The Kobayashi Metric of a Complex Ellipsoid in $\mathbf{C}^2$

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CONTENTS

The infinitesimal Kobayashi metric of an ellipsoid of the form

$$E_m = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\}$$

is calculated explicitly, modulo a parameter that is determined by solving a transcendental equation. Using this result, we show that the metric is C^1 on the tangent bundle away from the zero section. We also describe software that will calculate, using a Monte Carlo method, the infinitesimal Kobayashi metric on a domain of the form

$$\Omega_\rho = \{(z_1, z_2) \in \mathbf{C}^2 : \rho(z_1, z_2) < 0\},$$

where ρ is a real-valued polynomial. We compare results of computer calculations with those obtained from the explicit formula for the Kobayashi metric.

Invariant metrics such as the Kobayashi metric and the Carathéodory metric have become important tools in the study of holomorphic functions on bounded domains in complex Euclidean space. Nevertheless, many aspects of the behavior of these metrics, such as whether or not they are smooth, remain unknown. Of course, an explicit formula for such a metric would allow the determination of this behavior, but it is generally very difficult to compute these metrics explicitly. To date, the only domains for which the invariant metrics have been explicitly calculated are symmetric domains and Teichmüller space.

The purpose of this paper is to give an explicit formula for the Kobayashi metric on the *complex ellipsoid*

$$E_m = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\}$$

for real $m \geq \frac{1}{2}$. Because the domain E_m is convex for that range of m, the Carathéodory metric coincides with the Kobayashi metric. (In fact, E_m is *strictly convex*, that is, E_m contains the inte-

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rior of every line segment joining two points in the topological closure of E_m. A proof of the coincidence of the Carathéodory and Kobayashi metrics on strictly convex domains in $\mathbf{C}^n$ may be found in [Lempert 1981].) As an application of the formula obtained, we show that the Kobayashi metric on E_m is at least C^1.

The complex ellipsoid E_m, including the unit ball E_1 as a special case, has been a useful model in the study of bounded domains because of its noncompact automorphism group [Greene and Krantz 1986]. For $m \neq 1$, the ellipsoid E_m has the property that the orbit of a point $z \in E_m$ under the automorphism group accumulates at a weakly convex boundary point of E_m. In [Greene and Krantz 1986] the complex ellipsoid is used as a model to study pseudoconvex domains D with noncompact automorphism group and with the orbit accumulation property just mentioned. It should also be noted that the domains E_m, for m integral, are the only bounded pseudoconvex domains in $\mathbf{C}^2$ that have noncompact automorphism group and smooth boundary of finite type [Bedford and Pinchuk 1989; 1991].

Although the detailed procedure for discerning our formula for the Kobayashi metric E_m is too tedious to present, several basic ideas and techniques that are used in obtaining the formula are worth mentioning (as well as the influence of Poletskii's work [Poletskii 1983], to which we do not directly refer, on finding extremal maps). The *infinitesimal Kobayashi metric* $F : TE_m \to (0,\infty)$ is defined on the complexified tangent bundle as follows: For $z \in E_m, v \in \mathbf{C}^2$, and f a holomorphic map from the open unit disk Δ in $\mathbf{C}^1$ into E_m with $f(0) = z$ and $f'(0)$ a positive scalar multiple of v, we write $f'(0) = \lambda_f v$; then $F(z,v) = \inf\{\lambda_f^{-1}\}$. A holomorphic map $\psi = \psi_p : \Delta \to E_m$ is *extremal* for $p = (z,v) \in TE_m$ if $F(p) = \lambda_\psi^{-1}$. Lempert [1981] has shown that for each point in the tangent bundle of a strictly convex domain D there is a unique extremal map, and that the resulting extremal maps are proper isometric imbeddings of Δ with the Poincaré metric into D with the Kobayashi metric. Lempert's results apply in particular to E_m, and the formula for the Kobayashi metric can be obtained once we have the formulas for the extremal maps for each p in TE_m. We should also mention Kay's recent work [Kay 1991] on extremal disks for *genuine* ellipsoids in $\mathbf{C}^n$ (although there is no apparent overlap), as well as [Royden and Wong].

Of course, the domain E_1 is the unit ball, on which there are several easy techniques for calculating the Kobayashi metric. There is a fundamental difference in computing the extremal maps of E_m for $m \neq 1$. In this paper, we establish formulas for the extremal maps $\psi(z,v)$ in two cases that depend, for given z, on the direction of v. A class of extremal maps into E_m, for $m \neq 1$, is obtained by composing a branch of the many-valued inverse of the covering map $E_m \to E_1$ given by $(z_1,z_2) \to (z_1, z_2^m)$ with extremal maps into E_1 taking values in $E_1 - (\Delta \times \{0\})$. This gives extremal disks only for certain directions of v at each point z of E_m. For the other directions, we compute the extremal disks at the center $0 \in E_m$ using the following fact: The extremal disk of a circular convex domain at the center 0 is the disk obtained by intersecting the domain with the complex line in the direction v through 0. The extremal disks at other points are obtained by repositioning this extremal disk at 0 by the action of the automorphisms $(z_1,z_2) \to (z_1', z_2')$, where

$$z_1' = \frac{z_1 + a}{1 + \bar{a} z_1}, \qquad z_2' = \frac{(1-|a|^2)^{1/2m} z_2}{(1+\bar{a}z_1)^{1/m}}$$

for $a \in \Delta$.

It is not clear a priori whether the Kobayashi metric of E_m obtained from formulas in the two different cases of directions is differentiable. Lempert proved that on a strongly convex domain with smooth boundary, that is, a domain for which the function defining its boundary has positive definite Hessian in the tangential directions, the Kobayashi metric is smooth off the zero section. This is consistent with the special case of the unit ball E_1. However, if $m < 1$, the boundary of E_m is not C^2, and when $m > 1$, the ellipsoid E_m is not strongly convex and Lempert's result does not apply. As an application of the formula obtained here, we show that the metric F is at least C^1 on the tangent space away from the zero section. It remains an open question whether F is differentiable to a higher order.

This paper is divided into four sections. In Section 1, we give formulas for the extremal maps and for the Kobayashi metric of E_m. In Section 2, we supply a proof that the given formulas do indeed define extremal functions. We rely here on the characterization of extremal maps given in [Lempert 1981]. Verification of Lempert's criteria requires routine computations too lengthy to justify full publication, but we will provide enough steps for any reader who is so inclined to easily fill in missing details. In Section 3, we establish that the Kobayashi metric is C^1 as an application of the formulas presented in Section 1. In recognition that the computations carried out in Section 2 are very specific to the complex ellipsoid, we address in Section 4 the possibility of implementing software to obtain experimental information about the Kobayashi metric on domains for which we have no hope of performing explicit calculations at this time.

1. EXPLICIT FORMULAS

The formulas for the extremal maps and for the Kobayashi metric are given in Theorems 1 and 2 (see box on next page). Several remarks should be made.

We denote by $H(\Delta, E_m)$ the set of holomorphic maps from the unit disk Δ into the ellipsoid E_m. For $p = (z, v)$ in the complex tangent bundle TE_m, where $z \in E_m$ and $v \in \mathbf{C}^2$, we let $H(\Delta, E_m : p)$ denote the subset of $H(\Delta, E_m)$ consisting of elements f for which $f(0) = z$ and $f'(0) = \lambda_f v$ for $\lambda_f > 0$. The infinitesimal Kobayashi metric $F : TE_m \to \mathbf{R}^+$ for the domain E_m is defined by

$$F(p) = \inf\{\lambda_f^{-1} \mid f \in H(\Delta, E_m : p)\}, \qquad (1.1)$$

for $p \in TE_m$. An element $\psi = \psi_p$ of $H(\Delta, E_m : p)$ is extremal for $p \in TE_m$ if $F(p) = \lambda_\psi^{-1}$. As mentioned, it follows as a special case of a theorem of Lempert that ψ exists and is unique. The explicit formula for ψ_p in Theorem 1 is given for the point $z = (0, b)$, for $b \in \mathbf{C}$, from which the general formula follows by composition with Möbius transformations.

The computation of ψ is divided into two cases, according to the size of $u = u(v) = m|v_1^{-1}v_2|$ relative to $|b|$, where v_1 and v_2 are the components of v in $p = (z, v)$. The first case is defined by $u \le |b|$. In the second case, when $u > |b| > 0$, we will need two parameters $a(p)$ and $\lambda(p)$. Let $a = a(p)$ be defined by

$$a = \frac{2m|b|}{u + (u^2 + 4m(m-1)|b|^2)^{1/2}}, \qquad (1.2)$$

and let $\lambda = \lambda(p)$ be the unique positive solution of

$$\lambda^2 a^2 + \lambda^{2m}(1 - a^2)|b|^{2m} = 1. \qquad (1.3)$$

Observe that the parameter $a(p)$, used only in the case where $u > |b| > 0$, satisfies

$$0 < a(p) < \frac{2m|b|}{|b| + (|b|^2 + 4m(m-1)|b|^2)^{1/2}} = \frac{2m}{1 + |2m-1|} \le 1$$

for these p. It follows not only that (1.3) determines a unique positive solution $\lambda(p)$ when $u > |b|$, but also that $\lambda(p) > 1$. We will deliberately leave a third case undefined for the moment: When $u > |b| = 0$, then $a(p)$ vanishes and $\lambda(p)$ is not defined by (1.3). The reader should not be troubled by the possibility that the formulas in Theorems 1 and 2 are not meaningful in this case; we redress this situation after the statements of these theorems in preference to disturbing the relative simplicity of the formulas right away.

The assertion that our formulas are explicit is wholly true only part of the time. The exaggeration is to be found in the implicitly defined parameter λ in the case $u > |b|$. Even in this case, where the formula is semi-explicit, the formula is essentially as useful as if it were fully explicit. We have in mind, for example, the proof of Theorem 3 (see box). One may also use symbolic manipulation software to actually find λ in specific cases.

As observed above, the parameter $\lambda(p)$ used in the second case satisfies $\lambda(p) > 1$. Therefore, $b = 0$ is the only potential singularity in (1.7). Of course, such a singularity must be removable, and we will presently provide a formula, (1.11), that is equivalent to (1.7) but that clearly defines K at $b = 0$. Nevertheless, (1.7) does have the advantage of being much simpler than (1.11), and it is sensible away from $b = 0$. As b tends to 0, it is easy to verify that $a(p)$ tends to 0, and $\lambda(p)$ tends to infinity in such a way that $\lim_{b\to 0} |v_2|/(|b|\lambda)$ exists and equals $K(0, 0, v_1, v_2)$. An alternative approach, which we now sketch, is to use different parameters.

Notation

Let $m \in [\frac{1}{2}, \infty)$ be fixed, and let E_m be the ellipsoid $\{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\}$.

Let $p = ((0, b), (v_1, v_2)) \in TE_m$, with $b = |b|e^{i\theta} \in \Delta$ and $v_j = |v_j|e^{i\varphi_j} \in \mathbf{C}$ for $j = 1, 2$.

Let $u = m|v_1^{-1}v_2|$ and $w = u|b|^{m-1}(1 - |b|^{2m})^{-1/2}$.

Let $a = a(p)$ and $\lambda = \lambda(p)$ be defined by formulas (1.2) and (1.3).

Theorem 1 (Formula for the extremal map). *The extremal map* $\psi = (\psi_1, \psi_2) : \Delta \to \mathbf{C}^2$ *for* E_m *is given by*

$$\left.\begin{aligned}\psi_1(\xi) &= e^{i(\varphi_1-\varphi_2+\theta)}\xi(1-|b|^{2m})^{1/2}((1+w^2)^{1/2}+w\xi|b|^m)^{-1}\\ \psi_2(\xi) &= e^{i\theta}(w\xi+|b|^m(1+w^2)^{1/2})^{1/m}((1+w^2)^{1/2}+w\xi|b|^m)^{-1/m}\end{aligned}\right\}\quad \text{for } u \le |b|, \tag{1.4}$$

$$\left.\begin{aligned}\psi_1(\xi) &= e^{i(\varphi_1-\varphi_2+\theta)}a(\lambda^2-1)\xi(\lambda(1-a^2)+(1-\lambda^2a^2)\xi)^{-1}\\ \psi_2(\xi) &= \lambda b(1-a^2)^{1/m}(\lambda\xi+1)(\xi+\lambda)^{m^{-1}-1}(\lambda(1-a^2)+(1-\lambda^2a^2)\xi)^{-1/m}\end{aligned}\right\}\quad \text{for } u > |b|. \tag{1.5}$$

Theorem 2 (Formula for the Kobayashi metric). *The Kobayashi metric* K *for* E_m *is given by*

$$K(p) = \left(\frac{|v_1|^2}{1-|b|^{2m}} + \frac{m^2|b|^{2m-2}|v_2|^2}{(1-|b|^{2m})^2}\right)^{1/2} \quad \text{for } u \le |b|, \tag{1.6}$$

$$K(p) = \frac{|v_2|\lambda m(1-a^2)}{|b|(\lambda^2-1)(m(1-a^2)+a^2)} \quad \text{for } u > |b|. \tag{1.7}$$

Theorem 3. K *is* C^1, *for* $m > \frac{1}{2}$, *away from the zero section of the complex tangent bundle.*

We set

$$\alpha = \alpha(p) = \tfrac{1}{2}\big(|v_2| + (|v_2|^2 + 4(1-m^{-1})|b|^2|v_1|^2)^{1/2}\big) \tag{1.8}$$

and

$$\mu = \mu(p) = \lambda|b|\alpha^{-1}, \tag{1.9}$$

and use $\alpha(p)$ and $\mu(p)$ to replace the parameters $a(p)$ and $\lambda(p)$. We temporarily proceed on a formal basis. Note that $a = |b||v_1|\alpha^{-1}$. Substituting for a and λ in the defining equation for λ gives the implicit definition of μ as the unique positive solution of the equation

$$\mu^2|v_1|^2 + \mu^{2m}\alpha^{2m}(1 - |b|^2|v_1|^2\alpha^{-2}) = 1. \tag{1.10}$$

We now use (1.10) to define μ instead of (1.9). Observe that, when $u > |b| = 0$, we have $\alpha = |v_2|$ from (1.8), and μ is well-defined by (1.10). Similarly, substituting in (1.7) gives $K = K(p)$ by

$$K = \frac{|v_2|\mu\alpha m(1-|b|^2|v_1|^2\alpha^{-2})}{(\mu^2\alpha^2-|b|^2)\big(m(1-|b|^2|v_1|^2\alpha^{-2})+|b|^2|v_1|^2\alpha^{-2}\big)} \tag{1.11}$$

for $u > |b|$. When $u > |b| = 0$, we have $\alpha = |v_2|$ as noted, and (1.11) reduces to

$$K(p) = \frac{1}{\mu} \quad \text{for } p = ((0,0),(v_1,v_2)), \tag{1.12}$$

with $|v_2| > 0$. This, in view of (1.9), is in accordance with the asymptotic approach outlined above. Note that (1.7) and (1.12) are together equivalent to (1.11). See also (1.7′) in Section 3.

Of course, (1.8) and (1.9) can be used to provide a substitute for (1.5) that is valid for $u > |b|$. Since the resulting formula in full generality is cumbersome and since we need a formula only for $b = 0$, we record only this case:

$$\psi_p(\xi) = (\mu v_1\xi, \mu v_2\xi) \tag{1.13}$$

for $p = ((0,0),(v_1,v_2))$, with $\|v\| > 0$.

2. PROOFS OF THE FORMULAS

The proof of Theorem 1 is accomplished in several steps. The main work consists of verifying (1.4) and (1.5) at certain points. Then, composition with appropriate holomorphic maps gives Theorem 1 in full generality. Once the validity of Theorem 1 is established, that of Theorem 2 follows from a straightforward calculation that can safely be omitted.

For $\zeta \in \partial E_m$, let $\nu(\zeta)$ denote the exterior unit normal. We recall from [Lempert 1981, §4] that a proper holomorphic map $f \in H(\Delta, E_m : p)$ is said to be *stationary* for $p \in TE_m$ if there exist two $\mathrm{Lip}(\frac{1}{2})$ functions $\bar{f} : \bar{\Delta} \to \bar{E}_m$ and $\pi : \partial\Delta \to (0,\infty)$ and a continuous function $\tilde{f} : \bar{\Delta} \to \bar{E}_m$ such that (a) $\bar{f}|_\Delta = f$, (b) $\bar{f}(\partial\Delta) \subset \partial E_m$, (c) $\tilde{f}|_\Delta \in H(\Delta, E_m)$, and (d) $\tilde{f}(\zeta) = \zeta\,\pi(\zeta)\,\overline{\nu(f(\zeta))}$ for $\zeta \in \partial\Delta$.

From the proof of Proposition 1 of [Lempert 1981], it suffices to show that the maps ψ defined in Theorem 1 are stationary. Because of property (a), we will write ψ for $\bar{\psi}$ in verifying properties (a) through (d) for the function ψ given in Section 1. Also note that a smooth scaling of the vector ν can be accommodated by the positive function π; in the computations that follow, we therefore use the gradient to obtain *some* exterior normal ν.

Case 1. $p = ((0,b),(\cos\varphi,\sin\varphi)) \in TE_m$, where $m\tan\varphi \le b < 1$, $0 < b$, and $0 \le \varphi \le \frac{\pi}{2}$.

In the notation of Theorem 1, $\varphi = \varphi_1 = \varphi_2$, $\theta = 0$, $u = m\tan\varphi$, and $w = b^{m-1}(1-b^{2m})^{-1/2}u$. Let $\alpha = \arctan w$; then $\alpha \in [0,\frac{\pi}{2})$ and (1.4) becomes

$$\psi(\xi) = \left(\frac{\xi\cos\alpha(1-b^{2m})^{1/2}}{1+b^m\xi\sin\alpha}, \left(\frac{\xi\sin\alpha+b^m}{1+b^m\xi\sin\alpha}\right)^{1/m}\right) \tag{2.1}$$

for $\xi \in \Delta$. Of course, (2.1) defines ψ as well for $\xi = \zeta \in \partial\Delta$, and on $\partial\Delta$ we have

$$\begin{aligned}|\psi_1(\zeta)|^2 + |\psi_2(\zeta)|^{2m} &= \frac{1+b^{2m}\sin^2\alpha+\bar{\zeta}b^m\sin\alpha+\zeta b^m\sin\alpha}{|1+b^m\zeta\sin\alpha|^2}\\ &= 1,\end{aligned}$$

so that $\psi(\partial\Delta) \subset \partial E_m$. Let $\pi(\zeta) = |1+b^m\zeta\sin\alpha|^2$ and

$$\tilde{\psi}(\zeta) = (\tilde{\psi}_1(\zeta), \tilde{\psi}_2(\zeta)) = \zeta\,\pi(\zeta)\,\overline{\nu(\psi(\zeta))}$$

for $\zeta \in \partial\Delta$. Since $\overline{\nu(\psi(\zeta))}$ is given by

$$\left(\frac{(1-b^{2m})^{1/2}\bar{\zeta}\cos\alpha}{1+b^m\bar{\zeta}\sin\alpha},\; m\left(\frac{\bar{\zeta}\sin\alpha+b^m}{1+b^m\bar{\zeta}\sin\alpha}\right)\left(\frac{\zeta\sin\alpha+b^m}{1+b^m\zeta\sin\alpha}\right)^{1-1/m}\right),$$

it follows that for ζ in $\partial\Delta$ we have

$$\tilde{\psi}_1(\zeta) = (1+b^m\zeta\sin\alpha)(1-b^{2m})^{1/2}\cos\alpha, \tag{2.2}$$

$$\tilde{\psi}_2(\zeta) = m(\sin\alpha+\zeta b^m)(\zeta\sin\alpha+b^m)^{1-1/m} \times(1+\zeta b^m\sin\alpha)^{-1/m}. \tag{2.3}$$

That ψ is Hölder continuous of order $\frac{1}{2}$, like $\pi(\zeta)$, is an elementary consideration. In order to show that $\tilde{\psi}$ extends to a function holomorphic in Δ, it suffices, as we will see, to verify that $\sin\alpha \le b^m$. Now

$$\tan\alpha = \frac{mb^{m-1}\tan\varphi}{(1-b^{2m})^{1/2}},$$

whence

$$\frac{\sin\alpha}{\cos\alpha} \le \frac{b^m}{(1-b^{2m})^{1/2}}$$

for $b \ge m\tan\varphi$. Therefore, $\sin\alpha \le b^m$ and formulas (2.2) and (2.3) define a function $\tilde{\psi}$ on $\bar{\Delta}$ that is holomorphic in Δ. Moreover, unless $\sin\alpha = b^m$, it is clear that $\tilde{\psi}$ is continuous on $\bar{\Delta}$. In the case of equality, $\tilde{\psi}_2$ may be written as

$$\tilde{\psi}(\zeta)mb^{2m-1}(1+\zeta)^{2-1/m}(1+b^{2m}\zeta)^{-1/m},$$

and $\tilde{\psi}$ is continuous on $\bar{\Delta}$, provided $m \ge \frac{1}{2}$.

Case 2. $p = ((0,b),(\cos\varphi,\sin\varphi)) \in TE_m$, where $0 < b < m\tan\varphi$, $b < 1$, and $0 < \varphi \le \frac{\pi}{2}$.

In the notation of Theorem 1, $\varphi = \varphi_1 = \varphi_2$, $\theta = 0$, $u = m\tan\varphi$, $w = b^{m-1}(1-b^{2m})^{-1/2}u$, and

$$a = a(p) = 2mb(u+(u^2+4m(m-1)b^2)^{1/2})^{-1}.$$

Furthermore (1.5) becomes

$$\psi(\xi) = \left(\frac{a\xi(\lambda^2-1)}{\lambda(1-a^2)+\xi(1-\lambda^2a^2)},\; \frac{\lambda b(1-a^2)^{1/m}(\lambda\xi+1)(\xi+\lambda)^{-1+1/m}}{(\lambda(1-a^2)+\xi(1-\lambda^2a^2))^{1/m}}\right), \tag{2.4}$$

where $\xi \in \Delta$ and $\lambda = \lambda(p)$ is the unique positive solution of $\lambda^2a^2+\lambda^{2m}(1-a^2)b^{2m} = 1$. Once again, after extending ψ to $\bar{\Delta}$ by allowing ξ to belong to $\partial\Delta$ in (2.4), we find that $\psi(\partial\Delta) \subset \partial E_m$ by virtue of (1.13) and of the identity

$$a^2(\lambda^2-1)^2+(1+\lambda^2)(1-\lambda^2a^2)(1-a^2) = \lambda^2(1-a^2)^2+(1-\lambda^2a^2)^2.$$

For $\zeta \in \partial\Delta$, let $D(\zeta) = \lambda(1-a^2)+\zeta(1-\lambda^2a^2)$ and let $\pi(\zeta) = |D(\zeta)|^2$. Since $\lambda > 1$, it follows that $(\lambda-1)(1+a^2\lambda) > 0$ and therefore $\lambda(1-a^2) >$

$1-\lambda^2 a^2$. Thus, $\pi(\zeta) > 0$ for $\zeta \in \partial\Delta$. Let $c = c(p) = \lambda b(1-a^2)^{1/m}$. Then

$$\psi(\zeta) = \left(\frac{a\zeta(\lambda^2-1)}{D(\zeta)}, \frac{c(p)(\lambda\zeta+1)(\zeta+\lambda)^{-1+1/m}}{D(\zeta)^{1/m}}\right),$$

and, since $\nu(z_1,z_2) = (z_1, m|z_2|^{2m}\bar{z}_2^{-1})$, we see that

$$\overline{\nu(\psi(\zeta))} = \left(\frac{a\bar\zeta(\lambda^2-1)}{\overline{D(\zeta)}}, \; mc(p)^{2m-1}\frac{|\lambda\zeta+1|^{2m}|\zeta+\lambda|^{2-2m}D(\zeta)^{1/m}}{|D(\zeta)^2|(\lambda\zeta+1)(\zeta+\lambda)^{-1+1/m}}\right).$$

Moreover, since $|\lambda\zeta+1| = |\lambda+\zeta^{-1}| = |\lambda+\bar\zeta| = |\lambda+\zeta|$ for $\zeta\in\partial\Delta$, it follows that

$$\zeta\overline{\nu(\psi(\zeta))} = |D(\zeta)|^{-2}\left(a(\lambda^2-1)D(\zeta), \; mc(p)^{2m-1}\frac{\zeta|\zeta+\lambda|^2 D(\zeta)^{1/m}}{(\lambda\zeta+1)(\zeta+\lambda)^{-1+1/m}}\right),$$

and therefore

$$\tilde\psi(\zeta) = [a(\lambda^2-1)D(\zeta), \; mc(p)^{2m-1}(\zeta+\lambda)^{2-1/m}D(\zeta)^{1/m}).$$

All requirements for $\tilde\psi$ are now either evident or easily proved from this expression.

As discussed in Section 1, there is actually a third case: $u > b = 0$. In fact, in this case, ψ has the extremely simple form given by (1.3), and it is trivial to verify that it is stationary. The more general formulas in Section 1 now follow by composition with the obvious holomorphic maps.

3. SMOOTHNESS OF THE KOBAYASHI METRIC

In this section we assume that $m > \frac{1}{2}$. To show that K is C^1, consider a curve $\gamma(v) = (0,b,1,v)$ with $|b|>0$. When $u = mv \le |b|$, we have

$$K(v) = K(\gamma(v)) = \left(\frac{1}{1-|b|^{2m}} + \frac{|b|^{2m-2}u^2}{(1-|b|^{2m})^2}\right)^{1/2},$$

from which it follows that

$$\lim_{u\to|b|}\frac{DK}{dv} = \frac{m|b|^{2m-1}}{1-|b|^{2m}}.$$

In the second case, $0<|b|<u$, equation (1.2) is equivalent to $u = a^{-1}|b|(m-(m-1)a^2)$, equation (1.3) is equivalent to

$$1-a^2 = \frac{(\lambda^2-1)a^2}{1-\lambda^{2m}|b|^{2m}},$$

and (1.7) may be written more simply as

$$K(v) = \frac{a\lambda|v_1|}{1-\lambda^{2m}|b|^{2m}}. \tag{1.7$'$}$$

From (1.2) and (1.3), it is clear that

$$\lim_{u\to|b|} a(p) = 1 \quad\text{and}\quad \lim_{u\to|b|}\lambda(p) = 1. \tag{3.1}$$

Differentiating a^{-1} in (1.2) with respect to v, we obtain

$$\frac{da}{dv} = -\frac{a^2}{2|b|}(1+u(u^2+4m(m-1)|b|^2)^{1/2}),$$

and therefore

$$\lim_{u\to|b|}\frac{da}{dv} = \frac{-m}{|b|(2m-1)}. \tag{3.2}$$

On differentiating (1.3) with respect to v, taking the limit as u tends to $|b|$, and using (3.2), we get

$$\lim_{u\to|b|}\frac{d\lambda}{dv} = \frac{m(1-|b|^{2m})}{|b|(2m-1)}. \tag{3.3}$$

Finally, we find that, in agreement with the first case,

$$\lim_{u\to|b|}\frac{dK}{dv} = \lim_{u\to|b|}\frac{d}{dv}\left(\frac{a\lambda}{1-\lambda^{2m}|b|^{2m}}\right) = \frac{m|b|^{2m-1}}{1-|b|^{2m}},$$

by virtue of (1.7$'$), (3.1), (3.2), (3.3), and the quotient rule.

In the third case, $0=|b|<u$, the formula for $K(v)$ in (1.7$'$) makes sense only asymptotically. However, the Kobayashi indicatrix

$$\{v : K(0,0,v_1,v_2) = 1\}$$

at (0,0) is given by the equation $|v_1|^2 + |v_2|^{2m} = 1$ and, since the indicatrix is clearly C^1, the Kobayashi metric $K(0,0,v_1,v_2)$ must be a C^1 function of the variable v.

To complete the proof that K is C^1, it remains to show that

$$\lim_{b\to u^+}\frac{\partial K}{\partial b}(0,b,1,v) = \lim_{b\to u^-}\frac{\partial K}{\partial b}(0,b,1,v), \tag{3.4}$$

where $v \to 0$ and $u = mv$. To prove this, we will show that both sides exist and equal

$$u^{2m-1}(u^{2m} + 2m - 1)(1 - u^{2m})^{-2}.$$

In working first with the left side of (3.4), we will abbreviate $\lim_{b \to u^+}$ by lim. Since $K = K(0, b, 1, v)$ is given by

$$K = \left(\frac{1}{1 - b^{2m}} + \frac{b^{2m-2}u^2}{(1 - b^{2m})^2} \right)^{1/2} = \frac{\left(1 - b^{2m} + u^2 b^{2m-2}\right)^{1/2}}{1 - b^{2m}},$$

it is clear that

$$\lim K(0, b, 1, v) = (1 - u^{2m})^{-1}. \qquad (3.5)$$

To compute $\partial K/\partial b$, we differentiate both sides of

$$(1 - b^{2m})^2 K^2 = 1 - b^{2m} + u^2 b^{2m-2}.$$

After simplification, we get

$$(1 - b^{2m})^2 K \frac{\partial K}{\partial b} = m b^{2m-1}(2K^2(1 - b^{2m}) - 1) + u^2(m - 1)b^{2m-3}.$$

Taking the limit of both sides as $b \to u^+$ and using (3.5), we obtain, after simplification,

$$\lim \frac{\partial K}{\partial b} = u^{2m-1}(u^{2m} + 2m - 1)(1 - u^{2m})^{-2}.$$

Working with the right side of (3.4) is somewhat more difficult, reflecting the more complicated formula for the Kobayashi metric appearing in this case, the second listed in Theorem 1. We now write lim for $\lim_{b \to u^-}$. Recall that $K(0, b, 1, u) = a\lambda/(1 - (\lambda b)^{2m})$. Clearly,

$$\lim a = 1 \quad \text{and} \quad \lim \lambda = 1. \qquad (3.6)$$

We now compute $\lim \partial a/\partial b$ and $\lim \partial\lambda/\partial b$. From

$$a = \frac{2mb}{u + (u^2 + 4m(m-1)b^2)^{1/2}},$$

we obtain

$$\frac{\partial a}{\partial b} = \frac{2m(u + \sqrt{R}) - 2mb \cdot 4m(m-1)bR^{-1/2}}{(u + \sqrt{R})^2},$$

where $R = u^2 + 4m(m-1)b^2$. Note that $\lim R = (2m-1)^2 u^2$, so that

$$\lim \frac{\partial a}{\partial b} = \frac{4m^2 u - 8m^2(m-1)u^2((2m-1)u)^{-1}}{4m^2 u^2}.$$

After simplification,

$$\lim \frac{\partial a}{\partial b} = \frac{1}{(2m-1)u}. \qquad (3.7)$$

On differentiating both sides of

$$\lambda^2 a^2 + \lambda^{2m}(1 - a^2)b^{2m} = 1,$$

we get

$$2\lambda \frac{\partial \lambda}{\partial b} a^2 + \lambda^2(2a)\frac{\partial a}{\partial b} + m\lambda^{2m-1}\frac{\partial \lambda}{\partial b}(1 - a^2)b^{2m} + \lambda^{2m}\left(-2a\frac{\partial a}{\partial b}\right)b^{2m} + \lambda^{2m}(1 - a^2)2mb^{2m-1} = 0.$$

Combining this with (3.6) and (3.7), we obtain

$$\lim \frac{\partial \lambda}{\partial b} = \frac{u^{2m} - 1}{(2m-1)u} \qquad (3.8)$$

and, recalling that $K = a\lambda/(1 - (\lambda b)^{2m})$,

$$\frac{\partial K}{\partial b} = \frac{\frac{\partial a}{\partial b}\lambda + a\frac{\partial \lambda}{\partial b}}{1 - (\lambda b)^{2m}} + \frac{2am\lambda(\lambda b)^{2m-1}}{(1 - (\lambda b)^{2m})^2}\left(\frac{\partial \lambda}{\partial b} b + \lambda\right).$$

It then follows from (3.7), (3.8), and much simplification that

$$\lim \frac{\partial K}{\partial b} = \frac{u^{2m-1}}{(1 - u^{2m})^2}(u^{2m} + 2m - 1),$$

as required.

4. NUMERICAL APPROXIMATIONS

In view of the semicontinuity of the Kobayashi metric [Reiffen 1963; Greene and Krantz 1984], and especially the semicontinuity of extremal disks that follows from considerations of normal families, it is attractive to use numerical methods to search for extremal disks.

One seeks to implement an algorithm that takes as input the defining function ρ for the domain, the base point z, and the direction v, and that performs a search among polynomial mappings $\varphi : \Delta \to E_m$, with user-specified restrictions on the degree of the polynomial, the range of the coefficients, and the maximum size of $|\lambda_\varphi|$.

Even with the parameter space so restricted, the problem is too large to be computationally feasible. Helton et al. (personal communication) have addressed this infeasibility by studying a real-variable analogue of the Kobayashi metric. We address it by employing a Monte Carlo method that checks a prespecified number of randomly selected disks satisfying the parameter bounds indicated above. This method has been implemented in a C program `kobayashi.c` that is publically available (see information at the end of the article).

In effect, the program shows that, with a certain (unspecified) probability, the extremal disk and corresponding value for the Kobayashi metric correspond to a certain output. Repeated running of the program, of course, increases the probability. In several experiments that we have performed, the numerical answer obtained substantiates the explicit calculations performed in earlier sections of this paper. Alternatively, those calculations, which are of course self-contained as presented, may be viewed as a test of the algorithm's effectiveness (see Table 1).

p	v	iterations	K (Monte Carlo)	K (exact)
$(0,0)$	$(1,0)$	20,000	1.0	1.0
$(0,0)$	$(1,1)$	40,000	0.786	0.786
$(0,0.2)$	$(1,0)$	40,000	0.9992	0.9992
$(0,0.2)$	$(1,1)$	10,000	0.749	0.762
$(0,0.2)$	$(1,1)$	40,000	0.749	0.762
$(0,0.2)$	$(1,1)$	60,000	0.752	0.762
$(0,0.2)$	$(1,1)$	160,000	0.754	0.762
$(0,0.5)$	$(1,0)$	40,000	0.968	0.968
$(0,0.5)$	$(1,1)$	40,000	0.577	0.626

TABLE 1. Values of the Kobayashi metric K on the ellipsoid E_m with $m = 2$, as obtained from Theorem 2 using Mathematica (last column) and from the program `kobayashi.c` (fourth column). The first two columns indicate the base point and the direction of the sample vector, the third column the number of random disks used in running the program. The search for extremal disks is limited to polynomial functions of degree at most eight. Four entries refer to the same base point and sample vector, but were computed with different numbers of iterations; this gives an idea of how slowly the algorithm converges to the correct answer, and also of the complexity of the problem.

Armed with this information, one would like to use the software to obtain experimental information about the Kobayashi metric on domains for which we have no hope of performing explicit calculations at this time. The problem of determining the Kobayashi metric for an ellipsoid of the form

$$E = \{(z_1, \dots, z_n) : |z_1|^{2m_1} + \cdots + |z_n|^{2m_n} < 1\},$$

with the m_j's positive integers, should be tractable using the methods presented in this paper (though the details are sure to be unpleasant). When the m_j's are positive but nonintegral, the matter is still of great interest but our methods do not apply directly—especially if the m_j's are irrational. The software will be of use in determining the behavior of the metric near nonsmooth boundary points of such a domain.

Given the results of [Lempert 1981], it is also natural to consider convex domains. By modifying one line of the C program, one could consider domains of the form

$$\{(z_1, z_2) : \operatorname{Re} z_1 + f(|z_2|) < 0\},$$

with f a real-valued convex function.

One of the more interesting open problems of the subject is to determine whether a smoothly bounded, pseudoconvex domain is complete in the Kobayashi metric. All known partial results indicate that this is true. (But Fornaess and Krantz have an unpublished example of a smooth, pseudoconvex domain on which the Kobayashi metric does not blow up as the reciprocal of the distance to the boundary.) The software can be used to gather data to help support or refute the conjecture.

A bare-hands calculation of the Kobayashi metric, essentially by trying all possible disks, is computationally expensive. The problem is exponentially complex in the degree of the polynomials describing the extremal disk and also in the dimension of the ambient space. It would be interesting to use some of Lempert's ideas [Lempert 1981] to simplify the search in the convex case. It is also possible that the dual extremal methods of [Royden and Wong], which apply on a formal level in considerable generality, could be used to simplify the search. We intend to address these issues in future work.

Here is a sketch of the algorithm implemented by the software: To limit the order of complexity of the problem, this software calculates extremal mappings of the disk into domains in $\mathbf{C}^2$. The domain is specified as $\Omega = \{z \in \mathbf{C}^2 : \rho(z) < 0\}$. It is clear that in order to evaluate the infinitesimal

Kobayashi metric at a point $P = (p_1, p_2)$ in the direction $v = (v_1, v_2)$, it suffices to consider polynomial mappings $\varphi(\zeta) = (\varphi_1(\zeta), \varphi_2(\zeta))$ from the disk to the domain Ω. The zero- and first-order coefficients of these polynomials are determined (the latter up to a scalar multiple) by the base point P and the direction v.

The coefficients of φ are randomly perturbed, within the specified constraints, in an effort to push the value λ_φ toward $\lambda_{\max}$. At each stage, the software checks that the new configuration of φ has image lying in the domain Ω. In the spirit of a Markov process, the step-k perturbation takes into account information about the success or failure of the first $k-1$ perturbations. When successive configurations of $\varphi'(0)$ become and remain within a user-specified ε of each other, the program halts, flagging success. If, after a user-specified number of attempts, no such convergence is achieved, the program halts, flagging failure.

In general, we cannot hope that the extremal disk for a given problem will be a polynomial. But the polynomial achieving the extreme value in this software algorithm is of some interest and is presumably closely related to the true extremal mapping. The program exhibits the extremal data upon completion of its run.

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SOFTWARE AVAILABILITY

The program `kobayashi.c` described in Section 4 is available by ftp from the machine `jezebel.wustl.edu`. Use the account `guest` and the password `anyonehome`, and go to the directory `/home/jezebel/sk/programs`. Also in that directory are other files of interest to workers in several complex variables, including the program `levi.mac`, a Macsyma program that will take as input a defining function $\rho(z)$ and a point P satisfying $\rho(P) = 0$, and will calculate an orthonormal basis for the complex tangent space to the surface $M = \{z : \rho(z) = 0\}$ at the point P, and the values of the Levi form at P.

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