Partial Sums of $\zeta(\frac{1}{2})$ Modulo 1

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CONTENTS

Let $P_s(n) = \sum_{j=1}^{n} j^{-s}$. For fixed s near $s = \frac{1}{2}$, we divide the unit interval into bins and count how many of the partial sums $P_s(1)$, $P_s(2)$, ..., $P_s(N)$ lie in each bin mod 1. The properties of the histogram are predicted by a random model unless $s = \frac{1}{2}$. When $s = \frac{1}{2}$ the histogram is surprisingly flat, but has a few strong spikes. To explain the surprises at $s = \frac{1}{2}$, we use classical results about Diophantine approxmation, lattice points, and uniform distribution of sequences.

1. INTRODUCTION

The Riemann zeta function is defined by the following sum for $\operatorname{Re} s > 1$:

$$\zeta(s) = \sum_{j=1}^{\infty} j^{-s}.$$

The sum fails to converge when $\operatorname{Re} s \leq 1$, but its partial sums are well defined:

$$P_s(n) = \sum_{j=1}^{n} j^{-s}.$$

To study the distribution of the $P_s(n) \bmod 1$, we partition the unit interval into B equal bins, $I_j = \bigl[(j-1)/B,\, j/B\bigr)$, and count how many of the partial sums $P_s(1)$, $P_s(2)$, ..., $P_s(N)$ lie in each bin mod 1. Daniel Asimov observed that for fixed values of N and B, the histogram of these bin counts is wildly different for $s = \frac{1}{2}$ than for other nearby values of s. Histograms with $s = 0.50$ and $s = 0.51$ are shown in Figure 1 for same N and B. *The goal of this article is to explain the striking difference between the two parts of Figure* 1.

In Section 2 we estimate the number of partial sums which lie in each bin. This baseline is a positively sloped ramp function with a jump discontinuity down at $P_s(N) \bmod 1$. Section 3 introduces a random model and predicts how much bin counts may deviate from the baseline. When $s \neq \frac{1}{2}$, our

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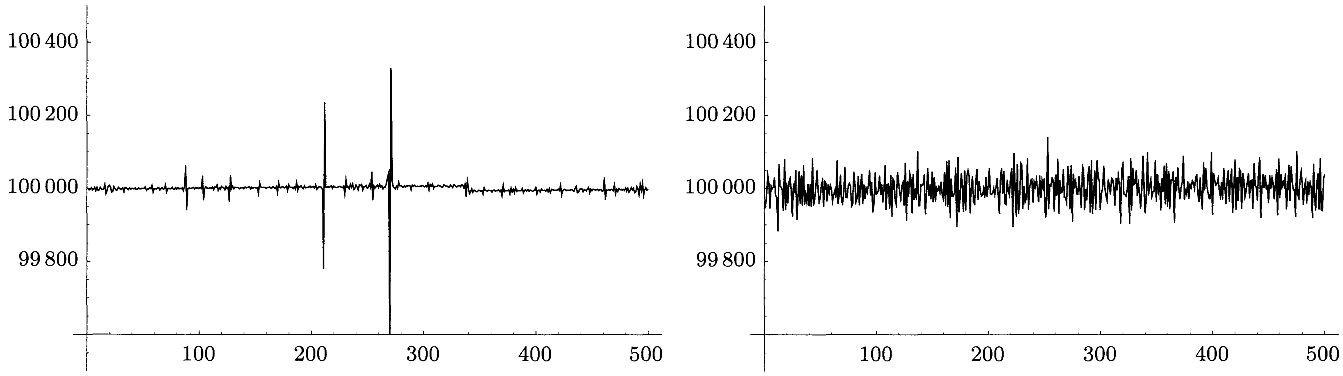

FIGURE 1. Histogram with 500 bins of the fractional parts of the first 5×10^7 partial sums of $\zeta(0.50)$ (left) and of $\zeta(0.51)$ (right).

numerical experiments agree closely with the predictions of the random model.

When $s = \frac{1}{2}$ there is structure not captured by the random model; see Figure 2. The staggering in this picture suggests that the partial sums will line up even better mod 2 than they do mod 1, as shown in Figure 3. To count how many times the partial sum is in a particular interval mod 2, draw vertical lines at the endpoints of the interval, and count how many of the following ordered pairs land inside:

$$\left(\{P_{1/2}(n)\}_2,\ \lfloor P_{1/2}(n)\rfloor_2\right), \quad \text{for } 1 \le n \le N.$$

Here $2\lfloor x\rfloor_2$ is the largest even integer less than or equal to x, and $\{x\}_2 = x - 2\lfloor x\rfloor_2$. The same symbols $\lfloor x\rfloor$ and $\{x\}$ without subscripts are just the usual integer and fractional parts of a real number x.

We now apply a transformation which unfolds Figure 3. The walls of the bins map to sloped lines. The ordered pairs $\left(\{P_{1/2}(n)\}_2,\ \lfloor P_{1/2}(n)\rfloor_2\right)$ map approximately to points on the integer lattice. Thus counting the number of partial sums in a particular interval mod 2 is related to counting the number of lattice points in a triangle.

The difference between the area of a triangle and the number of lattice points inside (approximately the size of the spike in the histogram) is related to three sums of the form

$$\sum_{k=0}^{M} \{k\alpha + \beta\} - \tfrac{1}{2},$$

where α is the reciprocal slope of one of the sides of the triangle. Closely related to these sums is the *discrepancy* of the sequence $\{k\alpha\}$, for $k = 1, 2, \ldots, M$. We present some of the classical results about discrepancy in Section 8. For almost every α, the discrepancy grows very slowly with M — hence the very flat parts of Figure 1, left. When α has large continued fraction coefficients (that is, when it is very close to a rational), the discrepancy can be larger and cause a spike.

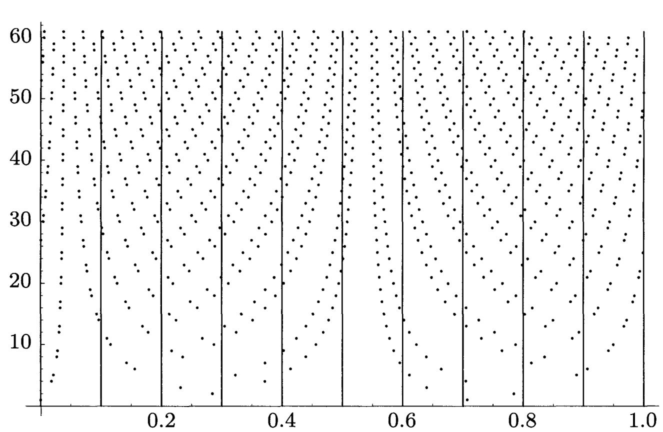

FIGURE 2. Ordered pairs $(\{P_{1/2}(n)\}, \lfloor P_{1/2}(n)\rfloor)$ and how they fall into 10 bins of equal size, for $n = 1, 2, 3, \ldots, 1000$.

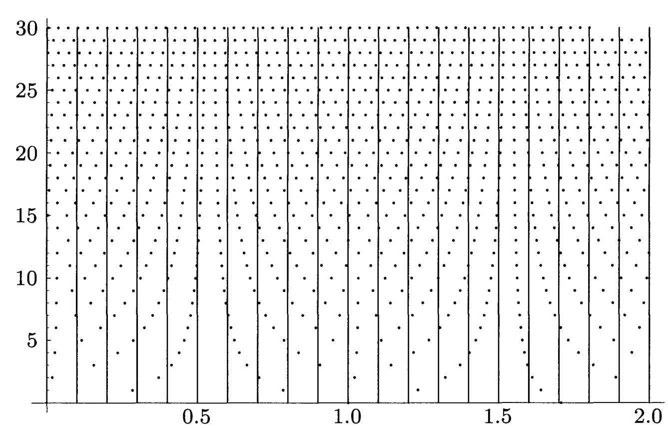

FIGURE 3. Ordered pairs $\left(\{P_{1/2}(n)\}_2,\ \lfloor P_{1/2}(n)\rfloor_2\right)$ and how they fall into 20 bins of equal size, for $n = 1, 2, 3, \ldots, 1000$.

As a concrete example, consider the large central spike in the histogram for $\zeta(0.50)$ (Figure 1, left), which chiefly occurs in the interval $[0.538, 0.540)$. Modulo 2, this corresponds to the intervals $[0.538, 0.540)$ and $[1.538, 1.540)$. When unfolded, the first interval corresponds to a triangle whose sloped edges have reciprocal slopes approximately 1.99835 and 2.00035. These reciprocal slopes have the continued fraction expansions:

$$1.99835 = [1, 1, 606, 1, 2, 1, \dots],$$
$$2.00035 = [2, 2820, 1, 4, 8, \dots]$$

The large terms appearing in the continued fraction expansion indicate that the reciprocal slopes are well approximated by a rational, and so the large spike is to be expected.

2. THE ESTIMATED COUNT FOR EACH BIN

We estimate the number of partial sums which land in each bin and call this estimate the baseline. As illustrated in Figure 4, bins just to the right of $P_s(N)$ mod 1 should contain about N^s/B fewer partial sums than bins just to the left. We assume that away from this jump discontinuity, the baseline is linear, and so it must have slope equal to N^s/B. The mean of the baseline is N/B.

The jump in the histogram for $\zeta(0.51)$ (Figure 1, right) at bin number 482 is of size 17. This jump discontinuity is small compared to the noise, which is of size around 50. In the histogram for $\zeta(0.50)$ (same figure, left) the jump discontinuity is of size 14, and is visible near bin number 340.

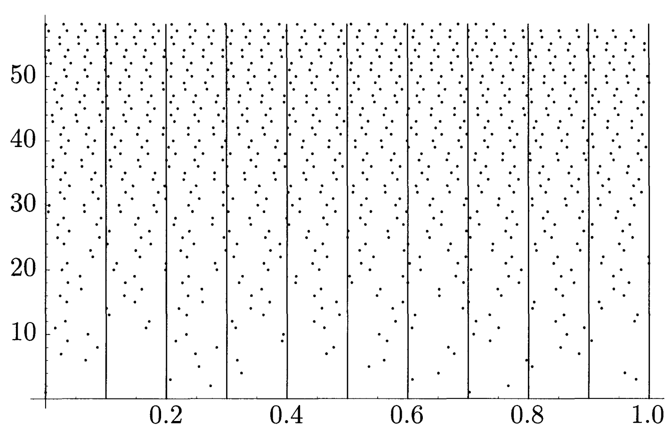

FIGURE 4. The ordered pairs $(\{P_{0.51}(n)\}, \lfloor P_{0.51}(n) \rfloor)$ and how they fall into 10 bins of equal size, for $n = 1, 2, 3, \dots, 1000$.

3. THE RANDOM MODEL

After subtracting the baseline from the bin counts, the mean will be zero. What do we predict for the variance?

Looking at Figure 4 we can say that each bin has been passed by about $P_s(N) \approx 58$ times. Suppose that on a pass by the bin I, the spacings n^{-s} are about

$$\frac{|I|}{k+\alpha} = \frac{1}{B(k+\alpha)},$$

where $k \in \mathbb{Z}$ and $0 \le \alpha < 1$. Then with probability α the bin is incremented $(k+1)$ times and with probability $(1-\alpha)$ the bin is incremented k times. We imagine that each bin count is a sum of independent random variables, each with its own variance $\alpha(1-\alpha)$. Averaging over alpha gives a variance of $\frac{1}{6}$ per pass. We therefore estimate that the variance on the bin counts, after subtracting the baseline, is $P_s(N)/6$. We expect that averaging over $\alpha \in (0, 1)$ is only valid if $N^s \gg B$. At the other extreme, if $B \gg N^s$ then on all the passes $k = 0$ and α is small, so averaging over α is not valid.

Other than the restriction $N^s \gg B$, the variance predicted by the random model does not depend on the width of the bin. If A_1 and A_2 are the counts for successive bins, then $\operatorname{Var}(A_1 + A_2) = \operatorname{Var}(A_1) = \operatorname{Var}(A_2)$. So the correlation of successive bins is

$$\rho = \frac{\operatorname{Cov}(A_1, A_2)}{\sqrt{\operatorname{Var}(A_1)\operatorname{Var}(A_2)}} = -\frac{1}{2}$$

In two numerical experiments, the observed variance and that predicted by the random model agree. When $s = 0.51$, $N = 500\,000$, and $B = 1000$, the predicted variance is 211 and the observed variance is 243. When $s = 0.51$, $N = 10^7$, and $B = 500$, the predicted variance is 915 and the observed variance is 928.

The random fluctuations have a standard deviation proportional to $N^{(1-s)/2}$. When $s > \frac{1}{3}$, the random fluctuations are smaller as $N \to \infty$ than the jump discontinuity in the baseline, which is of size N^s/B. According to the random model, the jump discontinuity will be the dominant feature as $N \to \infty$, even though it is just barely visible on the right half of Figure 1.

4. THE SPECIAL CASE $s = \frac{1}{2}$

We now focus on the case $s = \frac{1}{2}$. For the remainder of this note, we omit the subscript of $\frac{1}{2}$ from $P_{1/2}(n)$. The staggering in Figure 2 indicates that the fractional parts $\{P(n)\}$ line up differently depending on the parity of the integer part $\lfloor P(n) \rfloor$. The way to isolate these two competing patterns is to look at the partial sums $P(n) \bmod 2$ instead of mod 1. Indeed, the pattern in Figure 3 is much simpler.

Since the ordered pairs $\big(\{P(n)\}_2,\ \lfloor P(n) \rfloor_2\big)$ are arranged in an orderly fashion, we are able to approximate the partial sums $P(n)$. In this section we derive the approximations; in the next section we use these approximations to unfold Figure 3 so that the ordered pairs $\big(\{P(n)\}_2,\ \lfloor P(n) \rfloor_2\big)$ map approximately to lattice points.

Proposition 4.1. *For large $n < N$,*

$$\sum_{j=n+1}^{N} \frac{1}{\sqrt{j}} = 2(N^{1/2} - n^{1/2}) + \tfrac{1}{2}(N^{-1/2} - n^{-1/2}) - \tfrac{1}{24}(N^{-3/2} - n^{-3/2}) + O(n^{-5/2}).$$

Proof. The formula in the proposition is a telescoping sum. It suffices to show that

$$\frac{1}{\sqrt{n+1}} = 2\big((n+1)^{1/2} - n^{1/2}\big) + \tfrac{1}{2}\big((n+1)^{-1/2} - n^{-1/2}\big) - \tfrac{1}{24}\big((n+1)^{-3/2} - n^{-3/2}\big) + O(n^{-7/2}).$$

To approximate the terms on the right, we use Taylor expansions with remainder for the functions $x^{1/2}$, $x^{-1/2}$, and $x^{-3/2}$, expanding around $x = n+1$. □

Corollary 4.2. *There exists $\tau \in \mathbb{R}$ such that, for large n,*

$$P(n) = \tau + 2n^{1/2} + \tfrac{1}{2}n^{-1/2} - \tfrac{1}{24}n^{-3/2} + O(n^{-5/2}).$$

Numerically, $\tau \approx -1.46035450880959$.

Proof. Define the sequence τ_n by

$$\tau_n = P(n) - 2n^{1/2} - \tfrac{1}{2}n^{-1/2} + \tfrac{1}{24}n^{-3/2}.$$

By Proposition 4.1, τ_n is a Cauchy sequence and converges. If τ is the limiting value, then

$$\tau_n = \tau + O(n^{-5/2}).$$

We used Maple to evaluate τ numerically. □

Corollary 4.3. *Let $n = m^2 + k$, where $|k| < 4m$. Then*

$$P(n) = \tau + 2m + \frac{k}{m} - \frac{k^2}{4m^3} + \frac{1}{2m} + O(m^{-2}).$$

Proof. From the conclusion of Corollary 4.2, substitute $n = m^2 + k$ and for each term, use the Taylor expansion centered at $n = m^2$. □

5. COUNTING LATTICE POINTS

Locally, in any small patch of Figure 3, the ordered pairs

$$\big(\{P(n)\}_2,\ \lfloor P(n) \rfloor_2\big)$$

are arranged roughly in a lattice. We apply a transformation which unfolds Figure 3:

$$(x, y) \to \big((x - \tau)y - \big(\tfrac{1}{2} - \tfrac{1}{4}(x - \tau)^2\big),\ y\big).$$

The walls of the bins in Figure 3 are vertical lines, defined by the equation $x = \text{const}$. These bin walls are mapped to lines with slope $1/(x - \tau)$. By Corollary 4.3, the partial sums in Figure 3 are mapped very close to lattice points:

$$\big(P(m^2 + k) - 2m, m\big) \to \big(k + O(m^{-1}),\ m\big).$$

Counting the number of partial sums inside a bin modulo 2 is equivalent to counting the number of slightly displaced lattice points inside a triangle. In order to simplify the counting problem, we ignore the slight displacements. This introduces an error, which we model as two sums of random variables

$$\sum_{1}^{M} X_k,$$

one for each of the sloped sides of the triangle. Here X_k is 0 or 1 with probability $O(k^{-1})$ depending on whether or not the $O(k^{-1})$ displacement of the lattice point affects the number of lattice points in the triangle. We expect that this error is $O(\log M)$ as $M \to \infty$.

In the next section we count the number of lattice points inside a triangle. In particular, we are interested in the the extent to which the number of lattice points can differ from the area of the triangle. This difference is comparable to the size of the spike in the bin counts.

6. ESTIMATING A SUM

Given an interval $I = (a_0, a_1) \bmod 2$, we wish to count the number of lattice points inside the triangle given by

$$\begin{aligned} y &< \left\lfloor \tfrac{1}{2}(P(n) - a_1) \right\rfloor + \tfrac{1}{2} \equiv M + \tfrac{1}{2}, \\ x &> \alpha_0 y + \beta_0, \\ x &< \alpha_1 y + \beta_1. \end{aligned}$$

The α_i and β_i are defined by

$$\begin{aligned} \alpha_0 y + \beta_0 &\equiv (a_0 - \tau)y - \left(\tfrac{1}{2} - \tfrac{1}{4}(a_0 - \tau)^2\right), \\ \alpha_1 y + \beta_1 &\equiv (a_1 - \tau)y - \left(\tfrac{1}{2} - \tfrac{1}{4}(a_1 - \tau)^2\right) \end{aligned}$$

The bottom vertex of this triangle has y-coordinate equal to $y_0 \equiv (\beta_0 - \beta_1)/(\alpha_1 - \alpha_0)$. The number of lattice points inside is approximately the area of the triangle. The area will underestimate the count if lots of lattice points are just barely inside and will overestimate when lattice points are just barely outside. More precisely,

$$\begin{aligned} &\left|(k, m) \in \mathbb{Z}^2 \cap \triangle\right| \\ &\qquad = \operatorname{Area}(\triangle) + \sum_{y=\lceil y_0 \rceil}^{M} \left(\{\alpha_0 y + \beta_0\} - \tfrac{1}{2}\right) \\ &\qquad\quad - \sum_{y=\lceil y_0 \rceil}^{M} \left(\{\alpha_1 y + \beta_1\} - \tfrac{1}{2}\right) + \text{error}. \end{aligned}$$

The error term has absolute value less than

$$\frac{1}{8}|\alpha_1 - \alpha_0|.$$

This formula for counting lattice points motivates the following question:

Question 6.1. Given α and β, how does

$$\sum_{k=1}^{M} \{k\alpha + \beta\} - \tfrac{1}{2}$$

behave as $M \to \infty$, and how does this depend on α and β?

We will show in the following section that the growth of the sum in Question 6.1 is bounded by the discrepancy of the sequence α, 2α, 3α, $\ldots \bmod 1$.

7. THE DISCREPANCY OF A SEQUENCE

The discrepancy of a sequence mod 1 measures the extent to which that sequence fails to be uniformly distributed mod 1:

Defintion 7.1. Let u_1, u_2, u_3, $\ldots$ be a sequence of points on the torus $[0, 1)$. The *discrepancy* $D(m)$ is

$$\begin{aligned} D(m) &= \sup_{x<y} \left|\#\{j : x \le u_j < y,\ 1 \le j \le m\} - m(y - x)\right| \\ &= \sup_{x<y} \left|D(m; x, y)\right|. \end{aligned}$$

This definition is translation invariant on the torus. An alternate definition (which is not translation invariant) is $\sup_y D(m; 0, y)$. These two are equivalent up to multiplicative constants:

$$\frac{1}{2} D(m) < \sup_y |D(m; 0, y)| < D(m).$$

The sum in Question 6.1 may be expressed as $\sum_{k=1}^{M} f_\beta(k\alpha)$, where $f_\beta(x) = \{x + \beta\} - \frac{1}{2}$. Since f_β is of total variation 2 on the torus $[0, 1)$, Koksma's inequality ([Montgomery 1994] p.1-3) allows us to bound the sum in terms of the discrepancy of the sequence $\{k\alpha\}$.

Theorem 7.2 (Koksma's Inequality). *Let f be a function of bounded variation on the torus $[0, 1)$. Then*

$$\left|\sum_{j=1}^{m} f(u_j) - m \int_0^1 f(t)\, dt\right| < \tfrac{1}{2} D(m) \operatorname{Var}(f).$$

Proof. Since $dD(m; 0, t) = \sum_1^m \delta_{u_j} - m$, we have

$$\sum_{j=1}^{m} f(u_j) - m \int_0^1 f(t) dt = \int_0^1 f(t) dD(m; 0, t).$$

Now set

$$\begin{aligned} &\tilde{D}(m, t) \\ &\quad = D(m; 0, t) - \tfrac{1}{2}\left(\sup_y D(m; 0, y) + \inf_y D(m; 0, y)\right). \end{aligned}$$

But $dD(m; 0, t) = d\tilde{D}(m, t)$ and $\sup_t \left|\tilde{D}(m, t)\right| = \frac{1}{2} D(m)$. Integrating by parts we get

$$\sum_{j=1}^{m} f(u_j) - m \int_0^1 f(t)\, dt = -\int_0^1 df(t) \tilde{D}(m, t).$$

Now taking absolute values shows that the left-hand side of the inequality in the theorem is less than $\frac{1}{2} D(m) \int_0^1 \left|df(t)\right|$, which proves the result. □

Koksma's inequality bounds the sum in Question 6.1 in terms of the discrepancy of the sequence α, 2α, 3α, $\ldots \bmod 1$. However, it is possible for the discrepancy of the sequence to grow very large while the sum $\sum_{k=1}^{M} f_\beta(k\alpha)$ grows slowly. For example, let $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{4}$. The discrepancy of the sequence $\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, \dots$ grows linearly — which is as fast as possible. Meanwhile the sum $\sum_{k=1}^{M} f_\beta(k\alpha)$ does not grow: it alternates between $\frac{1}{4}$ and 0.

The discrepancy gives an upper bound for the sum $\sum_{k=1}^{M} f_\beta(k\alpha)$, but not a lower bound. In the next section we present several results about the growth rate of the discrepancy of $\alpha, 2\alpha, 3\alpha, \dots \bmod 1$ and how this depends on arithmetical properties of α.

8. THE DISCREPANCY OF THE SEQUENCE $\{k\alpha\}$

The continued fraction expansion of α encodes information about the discrepancy of the sequence $\alpha, 2\alpha, 3\alpha, \dots, M\alpha \bmod 1$ and how the discrepancy grows with M. For example, the continued fraction expansion of π is

$$\begin{aligned}\pi &= [3, 7, 15, 1, 292, 1, 1, \dots]\\ &= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}}},\end{aligned}$$

and its partial quotients are

$$\frac{p_1}{q_1} = \frac{3}{1}, \quad \frac{p_2}{q_2} = \frac{22}{7}, \quad \frac{p_3}{q_3} = \frac{333}{106}, \quad \frac{p_4}{q_4} = \frac{355}{113}, \quad \frac{p_5}{q_5} = \frac{103993}{33102}, \quad \frac{p_6}{q_6} = \frac{104348}{33215}, \quad \dots.$$

Figure 5 plots the sum $\sum_{j=1}^{m} \{j\pi\} - \frac{1}{2}$, which is a lower bound for $D(m)$, as a function of m. This sum oscillates on at least two different scales: one oscillation has period about 110 and another has period about 33000.

The periods of these oscillations seem, roughly, to be the denominators q_k of the partial quotients. The amplitudes are roughly $\frac{1}{8} q_k / q_{k-1}$. Only those oscillations with period $q_k = O(m)$ have had the opportunity to influence the sum $\sum_{j=1}^{m} \{j\pi\} - \frac{1}{2}$. At least once before M, say at m, we expect that all the oscillations up to $q_k \approx M$ are in phase and

$$\sum_{j=1}^{m} \{j\pi\} - \frac{1}{2} \approx \frac{1}{8}\Big(q_1 + \frac{q_2}{q_1} + \dots + \frac{q_k}{q_{k-1}}\Big).$$

If $D(m)$ has the same behavior as $\sum_{j=1}^{m} \{j\pi\} - \frac{1}{2}$, we have provided a heuristic derivation of the following theorem, attributed by Beck [1994, p. 453–454] to Hardy, Littlewood, and Ostrowski (though I have not been able to find it, at least in this form, in the original papers [Hardy and Littlewood 1922; Ostrowski 1922]). For convenience, we define

$$\triangle(\alpha, M) = \max_{1 \le m \le M} D(m).$$

Theorem 8.1. *Define a_{s+1} as q_{s+1}/q_s if $M > q_{s+1}$, as M/q_s if $q_s \le M < q_{s+1}$, and as zero otherwise. There are absolute constants $c_2 > c_1 > 0$ such that*

$$c_1 \sum_s a_s < \triangle(\alpha, M) < c_2 \sum_s a_s.$$

For a specific value of α, the growth of $\triangle(\alpha, M)$ depends very explicitly on the continued fraction expansion of α. By analyzing the growth of continued fraction coefficients for almost every α, Khintchine [1924, p. 125] was able to control the growth of $\triangle(\alpha, M)$ for almost every α, obtaining the next theorem.

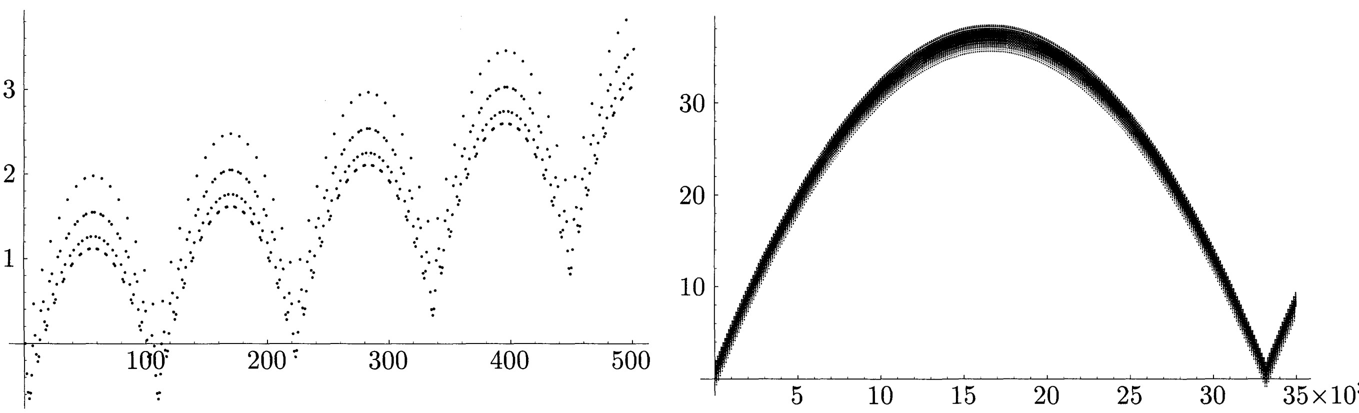

FIGURE 5. The sum $\sum_{j=1}^{m} \{j\pi\} - \frac{1}{2}$ for $m = 1, 2, \dots, 500$ (left) and for $m = 1, 2, \dots, 35000$ (right).

Theorem 8.2. *Suppose that $\varphi(M)$ is a positive function and that $\varphi(M)/M$ is monotone increasing. If*

$$\sum_{M=1}^{\infty} \frac{1}{\varphi(M)} < \infty, \tag{8–1}$$

that is, if the $\varphi(M)$ grow fast enough, then for almost every α,

$$\triangle(\alpha, M) = O\big(\varphi(\log M)\big) \tag{8–2}$$

as $M \to \infty$. The constant implied by O may depend on α. If the sum in 8–1 *does not converge, then for almost every α the Statement* 8–2 *is false.*

Corollary 8.3. *For almost all α, $\triangle(\alpha\, M)$ is*

$$O\left(\log M\,(\log\log M)^{1+\varepsilon}\right)$$

but not $O\,(\log M \log\log M)$).

Beck [1994] has extended this corollary to higher dimensions.

In order to return to the original problem, we conclude our brief tour of lattice points, discrepancy, and continued fractions.

9. APPLICATION TO THE ORIGINAL PROBLEM

We began this article with the intention of explaining the large spikes in the histogram of partial sums of $\zeta(\frac{1}{2})$ and the surprising flatness everywhere else. See Figure 6 for a closeup of the biggest spike.

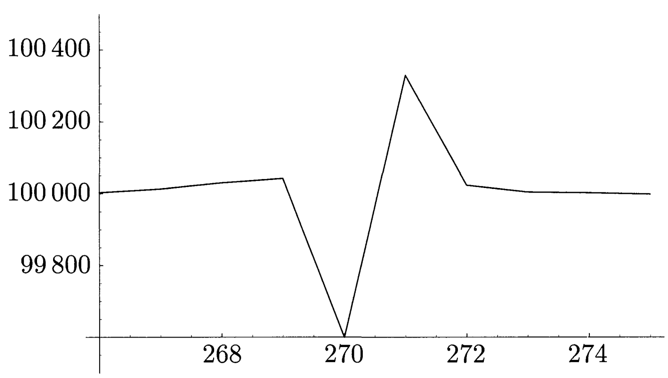

FIGURE 6. A closeup of the large, central spike in Figure 1, left.

The large downward part of the spike corresponds to bin number 270, which is the interval [0.538, 0.540). Modulo 2, this is the intervals [0.538, 0.540) and [1.538, 1.540). After unfolding, the reciprocal slopes (α_0, α_1) corresponding to the sides of the triangular regions are, respectively, (1.99835, 2.00035) and (2.99835, 2.00035). The continued fraction expansions of the first two (to get the others just add one to the first coefficients) are

$$\alpha_0 = [1, 1, 606, 1, 2, 1, \dots],$$
$$\alpha_1 = [2, 2820, 1, 4, 8, \dots].$$

These α have a very large term early in their continued fraction expansion. Using Theorem 8.1 for $N = 5 \times 10^7$ and $M = 7071$, we know that $\triangle(\alpha_1, M)$ will be comparable to $2826 = 1 + 2820 + 1 + 4$. We now retrace our steps. The sum in Question 6.1 can be large, so the number of lattice points inside the triangle can differ significantly from the area. Thus the number of partial sums in the bin [0.538, 540) can differ significantly from the baseline, causing a spike in the histogram.

Another spike is at bin 211, which is the interval [0.420, 0.422), corresponding to

$$(\alpha_0, \alpha_1) \approx (1.88035, 1.88235).$$

These reciprocal slopes have the continued fraction expansions

$$\alpha_0 = [1, 1, 7, 2, 1, 3, 1, 4, 1, 598, 1, 1, \dots],$$
$$\alpha_1 = [1, 1, 7, 1, 1, 2206, 1, 3, \dots].$$

We have observed that, except for the few strong spikes, the histogram of partial sums of $\zeta(0.50)$ (Figure 1, left) is surprisingly flat — much flatter than the random model of Section 3 would predict. Corollary 8.3 of Khintchine's Theorem offers a possible explanation. Indeed, pick a bin with random endpoints (a_0, a_1). The deviation of this bin count is related to sums of the form

$$\sum_{k=1}^{M} \{k\alpha_i + \beta_i\} - \frac{1}{2},$$

where $\alpha_i = a_i - \tau$. These sums are bounded by $\triangle(\alpha_i, M)$. Corollary 8.3 applies and, for almost all choices of (a_0, a_1), the value of $\triangle(\alpha_i, M)$ is dominated by

$$\log M\,(\log\log M)^{1+\varepsilon}$$

as $M \to \infty$. This is much slower than the $M^{1/2} \approx N^{(1-s)/2}$ growth rate predicted by the random model. To the extent that a random bin in has random endpoints, Corollary 8.3 explains why in most places the histogram of partial sums of $\zeta(0.50)$ is flat.

ACKNOWLEDGEMENTS

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REFERENCES

[Beck 1994] J. Beck, "Probabilistic Diophantine approximation, I: Kronecker sequences", *Ann. of Math.* (2) **140**:2 (1994), 449–502.

[Hardy and Littlewood 1922] G. H. Hardy and J. E. Littlewood, "Some problems of diophantine approximation: the lattice points of a right-angled triangle", *Abh. Math. Sem. Hamburg* **1** (1922), 212–248.

[Khintchine 1924] A. Khintchine, "Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen", *Math. Ann.* **92** (1924), 115–125.

[Montgomery 1994] H. Montgomery, *Ten lectures on the interface between analytic number theory and harmonic analysis*, CMBS Regional Conf. Ser. Math. **84**, Amer. Math. Soc., Providence, RI, 1994.

[Ostrowski 1922] A. Ostrowski, "Bemerkungen zur Theorie der Diophantischen Approximationen", *Abh. Math. Sem. Hamburg* **1** (1922), 77–101.

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