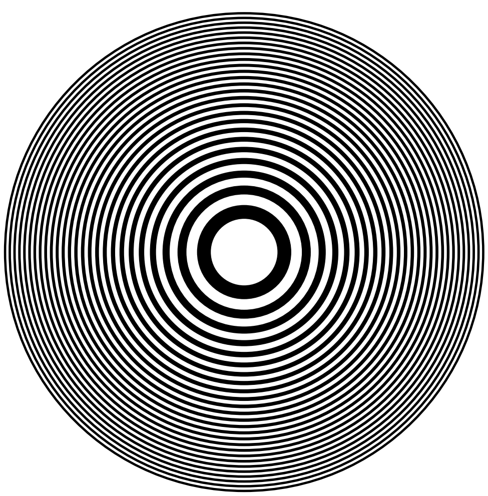
Fresnel Zones on the Screen

Pierre Goetgheluck

CONTENTS



For λ real, we consider the pattern given by the value modulo 2 of the integer part of $\lambda(x^2+y^2)$, where $(x,y) \in \mathbb{Z} \times \mathbb{Z}$. We study the periodicity and other geometric properties of this pattern, and show that it can provide, by visual inspection and an elementary computation, a diophantine approximation for λ. We conclude with similar results for other moduli.

1. INTRODUCTION

Fresnel zones arise from diffraction. They consist of alternating light and dark concentric rings whose radii increase as $\sqrt{n}$, for n a positive integer. In nature the boundary between the rings is not sharp—the brightness varies continuously with distance from the center—but we will consider the all-or-nothing approximation that appears on the left.

To describe this brightness function $f(x,y)$ we choose a scale coefficient, denoted $\sqrt{\lambda}$ for convenience. Then

$$f(x,y) = \begin{cases} 0 & \text{if} \quad \sqrt{2n} \le \sqrt{\lambda}\sqrt{x^2+y^2} < \sqrt{2n+1}, \\ 1 & \text{if} \ \sqrt{2n+1} \le \sqrt{\lambda}\sqrt{x^2+y^2} < \sqrt{2n+2}, \end{cases}$$

for some positive integer n. Equivalently,

$$f(x,y) = [(x^2+y^2)\lambda] \pmod 2, \tag{1.1}$$

where the brackets denote the floor function: $[a]$ is the greatest integer not exceeding a.

To plot the Fresnel zones on a computer screen, we must discretize the domain. From now on we regard f as a function defined on $\mathbb{Z}\times\mathbb{Z}$, and color a pixel (x,y) white if $f(x,y)=0$, black if $f(x,y)=1$. We let G_λ denote the pattern obtained in this way.

The figures on the next two pages, which show G_λ for several rational values of λ, contain some surprises. We get not one but several families of

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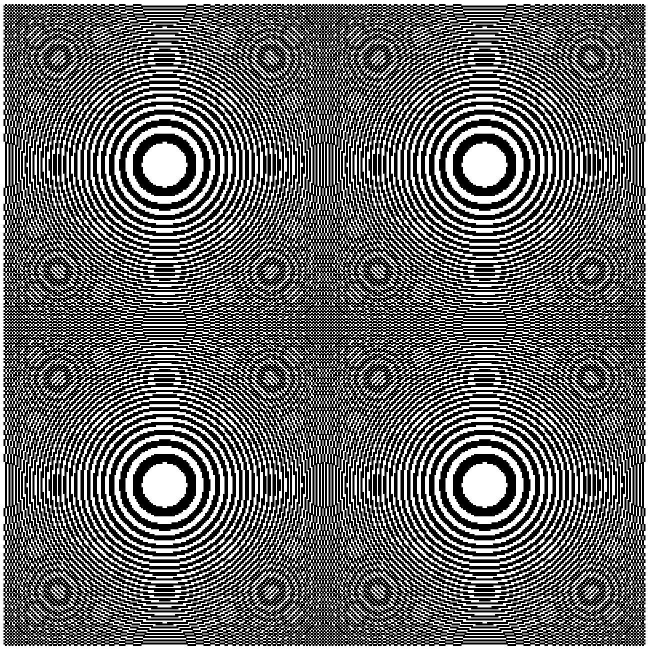

FIGURE 1. Region $[-100, 300] \times [-100, 300]$ for $\lambda = \frac{1}{200}$.

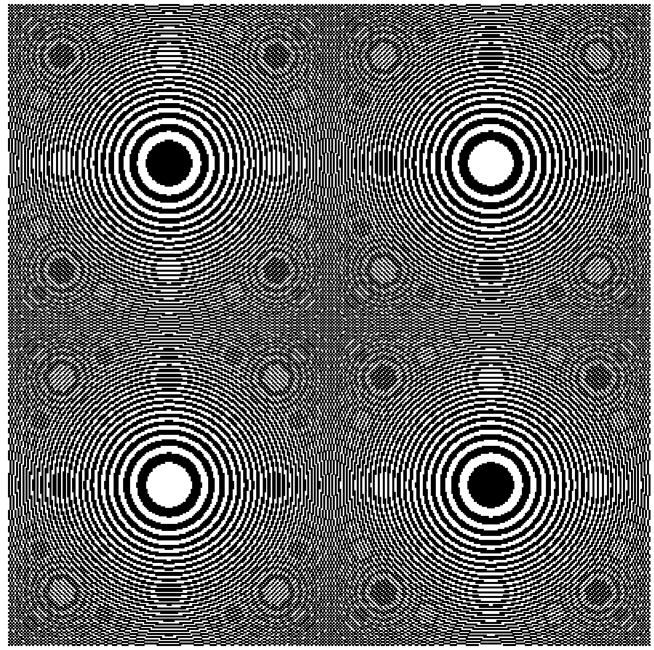

FIGURE 2. Region $[-100, 300] \times [-100, 300]$ for $\lambda = \frac{1}{201}$.

Fresnel rings (Figures 1 and 2); the pattern is periodic (Figures 1 and 3); and secondary systems of rings appear (Figures 1 and 2). The aim of this paper is to explain these phenomena.

In Section 3, we prove that G_λ is periodic if and only if λ is rational, and find its shortest period. In Section 4, we describe the geometrical structure of G_λ. In Section 5, we explain why secondary systems of rings arise, and where they are located. In Section 6, we show that one can find a rational approximation of λ by visual inspection of G_λ and an elementary calculation. Section 7 concludes with some generalizations.

Dewdney [1986] has discussed similar patterns, but to my knowledge there has been no mathematical treatment of them.

2. NOTATION AND CONVENTIONS

For λ a real number, we define f by (1.1), and denote by G_λ the associated pattern. When necessary we write f_λ instead of f. Clearly $f_\lambda = f_{\lambda+2}$, so by adding or subtracting a positive integer we can assume that $\lambda \in [0, 2)$ as far as f is concerned.

Convention. Whenever we write $\lambda = r/s$ we assume that r and s are relatively prime positive integers.

If there exists a positive integer T such that

$$f(x+T, y) = f(x, y) \quad \text{for all } x, y \in \mathbb{Z},$$

we say that f and G_λ are periodic of period T. In this case f is also periodic of period T in y, since f is symmetric. The *shortest period* of f (or of G_λ) is the smallest integer T such that f is periodic of period T.

Any real number $\theta \in [0, 2)$ can be written in base 2 in the form $\theta = a_0.a_1a_2a_3\ldots$, where $a_i = 0$ or 1 for all i. This is the same as writing

$$\theta = \sum_{0}^{\infty} \frac{a_i}{2^i}.$$

Convention. If θ is of the form $k2^{-j}$ for integers $j \geq 0$ and k, there are two binary expansions for θ, one of the form $\ldots a_{n-1}a_n1000\ldots$ and the other of the form $\ldots a'_{n-1}a'_n0111\ldots$. We will always use the former expansion: in other words, there is never an integer i_0 such that $a_i = 1$ for all $i \geq i_0$.

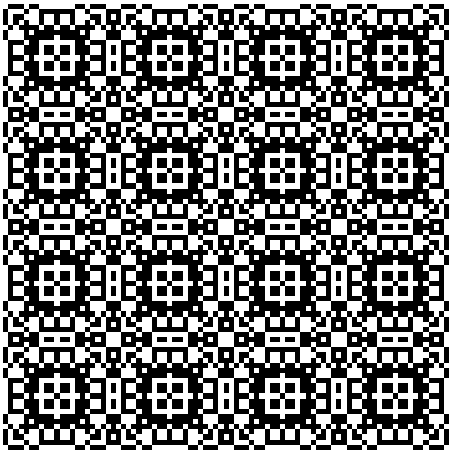

FIGURE 3. Region $[-43,43]\times[-43,43]$ for $\lambda=\frac{7}{22}$.

3. UNIQUENESS AND PERIODICITY

Lemma 3.1. *Let λ be a real number. Then f_λ is identically zero if and only if λ is an even integer.*

Proof. As already observed, we can assume that $\lambda\in[0,2)$. Suppose that f_λ vanishes identically, so that $[(p^2+q^2)\lambda]=0 \pmod 2$ for all $p,q\in\mathbb{Z}$. Let $\lambda=a_0.a_1a_2a_3\dots$ be the binary expansion of λ. For an arbitrary positive integer j, we plug in $p=2^j$ and $q=0$; then

$$[(p^2+q^2)\lambda]=\left[2^{2j}\sum_0^\infty \frac{a_i}{2^i}\right]=[a_{2j}] \pmod 2,$$

where for the second equality we have used the convention that there is never a position beyond which all the $a_i=1$. We conclude that $a_{2j}=0$ for all j. Then we plug in $p=2^j$ and $q=2^j$; this gives

$$[(p^2+q^2)\lambda]=\left[2^{2j+1}\sum_0^\infty \frac{a_i}{2^i}\right]=[a_{2j+1}] \pmod 2,$$

so that, likewise, $a_{2j+1}=0$ for all j. This shows that $\lambda=0$. □

This argument actually shows that the whole binary expansion $a_0.a_1a_2a_3\dots$ of a number $\lambda\in[0,2)$ can be recovered from f_λ: namely, $a_{2j}=f_\lambda(2^j,0)$ and $a_{2j+1}=f_\lambda(2^j,2^j)$. We thus have proved:

Proposition 3.2. *$G_\lambda=G_\mu$ (equivalently, $f_\lambda=f_\mu$) if and only if λ and μ differ by an even integer.* □

Remark. It is still possible to have G_λ coincide with G_μ *after a translation*, for distinct $\lambda,\mu\in[0,2)$. This happens when $\lambda=r/s$ with r odd and s is a multiple of four: then $G_{\lambda+1}$ is a translate of G_λ by the vector $(\frac12 s,\frac12 s)$, as a straightforward calculation shows.

Proposition 3.3. *G_λ is periodic if and only if λ is rational.*

Proof. If $\lambda=r/s$, we easily verify that $2s$ is a period of f. Conversely, assume that f is periodic of period T. This means that

$$[((x+pT)^2+(y+qT)^2)\lambda]=[(x^2+y^2)\lambda] \pmod 2$$

for any integers p,q. Taking $x=0$ and $y=0$ shows that $f_{T^2\lambda}$ is identically zero, so $T^2\lambda$ is an even integer by Lemma 3.1. Since T is an integer, λ is rational. □

Theorem 3.4. *If $\lambda=r/s$, the shortest period of G_λ is $2s$ if rs is odd, and s if rs is even.* (Recall that r and s are relatively prime positive integers.)

Lemma 3.5. *Let $\alpha,\beta\in\mathbb{R}$ be such that*

$$[\alpha+k\beta]=[\alpha] \pmod 2 \quad \textit{for any } k\in\mathbb{Z}. \tag{3.1}$$

Then β is an even integer.

Proof. Again we can obviously reduce to the case $\beta\in[0,2)$. We prove that $\beta=0$ by contradiction.

If $\beta=1$ then $[\alpha+\beta]=[\alpha]+1$, contradicting (3.1). If $0<\beta<1$, let n be the largest integer such that $[\alpha+n\beta]=[\alpha]$. Then $[\alpha+(n+1)\beta]=[\alpha]+1$, again contradicting (3.1). Finally, if $1<\beta<2$, the same reasoning applied to $2-\beta$ contradicts the equality

$$[\alpha-k(2-\beta)]=[\alpha] \pmod 2 \quad \text{for any } k\in\mathbb{Z},$$

which is equivalent to (3.1). □

Proof of the theorem. We know that f is periodic of period $2s$; let t be the shortest period. The proof of Proposition 3.3 shows that λt^2 is an even integer. We substitute $x=1$ and $y=0$ in the equation

$$((x+kt)^2+y^2)\frac{r}{s}=(x^2+y^2)\frac{r}{s} \pmod 2,$$

where k is any integer, and expand the square. Taking into account that $(r/s)t^2$ is an even integer, we obtain

$$\left[\frac{r}{s} + k\frac{2tr}{s}\right] = \left[\frac{r}{s}\right] \pmod 2$$

for all $k \in \mathbb{Z}$, and by Lemma 3.5 this implies that $2rt/s$ is an even integer. Since r and s are relatively prime, s divides t. But $2s$ is a period, and so a multiple of t. Therefore $t = s$ or $t = 2s$. Finally, the equality

$$\left[((x+s)^2 + y^2)\frac{r}{s}\right] = \left[(x^2+y^2)\frac{r}{s} + rs\right] \pmod 2,$$

obtained by expanding $(x+s)^2$, shows that s is a period if and only if rs is even. □

4. SYMMETRIES AND OTHER GEOMETRIC REMARKS

We now turn to the symmetries of G_λ. We start by observing that there are always eight symmetries fixing the origin: four rotations by multiples of 90°, and four reflections in the coordinate axes and in the diagonals $x = y$ and $x = -y$.

When λ is irrational, G_λ has no other symmetries.

When λ is rational, let t be the shortest period of G_λ. We already know that the translations $(t, 0)$ and $(0, t)$ preserve G_λ.

When rs is even, these two translations generate the group of translational symmetries of G_λ. Adjoining the symmetries about the origin we obtain the full group of symmetries of G_λ. Thus a point $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ has order-eight symmetry if and only if

$$(x, y) = (\tfrac{1}{2}pt, \tfrac{1}{2}qt) \quad \text{with } p, q \in \mathbb{Z} \text{ and } p + q \text{ even.}$$

Points of the form $(\frac{1}{2}pt, \frac{1}{2}qt)$, for $p + q$ odd, are fixed by four symmetries: reflections in horizontal and vertical lines, and 180° rotations.

When rs is odd, $(t, 0)$ and $(0, t)$ generate only a subgroup of index two in the group of translational symmetries of G_λ; the translation $(\frac{1}{2}t, \frac{1}{2}t)$ is also a symmetry. Adjoining this latter to the symmetries about the origin we get the full group of symmetries of G_λ. A point $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ has order-eight symmetry if and only if

$$(x, y) = (pt, qt) \quad \text{with } p, q \in \mathbb{Z}.$$

Points of the form $(\frac{1}{4}pt, \frac{1}{4}qt)$, for $p + q$ even, are fixed by four symmetries: reflections in diagonal lines and 180° rotations.

It is also interesting to consider transformations that don't quite leave G_λ invariant, but act in some simple way. For example, define a *semisymmetry* of G_λ as an isometry of $\mathbb{Z} \times \mathbb{Z}$ that interchanges black and white, or, more formally, that conjugates f to $1 - f$.

It is trivial to show that, if $\lambda = r/s$ with rs odd, a horizontal or vertical translation by $s = \frac{1}{2}t$ is a semisymmetry. In this case G_λ has a draughtboard pattern (Figure 4).

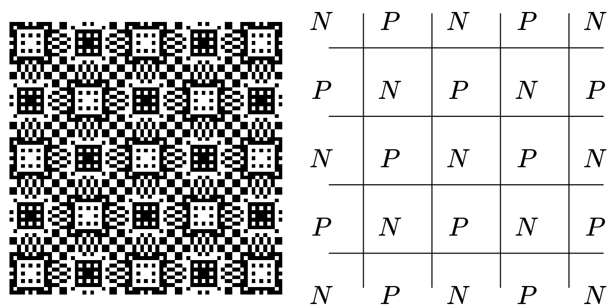

FIGURE 4. Left: Region $[-35, 35] \times [-35, 35]$ for $\lambda = 7/15$. Right: In general, for $\lambda = r/s$ with rs odd, G_λ can be divided into blocks of side s, arranged a draughtboard pattern (N and P denote complementary arrays).

For rs odd, the group of symmetries of G_λ described above has index two in the group of symmetries and semisymmetries combined. For rs even or λ irrational, there are no semisymmetries.

Yet another generalization of symmetries of G_λ is the following. If r is odd and s is even, every other pixel changes color under a diagonal translation by $(\frac{1}{2}t, \frac{1}{2}t)$, where $t = s$ is the shortest period. More precisely, this translation acts as a pixelwise exclusive-or with the filter

$$\begin{matrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{matrix},$$

where the origin combines with 1 (changes color) if $\frac{1}{2}s$ is odd and with 0 if $\frac{1}{2}s$ is even.

Finding r and s from G_λ

Proposition 3.2 says that a real number $\lambda \in [0, 2)$ is uniquely determined from G_λ. Here we assume that G_λ is periodic and spell out a procedure for finding $\lambda = r/s$.

First, find the shortest period t. If G_λ has the draughtboard structure, $s = \frac{1}{2}t$, otherwise $s = t$.

To find r, recall from the discussion preceding Proposition 3.2 that the $(2i)$-th bit in the binary expansion of λ is the color of the pixel $(2^i, 0)$, and the $(2i+1)$-th bit is the color of $(2^i, 2^i)$. Now choose j such that $2^j > s$, and find the bits $a_0, \ldots, a_j$. Since

$$\lambda = \frac{r}{s} = \sum_{i=0}^{j} \frac{a_i}{2^i} + \varepsilon \quad \text{with } 0 \le \varepsilon < \frac{1}{2^j}$$

and since $s(2^{-j} - \varepsilon) < 1$, we get

$$r = \left[s\left(\sum_{i=0}^{j} \frac{a_i}{2^i} + \frac{1}{2^j} \right) \right]. \tag{4.1}$$

We remark that this procedure requires the examination of $[\log_2 s] + 1$ pixels of G_λ.

5. THE RINGS

We observe in Figures 1 and 5 the surprising appearance of rings. In both cases we can remark that λ is close to a "simple" fraction: $\frac{1}{251}$ is close to $\frac{0}{1}$ and $\frac{72}{251}$ is close to $\frac{2}{7}$. The purpose of this section is to explain the following observation:

Observation. Rings are seen when $\lambda = r/s$ is close to a fraction a/b with small denominator. Main rings have center $(us/(2c), vs/(2c))$, where u and v are integers of same parity as ab, and $c = rb - as$.

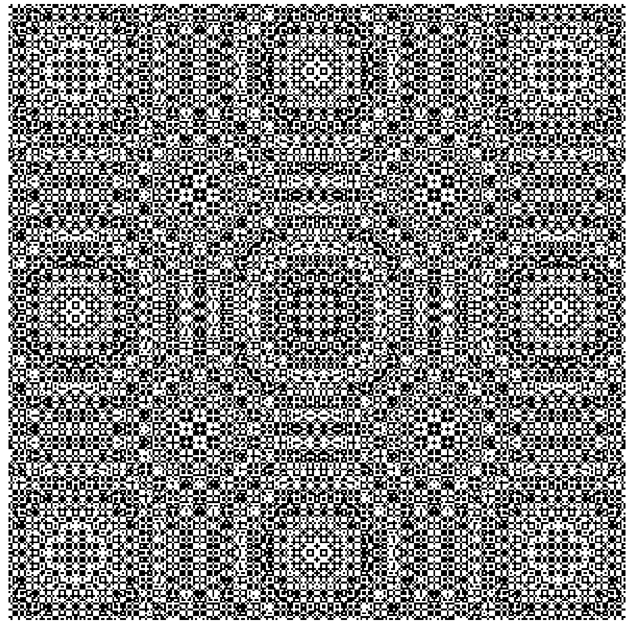

FIGURE 5. Region $[-160, 160] \times [-160, 160]$ for $\lambda = \frac{72}{251}$.

Explanation. Let r, s, a, b be positive integers, α and β real numbers, and set $x_0 = \alpha s$, $y_0 = \beta s$, $c = rb - as$. For any integer x and y, define ξ and η by $x = x_0 + \xi$ and $y = y_0 + \eta$. We have

$$x^2 + y^2 = 2(x_0 x + y_0 y) - (x_0^2 + y_0^2) + (\xi^2 + \eta^2),$$

and so

$$(x^2+y^2)\frac{c}{sb} = 2(\alpha x+\beta y)\frac{c}{b} - (\alpha^2+\beta^2)\frac{cs}{b} + (\xi^2+\eta^2)\frac{c}{sb}.$$

Substituting $c/(sb) = r/s - a/b$, we obtain

$$(x^2 + y^2)\frac{r}{s} = A(x, y) + (\xi^2 + \eta^2)\frac{c}{sb},$$

where

$$A(x, y) = (x^2 + y^2)\frac{a}{b} + 2(\alpha x + \beta y)\frac{c}{b} - (\alpha^2 + \beta^2)\frac{cs}{b}.$$

Let $(x_1, y_1) \in \mathbb{Z} \times \mathbb{Z}$ and let $z_1 \in [0, 2)$ be the residue of $A(x_1, y_1)$ modulo 2. Then all $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ satisfying $A(x, y) = z_1 \pmod 2$

- have the same color in every ring limited by consecutive circles with center (x_0, y_0) and radii of the form $\sqrt{(k - z_1)sb/c}$, where k is an integer $\ge z_1$ if $c > 0$ and $\le z_1$ if $c < 0$; and they

- change their color when passing from a ring to the next.

The same properties hold for all values of $A(x,y)$ modulo 2. In order to see the rings on the pattern G_λ it is necessary that the limit circles corresponding to these values be clearly distinct and have big enough radii: for example, the radii of the smallest circles should be ≥ 10, and the difference between two successive radii should be ≥ 2. Then

- sb/c must be big (> 100) and a/b must be close to r/s;
- the values of $A(x,y)$ modulo 2 must be few, which requires that α, β be rational and b be small.

Namely, in order that $(a(x^2+y^2)^2+2(\alpha x+\beta y)c)/b$ take only a few values, we must choose $2\alpha c$ and $2\beta c$ to be integers. Then, if $\lambda = r/s$ is close to a fraction a/b with a small denominator, we observe families of concentric rings with center at $(us/(2c), vs/(2c))$ for $u, v \in \mathbb{Z}$.

If we choose $\alpha = u/(2c)$ and $\beta = v/(2c)$, we have

$$(x^2+y^2)\frac{r}{s} = \frac{a(x^2+y^2)+(ux+vy)}{b} - (\alpha^2+\beta^2)\frac{cs}{b} + (\xi^2+\eta^2)\frac{c}{sb}.$$

Now

$$\frac{a(x^2+y^2)+(ux+vy)}{b} - (\alpha^2+\beta^2)\frac{cs}{b} \tag{5.1}$$

varies much faster than the last summand in the preceding equality. This means that near (x_0, y_0) we can obtain G_λ by modifying the pattern arising from the integer part of (5.1) (mod 2) with the help of the term $(\xi^2+\eta^2)c/(sb)$. Assume that ab is odd. It is easy to show that the shortest period of (5.1) is b if uv is odd and $2b$ otherwise. In the second case, the draughtboard structure with a small b gives a general impression of grey, and consecutive rings are indistinguishable: the rings are seen when ab is odd if u and v are odd. Similarly, if ab is even, the rings are seen if both u and v are even. □

In the particular case when r/s is small, that is, if $a = 0$, $b = 1$, $c = r$, the expression $2(\alpha x + \beta y)c$ takes only very few values modulo 2 if α and β are fractions with the same small denominator. We observe in this case families of rings with center at $(us/w, vs/w)$, for u and v integers and w a small positive integer (see Figure 3).

Application. Given a G_λ that shows rings, with $\lambda = r/s$, we can easily find a "simple" fraction a/b close to r/s as follows: count the number k of the most visible systems of rings whose centers belong to a horizontal segment of length s; then solve the equation $ry - sx = k$ in integers and select the solution with smallest $|x|$ and $|y|$. These two absolute values are a and b.

For example, in Figure 5, with $r = 72$ and $s = 251$, we see that $k = 2$. Solving $72y - 251x = 2$ gives $x = 2 + 72m$ and $y = 7 + 251m$, for m integer. Then $a = 2$ and $b = 7$. The error in the approximation is $\frac{2}{1757}$.

6. DIOPHANTINE APPROXIMATION USING G_λ

Nearby values of λ lead to patterns that differ but little near the origin: we will formalize this assertion shortly. Therefore, if a pattern G_λ is quasi-periodic—that is, periodic except at some exceptional points—in a neighborhood of the origin, this should mean that λ is close to a rational number r/s. This rational approximation can be found by the method at the end of Section 4.

We denote by $D(R)$ the open disk with center at $(0,0)$ and radius R. Suppose $\lambda \in [0,2)$ satisfies $\lambda = r/s + \varepsilon$, where r, s are positive integers, $r/s \in [0,2)$, and the real number ε is less than $1/s$ in absolute value. Let E_s be the set of $(x,y) \in \mathbb{Z} \times \mathbb{Z}$ such that $x^2 + y^2 = ks$ for some positive integer k; this exceptional set is where changes may occur.

For any $(x,y) \in \mathbb{Z} \times \mathbb{Z}$ such that $x^2 + y^2 < (|\varepsilon| s)^{-1}$, we have

$$[(x^2+y^2)\lambda] = \left[(x^2+y^2)\frac{r}{s}+(x^2+y^2)\varepsilon\right] = \left[(x^2+y^2)\frac{r}{s}\right]$$

unless $\varepsilon < 0$ and $(x,y) \in E_s$. Therefore:

- If $\varepsilon > 0$ and $R^2 < (\varepsilon s)^{-1}$, G_λ is identical to $G_{r/s}$ in $D(R)$.
- If $\varepsilon < 0$ and $R^2 < -(\varepsilon s)^{-1}$, G_λ is identical to $G_{r/s}$ in $D(R) \setminus E_s$.

Thus G_λ is quasiperiodic in $D(R)$. We note that $D(R) \cap E_s$ is small, since the number of integer solutions of the equation $x^2 + y^2 = n$ for integer n is

$$4 \sum_{\substack{d|n \\ d \text{ odd}}} (-1)^{(d-1)/2}$$

(see, for example, [Landau 1958, p. 138]), and this number is $O(n^\alpha)$ for any $\alpha > 0$ [Hua 1982, p. 120].

To allow the detection of a quasiperiod of a pattern G_λ, the window under examination should contain at least two shortest periods t of $G_{r/s}$, so that G_λ is identical to $G_{r/s}$ in $[-t,t]^2 \setminus E_s$. This would require $2s^3|\varepsilon| < 1$ if $t = s$ and $8s^3|\varepsilon| < 1$ if $t = 2s$. But experience shows that in most cases one can guess t when $s^3|\varepsilon| < 1$. In this case one can also conclude that r/s is a convergent of the continued fraction expansion of λ, since $|\lambda - r/s| < s^{-3} < \frac{1}{2}s^{-2}$ for $s > 2$ (for the continued fraction criterion, see [Hua 1982, p. 262], for example).

Given a pattern G_λ quasiperiodic around the origin, the shortest quasiperiod t can be easily measured, and from it s can be deduced. Finally, r can be computed using the method at the end of Section 4. So even if λ is not known one can use G_λ to find a rational approximation.

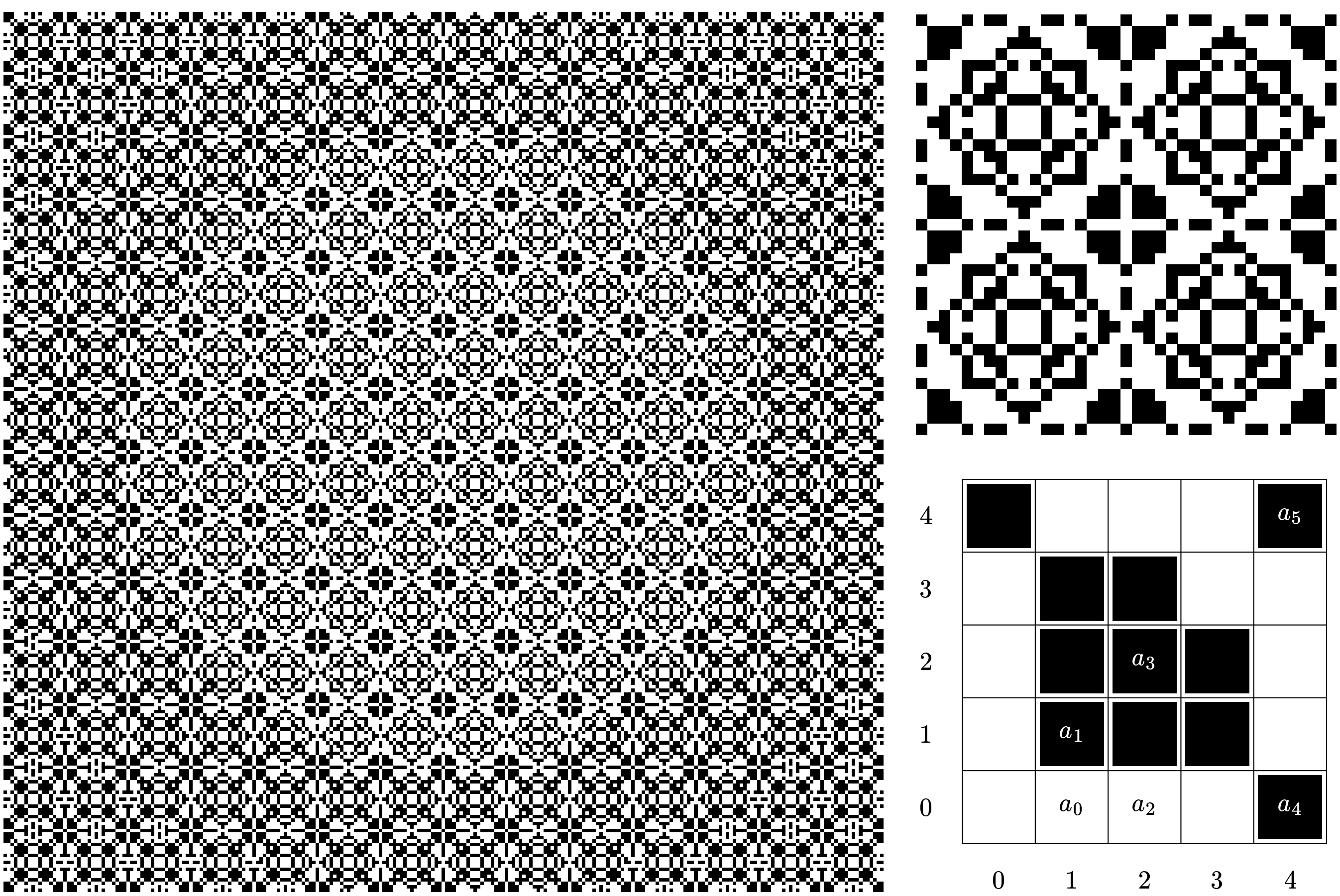

FIGURE 6. Regions $[-125,125] \times [-125,125]$, $[-18,18] \times [-18,18]$ and $[0,4] \times [0,4]$, for $\lambda = \sin 0.807$. The figure under high magnification shows the pixels relevant to the computation of the binary expansion of λ.

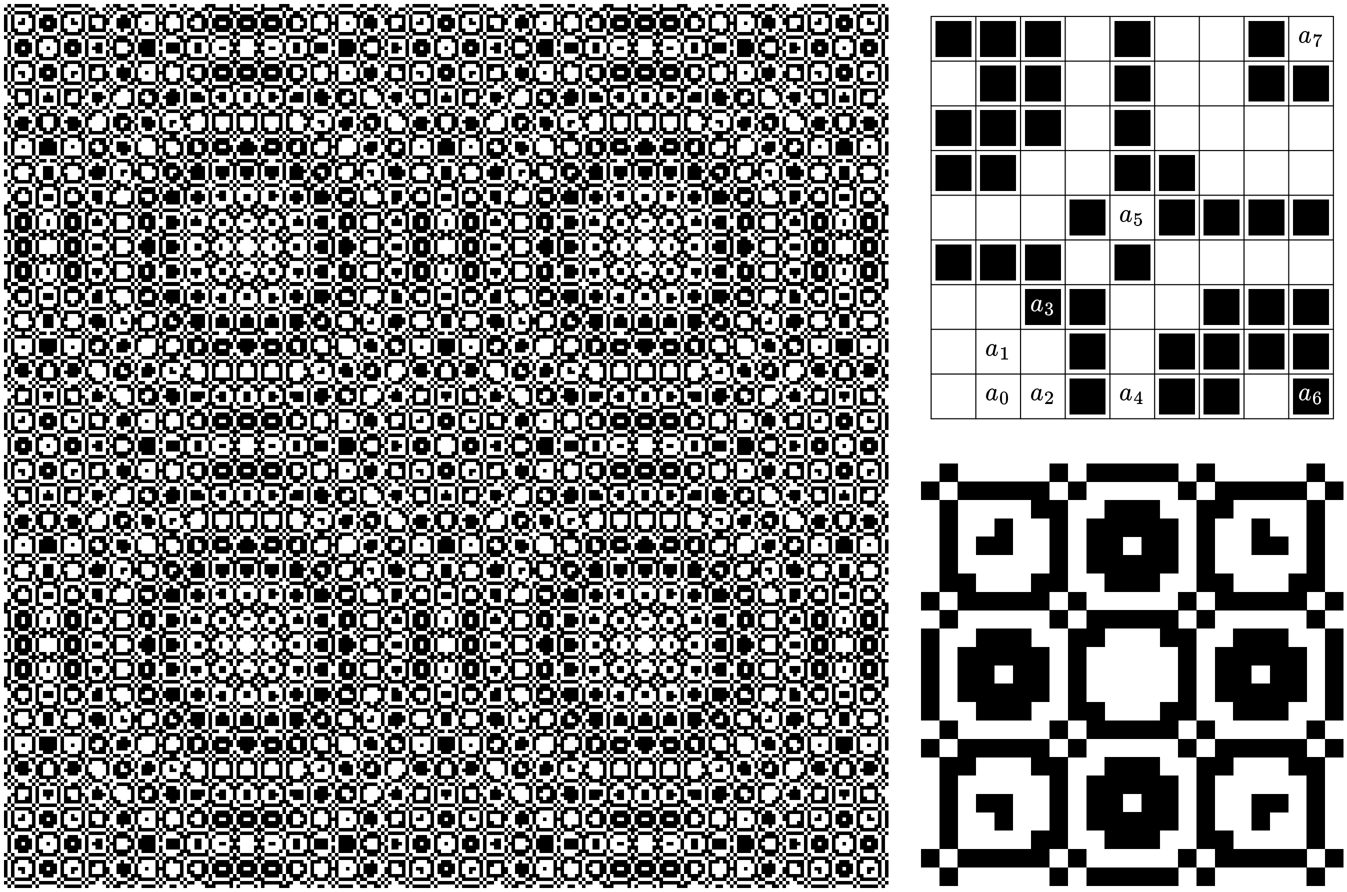

FIGURE 7. Regions $[-125, 125] \times [-125, 125]$, $[0, 16] \times [0, 16]$ and $[-10, 10] \times [-10, 10]$, for $\lambda = \pi - 3$.

Examples. Figure 6 shows G_λ for $\lambda = \sin 0.807$. From the top right diagram we see that $t = 18$, and so $s = 18$ since G_λ does not have the draughtboard structure. Since $2^4 < 18$ and $2^5 > 18$, we need the bits $a_0, \dots, a_5$ of the binary expansion of λ in order to compute r. From the bottom right diagram we read $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = 1$, $a_4 = 1$, $a_5 = 1$. Consulting (4.1) we then have $r = 13$. The difference $\lambda - r/s$ is in fact less than s^{-4} in this case.

Figure 7 shows G_λ for $\lambda = \pi - 3$, with the fairly large quasiperiod $t = 113$. Again, $s = t$, and r is computed by a binary calculation to have the value 16, and $\lambda - r/s < \frac{1}{2}s^{-3}$. We recover the well-known rational approximation $\pi = 3\frac{16}{113} = \frac{355}{113}$. Moreover, if we look near the origin we see another quasiperiodicity (Figure 7, bottom right), showing the draughtboard pattern. Here $s = 7$ and $r = 1$ with $\lambda - r/s < \frac{1}{2}s^{-3}$, again yielding a famous rational approximation for π.

7. GENERALIZATIONS

Similar results can be developed replacing the modulus 2 by any modulus $p > 2$, and using p different colors to draw the pattern. As an example we give without proof the result about the periodicity of the pattern.

We denote by $V_2(n)$ the exponent of 2 in the factorization of a positive integer n into primes.

Theorem 7.1. *For λ a real number and $p \geq 2$ an integer, let g be the function defined on $\mathbb{Z} \times \mathbb{Z}$ by*

$$g(x, y) = [(x^2 + y^2)\lambda] \pmod{p}.$$

Then g is periodic if and only if λ is rational. If $\lambda = r/s$ with r, s relatively prime positive integers, the shortest period t of g is

$$t = c\frac{ps}{\gcd(ps, 2r)},$$

where

$$c = \begin{cases} 1 & \text{if } V_2(ps) \neq V_2(2r), \\ 2 & \text{if } V_2(ps) = V_2(2r). \end{cases}$$

Surprisingly, the situation in one dimension is more complicated than in two:

Theorem 7.2. *For λ a real number and $p \geq 2$ an integer, let h be the function defined on $\mathbb{Z}$ by*

$$h(x) = [x^2\lambda] \pmod{p}.$$

Then h is periodic if and only if λ is rational. If $\lambda = r/s$ with r, s relatively prime positive integers, the shortest period t of h is given by the same formula as in the preceding theorem, except for the following combinations of p, r, s:

p	s	$r \pmod{ps}$	t
2	2	1	1
2	3	2	1
2	3	5	2
2	6	11	3
2	12	11 or 23	4
3	4	3	1
3	4	7 or 11	3

ACKNOWLEDGEMENT

I am greatly indebted to Dr. Silvio Levy for a correction, remarks and help in the preparation of the final version of this paper.

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Pierre Goetgheluck, Université de Paris-sud, Mathématiques, Bâtiment 425, 91405 Orsay CEDEX, France (goetghe@iut-orsay.fr)

Received May 13, 1993; accepted January 12, 1994