Empirically Determined Apéry-Like Formulae for $\zeta(4n+3)$

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CONTENTS

Some rapidly convergent formulae for special values of the Riemann zeta function are given. We obtain a generating function formula for $\zeta(4n+3)$ that generalizes Apéry's series for $\zeta(3)$, and appears to give the best possible series relations of this type, at least for $n < 12$. The formula reduces to a finite but apparently nontrivial combinatorial identity. The identity is equivalent to an interesting new integral evaluation for the central binomial coefficient. We outline a new technique for transforming and summing certain infinite series. We also derive a formula that provides strange evaluations of a large new class of nonterminating hypergeometric series.

[Editor's Note: The beautiful formulas in this paper are no longer conjectural. See note on page 194.]

1. INTRODUCTION

The Riemann zeta function is

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad \text{for } \operatorname{Re} s > 1. \tag{1–1}$$

The following equality, commonly called Apéry's formula because it was essential in his proof of the irrationality of $\zeta(3)$, goes back at least as far as [Hjortnaes 1954]:

$$\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}}. \tag{1–2}$$

Extensive computation has suggested that there is no analogous formula for $\zeta(5)$ or $\zeta(7)$. In other words, if there exist relatively prime integers a and b such that

$$\zeta(5) = \frac{a}{b} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5 \binom{2k}{k}},$$

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then b is astronomically large. Consider however, the following result of Koecher [1987]:

$$\zeta(5) = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5\binom{2k}{k}} - \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^2}. \tag{1–3}$$

Inspired by this result, one of us (Bradley), at the suggestion of the other, searched for additional zeta identities of this sort using high-precision arithmetic and Maple's lattice-based integer relations algorithms. Within the hour, we were rewarded with the following elegant new formula for $\zeta(7)$:

$$\zeta(7) = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^7\binom{2k}{k}} + \frac{25}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^4}. \tag{1–4}$$

Encouraged by this initial success, we searched for and found similar identities for $\zeta(9)$, $\zeta(11)$, $\zeta(13)$, etc. The representation for $\zeta(4n+3)$ has a convenient form in terms of a generating function (1–9), which is our main result (2–1). It is curious that there is apparently no analogous generating function in the $4n+1$ case. We refer the reader to the discussion at the end of Section 8. For now, it will be advantageous to exhibit the recursive nature of the formulae in the $4n+3$ case.

We denote the power sum symmetric functions $P_r := P_r^{(s)}(k)$ by $P_0 := 1$ and

$$P_r^{(s)}(k) := \sum_{j=1}^{k-1} j^{-rs}, \quad \text{for } r \geqslant 1.$$

Next, we define a two-place function

$$\lambda\Bigl(m, \prod_{j=1}^{n} P_{r_j}^{(s)}\Bigr) := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^m\binom{2k}{k}} \prod_{j=1}^{n} P_{r_j}^{(s)}(k).$$

In (1–3) and Section 9, $s=2$ is relevant, but for now we are only interested in the case $s=4$. Therefore, to minimize symbol clutter we shall occasionally repress the superscript, in which case $s=4$ should be assumed. With Maple's help, the following list was produced:

$$\begin{aligned}
\tfrac{2}{5}\zeta(\,3\,) &= \lambda(3, P_0),\\
\tfrac{2}{5}\zeta(\,7\,) &= \lambda(7, P_0) + 5\lambda(3, P_1),\\
\tfrac{2}{5}\zeta(11) &= \lambda(11, P_0) + 5\lambda(7, P_1) - \tfrac{15}{2}\lambda(3, P_2)\\
&\quad + \tfrac{25}{2}\lambda(3, P_1^2),\\
\tfrac{2}{5}\zeta(15) &= \lambda(15, P_0) + 5\lambda(11, P_1) - \tfrac{15}{2}\lambda(7, P_2)\\
&\quad + \tfrac{25}{2}\lambda(7, P_1^2) + \tfrac{130}{6}\lambda(3, P_3)\\
&\quad - \tfrac{225}{6}\lambda(3, P_1P_2) + \tfrac{125}{6}\lambda(3, P_1^3),
\end{aligned} \tag{1–5}$$

etc. The first of these equations is just a restatement of Apéry's formula (1–2), and the second is just a restatement of our formula (1–4). From the list, it became clear to us that the formula for $\zeta(4n+3)$ borrows the terms and coefficients from the formula for $\zeta(4n-1)$, except that the first argument of λ is increased by 4. The number of additional terms is equal to the number of partitions of n, and each combination of power sum symmetric functions that occurs corresponds to a specific partition of n. Thus, we were led to conjecture that

$$\tfrac{2}{5}\zeta(4n+3) = \sum_{j=0}^{n}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{4j+3}\binom{2k}{k}} \sum_{\alpha\vdash n-j} c_\alpha P_\alpha^{(4)}(k), \tag{1–6}$$

where the notation $\alpha \vdash n-j$ indicates that the inner sum is over all partitions

$$\alpha = (\alpha_1, \alpha_2, \ldots)$$

of $n-j$ (so that $\alpha_1+\alpha_2+\cdots = n-j$), the c_α are rational numbers to be determined, and

$$P_\alpha^{(s)}(k) := \prod_{r\geqslant 1} P_{\alpha_r}^{(s)}(k).$$

Since it seemed plausible that a generating function could simplify matters, we rewrote our conjecture (1–6) in the form

$$\begin{aligned}
&\frac{2}{5}\sum_{n=0}^{\infty} x^{4n}\zeta(4n+3)\\
&=\sum_{n=0}^{\infty} x^{4n}\sum_{j=0}^{n}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{4j+3}\binom{2k}{k}}\sum_{\alpha\vdash n-j} c_\alpha P_\alpha^{(4)}(k)\\
&=\sum_{j=0}^{\infty}\sum_{n=j}^{\infty} x^{4(n-j)}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{3}\binom{2k}{k}}\left(\frac{x}{k}\right)^{4j}\sum_{\alpha\vdash n-j} c_\alpha P_\alpha^{(4)}(k)\\
&=\sum_{s=0}^{\infty} x^{4s}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{3}\binom{2k}{k}}\frac{1}{1-x^4/k^4}\sum_{\alpha\vdash s} c_\alpha P_\alpha^{(4)}(k)\\
&=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{3}\binom{2k}{k}}\frac{E_k(x^4)}{1-x^4/k^4}, \qquad (1\text{–}7)
\end{aligned}$$

where

$$E_k(x) := \sum_{s=0}^{\infty} x^s \sum_{\alpha\vdash s} c_\alpha P_\alpha^{(4)}(k). \qquad (1\text{–}8)$$

For n a nonnegative integer, let $p(n)$ denote the number of partitions of n. By convention, $p(0) = 1$. We suspected that $E_k(x)$ had a closed form that might be revealed by determining enough of the coefficients in its power series. Fortunately, due to the recursive nature of the formulae we were able to extend the list (1–5) without unduly straining Maple's lattice algorithms. This was accomplished by introducing only $p(n)$ unknown coefficients for $\zeta(4n + 3)$, rather than $\sum_{j=0}^{n} p(j)$, the actual number of terms involved. Also, when the evidence warranted, we supplied the coefficients of as many of the additional $p(n)$ terms as we confidently could, based on our ability to recognize patterns and extrapolate from previously tabulated values. All identities so obtained were subsequently checked numerically, typically to 250 significant digits.

After having sufficiently extended the list (1–5), we were able to determine a good many of the coefficients c_α for partitions α of small positive integers, and hence the initial terms of the series expansion (1–8). Maple's convert(series, ratpoly) feature then produced the following evaluations:

$$\begin{aligned}
E_1(x) &= 1,\\
E_2(x) &= \frac{1+4x}{1-x},\\
E_3(x) &= \frac{(1+4x)(16+4x)}{(1-x)(16-x)},\\
E_4(x) &= \frac{(1+4x)(16+4x)(81+4x)}{(1-x)(16-x)(81-x)},
\end{aligned}$$

etc. Thus we were led to conjecture that

$$E_k(x) = \prod_{j=1}^{k-1}\frac{j^4+4x}{j^4-x},$$

and hence from (1–7) that

$$\sum_{n=0}^{\infty} x^{4n}\zeta(4n+3) = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\frac{1}{1-\frac{x^4}{k^4}}\prod_{j=1}^{k-1}\frac{j^4+4x^4}{j^4-x^4},$$

$$\text{for } |x|<1. \qquad (1\text{–}9)$$

We restate (1–9) in the next section in the form of a conjectured theorem, and discuss some of its implications in the subsequent sections.

2. A GENERATING FUNCTION FORMULA FOR $\zeta(4n + 3)$

Theorem 2.1 (conjectured). *Let z be a complex number. Then*

$$\sum_{k=1}^{\infty}\frac{1}{k^3(1-z^4/k^4)} = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\frac{1}{1-z^4/k^4}\prod_{j=1}^{k-1}\frac{1+4z^4/j^4}{1-z^4/j^4}. \qquad (2\text{–}1)$$

Remark. Taking coefficients of z^4 in (2–1) yields our formula (1–4) for $\zeta(7)$. Setting $z = 0$ in (2–1) yields Apéry's formula (1–2) for $\zeta(3)$. In general, taking coefficients of z^{4n} in (2–1) yields a rapidly convergent expansion for $\zeta(4n + 3)$, the k-th term of which is a rational function of k whose denominator is a power of k times the central binomial coefficient, and whose numerator is a symmetric function of partial harmonic sums in $1/j^4$.

More precisely, we denote the elementary symmetric functions by

$$e_r^{(s)}(k) := [t^r]\prod_{j=1}^{k-1}(1+j^{-s}t) = \sum_{1\leqslant j_1<j_2<\cdots<j_r\leqslant k-1} (j_1j_2\cdots j_r)^{-s}$$

and the complete monomial symmetric functions by

$$h_r^{(s)}(k) := [t^r]\prod_{j=1}^{k-1}(1-j^{-s}t)^{-1},$$

where, as customary, $[t^r]$ means take the coefficient of t^r. Then, by extracting the coefficient of z^{4n} from each side of (2–1), we have:

Corollary 2.2 (equivalent to conjectured Theorem 2.1). *Let n be a positive integer. Then*

$$\zeta(4n+3) = \frac{5}{2}\sum_{j=0}^{n}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{4j+3}\binom{2k}{k}}\sum_{r=0}^{n-j}4^r h_{n-j-r}^{(4)}(k)e_r^{(4)}(k). \quad (2\text{–}2)$$

In light of the fact that both the complete symmetric functions and the elementary symmetric functions can be expressed as rational linear combinations of the power sum symmetric functions, it is possible to rewrite (2–2) in terms of the P_α of Section 1, as in (1–6). However, the formula for the coefficients c_α appears to be very complicated. Thus, we have replaced the sum over partitions in (1–6) with a much more manageable sum, at the expense of introducing additional symmetric functions into the summand.

An additional consequence of (2–1) is an attractive formula that provides strange evaluations for a large new class of nonterminating hypergeometric series.

Corollary 2.3 (equivalent to conjectured Theorem 2.1). *For all positive integers n, we have the formula*

$${}_6F_5\left(\left.\begin{matrix} n{+}1,\ n{+}1,\ 2n{+}in,\ 2n{-}in,\ in,\ -in \\ n{+}\frac{1}{2},\ n,\ 2n{+}1,\ n{+}1{+}in,\ n{+}1{-}in\end{matrix}\right| -\tfrac{1}{4}\right) = \frac{2}{5}\binom{2n}{n}\prod_{j=1}^{n-1}\frac{n^4-j^4}{4n^4+j^4}. \quad (2\text{–}3)$$

Aside. Throughout, we adhere to the standard notation

$${}_pF_q\left(\left.\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q\end{matrix}\right| z\right) := \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\frac{z^k}{k!},$$

where, as customary,

$$(a)_k := \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\cdots(a+k-1).$$

Proof of the equivalence of Corollary 2.3 and Theorem 2.1. We can rewrite (2–1) as a formula for a nonterminating ${}_6F_5$:

$$\left(\frac{1}{1-z^4}\right){}_6F_5\left(\left.\begin{matrix} 2,\ 2,\ 1{+}z{+}iz,\ 1{+}z{-}iz,\ 1{-}z{+}iz,\ 1{-}z{-}iz \\ \frac{3}{2},\ \ 2+z,\ \ 2-z,\ \ 2+iz,\ \ 2-iz\end{matrix}\right| -\tfrac{1}{4}\right) = \frac{4}{5}\sum_{k=1}^{\infty}\frac{1}{k^3(1-z^4/k^4)}. \quad (2\text{–}4)$$

We note that both sides of (2–4) are meromorphic functions with simple poles at $z=\pm n$ and $z=\pm in$, where n is a positive integer. We shall see that Corollary 2 is a consequence of equating residues of both sides of (2–4) at the simple pole $z=n$. If we denote the requisite residue by R_n, then from the right side of (2–4), it is clear that

$$R_n = -\frac{1}{5n^2}. \quad (2\text{–}5)$$

The residue calculation for the left side of (2–4) is more difficult. We have

$$
\begin{aligned}
R_n &= -\frac{1}{4z^3}\sum_{k=n-1}^{\infty}\frac{(2)_k\,(2)_k\,(1\pm z\pm iz)_k}{(3/2)_k\,(1)_k\,(-4)^k}\prod_{\substack{j=1\\ j\neq n}}^{k+1}\frac{1}{j^4-z^4}\bigg|_{z=n}\\
&= -\frac{n^4-z^4}{4z^3}\sum_{k=0}^{\infty}\frac{(2)_{k+n-1}\,(2)_{k+n-1}\,(1\pm z\pm iz)_{k+n-1}}{(3/2)_{k+n-1}\,(1)_{k+n-1}\,(-4)^{k+n-1}}\prod_{j=1}^{k+n}\frac{1}{j^4-z^4}\bigg|_{z=n}\\
&= -\frac{(n^4-z^4)(2)_{n-1}\,(2)_{n-1}\,(1\pm z\pm iz)_{n-1}}{4z^3(3/2)_{n-1}\,(1)_{n-1}\,(-4)^{n-1}}\prod_{j=1}^{n}\frac{1}{j^4-z^4}\sum_{k=0}^{\infty}\frac{(n+1)_k\,(n+1)_k\,(n\pm z\pm iz)_k}{(n+1/2)_k\,(n)_k\,(-4)^k}\prod_{j=n+1}^{n+k}\frac{1}{j^4-z^4}\bigg|_{z=n}\\
&= -\frac{(2)_{n-1}\,(2)_{n-1}\,(1\pm n\pm in)_{n-1}}{n^3(3/2)_{n-1}\,(1)_{n-1}\,4^n}\prod_{j=1}^{n-1}\frac{1}{n^4-j^4}\,{}_6F_5\left(\begin{matrix}n+1,\ n+1,\ 2n+in,\ 2n-in,\ in,\ -in\\ n+\frac12,\ n,\ 2n+1,\ n+1+in,\ n+1-in\end{matrix}\;\middle|\;-\frac14\right).
\end{aligned}
\tag{2–6}
$$

Comparing (2–5) and (2–6), it follows that

$$
\begin{aligned}
{}_6F_5\left(\begin{matrix}n+1,\ n+1,\ 2n+in,\ 2n-in,\ in,\ -in\\ n+\frac12,\ n,\ 2n+1,\ n+1+in,\ n+1-in\end{matrix}\;\middle|\;-\frac14\right) &= \frac{n4^n(3/2)_{n-1}\,(1)_{n-1}}{5\,(2)_{n-1}\,(2)_{n-1}}\prod_{j=1}^{n-1}\frac{n^4-j^4}{4n^4+j^4}\\
&= \frac25\left(\frac{4^n(1/2)_n}{n!}\right)\frac{n(1)_{n-1}}{(2)_{n-1}}\prod_{j=1}^{n-1}\frac{n^4-j^4}{4n^4+j^4}\\
&= \frac25\binom{2n}{n}\prod_{j=1}^{n-1}\frac{n^4-j^4}{4n^4+j^4},
\end{aligned}
$$

as required. Thus we have shown that Corollary 2.3 follows from the conjectured Theorem 2.1. That Corollary 2.3 implies Theorem 2.1 now follows from Mittag-Leffler's Theorem. ☐

When $n=1$, the ${}_6F_5$ in Corollary 2.3 reduces to a ${}_4F_3$, and we obtain:

Corollary 2.4. $${}_4F_3\left(\begin{matrix}2,\ 2,\ -i,\ i\\ \frac32,\ 1,\ 3\end{matrix}\;\middle|\;-\frac14\right)=\frac45.$$

Corollary 2.4 is true, and we have a proof. However, since Corollary 2.4 is only a minor consequence of our conjectures, we delay the proof to the end of Section 6, where the proof is used to illustrate some remarks we have to make on our methods.

3. REDUCTION TO A FINITE IDENTITY

As we have said, (2–1) was originally a conjecture based on heavy experimental data. However, in the end, we managed to reduce the problem to that of proving a finite combinatorial identity that is beautiful in and of itself, and that we have, thus far, been unable to prove. It is

$$
\frac52\sum_{k=1}^{n}\binom{2k}{k}\frac{k^2}{4n^4+k^4}\prod_{j=1}^{k-1}\frac{n^4-j^4}{4n^4+j^4}=\frac{1}{n^2},\quad \text{for integer } n\geqslant 1. \tag{3–1}
$$

The marvelous connection between this identity and the conjectured generating function formula (2–1) is presented in the reduction below.

Reduction. By partial fractions we have, for each positive integer k,

$$
\frac{1}{1-z^4/k^4}\prod_{j=1}^{k-1}\frac{1+4z^4/j^4}{1-z^4/j^4}=\sum_{j=1}^{k}\frac{c_j(k)}{1-z^4/j^4}, \tag{3–2}
$$

where

$$c_n(k) = \prod_{j=1}^{k-1}(1+4n^4/j^4) \Bigg/ \prod_{\substack{j=1\\ j\neq n}}^{k}(1-n^4/j^4), \qquad \text{for } 1 \leqslant n \leqslant k. \tag{3–3}$$

Substituting (3–2) into the right hand side of (2–1) and interchanging order of summation shows that (2–1) is equivalent to

$$\sum_{k=1}^{\infty}\frac{1}{k^3(1-z^4/k^4)} = \frac{5}{2}\sum_{j=1}^{\infty}\frac{1}{1-z^4/j^4}\sum_{k=j}^{\infty}\frac{(-1)^{k+1}c_j(k)}{k^3\binom{2k}{k}}, \qquad \text{for } z\in\mathbb{C}.$$

Clearly, it suffices to prove that for all positive integers n,

$$\sum_{k=n}^{\infty} t_n(k) = \frac{1}{n^3}, \tag{3–4}$$

where

$$t_n(k) := \frac{5(-1)^{k+1}c_n(k)}{2k^3\binom{2k}{k}} \qquad 1 \leqslant n \leqslant k \in \mathbb{Z}. \tag{3–5}$$

Our method of attack is to transform the infinite sum (3–4) into a purely finite combinatorial identity. This is accomplished via analytic continuation of the summand combined with a process that might aptly be referred to as "Gosper reflection".

Let n be a fixed positive integer. We wish to extend the definition (3–5) to include values of k less than n. One approach is to convert the products implicit in (3–5) into gamma functions. Abbreviating $\Gamma(a+b+c)\Gamma(a+b-c)\Gamma(a-b+c)\Gamma(a-b-c)$, as $\Gamma(a\pm b\pm c)$, we evidently have

$$\begin{aligned} c_n(k) &= \lim_{x\to n}\frac{1-x^4/n^4}{1-x^4}\prod_{j=1}^{k-1}\frac{1+4x^4/j^4}{1-x^4/(j+1)^4} \\ &= \lim_{x\to n}\frac{\Gamma(k\pm x\pm ix)}{\Gamma(1\pm x\pm ix)}\frac{\Gamma^4(k+1)}{\Gamma^4(k)}\frac{\Gamma^4(n)}{\Gamma^4(n+1)} \\ &\qquad\times\frac{\Gamma(n+1\pm x)}{\Gamma(k+1\pm x)}\frac{\Gamma(n+1\pm ix)}{\Gamma(k+1\pm ix)}\frac{\Gamma(1\pm x)}{\Gamma(n\pm x)}\frac{\Gamma(1\pm ix)}{\Gamma(n\pm ix)} \\ &= \frac{\Gamma(k\pm n\pm in)}{\Gamma(1\pm n\pm in)}\frac{k^4}{n^4}\frac{\Gamma(2n+1)}{\Gamma(n+k+1)}\frac{1}{\Gamma(k+1-n)} \\ &\qquad\times\frac{\Gamma(n+1\pm in)}{\Gamma(k+1\pm in)}\frac{\Gamma(n+1)}{\Gamma(2n)}\frac{(-1)^{n-1}}{(n-1)!}\frac{\Gamma(1\pm in)}{\Gamma(n\pm in)}. \end{aligned} \tag{3–6}$$

It follows that, for real k, one can define

$$\begin{aligned} t_n(k) = -\frac{5}{2}&\frac{e^{\pi ik}\Gamma^2(k+1)}{\Gamma(2k+1)}\frac{\Gamma(k\pm n\pm in)}{\Gamma(1\pm n\pm in)}\frac{k}{n^4} \\ &\times\frac{\Gamma(2n+1)}{\Gamma(n+k+1)}\frac{1}{\Gamma(k+1-n)}\frac{\Gamma(n+1\pm in)}{\Gamma(k+1\pm in)} \\ &\times\frac{\Gamma(n+1)}{\Gamma(2n)}\frac{(-1)^{n-1}}{(n-1)!}\frac{\Gamma(1\pm in)}{\Gamma(n\pm in)}. \end{aligned} \tag{3–7}$$

Since $1/\Gamma(k+1-n) = 0$ when k is an integer less than n, in view of (3–4) it is necessary and sufficient to show that for all positive integers n,

$$\sum_{k=0}^{\infty} t_n(k) = \frac{1}{n^3}. \tag{3–8}$$

To carry out the reflection process, we need to evaluate $t_n(k)$ when k is a negative integer. We shall see that when k is a negative integer, the rather forbidding expression in (3–7) takes a most attractive form. From (3–7), $t_n(-1)$ equals

$$\begin{aligned} &-\frac{5}{2n^4}\frac{(-1)^{n-1}}{(n-1)!}\frac{2n(n\pm in)(\pm in)n}{(\pm n\pm in)(-1\pm n\pm in)} \\ &\qquad\qquad\times\lim_{k\to -1}\frac{\Gamma^2(k+1)}{\Gamma(2k+1)\Gamma(k+1-n)} \\ &= \frac{5}{2n^2(1+4n^4)}\frac{(-1)^{n-1}}{(n-1)!} \\ &\quad\times\lim_{k\to -1}\frac{\Gamma^2(k+2)}{(k+1)^2}\frac{(2k+2)(2k+1)}{\Gamma(2k+3)}\frac{(k+1)}{\Gamma(k+2)}\prod_{j=0}^{n-1}(k-j) \\ &= -\frac{5}{n(1+4n^4)}. \end{aligned}$$

Let j be a positive integer. One can of course evaluate $t_n(-j)$ directly from (3–7) by taking the limit as $k \to -j$, just as we evaluated $t_n(-1)$ above. However, it is preferrable to introduce the following labour saving device. For positive integer $k \geqslant n$, define

$$\alpha_n(k) := \frac{t_n(k)}{t_n(k+1)} = \frac{-2k(2k+1)((k+1)^4 - n^4)}{(k+1)^2(k^4+4n^4)}.$$

For other values of k, define $\alpha_n(k)$ by the above expression on the far right-hand side. Then for positive integer k,

$$\begin{aligned} t_n(-k) &= \alpha_n(-k)\alpha_n(1-k)\cdots\alpha_n(-2)t_n(-1) \\ &= -\frac{5}{n(1+4n^4)}\prod_{j=2}^{k}\frac{2j(1-2j)}{(j-1)^2}\frac{(j-1)^4-n^4}{4n^4+j^4} \\ &= -\frac{5}{n(1+4n^4)}\frac{(2k)!}{2(k-1)!^2}\frac{n^4-1}{4n^4+k^4}\prod_{j=2}^{k-1}\frac{n^4-j^4}{4n^4+j^4} \\ &= -\frac{5}{2n}\binom{2k}{k}\frac{k^2}{4n^4+k^4}\prod_{j=1}^{k-1}\frac{n^4-j^4}{4n^4+j^4}. \end{aligned} \tag{3–9}$$

Set

$$\begin{aligned} p_n(k) &:= k^2\prod_{j=1}^{n-1}(k-j)(j+k+in)(j+k-in), \\ q_n(k) &:= (k-n+in)(k-n-in), \\ r_n(k) &:= -2(k+n)(2k-1). \end{aligned}$$

Then for all integers k,

$$\frac{t_n(k+1)}{t_n(k)} = \frac{p_n(k+1)}{p_n(k)}\frac{q_n(k)}{r_n(k+1)}.$$

Furthermore, q_n and r_n share no linear factors differing by an integer. According to Gosper's algorithm[1] [Graham et al. 1989], there exists a polynomial s_n of degree no greater than $3n-3$ that satisfies

$$p_n(k) = s_n(k+1)q_n(k) - r_n(k)s_n(k) \tag{3–10}$$

[1] We were led to consider Gosper's algorithm when the first author attempted to get Maple to evaluate the sum (3–4) symbolically. Mistyping 'infinity' revealed that Maple could evaluate the resulting indefinite sum for specific instances of the parameter n.

for all integers k. Define

$$T_n(k) := \frac{r_n(k)s_n(k)t_n(k)}{p_n(k)}. \tag{3–11}$$

Using (3–10), it is not hard to show that

$$T_n(k+1) - T_n(k) = t_n(k)$$

for all integers k. Note that since $t_n(-n)$ is finite and $p_n(-n) \neq 0 = r_n(-n)$, we have $T_n(-n) = 0$. It follows that

$$T_n(m) = \sum_{j=-n}^{m-1} t_n(j), \qquad m-1 \geqslant -n.$$

Also, it is clear from (3–11) and (3–5) that

$$\lim_{k\to\infty} T_n(k) = 0.$$

Thus (3–8) is equivalent to

$$T_n(0) = \sum_{j=-n}^{-1} t_n(j) = -\frac{1}{n^3}. \tag{3–12}$$

Ideally, one would like to prove (3–12) using (3–11). Unfortunately, we do not know enough about the polynomials s_n to infer the value $s_n(0)$ in general. For specific values of n, we can use (3–10) to solve for the unknown polynomial s_n and hence, at least in principle, prove (3–12) for any specific value of n. However, using this approach to prove (3–12) in general would require an explicit formula for the constant coefficient of the possibly degree $3n-3$ polynomial s_n. Of course, such a formula can be inferred by assuming (3–12), but to us, at least, proving the formula directly seems a formidable task. However, substituting (3–9) into (3–12), it is readily apparent that we need only prove the beautiful combinatorial identity

$$\frac{5}{2}\sum_{k=1}^{n}\binom{2k}{k}\frac{k^2}{4n^4+k^4}\prod_{j=1}^{k-1}\frac{n^4-j^4}{4n^4+j^4} = \frac{1}{n^2}, \quad \text{for } n \in \mathbb{Z}^+. \tag{3–13}$$

(This identity is apparently nontrivial. All our attempts to prove it using software based on the Wilf–Zeilberger method [Petkovšek et al. 1996]—in particular, Zeilberger's marvelous package EKHAD and Peter Paule's corresponding Mathematica implementation zb_alg.m (available by request from Peter.Paule@risc.uni-linz.ac.at)—have failed. In a personal communication date May 10, 1996, Professor Zeilberger kindly informed us that neither (3–13) nor its equivalent hypergeometric formulation (6–1) fall under the scope of identities provable via the WZ method.)

We discuss the identity (3–13) and some related results in the next section. In Section 6, we examine the process of Gosper reflection in greater detail, where it is revealed that identity (3–13) and our conjectured generating function formula (2–1) are in fact equivalent.

4. A COMBINATORIAL IDENTITY

Lemma 4.1 (equivalent to conjectured Theorem 2.1). *For all positive integers n,*

$$\sum_{k=1}^{n} \frac{5}{2} n^2 k^2 \binom{2k}{k} \frac{1}{4n^4+k^4} \prod_{j=1}^{k-1} \frac{n^4-j^4}{4n^4+j^4} = 1.$$

Although we have verified Lemma 4.1 for all positive integers $n < 300$, we have so far been unable to find a proof. The following equivalent proposition suggests one possible approach.

Proposition 4.2 (equivalent to conjectured Theorem 2.1). *For each positive integer n, there exists an even polynomial f_n of degree $2n$ such that*

$$f_n(x) \prod_{j=1}^{n} \frac{x^2-j^2}{4x^4+j^4} = 1 - \frac{5}{2} \sum_{k=1}^{n} \binom{2k}{k} \frac{x^2k^2}{4x^4+k^4} \prod_{j=1}^{k-1} \frac{x^4-j^4}{4x^4+j^4}.$$

Clearly, Lemma 4.1 can be obtained from Proposition 4.2 if one sets $x = \pm 1, \pm 2, \pm 3, \ldots \pm n$. To see how we arrived at Proposition 4.2, let

$$\sigma_k(x) := \frac{5}{2} x^2 k^2 \binom{2k}{k} \prod_{j=1}^{k-1} (x^2+j^2) \tag{4–1}$$

and define a sequence of functions g_k recursively. Put $g_0(x) = 1$ for all x and for $k > 0$ let

$$g_{k-1}(x) - g_k(x) = \frac{\sigma_k(x)}{4x^4+k^4} \prod_{j=1}^{k-1} \frac{x^2-j^2}{4x^4+j^4}. \tag{4–2}$$

Telescoping (4–2) would prove Lemma 4.1 if we could show that $g_n(n) = 0$. Define

$$f_k(x) := g_k(x) \prod_{j=1}^{k} \frac{4x^4+j^4}{x^2-j^2}. \tag{4–3}$$

Then

$$g_k(x) = f_k(x) \prod_{j=1}^{k} \frac{x^2-j^2}{4x^4+j^4}. \tag{4–4}$$

Clearly, $g_n(n) = 0$ if $f_n(n)$ is finite. In fact, the evidence strongly suggests that each f_k is a polynomial. From (4–2) and (4–4) it follows that

$$(4x^4+k^4) f_{k-1}(x) - (x^2-k^2) f_k(x) = \sigma_k(x), \quad \text{for } k > 0. \tag{4–5}$$

In particular, (4–2) and (4–5) imply that for all x, $f_0(x) = 1$, $f_1(x) = 4x^2 - 1$, $f_2(x) = 16x^4 + 4$, etc. Now Proposition 4.2 is obtained by telescoping (4–2) and writing g_n in terms of f_n.

We remark that standard telescoping techniques prove a superficially similar identity:

Proposition 4.3. *For each positive integer n,*

$$\frac{5}{4} \sum_{k=1}^{n} \frac{k^4 4^k}{4n^4+k^4} \prod_{j=1}^{k-1} \frac{n^4-j^4}{4n^4+j^4} = 1.$$

Proof. Use

$$\sum_{k=1}^{n} (a_{k-1} - b_k) \prod_{j=1}^{k-1} \frac{b_j}{a_j} = a_0 - b_n \prod_{j=1}^{n-1} \frac{b_j}{a_j} \tag{4–6}$$

with $a_k = \frac{1}{4}(4n^4 + (k+1)^4)$ and $b_k = n^4 - k^4$, for $k \geqslant 0$. Standard telescoping proves (4–6) for any sequences of a's and b's. In our case, we have

$$a_{k-1} - b_k = \tfrac{5}{4} k^4, \qquad b_n = 0,$$

and so

$$\frac{5}{4}\sum_{k=1}^{n} k^4 4^{k-1} \prod_{j=1}^{k-1} \frac{n^4 - j^4}{4n^4 + (j+1)^4} = \frac{4n^4+1}{4}.$$

Now cross multiply and obtain

$$\frac{5}{4}\sum_{k=1}^{n} \frac{k^4 4^k}{4n^4+1} \prod_{j=1}^{k-1} \frac{n^4 - j^4}{4n^4 + (j+1)^4} = 1,$$

from which the claimed identity easily follows. □

If we try to play the same game using (4–6) to prove Lemma 4.1, it seems most natural to define $a_k := a_k(n) = 4n^4 + (k+1)^4$ for $k \geqslant 0$, and then choose $b_k := b_k(n)$ so as to satisfy the recursion

$$(a_{k-1} - b_k) \prod_{j=1}^{k-1} \frac{b_j}{a_j} = \tfrac{5}{2} n^2 k^2 \binom{2k}{k} \prod_{j=1}^{k-1} \frac{n^4 - j^4}{4n^4 + (j+1)^4}, \quad \text{for } k \geqslant 1. \tag{4–7}$$

If we can somehow show that $b_n(n) = 0$, then (4–6) implies that

$$\sum_{k=1}^{n} \frac{5}{2} n^2 k^2 \binom{2k}{k} \prod_{j=1}^{k-1} \frac{n^4 - j^4}{4n^4 + (j+1)^4} = 4n^4 + 1,$$

which, after cross multiplying, is easily seen to be equivalent to Lemma 4.1. Now the recursion (4–7) is equivalent to

$$(a_{k-1}(n) - b_k(n)) \prod_{j=1}^{k-1} \frac{b_j(n)}{n^4 - j^4} = \frac{5}{2} n^2 k^2 \binom{2k}{k}.$$

Thus, $b_n(n) = 0$ is equivalent to

$$a_{n-1}(n) \prod_{j=1}^{n-1} \frac{b_j(n)}{n^4 - j^4} = \frac{5}{2} n^4 \binom{2n}{n}.$$

that is,

$$\prod_{j=1}^{n-1} \frac{b_j(n)}{n^4 - j^4} = \frac{1}{2}\binom{2n}{n},$$

which is an equivalent formulation of Lemma 4.1.

5. AN INTEGRAL IDENTITY

Here, we give an exquisite integral evaluation for the central binomial coefficient that is equivalent to Lemma 4.1 (3–13) and hence equivalent to our main conjecture.

Corollary 5.1 (equivalent to conjectured Theorem 2.1). *For all positive integers n, we have*

$$\frac{1}{\pi}\int_0^\infty \frac{dy}{1+y^2} \prod_{j=0}^{n-1} \frac{4y^2 - (j/n)^4}{y^2 + (j/n)^4} = \binom{2n}{n}.$$

The equivalence of Corollary 5.1 and Theorem 2.1 relies on the following conjecture of Wenchang Chu (personal communication):

Lemma 5.2 (equivalent to conjectured Theorem 2.1). *For all positive integers n,*

$$\sum_{k=1}^{n} \frac{2n^2}{k^2} \prod_{j=1}^{n-1} (j^4 + 4k^4) \Big/ \prod_{\substack{j=1\\ j\neq k}}^{n} (k^4 - j^4) = \binom{2n}{n}.$$

Proof. We'll show that Lemma 5.2 and Lemma 4.1 are inverse pairs. This fact is a special case of an inverse pair relationship given in [Krattenthaler 1996], equivalent to

$$f(n) = \sum_{k=r}^{n} \frac{a_n d_n + b_n c_n}{d_k} \frac{\varphi(c_k/d_k; n)}{\psi_k(-c_k/d_k; n+1)}\, g(k) \tag{5–1}$$

if and only if

$$g(n) = \sum_{k=r}^{n} \frac{\psi(-c_n/d_n; k)}{\varphi(c_n/d_n; k+1)}\, f(k), \tag{5–2}$$

where

$$\varphi(x;k) := \prod_{j=0}^{k-1} (a_j + x b_j), \quad \psi(x;k) := \prod_{j=0}^{k-1} (c_j + x d_j),$$

and

$$\psi_m(x;k) := \prod_{\substack{j=0\\ j\neq m}}^{k-1} (c_j + x d_j).$$

Setting $r = 1$, $a_j = j^4$, $b_j = 4$, $c_j = j^4$, $d_j = 1$, $f(k) = 10k^2\binom{2k}{k}(-1)^k$, $g(n) = 1/n^2$ in the inverse pair (5–1), (5–2) yields the claimed inverse pair relationship between Lemma 4.1 and Lemma 5.2. □

We now proceed to show the equivalence of Corollary 5.1 and Lemma 5.2. By a suitable change of variable, the integral identity in Corollary 5.1 can be rewritten in the form

$$\frac{4n^2}{\pi}\int_0^\infty dx \prod_{j=1}^{n-1}(4x^2-j^4)\Big/\prod_{j=1}^{n}(x^2+j^4)=\binom{2n}{n}. \tag{5–3}$$

In view of the partial fraction expansion (3–2), we can rewrite the integrand of (5–3), obtaining the equivalent identity

$$\begin{aligned}\binom{2n}{n}&=\frac{4n^2}{\pi}\int_0^\infty(-1)^{n+1}\sum_{k=1}^{n}\frac{k^4}{n^4}\frac{c_k(n)}{k^4+x^2}\,dx\\&=(-1)^{n+1}\sum_{k=1}^{n}\frac{2k^2}{n^2}c_k(n),\end{aligned}$$

which, in view of the definition (3–3) of the numbers $c_k(n)$, is precisely the statement of Lemma 5.2.

6. SOME REMARKS ON REFLECTION

We can rewrite Lemma 4.1 or (3–13) in hypergeometric notation as

$$ {}_6F_5\left(\begin{matrix}2,\ 3/2,\ 1{+}n,\ 1{-}n,\ 1{+}in,\ 1{-}in\\ 1,\ 2{+}n{+}in,\ 2{+}n{-}in,\ 2{-}n{+}in,\ 2{-}n{-}in\end{matrix}\,\middle|\,-4\right)=\frac{4n^4+1}{5n^2}, \tag{6–1}$$

an strange evaluation, apparently new, of a terminating ${}_6F_5$. We can also rewrite (2–1) as a formula for a nonterminating ${}_6F_5$:

$$ {}_6F_5\left(\begin{matrix}2,\ 2,\ 1{+}z{+}iz,\ 1{+}z{-}iz,\ 1{-}z{+}iz,\ 1{-}z{-}iz\\ 3/2,\quad 2{+}z,\quad 2{-}z,\quad 2{+}iz,\quad 2{-}iz\end{matrix}\,\middle|\,-\tfrac14\right)=\frac45\sum_{k=1}^{\infty}\frac{1-z^4}{k^3(1-z^4/k^4)}. \tag{6–2}$$

Observe the dual nature of (6–1) and (6–2). Our process of Gosper reflection has taken a nonterminating ${}_6F_5$ at $-1/4$, and transformed it into a terminating ${}_6F_5$ at -4, in which certain of the numerator parameters and denominator parameters have been exchanged and shifted.

We can see the dual results of reflection in another way. Let $z^4=-n^4/4$ in (2–1). The right-hand side terminates, yielding

$$\sum_{k=1}^{\infty}\frac{4k}{4k^4+n^4}=\frac52\sum_{k=1}^{n}\frac{4^k}{\binom{2k}{k}}\frac{k}{n^4+4k^4}\prod_{j=1}^{k-1}\frac{n^4-j^4}{n^4+4j^4}. \tag{6–3}$$

On the other hand, standard techniques show that

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{4k}{4k^4+n^4}&=\frac{1}{2in^2}\left(\psi\Big(1-in\Big(\frac{1+i}{2}\Big)\Big)+\psi\Big(1+in\Big(\frac{1+i}{2}\Big)\Big)-\psi\Big(1+n\Big(\frac{1+i}{2}\Big)\Big)-\psi\Big(1-n\Big(\frac{1+i}{2}\Big)\Big)\right)\\&=\frac{1}{2n}\sum_{k=1}^{n}\frac{1}{(k-n/2)^2+n^2/4}.\end{aligned} \tag{6–4}$$

Comparing (6–3) and (6–4) yields the following identity:

$$\frac52\sum_{k=1}^{n}\frac{4^k}{\binom{2k}{k}}\frac{k}{n^4+4k^4}\prod_{j=1}^{k-1}\frac{n^4-j^4}{n^4+4j^4}=\frac{1}{2n}\sum_{k=1}^{n}\frac{1}{(k-n/2)^2+n^2/4}. \tag{6–5}$$

Now compare the left-hand sides of (6–5) and (3–13).

The astute reader will observe a close relationship between the right side of (2–3) in Corollary 2.3 and the summand of Lemma 4.1. In fact, Gosper reflection applied to Corollary 2.3 yields the identity in Lemma 4.1. Since the proof of this mirrors the development of Section 3, we omit the details. We remark however, that Gosper reflection easily proves any specific instance of Corollary 2.3. For the sake of brevity, we illustrate this assertion in the case $n = 1$, which is Corollary 2.4. Writing $t(k)$ for the summand of Corollary 2.4, we have

$$t(k) = \frac{(2)_k(2)_k(\pm i)_k(-1/4)^k}{(3/2)_k(1)_k(1)_k(3)_k} = \frac{(k+1)^2\Gamma(k\pm i)\Gamma(1/2)(-1/4)^k}{\Gamma(\pm i)\Gamma(3/2+k)\Gamma(k+3)}.$$

It follows that $t(-1) = 0$ and $t(-k) = 0$ for integer $k \geqslant 3$. Since

$$\sum_{k=-\infty}^{\infty} t(k) = 0,$$

it follows that

$$\sum_{k=0}^{\infty} t(k) = -t(-2) = \tfrac{4}{5},$$

which proves Corollary 2.4.

To conclude this section, we'd like to offer evidence in support of our claim that evaluations (6–1) and Corollary 2.3 are indeed new. After surveying the standard references, such as [Gessel and Stanton 1982; Bailey 1935; Slater 1966], in the vast hypergeometric literature, and after consulting many of the experts in this area, we have been unable to uncover anything remotely like (6–1) or Corollary 2.3. Hypergeometric summations in which the main argument is different from 1 are rare enough. Exceedingly rare are summations with complex parameters such as in (6–1) or Corollary 2.3, and neither of our evaluations appears to have a natural generalization. For example, there appears to be no generalization of either formula in which in is replaced by a general parameter m for example.

7. ALGORITHMS AND COMPLEXITY

The formulae developed here lend themselves easily to numerical computation. Algorithms based on Apéry's formula (1–2), Koecher's formula (1–3), and Bradley's formula (1–4) are particularly simple and are given below.

Algorithm 7.1. Given d, compute $\zeta(3)$ to d digits using (1–2). Computations are performed to d digits.

$N \leftarrow 1 + \lfloor 5d/3 \rfloor$; $c \leftarrow 2$; $s \leftarrow 0$;
for n from 1 to N do
    $s \leftarrow s + (-1)^{n+1}/(n^3 c)$; $c \leftarrow c(4n+2)/(n+1)$;
Return$(5s/2)$;

Algorithm 7.2. Given d, compute $\zeta(5)$ to d digits using (1–5). Computations are performed to d digits.

$N \leftarrow 1 + \lfloor 5d/3 \rfloor$; $a \leftarrow 0$; $c \leftarrow 2$; $s \leftarrow 0$;
for n from 1 to N do
    $g \leftarrow 1/n^2$; $s \leftarrow s + (-1)^{n+1}(4g - 5a)/(n^3 c)$;
    $c \leftarrow c(4n+2)/(n+1)$; $a \leftarrow a + g$;
Return$(s/2)$;

Algorithm 7.3. Given d, compute $\zeta(7)$ to d digits using (1–4). Computations are performed to d digits.

$N \leftarrow 1 + \lfloor 5d/3 \rfloor$; $a \leftarrow 0$; $b \leftarrow 0$; $c \leftarrow 2$; $s \leftarrow 0$;
for n from 1 to N do
    $g \leftarrow 1/n^4$; $s \leftarrow s + (-1)^{n+1}(5a + g)/(n^3 c)$;
    $c \leftarrow c(4n+2)/(n+1)$; $a \leftarrow a + g$;
Return$(5s/2)$;

By Stirling's asymptotic formula for the gamma function, it readily follows that

$$\binom{2k}{k} \sim \frac{4^k}{\sqrt{\pi k}} \qquad \text{as } k \to \infty,$$

and thus all formulae we have discussed yield two binary digits per term asymptotically, or slightly better than 1.2 decimal digits per term, given that $\log 4/\log 10 \approx 3/5$. This should be contrasted with the definition (1–1), which is asymptotically useless, yielding 0 digits per term. For example, computing $\zeta(3)$ from the definition (1–1) and applying the integral test to the tail of the series shows that the n-th tail drops off like $O(1/n^2)$. Thus each successive digit requires computing $\sqrt{10}$ times as many terms as its predecessor. To get d digits, $O(10^{d/2})$ operations are involved. On the other hand, it's not hard to see that the algorithms we have presented require only $O(d)$ operations for d digits.

These considerations have ignored the size of the numbers being operated on. A more realistic evaluation of run times must take this into account. If we take as given that the cost of multiplying two d digit numbers is $O(d(\log d)\log\log d)$, a crude upper bound on the run time for computing d digits using our Apéry-like algorithms is

$$O\biggl(\sum_{j=1}^{d} j(\log j)\log\log j\biggr) = O(d^2(\log d)\log\log d).$$

However, it is possible to adapt these algorithms using the method of [Karatsuba 1993] to yield the highly respectable run time

$$O(d(\log d)^3\log\log d).$$

We coded our Apéry-like algorithms (without Karatsuba's optimization) in Maple V Release 3 and ran them on an Indy R4600PC 100 MHz Silicon Graphics Workstation. The following table compares the run times in cpu seconds with Maple's built-in implementation of the Riemann zeta function.

	$\zeta(3)$	$\zeta(5)$	$\zeta(7)$	Digits
Apéry-like	0.4561	1.8720	2.8141	200
Maple	8.1720	8.4600	8.3462	200
Apéry-like	1.1401	5.5019	8.0399	300
Maple	28.0742	28.1819	28.3860	300

8. OTHER DIRICHLET SERIES

For all positive integers n and all real k, let

$$d_n(k) := \frac{5n^3}{2k^3}c_n(k).$$

Then (3–4) becomes

$$\sum_{k=n}^{\infty}\frac{(-1)^{k+1}}{\binom{2k}{k}}d_n(k) = 1,$$

for integer $n \geqslant 1$. Thus, for any sequence $a_1, a_2, \ldots$, we may write

$$\frac{a_n}{n^s} = \frac{a_n}{n^s}\sum_{k=n}^{\infty}\frac{(-1)^{k+1}}{\binom{2k}{k}}d_n(k). \tag{8–1}$$

Let's suppose that $\sum n^{-s}a_n$ is absolutely convergent. Summing (8–1) on n and interchanging the order of summation, we get

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s} = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\binom{2k}{k}}\sum_{j=1}^{k}\frac{a_j}{j^s}d_j(k). \tag{8–2}$$

This gives a "formula" for any absolutely convergent Dirichlet series. However, (8–2) does not appear to be of much use, except in special cases where we can take advantage of known properties of the numbers $d_n(k)$. For example, since

$$\sum_{j=1}^{k} c_j(k) = 1$$

for all $1 \leqslant j \leqslant k \in \mathbb{Z}$, putting $s = 3$ and $a_1 = a_2 = \cdots = 1$ in (8–2) recovers Apéry's formula (1–2).

Unfortunately, there seems to be no way to make use of (8–2) or the ideas of Section 3 to obtain a generating function analogue of our result (2–1) for $\zeta(4n+1)$. Since (2–1) started with Apéry's formula (1–2) for $\zeta(3)$, one might expect that a generating function analogue of (2–1) for $\zeta(4n+1)$ would be based on Koecher's formula (1–3) for $\zeta(5)$ and derive from recurrence properties akin to those implicit in the list (1–5). However, none of the formulae for $\zeta(9)$ that we discovered (and we have good reason to believe there are no others) bears the necessary relationship to (1–3).

We should also point out that even in the $4n+3$ case, much work remains to be done, as there are several Apéry-like formulae for $\zeta(7), \zeta(11), \dots$ that do not arise from our generating function (2–1). In the $4n+1$ case, the proliferation of formulae appears to be even greater. We have created code for systematically listing the formulae for $\zeta(13)$, and ran the code for two months or so. The resulting file is over three thousand lines long and contains hundreds and hundreds of independent formulae, all having the characteristic power of k and central binomial coefficient in the denominator, accompanied by harmonic-like sums in the numerator. Classifying the myriad relations and interrelations amongst these sums for the various even/odd zeta values would be a huge project indeed.

9. ADDENDUM

As we later learned, Koecher [1980] had given a very simple proof of the following generating function for $\zeta(2n+1)$, namely

$$\sum_{k=1}^{\infty} \frac{1}{k^3(1-z^2/k^2)} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\left(\frac{1}{2}+\frac{2}{1-z^2/k^2}\right)\prod_{j=1}^{k-1}(1-z^2/j^2). \tag{9–1}$$

If n is a nonnegative integer, extracting the coefficient of z^n from each side of (9–1) produces the formula

$$\zeta(2n+3) = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}}(-1)^n e_n^{(2)}(k) + 2\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2j+3}\binom{2k}{k}}(-1)^{n-j} e_{n-j}^{(2)}(k), \tag{9–2}$$

where the $e_r^{(s)}(k)$ are the elementary symmetric functions defined in Section 2. Equations (1–2) and (1–3) follow as special cases.

Despite the fact that Koecher's generating function (9–1) gives formulae for all odd zeta values, there is a very real sense in which (9–1) is inferior to our generating function (2–1). In (9–1), among other things, the fourth powers that feature in (2–1) are replaced by squares. This results in redundant terms in his zeta formula (9–2) for $n>1$. For example, $n=2$ in (9–2) yields

$$\zeta(7) = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^7\binom{2k}{k}} - 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^2} + \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{1\leqslant j<l\leqslant k-1}\frac{1}{j^2l^2}, \tag{9–3}$$

which should be compared with our more compact formula (1–4). To enable a more detailed comparison, we rewrite (9–3) in the notation of Section 1. Then (9–3) becomes

$$\zeta(7) = 2\lambda(7,P_0^{(2)}) - 2\lambda(5,P_1^{(2)}) + \tfrac{5}{4}\lambda(3,P_1^{(2)}P_1^{(2)}) - \tfrac{5}{4}\lambda(3,P_2^{(2)}), \tag{9–4}$$

whereas (1–4) is simply

$$\zeta(7) = \tfrac{5}{2}\lambda(7,P_0^{(4)}) + \tfrac{25}{2}\lambda(3,P_1^{(4)}). \tag{9–5}$$

Since $P_0^{(4)} = P_0^{(2)} = 1$ and $P_1^{(4)} = P_2^{(2)}$, the middle two terms of (9–4) are redundant. Indeed, lattice-based reduction shows that

$$2\lambda(7,P_0^{(2)}) + 8\lambda(5,P_1^{(2)}) - 5\lambda(3,P_1^{(2)}P_1^{(2)}) + 55\lambda(3,P_2^{(2)}) = 0.$$

As far as we can tell, in contrast with the formulae derived from (9–2), there are no redundant terms in our formulae for $\zeta(4n+3)$ that come from (2–1), at least for $n<12$. It goes without saying that, despite our best efforts, Koecher's proof of (9–1) apparently cannot be adapted to prove (2–1). It seems that (1–4), and more generally (2–1), is a much deeper result. We should also point out that merely bisecting Koecher's generating function (9–1) will not yield (2–1), nor any new zeta formulae.

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NOTE ADDED IN PROOF. Gert Almkvist and Andrew Granville have recently provided an elegant proof of our Lemma 5.2 and hence all our conjectured results are now proved. Their preprint is available at http://www.math.uga.edu/~andrew/Postscript/BorBrad.ps.