Patterns in 1-Additive Sequences

Steven R. Finch

CONTENTS

Queneau observed that certain 1-additive sequences (defined by Ulam) are regular in the sense that differences between adjacent terms are eventually periodic. This paper extends Queneau's work and my recent work toward characterizing periods and fundamental differences of all regular 1-additive sequences. Relevant computer investigations of associated nonlinear recurring sequences give rise to unexpected evidence suggesting several conjectures.

1. INTRODUCTION

Starting with two relatively prime positive integers $u < v$, Ulam [1964, p. ix] defined the 1-*additive sequence with base* $\{u, v\}$ as the infinite sequence

$$(u, v) = a_1, a_2, a_3, a_4, \ldots,$$

where $a_1 = u$, $a_2 = v$ and subsequent terms are recursively defined by the condition that a_n is the smallest integer greater than a_{n-1} and having a unique representation $a_i + a_j$, for $i < j$. For example, when $u = 2$ and $v = 3$, the next fifty terms of the sequence are

5, 7, 8, 9, 13, 14, 18, 19, 24, 25, 29, 30, 35, 36, 40, 41, 46, 51, 56, 63, 68, 72, 73, 78, 79, 83, 84, 89, 94, 115, 117, 126, 153, 160, 165, 169, 170, 175, 176, 181, 186, 191, 212, 214, 230, 235, 240, 245, 266, 273.

These numbers do not appear to follow any recognizable pattern, and no pattern seems to emerge even after computation of the first several thousand terms. Such erratic behavior characterizes many 1-additive sequences [Guy 1981, Problem C4]. In contrast, when $u = 4$ and $v = 5$, a pattern appears [Queneau 1972; Finch 1991]. The sequence in this case breaks naturally into segments of 32 terms each, except for three extra terms in the initial segment, shown in boldface:

1058-6458/92 $0.50 per page

4, 5, 9, 13, **14**, 17, 19, 21, **24**, 25, 27, 35, 37, 43, 45, 47, 57, 67, 69, 73, 77, 83, 93, 101, 105, 109, 113, 115, 123, 125, 133, 149, 153, 163, 173,

197, 201, 205, 209, 211, 213, 217, 219, 227, 229, 235, 237, 239, 249, 259, 261, 265, 269, 275, 285, 293, 297, 301, 305, 307, 315, 317, 325, 341, 345, 355, 365,

389, 393, 397, ...

A 1-additive sequence is *regular* if successive differences $a_{n+1} - a_n$ are eventually periodic, that is, if there is a positive integer N such that $a_{N+n+1} - a_{N+n} = a_{n+1} - a_n$ for all sufficiently large n. The smallest such N is called the *period*, and the value $D = a_{N+n} - a_n$ for large n is called the *fundamental difference*. The asymptotic density of a regular 1-additive sequence (relative to the positive integers) is clearly N/D. Hence, the sequence $(4, 5)$ is regular with $N = 32$, $D = 192$ and asymptotic density equal to $\frac{1}{6} \approx 0.1667$.

The first examples of regular 1-additive sequences were discovered by Queneau [1972], specifically, the sequences $(2, v)$ for $v = 5$, 7 and 9. In [Finch 1991] I determined a condition sufficient for an arbitrary 1-additive sequence (u, v) to be regular (Section 2). In [Finch 1992] I found an approximate formula (with error bound) for N in terms of D for the special case when $u = 2$ and $v \geq 5$. This formula is subject to the truth of a highly plausible conjecture (Section 3) and is based on the distribution properties of linear recurring sequences in finite fields given in [Niederreiter 1976]. The present paper summarizes computer investigations of N and D for certain 1-additive sequences (u, v) with $u \geq 3$, from which follow several conjectures.

2. CONDITION FOR REGULARITY

The proof of Theorem 1 is provided in [Finch 1991]; it is included here for the sake of completeness.

Theorem 1. *A 1-additive sequence having only finitely many even terms is regular.*

Proof: Let e denote the number of even terms in the 1-additive sequence $a_1, a_2, \ldots$. Let $x_1 < x_2 < \cdots < x_e$ be the even terms and let $y_k = \frac{1}{2}x_k$ for each k, where $1 \leq k \leq e$. Given an integer $n \geq y_e$, let b_n be the number of representations of $2n + 1$ as a sum $a_i + a_j$, for $i < j$. Observe that $a_i + a_j = 2n + 1$ only if either a_i or a_j is equal to some x_k, since a sum of two integers is odd if and only if the integers have different parities. This observation gives rise to the recursive formula

$$b_n = \sum_{k=1}^{e} \delta(b_{n-y_k} - 1), \tag{1}$$

where $\delta(0) = 1$ and $\delta(r) = 0$ for $r \neq 0$. The summation simply counts the number of times (out of e) that $2n - x_k + 1$ is a term in $a_1, a_2, \ldots$.

Define now for each $n \geq x_e$ a vector of y_e components,

$$\beta_n = (b_{n-y_e}, b_{n-y_e+1}, b_{n-y_e+2}, \ldots, b_{n-1}).$$

Regularity of the 1-additive sequence $a_1, a_2, \ldots$ is clearly equivalent to eventual periodicity of the vector sequence $\beta_{x_e}, \beta_{x_e+1} \ldots$. The components of β_n obviously do not exceed e. Since the number of integer vectors of length y_e whose components are non-negative and bounded by e is finite, some β_n must recur, which in turn brings about periodicity by the recursive formula. □

A wide variety of 1-additive sequences (u, v) appear to satisfy the hypothesis of Theorem 1, though a proof is not known. Remember that u and v are always assumed to be relatively prime and that $u < v$.

Conjecture 1. (a) *These 1-additive sequences have finitely many even terms:* $(2, v)$, *for* $v \geq 5$; $(4, v)$; $(5, 6)$; (u, v), *for even* $u \geq 6$; *and* (u, v), *for odd* $u \geq 7$ *and even* v.

(b) *All other 1-additive sequences have an infinite number of even terms.*

Evidence supporting both parts of Conjecture 1 is entirely empirical in nature. We proceed now to formulate more refined versions of Conjecture 1 for each of the above five regular cases.

3. THE CASE $(2, v)$ for $v \geq 5$

Computer data and formulas derived in [Queneau 1972] suggest that the following is true.

Conjecture 2. *The sequence* $(2, v)$, *for* $v \geq 5$, *has precisely two even terms,* 2 *and* $2v + 2$.

This case was examined in detail in [Finch 1992], so a brief summary will suffice here. If Conjecture 2 is true, formula (1) becomes

$$b_n = \delta(b_{n-1} - 1) + \delta(b_{n-v-1} - 1), \tag{2}$$

v	period N	fundamental difference D
5	32	$126 = 2(2^6 - 1)$
$7 = 2^3 - 1$	$26 = 3^3 - 1$	$126 = 2(2^6 - 1)$
9	444	$1778 = 2(2^3 - 1)(2^7 - 1)$
11	1628	$6510 = 2(2^3 - 1)(2^4 - 1)(2^5 - 1)$
13	5906	$23{,}622 = 2(2^2 - 1)(2^5 - 1)(2^7 - 1)$
$15 = 2^4 - 1$	$80 = 3^4 - 1$	$510 = 2(2^8 - 1)$
17	126,960	$507{,}842 = 2(2^5 - 1)(2^{13} - 1)$
19	380,882	$1{,}523{,}526 = 2(2^2 - 1)(2^5 - 1)(2^{13} - 1)$
21	2,097,152	$8{,}388{,}606 = 2(2^{22} - 1)$
23	1,047,588	$4{,}194{,}302 = 2(2^{21} - 1)$
25	148,814	$597{,}870 = 2(2^9 - 1)(2^{12} - 1)/7$
27	8,951,040	$35{,}791{,}394 = 2(2^{28} - 1)/15$
29	5,406,720	$21{,}691{,}754 = 2(2^{30} - 1)/99$
$31 = 2^5 - 1$	$242 = 3^5 - 1$	$2046 = 2(2^{10} - 1)$
33	127,842,440	$511{,}305{,}630 = 2(2^4 - 1)(2^{30} - 1)/63$
35	11,419,626,400	$45{,}678{,}505{,}642 = 2(2^9 - 1)(2^{10} - 1)(2^{17} - 1)/3$
37	12,885,001,946	$51{,}539{,}607{,}546 = 2(2^2 - 1)(2^{33} - 1)$
$63 = 2^6 - 1$	$728 = 3^6 - 1$	$8190 = 2(2^{12} - 1)$
$127 = 2^7 - 1$	$2186 = 3^7 - 1$	$32{,}766 = 2(2^{14} - 1)$
$255 = 2^8 - 1$	$6560 = 3^8 - 1$	$131{,}070 = 2(2^{16} - 1)$
$511 = 2^9 - 1$	$19{,}682 = 3^9 - 1$	$524{,}286 = 2(2^{18} - 1)$
$1023 = 2^{10} - 1$	$59{,}048 = 3^{10} - 1$	$2{,}097{,}150 = 2(2^{20} - 1)$

TABLE 1. Parameters for the 1-additive sequence $(2, v)$, obtained on the assumption that Conjecture 2 holds.

with initial data

$$\beta_{(v+1)/2} = (b_{-(v+1)/2}, b_{-(v-1)/2}, \ldots, b_{(v-3)/2}, b_{(v-1)/2}) = (0, 0, \ldots, 0, 1).$$

Replacing the counter variable b_n by the indicator variable b_n^*, defined to be 1 if $2n+1$ is a term of $a_1, a_2, \ldots$ and 0 otherwise, and working modulo 2, we can simplify formula (2) and obtain

$$b_n^* = b_{n-1}^* + b_{n-v-1}^* \pmod 2,$$

a homogeneous binary linear recurring sequence. Such simplification gives rise to fast computer algorithms to compute the period $N(v)$ and the fundamental difference $D(v)$ of $(2, v)$ (Table 1), assuming Conjecture 2 is true. Only $v + 1$ bits of storage are required at any time, unlike the rapidly accumulating storage required when directly computing terms of $a_1, a_2, \ldots$ according to their definition. One can demonstrate that $\frac{1}{2}D$ is the smallest positive integer k satisfying the equation

$$\begin{pmatrix} 0 & I_v \\ 1 & \varepsilon_v^T \end{pmatrix}^k = I_{v+1}$$

over $\mathbf{Z}_2$, where I_p is the $p \times p$ identity matrix and $\varepsilon_v = (0, \ldots, 0, 1)$. One can also show, using techniques developed in [Niederreiter 1976], that N and D are related by

$$|N(v) - \tfrac{1}{4}D(v)| \le 2^{(v-1)/2}.$$

We emphasize that these results depend on the truth of Conjecture 2, which remains unproven.

Another unsolved problem regards the pattern that $N(v)$ and $D(v)$ exhibit at the values $v = 2^m - 1$ (Table 1). Is $N(2^m - 1) = 3^m - 1$ and $D(2^m - 1) = 2(2^{2m} - 1)$ for all $m \ge 3$?

4. THE CASE $(4, v)$

This is perhaps the most interesting of the five regular cases. All discussion within this section rests on the truth of the following conjecture:

Conjecture 3. (a) *The sequence $(4, v)$ has precisely three even terms, 4, $2v + 4$ and $4v + 4$, when $v \ne 2^m - 1$ for any $m \ge 3$.*

(b) *When $v = 2^m - 1$ for some $m \ge 3$, the sequence $(4, v)$ has precisely four even terms, 4, $2v + 4$, $4v + 4$ and $2(2v^2 + v - 2)$.*

v	N	D	v	N	D
5	32	$192 = 2^5(5+1)$	35	826	5326
$7 = 2^3 - 1$	1,927,959	11,301,098	37	776	$9728 = 2^8(37+1)$
9	88	$640 = 2^6(9+1)$	39	108,966	620,796
11	246	1318	41	824	$10{,}752 = 2^8(41+1)$
13	104	$896 = 2^6(13+1)$	43	632	5632
$15 = 2^4 - 1$	$*$	$*$	45	856	$11{,}776 = 2^8(45+1)$
17	248	$2304 = 2^7(17+1)$	47	7,226,071	41,163,940
19	352	2560	49	896	$12{,}800 = 2^8(49+1)$
21	280	$2816 = 2^7(21+1)$	51	1488	13,312
23	5173	29,858	53	928	$13{,}824 = 2^8(53+1)$
25	304	$3328 = 2^7(25+1)$	55	856	7168
27	10,270	57,862	57	952	$14{,}848 = 2^8(57+1)$
29	320	$3840 = 2^7(29+1)$	59	97,150,536	553,730,584
$31 = 2^5 - 1$	$*$	$*$	61	968	$15{,}872 = 2^8(61+1)$
33	712	$8704 = 2^8(33+1)$	$63 = 2^6 - 1$	$*$	$*$

TABLE 2. Parameters for the 1-additive sequence $(4, v)$, assuming the truth of Conjecture 3. No periodicity was detected up to 3.65×10^9 terms for sequences marked with asterisks.

Assume first that $v \neq 2^m - 1$ for any m. In this case, formula (1) becomes

$$b_n = \delta(b_{n-2} - 1) + \delta(b_{n-v-2} - 1) + \delta(b_{n-2v-2} - 1), \tag{3}$$

with initial data

$$\begin{aligned}\beta_{(v+1)/2} &= (b_{-(3v+3)/2}, b_{-(3v+1)/2}, \ldots, b_{(v-3)/2}, b_{(v-1)/2}) \\ &= (0, 0, \ldots, 0, 1).\end{aligned}$$

One can, as in Section 3, simplify formula (3) by suppressing some of the information in b_n. Working modulo 3, we get

$$\begin{aligned}b^*_n = 2(b^*_{n-2}(b^*_{n-2}+1) &+ b^*_{n-v-2}(b^*_{n-v-2}+1) \\ &+ b^*_{n-2v-2}(b^*_{n-2v-2}+1)) \pmod 3,\end{aligned}$$

a ternary quadratic recurring sequence. This simplification, however, offers no known theoretical advantage, since no results for quadratic recurring sequences have been proved that parallel those for linear recurring sequences.

If we assume instead that $v = 2^m - 1$ for some m, a recursive formula analogous to formula (3) is obtained with four terms instead of three. One could, as above, rewrite the recursive formula as a certain quaternary cubic recurring sequence but, again, at no known advantage.

Table 2 presents results of N and D computations for $(4, v)$ and was made assuming Conjecture 3 holds. An asterisk denotes those sequences for which no periodicity was detected up to 3.65×10^9 terms; such sequences possess either very long periods or very long transient phases (initial stretches before periodicity begins). For comparison's sake, we note that the transient phase of $(4, 7)$ contains approximately 1.36×10^7 terms.

The most striking feature of Table 2 is the manner in which N and D behave according to the residue of v modulo 4. When $v \equiv 3 \bmod 4$, no trends are evident. In contrast, when $v \equiv 1 \bmod 4$, the following behavior appears to hold:

Conjecture 4. *If $v \equiv 1 \bmod 4$, the fundamental difference of the 1-additive sequence $(4, v)$ is*

$$D(v) = 2^{m+3}(v+1),$$

where m is the largest integer satisfying $2^m < v$.

No similar formula for $N(v)$ appears to be valid; an approximate relationship to $D(v)$ analogous to that for $(2, v)$ is all that can be hoped for.

5. THE CASE $(5, 6)$

For this single unusual regular 1-additive sequence, we have:

Conjecture 5. *The sequence $(5, 6)$ has precisely thirteen even terms:* 6, 16, 26, 36, 80, 124, 144, 172, 184, 196, 238, 416 *and* 448.

	$u = 6$		$u = 8$		$u = 10$	
v	N	D	N	D	N	D
7	9365	62,450	–	–	–	–
9	–	–	180	1440	–	–
11	218	1408	299,214	2,183,224	1782	15,312
13	252	1664	232,025	1,689,694	314	2496
15	–	–	2,287,191	16,687,270	–	–
17	14,089,505	93,609,388	306	2720	1618	13,056

TABLE 3. Parameters for the 1-additive sequence (u, v), for even $u \geq 6$ and odd v, assuming the truth of Conjecture 6. Dashes denote pairs (u, v) for which either $u \geq v$ or u and v are not relatively prime.

By formula (1) and assuming Conjecture 5, one determines that $N = 208$ and $D = 1720$ after a transient phase exceeding 1.56×10^5 terms.

6. THE CASE (u, v) FOR EVEN $u \geq 6$

This is the case for which the least is known. Transient phases appear to be particularly lengthy for sequences of this type—over 2×10^8 terms for the sequence $(8, 17)$—resulting in very long computation times. The starting point for these computations is this:

Conjecture 6. *The sequence (u, v), for even $u \geq 6$, has $2 + \frac{1}{2}u$ even terms, namely, $u + 2pv$ for $0 \leq p \leq \frac{1}{2}u$ and $(2u + 4)v$.*

Table 3 summarizes periods and fundamental differences for this case. Only more extended computations will reveal possible formulas or approximate relationships for N and D.

7. THE CASE (u, v) FOR ODD $u \geq 7$ AND EVEN v

This case contrasts unexpectedly with the other four cases. For fixed u, both N and D are evidently simple linear expressions in v, under the assumption that v is sufficiently large relative to u. The transient phases here are not quite as long as in Section 6, making the computer-aided discovery of these linear expressions possible.

Conjecture 7. *The sequence (u, v), for odd $u \geq 7$ and even v, has $2 + \frac{1}{2}v$ even terms, namely, $2qu + v$ for $0 \leq q \leq \frac{1}{2}v$ and $u(2v + 4)$.*

Table 4 summarizes periods and fundamental differences for this case. Based on this table and on similar results for $u = 13$, 15, 17 and 19, we have:

Conjecture 8. *Let $u \geq 7$ be an odd integer. There exists an integer v_0 such that, for even integers $v \geq v_0$, periods N and fundamental differences D for the 1-additive sequence (u, v) are necessarily of the form*

$$N(v) = f_N(u)v + g_N(u),$$
$$D(v) = f_D(u)v + g_D(u).$$

The corresponding coefficients for the first few values of u are as follows:

u	f_N	g_N	f_D	g_D	v_0
7	6	10	112	224	38
9	14	−2	144	288	56
11	64	194	704	1408	30
13	19	−56	208	416	56
15	42	672	480	960	52
17	28	−180	272	544	118
19	48	−60	608	1216	120

Apart from evidence that g_D is always equal to $2f_D$, no trends in the coefficients are apparent. The simplicity of these formulas for N and D suggest that a proof of Conjecture 8 for fixed u may be possible by directly listing the terms $a_1, a_2, \ldots$ as functions of (large) v. However, since transient phases are quite large—more than $90v + 828$ terms for $(7, v)$, for example—such a proof would almost certainly demand the use of computers.

8. A GENERALIZATION

Starting with three relatively prime positive integers $u < v < w$, Queneau [1972] defined the $(1, 3)$-*additive sequence with base* $\{u, v, w\}$ as the infinite sequence

$$(u, v, w) = a_1, a_2, a_3, a_4, \ldots,$$

	$u=7$		$u=9$		$u=11$	
v	N	D	N	D	N	D
8	5874	42,758	–	–	–	–
10	830	6594	80,240	630,818	–	–
12	182	1568	–	–	272	2464
14	–	–	258	2304	164	1408
16	124	1008	546	5184	670	6336
18	228	2240	–	–	708	7040
20	156	1232	300	3168	842	7744
22	140	1344	334	3456	–	–
24	310	2912	–	–	488	4576
26	532	5488	292	4032	1506	21,560
28	–	–	288	4320	614	5280
30	132	1792	–	–	2114	22,528
32	326	4284	984	9792	2242	23,936
34	326	4032	636	5184	2370	25,344
36	364	4256	–	–	2498	26,752
38	238	4480	870	11,520	2626	28,160
40	250	4704	464	6048	2754	29,568
42	–	–	–	–	2882	30,976
44	274	5152	584	6624	–	–
46	286	5376	628	6912	3138	33,792
48	298	5600	–	–	3266	35,200
50	310	5824	780	7488	3394	36,608
52	322	6048	614	7776	3522	38,016
54	334	6272	–	–	3650	39,424
56	–	–	782	8352	3778	40,832
58	358	6720	810	8640	3906	42,240
60	370	6944	–	–	4034	43,648
62	382	7168	866	9216	4162	45,056
64	394	7392	894	9504	4290	46,464

TABLE 4. Parameters for the 1-additive sequence (u, v) for odd $u \geq 7$ and even v, assuming the truth of Conjecture 7. Dashes denote pairs (u, v) for which either $u \geq v$ or u and v are not relatively prime.

where $a_1 = u$, $a_2 = v$, $a_3 = w$ and subsequent terms are recursively defined by the condition that a_n is the smallest integer greater than a_{n-1} and having a unique representation $a_i + a_j + a_k$, for $i < j < k$. (A further generalization to $(1, t)$-additivity, for any $t > 1$, is briefly discussed in [Finch 1991].) Broad conditions sufficient for a $(1, 3)$-additive sequence to be regular are not known. No fast recursive formulas as in the proof of Theorem 1 are available, even conjecturally, so we have little choice but to directly compute terms of $a_1, a_2, \ldots$ from their definition.

We show here that examples of regular $(1, 3)$-additive sequences exist; in fact, we exhibit classes of regular $(1, 3)$-additive sequences for which formulas for periods N and fundamental differences D can be proved, owing to the shortness of transient phases.

Theorem 2. *The periods N and fundamental differences D for the $(1, 3)$-additive sequence $(1, v, w)$ are given by*

$$N = w + 1 \quad \textit{and} \quad D = 7w + 1$$

when $v = 2$ and $w \equiv 0 \bmod 6$ and $w > 24$;

$$N = \tfrac{1}{3}(7w + 9) \quad \textit{and} \quad D = 21w + 1,$$

when $v = 2$ and $w \equiv 3 \bmod 6$ and $w > 45$;

$$N = w + 1 \quad and \quad D = 7(w+1),$$

when $v = 3$ *and* $w \equiv 0 \bmod 2$ *and* $w > 22$; *and*

$$N = \tfrac{1}{4}(w+3) \quad and \quad D = 5w + 9,$$

when $v = 3$ *and* $w \equiv 1 \bmod 4$ *and* $w > 17$.

Proof: For the sake of brevity, we prove only the last case, omitting the straightforward, if tedious, details. The initial terms of the sequence $(1, 3, w)$, when $17 < w \equiv 1 \bmod 4$, are $a_1, a_2, \ldots, a_K$, where $K = \frac{1}{4}(w + 31)$ and where

$$\begin{aligned} a_1 &= 1, \\ a_2 &= 3, \\ a_i &= w + 4(i-3) \quad \text{for } 3 \le i \le K - 4, \\ a_{K-3} &= 2w + 5, \\ a_{K-2} &= 2w + 11, \\ a_{K-1} &= 6w + 9, \\ a_K &= 6w + 11. \end{aligned}$$

(The expression for each term is found by summing all triples of distinct preceding terms, keeping track of which sums are obtained uniquely, and then determining the least such sum not already listed as a term of the sequence.) Subsequent terms a_{K+1+j} are of the form $c_{j \bmod N} + D\lfloor j/N \rfloor$, where

$$\begin{aligned} c_j &= 6w + 19 + 4j \quad \text{for } 0 \le j \le N - 4, \\ c_{N-3} &= 11w + 18, \\ c_{N-2} &= 11w + 20, \\ c_{N-1} &= 11w + 24, \end{aligned}$$

and where $N = \frac{1}{4}(w + 3)$ and $D = 5w + 9$. This completes the proof. □

The approach used in the proof of Theorem 2 might also be helpful in proving results on the 1-additive sequences in Section 7; but, as discussed there, any such proof would almost certainly depend on computers.

REFERENCES

[Finch 1991] S. R. Finch, "Conjectures about s-additive sequences", *Fibonacci Quart.* **29** (1991), 209–214.

[Finch 1992] S. R. Finch, "On the regularity of certain 1-additive sequences", *J. Combin. Theory* A**60** (1992), 123–130.

[Guy 1981] R. K. Guy, *Unsolved Problems in Number Theory*, Springer-Verlag, New York, 1981.

[Niederreiter 1976] H. Niederreiter, "On the cycle structure of linear recurring sequences", *Math. Scand.* **38** (1976), 53–77.

[Queneau 1972] R. Queneau, "Sur les suites s-additives", *J. Combin. Theory* A**12** (1972), 31–71. English summary in *Math. Rev.* **46**, 1741.

[Ulam 1964] S. M. Ulam, *Problems in Modern Mathematics*, Interscience, New York, 1964.

NOTE ADDED IN PROOF

James H. Schmerl and Eugene Spiegel report, in a personal communication, that they have proved Conjecture 2.

Steven R. Finch, 6 Foster Street, Wakefield, MA 01880, tel. 617-245-4980

Received September 24, 1991; revised March 11, 1992