0	-1	2
$R_{12}; \overline{G}_{21}$	1	1	2	1	2	2	1	2	1	2	1	2	1	2	1	1	1	2	2	2	1
$O_{10}; \delta_{\{5\}}$	-1	-1	-1	0	1	2	-1	0	-1	1	2	0	-1	1	2	1	1	1	0	1	-2
$R_{13}; \overline{G}_{20}$	2	2	2	1	2	1	1	1	2	1	1	1	2	1	1	1	1	1	2	1	2
$O_{11}; \delta_{\{\underline{3},5\}}$	-1	1	-1	0	1	2	1	0	-1	1	2	0	1	-1	-2	1	1	1	0	1	-2
$R_{14}; \overline{G}_{19}$	2	1	2	1	2	1	2	1	2	1	1	2	1	2	2	1	1	1	2	1	2
$O_{12}; \delta_{\{\underline{1},7\}}$	1	1	1	0	-1	2	-1	0	1	1	-2	0	1	1	-2	-1	1	-1	0	1	2
$R_{15}; \overline{G}_7$	1	1	1	1	1	1	1	1	1	1	2	1	1	1	2	2	1	2	1	1	1
$O_{13}; \delta_{\{\underline{7},4,1\}}$	1	1	-1	0	-1	2	-1	0	1	1	-2	0	1	1	-2	1	-1	1	0	1	2
$R_{16}; \overline{G}_8$	1	1	2	1	1	1	1	2	1	1	2	2	1	1	2	1	2	1	1	1	1
$O_{14}; \delta_{\{\underline{6},4\}}$	-1	-1	1	0	-1	2	-1	0	1	-1	2	0	1	-1	2	1	1	-1	0	-1	2
$R_{17}; \overline{G}_{18}$	2	2	1	2	1	1	1	2	1	2	1	2	1	2	1	1	1	2	2	2	1
O_{15}	-1	-1	-2	1	1	2	0	-1	1	1	2	-2	1	1	1	2	2	3	-1	-2	-2
$R_{18}; \overline{G}_{14}$	2	2	2	2	2	1	1	1	1	1	1	2	1	1	2	1	1	1	1	2	2
$O_{16}; \delta_{\{\underline{5},3\}}$	-1	1	-2	-1	1	2	0	-1	-1	1	2	2	1	-1	-1	-2	2	3	1	2	-2
$R_{19}; \overline{G}_{15}$	2	1	2	1	2	1	2	1	2	1	1	1	1	2	1	2	1	1	2	1	2
$O_{17}; \delta_{\{\underline{5},4\}}$	-1	-1	2	-1	1	2	0	1	-1	1	2	2	-1	1	1	2	-2	-3	1	2	-2
$R_{20}; \overline{G}_{17}$	2	2	1	1	2	1	1	2	2	1	1	1	2	1	2	1	2	2	2	1	2
$O_{18}; \delta_{\{\underline{7},2,6\}}$	-1	1	2	-1	-1	2	0	1	-1	-1	2	-2	1	1	-1	2	2	-3	-1	2	2
$R_{21}; \overline{G}_{13}$	2	1	1	1	1	1	2	2	2	2	1	2	1	1	1	1	1	2	1	1	1
$O_{19}; \delta_{\{\underline{7},4,1\}}$	1	1	-2	-1	-1	2	0	1	1	1	-2	2	1	1	-1	-2	-2	3	-1	2	2
$R_{22}; \overline{G}_6$	1	1	2	1	1	1	1	2	1	1	2	1	1	1	1	2	2	1	1	1	1
$O_{20}; \delta_{\{\underline{1},7\}}$	1	1	2	-1	-1	2	0	-1	1	1	-2	-2	1	1	-1	2	2	-3	-1	2	2
$R_{23}; \overline{G}_2$	1	1	1	1	1	1	1	1	1	1	2	2	1	1	1	1	1	2	1	1	1
$O_{21}; \delta_{\{\underline{4},5,6\}}$	-1	-1	2	-1	-1	2	0	1	-1	-1	2	2	-1	-1	1	2	2	-3	-1	2	2
$R_{24}; \overline{G}_{16}$	2	2	1	1	1	1	1	2	2	2	1	1	2	2	2	1	1	2	1	1	1
O_{22}	-1	-1	-2	1	2	3	-1	-2	1	2	3	-2	1	2	3	2	3	5	-2	-3	-5
$R_{25}; \overline{G}_{11}$	1	1	2	2	1	1	1	2	2	1	1	2	2	1	1	1	2	1	2	2	2
$O_{23}; \delta_{\{\underline{3},6\}}$	-1	1	-2	1	-2	3	1	-2	1	-2	3	2	-1	2	-3	2	-3	5	2	-3	5
$R_{26}; \overline{G}_{10}$	1	2	2	2	2	1	2	2	2	2	1	1	1	1	2	1	1	1	1	2	1
$O_{24}; \delta_{\{\underline{7},4\}}$	-1	-1	2	1	2	-3	-1	2	1	2	-3	2	1	2	-3	-2	-3	5	-2	3	5
$R_{27}; \overline{G}_9$	1	1	1	2	1	2	1	1	2	1	2	1	2	1	2	2	1	1	2	1	1
$O_{25}; \delta_{\{5\}}$	-1	-1	-2	-1	2	3	-1	-2	-1	2	3	-2	-1	2	3	-2	3	5	2	3	-5
$R_{28}; \overline{G}_{12}$	1	1	2	1	1	1	1	2	1	1	1	2	1	1	1	2	2	1	1	1	2
$O_{26}; \delta_{\{\underline{5},4\}}$	-1	-1	2	-1	2	3	-1	2	-1	2	3	2	-1	2	3	2	-3	-5	2	3	-5
$R_{29}; \overline{G}_3$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	1	1	2

TABLE 1. Nonhypermetric facets of CUT_7 and noncut extreme rays of HYP_7.

yields a highly redundant set of inequalities. We remove the redundant inequalities using an invariant group (of permutations preserving the cone C_i) and linear programming (see polyhedral FAQ[2] in [Fukuda 03]).

5. For each of 26 subcones, we found, by computation, a set of 21 nonredundant facets, i.e., each of the subcones C_i is a simplex. One gets 21 extreme rays for each of these 26 subcones.

6. We remove the 20 extreme rays, which correspond to cuts, from each list and get, for each of these subcones, exactly one noncut extreme ray.

So, one gets an upper bound 26 for the number of noncut orbits of extreme rays. But [Deza et al. 92] gave, in fact, a lower bound of 26 for this number. So, one gets:

[2]http://www.ifor.math.ethz.ch/ fukuda/polyfaq/polyfaq.html.

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	Inc.	$\|R_i\|$
R_1	90	150	150	180	20	15	15	180	30	30	180	180	120	120	1460	7
R_2	80	130	80	220	20	60	0	180	40	10	240	100	320	10	1490	21
R_3	75	126	96	180	18	36	12	156	30	12	162	132	240	84	1359	35
R_4	13	7	0	0	0	0	0	0	0	0	0	0	0	0	20	2520
R_5	14	6	0	0	0	0	0	0	0	0	0	0	0	0	20	2520
R_6	13	7	0	0	0	0	0	0	0	0	0	0	0	0	20	2520
R_7	14	5	0	0	1	0	0	0	0	0	0	0	0	0	20	2520
R_8	15	4	0	0	1	0	0	0	0	0	0	0	0	0	20	1260
R_9	14	5	0	0	1	0	0	0	0	0	0	0	0	0	20	1260
R_{10}	15	5	0	0	0	0	0	0	0	0	0	0	0	0	20	252
R_{11}	11	7	1	1	0	0	0	0	0	0	0	0	0	0	20	2520
R_{12}	11	7	0	2	0	0	0	0	0	0	0	0	0	0	20	2520
R_{13}	12	6	2	0	0	0	0	0	0	0	0	0	0	0	20	2520
R_{14}	11	7	0	2	0	0	0	0	0	0	0	0	0	0	20	2520
R_{15}	12	7	0	1	0	0	0	0	0	0	0	0	0	0	20	1260
R_{16}	12	6	0	2	0	0	0	0	0	0	0	0	0	0	20	1260
R_{17}	12	6	2	0	0	0	0	0	0	0	0	0	0	0	20	630
R_{18}	10	6	0	2	1	0	0	1	0	0	0	0	0	0	20	2520
R_{19}	11	5	1	1	1	0	0	1	0	0	0	0	0	0	20	2520
R_{20}	10	6	0	2	1	1	0	0	0	0	0	0	0	0	20	1260
R_{21}	10	6	1	1	1	1	0	0	0	0	0	0	0	0	20	840
R_{22}	11	6	1	0	1	0	0	1	0	0	0	0	0	0	20	840
R_{23}	11	6	0	2	1	0	0	0	0	0	0	0	0	0	20	420
R_{24}	11	6	2	0	0	0	1	0	0	0	0	0	0	0	20	420
R_{25}	7	6	1	3	0	1	0	1	0	0	0	0	1	0	20	840
R_{26}	8	5	2	2	0	0	0	2	0	0	1	0	0	0	20	630
R_{27}	8	6	0	3	0	0	0	2	0	0	0	1	0	0	20	420
R_{28}	8	6	4	0	0	0	1	0	0	0	0	0	0	1	20	210
R_{29}	8	6	0	4	0	2	0	0	0	0	0	0	0	0	20	105
$\|F_i\|$	105	210	210	420	35	105	21	420	105	42	630	420	840	210	3773	37170

TABLE 2. Orbitwise incidence between extreme rays and facets of HYP_7.

Proposition 3.1. *The hypermetric cone HYP_7 has 37,170 extreme rays, divided into 3 orbits, corresponding to nonzero cuts, and 26 orbits, corresponding to hypermetrics on 7-vertex affine bases of the Schläfli polytope.*

Note that the above computation proves again that the list of 14 orbits of hypermetric facets is complete. If not, there would exist an hypermetric facet, which is violated by one extreme ray belonging to the 29 found orbits, but this would imply that the Schläfli polytope (or the 1-simplex) has interior lattice points, which is false.

The actual implementation, using the program `cdd` [Fukuda 03], of our computational techniques is available from [Dutour 03b].

Corollary 3.2. *The only extreme Delaunay polytopes of dimension at most six are the 1-simplex and the Schläfli polytope.*

This method computes precisely the difference between HYP_7 and CUT_7.

The observed correspondence between the 37,107 nonhypermetric facets of CUT_7 and the 37,107 noncut extreme rays of HYP_7 is presented in Table 1.

The first line of Table 1 indicates the ij position of the vector, defining facets, and generators of extreme rays. Using double lines, we separate 26 pairs (facet and corresponding extreme ray) into five *switching* classes. Two facets F and F' of CUT_7 are called *switching equivalent* if there exist

$$S \subset \{1, \dots, 7\}, \text{ such that } F(\delta_S) = 0,$$
$$F_{ij} = -F'_{ij} \text{ for } |S \cap \{i, j\}| = 1 \text{ and}$$
$$F_{ij} = F'_{ij}, \text{ otherwise.}$$

See Section 9 of [Deza et al. 92] for details on the switching in this case. The first column of Table 1 gives, for each of five switching classes, the cut δ_S, such that the corresponding facet is obtained by the switching by δ_S from the first facet of the class. The nonhypermetric orbits of facets of CUT_7 are indicated by O_i and the corresponding noncut orbits of extreme rays of HYP_7 are indicated by R_{i+3}. For any extreme ray, we indicate also the corresponding graph $\overline{G}_j$ (in terms of [Deza et al. 92] and [Deza and Laurent 97, Chapter 16]).

The five switching classes of Table 1 correspond, respectively, to the following five classes of nonhypermetric facets of CUT_7, in terms of [De Simone et al. 94] and [Deza and Laurent 97, Chapter 30]: *parachute* facets $P1 - P3$; *cycle* facets $C1, C4 - C6$; *Grishukhin* facets $G1 - G7$; cycle facets $C2, C7 - C12$; and cycle facets $C3, C13 - C16$.

[Deza and Grishukhin 93] considered extreme rays of HYP_n, which correspond, moreover, to the path-metric of a graph; the Delaunay polytope, generated by such hypermetrics, belongs to an integer lattice and, moreover, to a root lattice. They found, among the 26 noncut orbits of extreme rays of HYP_7, exactly 12 that are graphic: R_4, R_5, R_8, R_9, R_{10}, R_{15}, R_{16}, R_{17}, R_{22}, R_{23}, R_{24}, R_{29}. For example, R_{10}, R_{23}, and R_{29} correspond to graphic hypermetrics on $K_7 - C_5$, $K_7 - P_4$, and $K_7 - P_3$, respectively. Three of the above 12 extreme hypermetrics correspond to polytopal graphs: the 3-polytopal graphs, corresponding to R_4, and 4-polytopal graphs $K_7 - C_5$, $K_7 - P_4$. Note also that the footnote and figures in [Deza and Laurent 97, pages 242–243] mistakenly attribute the graph $\overline{G}_{18}$ to the class $q = 11$ (fourth class, in our terms); in fact, it belongs to the class $q = 12$ (our third class) as it was originally correctly given in [Deza et al. 92].

4. ADJACENCY PROPERTIES OF THE SKELETON OF HYP_7 AND OF ITS DUAL

We start with Table 2, giving incidence between extreme rays and facets, i.e., the number of facets from the orbit F_j, containing a fixed extreme ray from the orbit R_i, is given at postion ij.

It turns out, curiously, that each of 19-dimensional hypermetric faces F_{11}–F_{14} of the 21-dimensional cone CUT_7 (which became simplex facets in HYP_7) is the intersection of a triangle facet and a cycle facet, corresponding, respectively, to orbits O_{23}, O_{24}, O_{22}, and O_{25} of Table 1.

The *skeleton graph* of HYP_7 is the graph whose nodes are the extreme rays of HYP_7 and whose edges are the pairs of adjacent extreme rays. The *ridge graph* of HYP_7 is the graph whose node set is the set of facets of HYP_7 and with an edge between two facets if they are adjacent on HYP_7. Those graphs can be computed easily, since we know all extreme rays and facets of HYP_7. Then the `nauty` program (see [McKay 03]) finds the symmetry group $Sym(7)$ for the ridge graph of HYP_7. The only symmetries preserving all facets of a cone, which is not a simplex, are the homotheties $v \mapsto \lambda v$ with $\lambda > 0$. So, the symmetry group of HYP_7 is $Sym(7) \times \mathbb{R}^*_+$.

Proposition 4.1. *One has the following properties of adjacency of extreme rays of HYP_7:*

(i) The restriction of the skeleton of HYP_7 on the union of cut orbits $R_1 \cup R_2 \cup R_3$ is the complete graph.

(ii) Every noncut extreme ray of HYP_7 has adjacency 20 (namely, it is adjacent to 20 cuts lying on corresponding nonhypermetric facets of CUT_7); see the distribution of those 20 cuts amongst the cut orbits in Table 3.

	R_1	R_2	R_3	Adj.	$\|R_i\|$
R_1	6	21	35	15662	7
R_2	7	20	35	12532	21
R_3	7	21	34	10664	35
R_4	3	6	11	20	2520
R_5	4	7	9	20	2520
R_6	3	7	10	20	2520
R_7	3	7	10	20	2520
R_8	4	7	9	20	1260
R_9	3	8	9	20	1260
R_{10}	5	5	10	20	252
R_{11}	3	6	11	20	2520
R_{12}	2	8	10	20	2520
R_{13}	4	5	11	20	2520
R_{14}	2	8	10	20	2520
R_{15}	4	7	9	20	1260
R_{16}	3	9	8	20	1260
R_{17}	4	4	12	20	630
R_{18}	2	8	10	20	2520
R_{19}	3	7	10	20	2520
R_{20}	1	10	9	20	1260
R_{21}	2	9	9	20	840
R_{22}	4	6	10	20	840
R_{23}	4	7	9	20	420
R_{24}	5	1	14	20	420
R_{25}	1	9	10	20	840
R_{26}	2	8	10	20	630
R_{27}	3	6	11	20	420
R_{28}	5	1	14	20	210
R_{29}	2	12	6	20	105
$\|R_j\|$	7	21	35		37170

TABLE 3. Orbitwise adjacency between extreme rays of HYP_7.

(iii) Any two simplex extreme rays are nonadjacent; any simplex extreme ray (i.e., noncut ray) has a local graph (i.e., the restriction of the skeleton on the set of its neighbors) K_{20}.

(iv) The diameter of the skeleton graph of HYP_7 is 3.

One also has the following properties of adjacency of facets of HYP_7:

(i') See Table 4, where the number of facets (from orbit F_j), which are adjacent to the fixed facet of orbit F_i, are given by the ijth position.

(ii') Any two simplex facets are nonadjacent; any simplex facet (i.e., one amongst F_9–F_{14}) has a local graph K_{20}.

(iii') The diameter of the ridge graph of HYP_7 is 3.

5. FINAL REMARKS

In order to find extreme rays of HYP_8, the same methods will probably work with more computational difficulties; in dimension $n \geq 9$, polyhedral methods may fail.

The list of 374 orbits of noncut extreme rays of HYP_8 (containing 7,126,560 extreme rays), found in [Dutour 03a], will be compared with the list of at least 2,169 orbits

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	Adj.	$\|F_i\|$
F_1	86	168	110	216	35	56	13	196	14	6	54	36	64	18	1072	105
F_2	84	116	62	114	3	5	1	18	0	0	15	12	24	6	460	210
F_3	55	62	9	20	1	1	1	4	1	1	6	0	4	4	169	210
F_4	54	57	10	25	2	2	0	6	1	0	3	3	6	0	169	420
F_5	105	18	6	24	0	3	0	12	0	0	0	0	0	0	168	35
F_6	56	10	2	8	1	2	0	8	0	0	0	0	8	0	95	105
F_7	65	10	10	0	0	0	0	0	0	0	0	0	0	10	95	21
F_8	49	9	2	6	1	2	0	5	0	0	3	2	2	0	81	420
F_9	14	0	2	4	0	0	0	0	0	0	0	0	0	0	20	105
F_{10}	15	0	5	0	0	0	0	0	0	0	0	0	0	0	20	42
F_{11}	9	5	2	2	0	0	0	2	0	0	0	0	0	0	20	630
F_{12}	9	6	0	3	0	0	0	2	0	0	0	0	0	0	20	420
F_{13}	8	6	1	3	0	1	0	1	0	0	0	0	0	0	20	840
F_{14}	9	6	4	0	0	0	1	0	0	0	0	0	0	0	20	210
$\|F_j\|$	105	210	210	420	35	105	21	420	105	42	630	420	840	210		3773

TABLE 4. Orbitwise adjacency between facets of HYP_7.

of facets of CUT_8, found in [Christof and Reinelt 98]. Exactly 55 of the above 374 orbits correspond to path-metrics of a graph. It was shown in [Deza and Grishukhin 93] that any graph whose path-metric lies on an extreme ray of a HYP_n is a subgraph of the skeleton of Gosset or Schläfli polytopes.

It turns out (it can also be found in [Deza and Laurent 97, Chapter 28]) that exactly 26 of those orbits consist of hypermetric inequalities; 10 are 0-extensions of the hypermetric facets of CUT_7 and 16 come from the following vectors b (see (1–1)):

$$(2, 1, 1, 1, -1, -1, -1, -1),$$
$$(3, 1, 1, 1, -1, -1, -1, -2),$$
$$(2, 2, 1, 1, -1, -1, -1, -2),$$
$$(4, 1, 1, -1, -1, -1, -1, -1),$$
$$(3, 2, 2, -1, -1, -1, -1, -2),$$

representing, respectively, switching classes of sizes 2, 4, 3, 2, 5.

There is a one-to-one correspondence between nonhypermetric facets of CUT_7 and noncut extreme rays of HYP_7; namely, every such facet is violated by exactly one such ray. There is also a one-to-one correspondence between 10 noncut extreme rays (in fact, 10 permutations of the path-metric $K_{2\times 3}$) for MET_5 and 10 nontriangle facets (in fact, 10 permutations of $b = (1, 1, 1, -1, -1)$) for CUT_5. There is no such connection between MET_n and CUT_n for $n > 5$ in general, but we hope, that similar correspondence exist for CUT_8 and HYP_8.

Another direction for further study is to find *all* faces of HYP_7. While the extreme rays of HYP_n yield the extreme Delaunay polytopes of dimension $n-1$, the study of all faces of HYP_n will provide the list of all (combinatorial types of) Delaunay polytopes of dimension less than or equal to $n-1$. See [Fedorov 85] for the three-dimensional case, [Ryshkov and Erdahl 87] and [Ryshkov and Erdahl 88] for the four-dimensional case, [Kononenko 99] and [Kononenko 02] for the five-dimensional case, and [Dutour 02] for the six-dimensional case.

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