

Patterns in Knot Cohomology, I

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We discuss Dror Bar-Natan's experimental data on cohomology groups of all prime knots with 11 or fewer crossings.

1. NOTATION

The Jones polynomial is determined by the skein relation

$$q^2 J(L_1) - q^{-2} J(L_2) = (q - q^{-1}) J(L_3),$$

where L_i are depicted in Figure 1, and by the normalization $J(\text{unknot}) = 1$. This standard normalization is different from the one in [Khovanov 99, Khovanov 02].

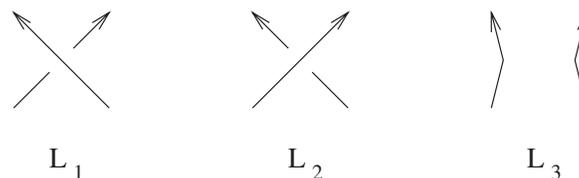


FIGURE 1.

Familiarity with [Khovanov 99, Khovanov 02] or [Bar-Natan 02] is assumed. *Note:* We use the grading conventions of [Khovanov 02], and the cohomology group that we denote by $\mathcal{H}^{i,j}$ is denoted $\mathcal{H}^{i,-j}$ in [Bar-Natan 02] and [Khovanov 99]. Let $h^{i,j}(K)$ (or simply $h^{i,j}$) be the rank of $\mathcal{H}^{i,j}(K)$. Ranks of cohomology groups satisfy (notice q^{-j} , rather than q^j)

$$(q + q^{-1})J(K) = \sum_{i,j} (-1)^i q^{-j} h^{i,j}(K). \quad (1-1)$$

We use the Rolfsen enumeration for knots with 10 or fewer crossings. Knots with more than 10 crossings are enumerated as in *Knotscape*, for instance, 11_{77}^n denotes the 77th nonalternating 11-crossing knot.

2. INITIAL OBSERVATIONS

There are 249 prime unoriented knots with at most 10 crossings (not counting mirror images). From Bar-Natan

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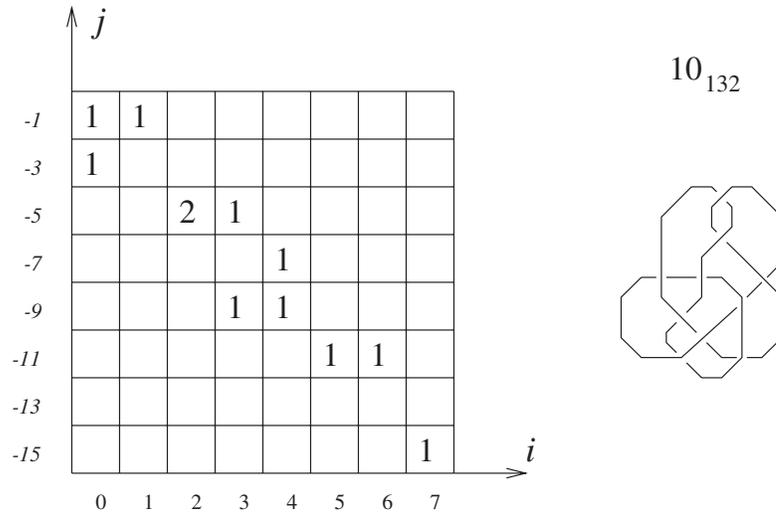


FIGURE 3. 10_{132} and ranks of its cohomology groups.

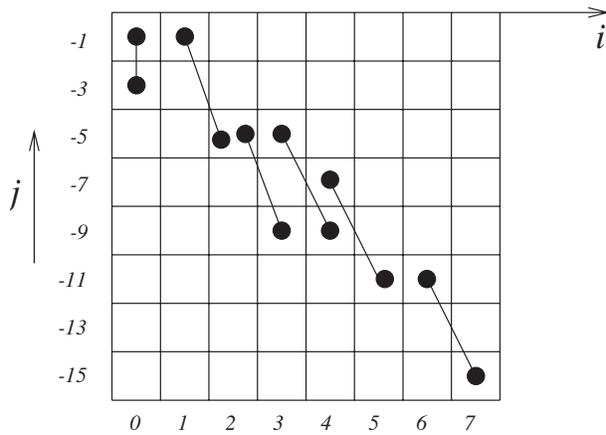


FIGURE 4. Cohomology of 10_{132} arranged in pairs.

(because of our choice of variable q , the right-hand side is $J_{\sqrt{-1}}(K)$ rather than the more common $J_{-1}(K)$).

Coefficient-wise, with notations from (iii), (iv),

$$\sum_i (-1)^i a_i = \sum_i (-1)^i c_i.$$

Since Jones and Alexander polynomials of H-thin knots with at most 10 crossings are alternating, for these knots we obtain

$$\sum_i |a_i| = \sum_i |c_i|.$$

Properties (i) and (ii) imply that, in addition,

$$\text{rank}\mathcal{H}(K) - 1 = \sum_i |c_i|,$$

where $\text{rank}\mathcal{H}(K) = \sum_{i,j} h^{i,j}$ is the rank of total cohomology of the knot. To summarize, H-thin knots with at

most 10 crossings satisfy

$$\text{rank}\mathcal{H}(K) - 1 = \sum_i |c_i| = \sum_i |a_i|. \quad (2-2)$$

What about the 12 H-thick knots? For each of them, the inequalities hold

$$\sum_i |a_i| \leq \text{rank}\mathcal{H}(K) - 3 \leq \sum_i |c_i| \quad (2-3)$$

(note that $\sum_i |a_i|$ and $\sum_i |c_i|$ are odd for any knot and $\text{rank}(\mathcal{H})$ is even).

Alternating knots with at most 10 crossings are H-thin, and it was conjectured in [Bar-Natan 02] and [Garoufalidis 01] that all alternating knots are H-thin. This conjecture is now a theorem, due to Eun Soo Lee [Lee 02]:

Theorem 2.1. *Nonsplit alternating links are H-thin.*

We now look at the data for 11-crossing knots [Bar-Natan 02]. There are 367 alternating and 185 nonalternating prime knots with 11 crossings. H-thick knots among them number 41.

Properties (i)–(iv) continue to hold for H-thin knots with 11 crossings. There are several 11-crossing knots with nonalternating Jones or Alexander polynomial. All of them are H-thick. Likewise, 11-crossing knots with a gap in the Alexander polynomial are H-thick.

Problem 2.2. Explain why so many nonalternating knots with 11 or fewer crossings are H-thin.

3. A-MODULE STRUCTURE OF KNOT COHOMOLOGY

In this section, we work over \mathbb{Q} (rather than over \mathbb{Z} , as in [Khovanov 99, Section 7]). In particular, the base ring is $\mathcal{A} = \mathbb{Q}[X]/(X^2)$ and the chain complex $\mathcal{C}(D)$ associated to a plane diagram D of a knot K is a complex of \mathbb{Q} -vector spaces. Cohomology groups $\mathcal{H}^{i,j}(D)$ are finite-dimensional \mathbb{Q} -vector spaces; only finitely many of them are nontrivial. Dimensions $h^{i,j}$ of these groups are invariants of K .

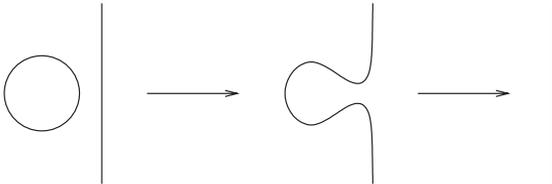


FIGURE 5. Cobordism between circle $\cup D$ and D .

Choose a segment I of D that does not contain crossings. Place an unknotted circle next to I and consider the cobordism that merges the circle and I (Figure 5). This cobordism induces a map of complexes $\mathcal{A} \otimes \mathcal{C}(D) \rightarrow \mathcal{C}(D)$ and makes $\mathcal{C}(D)$ into a complex of graded \mathcal{A} -modules. A Reidemeister move from D to D' that happens away from I induces a chain homotopy equivalence between complexes of \mathcal{A} -modules $\mathcal{C}(D)$ and $\mathcal{C}(D')$. Given two diagrams D_1 and D_2 of K and two segments I_1 and I_2 in them, there is a sequence of Reidemeister moves that takes (D_1, I_1) to (D_2, I_2) such that all moves happen away from I_1 . Instead of moving an arc over or under I_1 , we can move it across the rest of the plane (or S^2). In other words, there are as many knots as one-component $(1, 1)$ -tangles.

We obtain an invariant of K , the complex $\mathcal{C}(D)$ of free \mathcal{A} -modules up to chain homotopy equivalence. The Krull-Schmidt theorem, valid for bounded complexes of finite-dimensional modules over finite-dimensional algebras, tells us that $\mathcal{C}(D)$ decomposes (uniquely up to an isomorphism) as the direct sum of an acyclic complex and indecomposable complexes with nontrivial cohomology. The multiplicity of each indecomposable complex in this decomposition is an invariant of K . Denote by C_n the complex

$$0 \longrightarrow \mathcal{A} \xrightarrow{X} \mathcal{A}\{-2\} \xrightarrow{X} \dots \xrightarrow{X} \mathcal{A}\{-2n+2\} \xrightarrow{X} \mathcal{A}\{-2n\} \longrightarrow 0, \tag{3-1}$$

where the leftmost \mathcal{A} is in cohomological degree 0.

Proposition 3.1. *A nonacyclic indecomposable complex of free graded \mathcal{A} -modules is isomorphic to $C_n[i]\{j\}$ for a unique triple $(n, i, j), n \geq 0$.*

Example 3.2. If K is a $(2, 2m + 1)$ -torus knot, $\mathcal{C}(K)$ is a direct sum of $C_0\{2m\}$ and $C_1[2i + 1]\{4i + 2m + 2\}, 1 \leq i \leq m$.

Proposition 3.3. $\mathcal{C}(K_1 \# K_2) \cong \mathcal{C}(K_1) \otimes_{\mathcal{A}} \mathcal{C}(K_2)$.

Proof: Obvious. □

Define *homological width* of K , denoted $hw(K)$, as the minimal number m such that cohomology of K lie on m adjacent diagonals. The homological width of a knot is at least 2, since cohomology groups of indecomposable complexes C_n lie on 2 adjacent diagonals, and any knot has nontrivial cohomology (since the Jones polynomial does not vanish). According to our definitions, a knot is *H-thin* if and only if it has homological width 2.

Proposition 3.3 implies

Proposition 3.4. $hw(K_1 \# K_2) = hw(K_1) + hw(K_2) - 2$.

Corollary 3.5. $K_1 \# K_2$ is *H-thin* if and only if both K_1 and K_2 are *H-thin*.

3.1 Reduced Cohomology

Let $Q = \mathcal{A}/X\mathcal{A}$ be the one-dimensional representation of \mathcal{A} . Define the *reduced complex* of D by

$$\tilde{\mathcal{C}}(D) = \mathcal{C}(D) \otimes_{\mathcal{A}} Q.$$

This is a complex of graded \mathbb{Q} -vector spaces. We call its cohomology the *reduced cohomology* of D (and K) and denote by $\tilde{\mathcal{H}}(D)$ and $\tilde{\mathcal{H}}(K)$; the latter are defined up to isomorphism. Ranks of cohomology groups $\tilde{\mathcal{H}}^{i,j}(D)$ are invariants of K . The Euler characteristic of $\tilde{\mathcal{H}}$ is the Jones polynomial (compare to (1-1)):

$$J(K) = \sum_{i,j} (-1)^i q^{-j} \text{rank}(\tilde{\mathcal{H}}^{i,j}(K)),$$

therefore,

$$\text{rank} \tilde{\mathcal{H}}(K) \geq |J_{\sqrt{-1}}(K)| = |\Delta_{-1}(K)|.$$

Proposition 3.6. *Reduced cohomology groups $\tilde{\mathcal{H}}^{i,j}(K)$ lie on one diagonal ($2i + j$ is constant) if and only if K is *H-thin*.*

Corollary 3.7. *The Jones polynomial of an H -thin knot is alternating. The absolute values of its coefficients are dimensions of reduced cohomology groups.*

3.2 H-Restricted Knots

Properties (ii), (ii') admit a homological interpretation. We say that a knot K is H -restricted if nonacyclic indecomposable summands of the \mathcal{A} -module complex $\mathcal{C}(K)$ are one $\mathcal{A}\{i\}$, for some i , and one or several $C_1[j]\{k\}$, for $j, k \in \mathbb{Z}$. Cohomology groups of an H -restricted knot can be paired up as in (ii'). Existence of such pairing, however, does not imply that a knot is H -restricted.

$(2, 2m + 1)$ -torus knots are H -restricted. The figure-eight knot is H -restricted.

Proposition 3.8. *If K_1 and K_2 are H -restricted, then $K_1 \# K_2$ is H -restricted.*

Conjecture 3.9. *All knots are H -restricted.*

This is a homological counterpart of Conjecture 1 in [Bar-Natan 02]] about $Kh_{\mathbb{Q}}$.

Proposition 3.10. *If K is H -restricted, then $\text{rank}\mathcal{H}(K) = \text{rank}\tilde{\mathcal{H}}(K) - 1$.*

4. COHOMOLOGY WITH \mathbb{Z}_2 -COEFFICIENTS

Let us now work over \mathbb{Z} rather than \mathbb{Q} , so that $\mathcal{A} = \mathbb{Z}[X]/(X^2)$. A computation in [Khovanov 99, Section 6.2] implies that $\mathcal{C}(K)$, where K is a $(2, 2m + 1)$ -torus knot, is isomorphic to the direct sum (modulo acyclic complexes) of the complex $0 \rightarrow \mathcal{A}\{2m\} \rightarrow 0$ and m complexes C'_1

$$0 \rightarrow \mathcal{A} \xrightarrow{2X} \mathcal{A}\{-2\} \rightarrow 0 \tag{4-1}$$

with various shifts.

Cohomology of $C'_1 \otimes_{\mathbb{Z}} \mathbb{Q}$ is two-dimensional (over \mathbb{Q}), and is a matching pair of cohomology groups in bidegrees that differ by $(1, -4)$.

Now change the base field to \mathbb{Z}_2 . In characteristic 2, the differential in (4-1) is 0, and the dimension of cohomology groups of $C'_1 \otimes_{\mathbb{Z}} \mathbb{Z}_2$ is 4 (as a \mathbb{Z}_2 -vector space); see Figure 6.

According to the tables in Bar-Natan [Bar-Natan 02], the same patterns relate rational and \mathbb{Z}_2 -cohomology of any prime knot with at most seven crossings. Pair up the rational cohomology groups as in (ii), so that all but one pair look as on the left-hand side of Figure 6, and change each on them to the quadruple of 1s on the right-hand side. We get ranks of \mathbb{Z}_2 -cohomology groups.

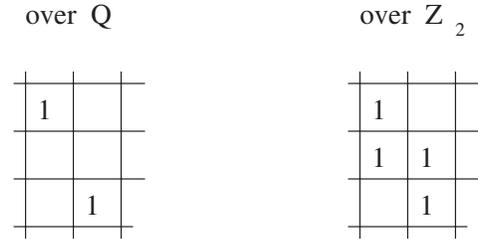


FIGURE 6. Dimensions of cohomology of C'_1 over \mathbb{Q} and \mathbb{Z}_2 .

It is likely that for any knot K with at most 7 crossings, $\mathcal{C}(K)$ decomposes as a direct sum of

- an acyclic complex,
- complex $0 \rightarrow \mathcal{A}\{i\} \rightarrow 0$ for some $i \in \mathbb{Z}$,
- complexes $0 \rightarrow \mathcal{A}\{j\} \xrightarrow{2kX} \mathcal{A}\{j - 2\} \rightarrow 0$ for $j, k \in \mathbb{Z}$.

This would explain the observed relationship between rational and \mathbb{Z}_2 -cohomology of these knots.

5. COHOMOLOGY OF ADEQUATE KNOTS

For a link diagram D , denote by s_+D (respectively, s_-D) the diagram obtained by taking 0-resolution (respectively, 1-resolution) of each crossing of D ; see Figure 7.

We say that D is *adequate* if

- for any crossing of D the two segments of s_+D that replace this crossing belong to distinct components of s_+D ;
- for any crossing of D the two segments of s_-D that replace this crossing belong to distinct components of s_-D .

A reduced alternating link diagram is adequate. A link admitting an adequate diagram is called *adequate*. For further information about adequate links, see Thistlethwaite [Thistlethwaite 88] and Lickorish [Lickorish 97, Chapter 5].

Proposition 5.1. *Adequate nonalternating knots are H -thick.*

Proof: Assume D is an adequate nonalternating diagram of a knot K . We continue to use cohomology with integer coefficients. Recall from [Khovanov 99, Chapter 7] that

$$\overline{\mathcal{H}}^0(D) \neq 0 \neq \overline{\mathcal{H}}^n(D),$$

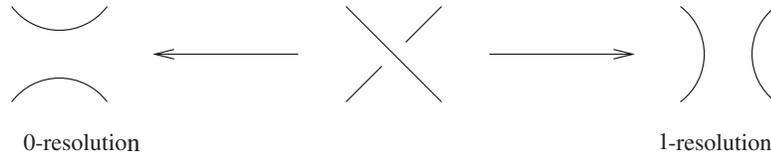


FIGURE 7. Two resolutions of a crossing.

where n is the number of crossing of D . More precisely,

$$\overline{\mathcal{H}}^{0,|s_+D|}(D) \cong \mathbb{Z}, \quad \overline{\mathcal{H}}^{0,i}(D) \cong 0, \text{ if } i > |s_+D|, \quad (5-1)$$

$$\overline{\mathcal{H}}^{n,-|s_-D|-n}(D) \cong \mathbb{Z}, \quad \overline{\mathcal{H}}^{0,i}(D) \cong 0, \text{ if } i < -|s_-D| - n, \quad (5-2)$$

where $|s_+D|$ is the number of components of s_+D , etc.

From discussion in Section 3, we know that rational cohomology groups come in pairs (complex (3-1) contributes $\mathbb{Q} \oplus \mathbb{Q}$ to cohomology, in two degrees that differ by $(n, -2n - 2)$). The companion of $\overline{\mathcal{H}}^{0,|s_+D|}(D) \otimes \mathbb{Q} \cong \mathbb{Q}$ will lie one diagonal below it, while the companion of $\overline{\mathcal{H}}^{n,-|s_-D|-n} \otimes \mathbb{Q}$ will lie one diagonal above it. This is illustrated in Figure 8, which unintentionally shows the case $n = |s_+D| + |s_-D|$.

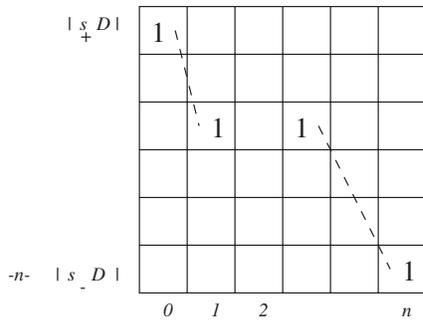


FIGURE 8.

If K is H -thin, these two pairs of cohomology groups must lie on two adjacent diagonals. This implies $n + 2 = |s_+D| + |s_-D|$.

Lemma 5.2. *If D is adequate, nonalternating, and prime, then $n + 2 > |s_+D| + |s_-D|$. If D is alternating, then $n + 2 = |s_+D| + |s_-D|$.*

This lemma is proved in [Lickorish 97, Chapter 5].

Therefore, if D is prime, K is H -thick. The case of composite D follows from Corollary 3.5. \square

There are no adequate nonalternating knots with 9 or fewer crossings; 3 adequate nonalternating knots with 10

crossings: 10_{152} , 10_{153} , 10_{154} ; and 15 adequate nonalternating 11-crossing knots.

6. COHOMOLOGY OF POSITIVE AND BRAID POSITIVE KNOTS

6.1 Positive Knots

We say that a knot is *positive* if it has a diagram with only positive crossings (Figure 9).

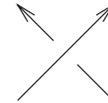


FIGURE 9. A positive crossing.

Proposition 6.1. *If K is a positive knot, then $\mathcal{H}^{i,j}(K) = 0$ if $i < 0$,*

$$\mathcal{H}^{0,j}(K) = \begin{cases} \mathbb{Z} & \text{if } j = s - n - 1 \pm 1 \\ 0 & \text{otherwise,} \end{cases}$$

and $\mathcal{H}^{i,j} = 0$ if $i > 0$ and $j \geq s - n$, where s is the number of Seifert circles and n the number of crossings in a positive diagram of K .

Proof: Left to the reader. \square

Note that $\frac{n-s+1}{2}$ is the genus of K .

6.2 Braid Positive Knots

8_{19} is a $(3, 4)$ -torus knot; 10_{124} is a $(3, 5)$ -torus knot. Both are H -thick. If n, m are odd, the (n, m) -torus knot is H -thick since its Jones polynomial is not alternating. We expect that (n, m) -torus knots, $2 < n < m$, are H -thick.

Torus knots are examples of *braid positive* knots, i.e., knots that are closures of positive braids.

Braid positive prime knots with at most 10 crossings are $(2, n)$ -torus knots, for $n \in \{3, 5, 7, 9\}$, and the four H -thick knots 8_{19} , 10_{124} , 10_{139} , 10_{152} .

There are two braid positive prime 11-crossing knots: the $(2, 11)$ -torus knot and 11^n_{77} , the closure of the braid $\sigma_1^2 \sigma_2^2 \sigma_1 \sigma_3 \sigma_2^3 \sigma_3^2$. The latter is H-thick [Bar-Natan 02].

There are 7 braid positive prime knots with 12 crossings. All of them are H-thick, since their Jones polynomials are not alternating.

Not counting the $(2, 13)$ -torus knot, there are 12 braid positive prime 13-crossing knots. At least 10 are H-thick (the Jones polynomial is not alternating). We don't know if the remaining knots 13^n_{4587} and 13^n_{5016} are H-thick.

There are 17 braid positive prime knots with 14 crossings. All but 3 have nonalternating Jones polynomial.

Problem 6.2. Are all braid positive prime knots other than $(2, n)$ -torus knots H-thick?

Problem 6.3. If K is braid positive, is $\mathcal{H}^{1,j}(K) = 0$ for all j ?

7. ALEXANDER POLYNOMIAL AND COHOMOLOGY

We say that a prime knot is *Ap-special* if its Alexander polynomial is not alternating or has a gap. A well-known theorem of Murasugi [Murasugi 59] can be restated as

Proposition 7.1. *Ap-special knots are not alternating.*

Few small knots are Ap-special, and all or nearly all small Ap-special knots are H-thick:

- There are 9 Ap-special knots with at most 10 crossings. All of them are H-thick.
- There are 19 Ap-special knots with 11 crossings. All of them are H-thick.
- There are 104 Ap-special knots with 12 crossings. For all but 8 of them the Jones polynomial is not alternating, so that at least 96 of these knots are H-thick.
- There are 115 knots with 13 crossings and a gap in the Alexander polynomial. All but 13 have nonalternating Jones polynomial, thus, at least 102 of these knots are H-thick.

Problem 7.2. Is any Ap-special knot H-thick?

Knots with nonalternating Jones polynomial are a minority among nonalternating knots with at most 14 crossings, as seen in Table 1.

crossings	≤ 9	10	11	12	13	14
nonalternating	11	42	185	888	5110	27110
Jones not alternating	0	7	26	169	1154	7075
H-thick	2	10	41	≥ 169	≥ 1154	≥ 7075

TABLE 1.

For instance, the fifth column says that there are 888 prime nonalternating knots with 12 crossings (not distinguishing mirror images); among them, 169 have nonalternating Jones polynomial, and, therefore, at least 169 are H-thick. On the other hand, there is no doubt that for large n , most n -crossing knots are H-thick.

The following examples provide another experimental relationship between the Alexander polynomial and knot cohomology.

1. The only knot with the trivial Alexander polynomial and at most 10 crossings is the unknot. There are 2 11-crossing, 2 12-crossing, 15 13-crossing and 36 14-crossing knots with the trivial Alexander polynomial. All of them are H-thick (since their Jones polynomials are not alternating).
2. The Alexander polynomial of the trefoil is $t^{-1} - 1 + t$. There are no other knots with at most 12 crossings and this Alexander polynomial. There are 8 13-crossing knots and 17 prime 14-crossing knots with this Alexander polynomial. All of them are H-thick (for the same reason).
3. The figure-eight knot is the only one with less than 13 crossings and Alexander polynomial $-t^{-1} + 3 - t$. There are 2 13-crossing knots and 15 14-crossing knots with this Alexander polynomial. All are H-thick.
4. $\Delta(5_2) = 2t^{-1} - 3 + 2t$. There are no other knots with this Alexander polynomial and less than 12 crossings. Four 12-crossing, 3 13-crossing, and 9 14-crossing knots have Alexander polynomial $2t^{-1} - 3 + 2t$. All of these knots are H-thick.
5. Consider knots with at most 14 crossings and Alexander polynomial $-2t^{-1} + 5 - 2t$. Four of them: 6_1 , 9_{46} , 11^n_{139} , and 13^n_{3523} are H-thin (these knots are examples of $(n, -3, 3)$ -pretzel knots; any $(n, -3, 3)$ -pretzel knot is slice, H-thin, and its cohomology has rank 10). The remaining 2 11-crossing knots, 4 12-crossing knots, 11 13-crossing knots, and 50 14-crossing knots with this polynomial are H-thick.

6. $\Delta(5_1) = \Delta(10_{132}) = t^{-2} - t^{-1} + 1 - t + t^2$. 5_1 is the $(2, 5)$ -torus knot and is H-thin. 10_{132} is H-thick. There are no 11- and 12-crossing knots with this Alexander polynomial. Two 13-crossing knots and 12 14-crossing knots have this Alexander polynomial. All are H-thick.
7. 10_{153} is the only knot with at most 11 crossings and Alexander polynomial $t^{-3} - t^{-2} - t^{-1} + 1 - t - t^2 + t^3$. Four 12-crossing, 7 13-crossing, and 19 14-crossing knots have this Alexander polynomial. All are H-thick. Unlike other examples, this Alexander polynomial is not alternating.

These examples suggest that knots with small Alexander polynomial relative to the crossing number tend to be H-thick.

8. VOLUME AND COHOMOLOGY

H-thick knots with few crossings tend to have small hyperbolic volume or to be nonhyperbolic:

- 8_{19} is the only H-thick knot with 8 crossings and the only nonhyperbolic knot with 8 crossings (it is the $(3, 4)$ -torus knot).
- The H-thick knot 10_{124} is the $(3, 5)$ -torus knot and the only nonhyperbolic 10-crossing knot.
- 9_{42} , the only H-thick, 9-crossing knot, has the second smallest volume (≈ 4.05686) among all 48 hyperbolic knots with 9 crossings (and the smallest determinant (= 7) among all 9-crossing knots). 9_{42} has the same volume as 10_{132} , another H-thick knot. The latter has the smallest volume among all hyperbolic knots with 10 crossings. Among known pairs of knots with the same volume, $(9_{42}, 10_{132})$ is the pair with the second smallest volume. The pair with the smallest volume (≈ 2.8281) consists of 5_2 and the famous $(-2, 3, 7)$ -pretzel knot. Knot 5_2 is H-thin, while the $(-2, 3, 7)$ -pretzel knot is H-thick, since its Jones polynomial is not alternating.
- Three out of the four hyperbolic 10-crossing knots with the smallest volumes are H-thick, even though among 164 hyperbolic 10-crossing knots only 9 are H-thick.

Determinant $\det(K)$ of a knot K is the determinant of the matrix $M + M^T$, where M is a Seifert matrix of K .

Determinant is a common specialization of the Alexander and Jones polynomials:

$$\det(K) = \Delta_{-1}(K) = J_{\sqrt{-1}}(K).$$

$|\det(K)|$ is also the number of elements in the first homology group of the double cover of \mathbb{S}^3 branched over K .

Nathan Dunfield documents a fascinating relationship between determinants and volumes of hyperbolic knots [Dunfield 01]. First, he plots $\log |\det(K)|$ versus the volume of K for all alternating knots K with a fixed number of crossings. Amazingly, the points cluster around a straight line. Next, he combines the pictures into one by plotting $\frac{\log |\det(K)|}{\log(\deg J(K))}$ versus the volume of K for all alternating knots with at most 13 crossings and samples of 14–16 crossing alternating knots. Again, all points stay close to a straight line.

Dunfield comments: “ $\log(J(-1))$ is one of the first terms in Kashaev’s conjecture about the relationship between the colored Jones polynomial and hyperbolic volume. However, the above doesn’t appear to simply be saying that you have fast convergence in Kashaev’s conjecture as the slope of the line is not what you would expect.”

When nonalternating knots are included, the plots become less impressive. The majority of points still lie close to the coveted straight line, but there are defections. For instance, there are hyperbolic knots with $\det(K) = \pm 1$, and points assigned to them will lie on the x-axis, far away from where we would like them to. This is illustrated in Figures 10 and 11, where we plot $(\text{vol}(K), \log |\det(K)|)$ for all hyperbolic nonalternating knots with 10- and 11-crossings (for 12-, 13-crossings, consult [Dunfield 01]).

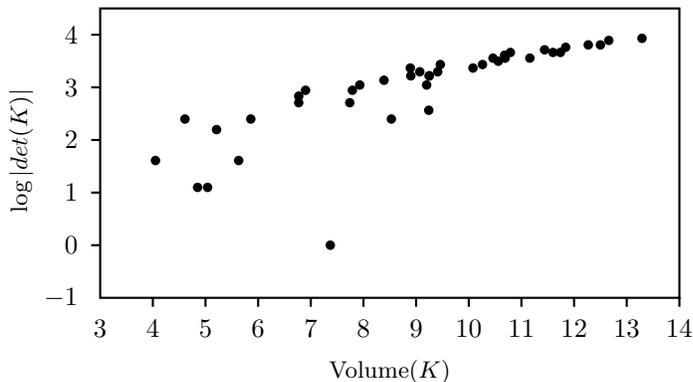


FIGURE 10. Volume versus $\log |\det(K)|$ for 10-crossing nonalternating knots.

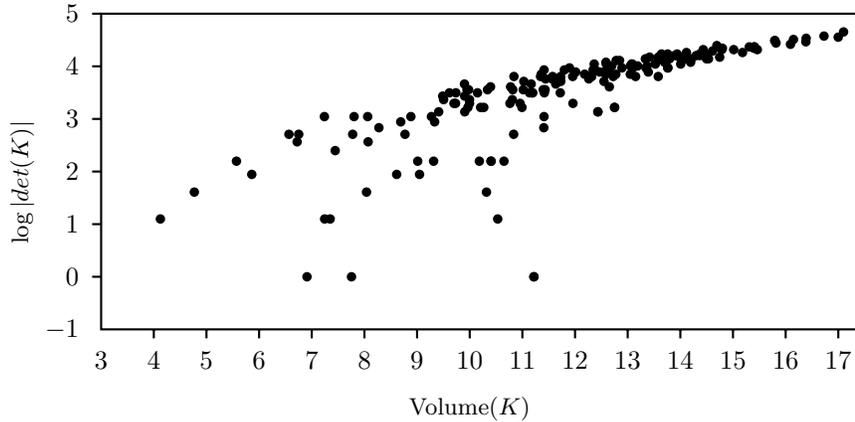


FIGURE 11. Volume versus $\log |det(K)|$ for 11-crossing nonalternating knots.

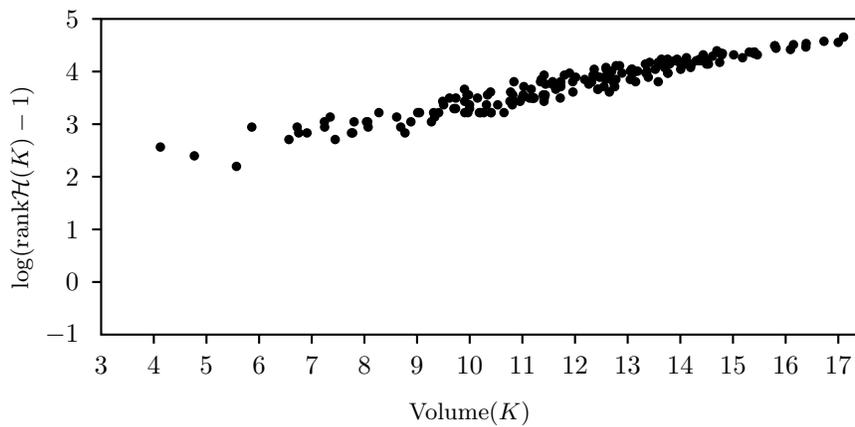


FIGURE 12. Volume versus $\log(\text{rank}\mathcal{H}(K) - 1)$ for 11-crossing nonalternating knots.

To save the day, we change from $det(K)$ to the rank of the reduced cohomology group of K . The inequality

$$\text{rank}\tilde{\mathcal{H}}(K) \geq |det(K)|$$

is valid for all knots, and turns into equality for H -thin knots. If the knot is H -restricted (and we expect that all knots are), $\text{rank}\tilde{\mathcal{H}}(K) = \text{rank}\mathcal{H}(K) - 1$. In Figures 12 and 13, we plot $(\text{vol}(K), \log(\text{rank}\mathcal{H}(K) - 1))$ for all hyperbolic nonalternating knots with 11- and 10- crossings (there are 185, respectively, 41, such knots).

Clearly, for nonalternating knots with 10- and 11-crossings, the correlation between the volume and the rank of cohomology is even better than the one between the volume and the determinant. Somehow $\text{vol}(K)$ and $\text{rank}\mathcal{H}(K)$ are successful in spying on each other. We have no explanation for this behaviour.

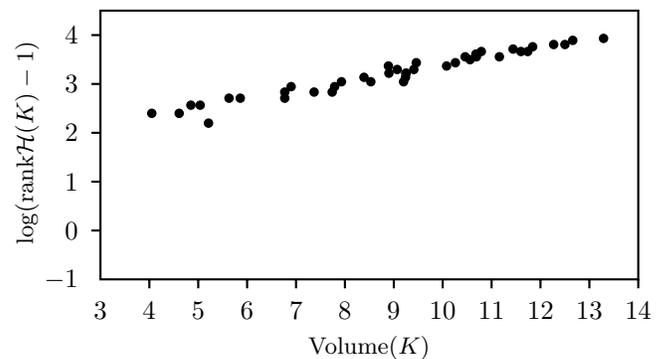


FIGURE 13. Volume versus $\log(\text{rank}\mathcal{H}(K) - 1)$ for 10-crossing nonalternating knots.

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