A Counterexample in the Theory of Local Zeta Functions

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CONTENTS

The generalized Igusa local zeta function $Z_{\mathbb{Q}_p}(s)$ associated to (SL_7, ρ), where ρ is the Cartan product of the first, third and fifth fundamental representations of SL_7, is explicitly computed and shown not to satisfy the expected functional equation

$$Z_{\mathbb{Q}_p}(s)|_{p\mapsto p^{-1}} = p^{-7s} Z_{\mathbb{Q}_p}(s).$$

1. INTRODUCTION

Igusa [1974; 1975] inaugurated the study of local zeta functions of Igusa type. In [Igusa 1989], he introduced the generalized Igusa local zeta function and accompanied this introduction with several conjectures about the general properties of such functions. In particular, he conjectured that such functions satisfy functional equations of a certain form. This conjecture is disproved here. By way of comparison, we note that Igusa local zeta functions defined, unlike the generalized Igusa local zeta function, with respect to translation-invariant measures have been shown to satisfy functional equations of the expected form [Denef and Meuser 1991].

Let $\mathbb{Q}_p$ denote the field of p-adic numbers, where p is chosen such that $\mathbb{Q}_p$ contains the seventh roots of unity. Let $\mathbb{Z}_p$ denote the ring of integers of $\mathbb{Q}_p$, and $p\mathbb{Z}_p$ the ideal of nonunits of $\mathbb{Z}_p$. By definition, $\mathbb{Z}_p/p\mathbb{Z}_p$ has p elements. Let $|\ |_p$ be the absolute value on $\mathbb{Q}_p$, normalized as $|p|_p = p^{-1}$. Let $G' = \mathrm{SL}_7$, and set $\rho = \Lambda_1 * \Lambda_3 * \Lambda_5$, the Cartan product of the first, third and fifth fundamental representations of G' as indexed by the Dynkin diagram of G'. By definition, ρ is the irreducible

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subrepresentation of $\Lambda_1 \otimes \Lambda_3 \otimes \Lambda_5$ containing highest weight equal the sum of the highest weights of each of Λ_1, Λ_3 and Λ_5. For ease of computation, represent SL_7 in the standard basis $v_1, \ldots, v_7$ of $\mathbb{Q}_p^7$ so that Λ_3 has a basis of elements of the form $v_{j_1} \wedge \cdots \wedge v_{j_3}$, for $1 \le j_1 < \cdots < j_3 \le 7$, and Λ_5 has one of elements of the form $v_{k_1} \wedge \cdots \wedge v_{k_5}$, for $1 \le k_1 < \cdots < k_5 \le 7$.

Let $\mathbb{G}_a = \mathrm{Aff}^1$ denote the one-dimensional affine group, and $\mathbb{G}_m = \mathrm{GL}_1$ the one-dimensional linear group. The almost direct product

$$G = \rho(G')(\mathbb{G}_m 1_{\dim \rho})$$

is a subgroup of $\mathrm{GL}_{\dim \rho}$ not contained in $\mathrm{SL}_{\dim \rho}$. Let T be the maximal torus of G. We note here the useful fact that, with R denoting the root system of G', the group G is generated by T and the distinguished one-dimensional subgroups $\theta_\alpha(\mathbb{G}_a)$, where $\alpha \in R$ and the homomorphisms $\theta_\alpha : \mathbb{G}_a \to G$ are such that $\mathbb{G}_a \cong \theta_\alpha(\mathbb{G}_a)$ and $t\theta_\alpha(u)t^{-1} = \theta_\alpha(\alpha(t)u)$ for all $t \in T$ and $u \in \mathbb{G}_a$. Define f as the positive generator of $\mathrm{Hom}(G, \mathbb{G}_m)$ and $G^0 = G_{\mathbb{Q}_p} \cap \mathrm{Mat}_{\dim \rho}(\mathbb{Z}_p)$, where $G_{\mathbb{Q}_p}$ denotes the $\mathbb{Q}_p$-rational points of G.

Definition [Igusa 1989; Martin 1996]. The generalized Igusa local zeta function associated to (G', ρ) is

$$Z_{\mathbb{Q}_p}(s) = \int_{G^0} |f(g)|_p^s \mu_c(g),$$

where μ_c is Serre's canonical measure defined on $G_{\mathbb{Q}_p}$, and s is a complex number with $\mathrm{Re}\, s > 0$.

It is conjectured in [Igusa 1989] that $Z_{\mathbb{Q}_p}(s)$ has a finite form that expresses $Z_{\mathbb{Q}_p}(s)$ as a rational function $Z(p^{-1}, p^{-s})$ satisfying the functional equation

$$Z(p^{-1}, p^{-s})|_{p \mapsto p^{-1}} = p^{-\deg(f)s} Z(p^{-1}, p^{-s}). \quad (1.1)$$

We disprove this conjecture by finding the rational expression for $Z_{\mathbb{Q}_p}(s)$ and showing that (1.1) fails.

2. REDUCTIONS

We follow the setup of Igusa. The details are in [Martin 1992b]. Since the measure μ_c is difficult to work with, define a $\mathbb{Q}_p^\times(G_{\mathbb{Q}_p} \cap \mathrm{GL}_{\dim \rho}(\mathbb{Z}_p))$ bi-invariant function Φ on $G_{\mathbb{Q}_p}$ by

$$\Phi(g) = \frac{\mu_c(g(G_{\mathbb{Q}_p} \cap \mathrm{GL}_{\dim \rho}(\mathbb{Z}_p)))}{\mu(G_{\mathbb{Q}_p} \cap \mathrm{GL}_{\dim \rho}(\mathbb{Z}_p))},$$

where μ is the Haar measure on $G_{\mathbb{Q}_p}$ normalized to be canonical measure on $G_{\mathbb{Q}_p} \cap \mathrm{GL}_{\dim \rho}(\mathbb{Z}_p)$. It follows that $\mu_c(g) = \Phi(g)\mu(g)$ and, thus, that

$$Z_{\mathbb{Q}_p}(s) = \int_{G^0} |f(g)|_p^{s+(\dim G/\deg f)} \frac{\Phi(g)}{|f(g)|_p^{\dim G/\deg f}} \mu(g).$$

Let $S = \{\alpha_1, \ldots, \alpha_6\}$ be the standard basis for the root system R. With respect to this basis, R decomposes into a direct sum of positive and negative roots as $R = R^+ \amalg R^-$. Define positive integers $a_1, \ldots, a_6$ by

$$\prod_{\alpha \in R^+} \alpha = \prod_{i=1}^{6} \alpha_i^{a_i}.$$

Let ϖ denote the highest weight of the irreducible representation of G in affine n-space $\mathbb{A}^n$ defined by $g \mapsto {}^t g^{-1}$. Set $\alpha_0 = f|_T$. By construction, $\alpha_0, \ldots, \alpha_6$, and ϖ generate $\mathrm{Hom}(T, \mathbb{G}_m)$ with relation

$$\varpi^{\deg f} = \alpha_0^{-1} \prod_{i=1}^{6} \alpha_i^{b_i},$$

defining positive integers $b_1, \ldots, b_6$.

Choose $\xi_0, \ldots, \xi_6 \in \mathrm{Hom}(\mathbb{G}_m, T)$ satisfying the conditions $\langle \alpha_i, \xi_j \rangle = \delta_{ij}$ for $0 \le i, j \le 6$. It follows that $\xi_0, \ldots, \xi_6$ generate $\mathrm{Hom}(\mathbb{G}_m, T)$. In a common abuse of notation, the Weyl group of G' will be denoted by $\mathrm{Sym}(7)$. The action of an element $w \in \mathrm{Sym}(7)$ on an element $\alpha \in R$ is given by $w(\alpha)(t) = \alpha(w^{-1}(t))$, for $t \in T$.

From the p-adic Bruhat decomposition and the properties of the length function λ for the Weyl group of SL_7 [Igusa 1989, pp. 703–6; Martin 1992a], we have

$$Z_{\mathbb{Q}_p}(s) = \mu(G_{\mathbb{Q}_p} \cap \mathrm{GL}_{\dim \rho}(\mathbb{Z}_p)) \cdot \sum_{w \in \mathrm{Sym}(7)} p^{-\lambda(w)} \cdot \left(\sum p^{\tau(n_0, \ldots, n_6)} \frac{\Phi(\xi(\pi))}{|f(\xi(\pi))|_p^{\dim G/\deg f}} \right), \quad (2.1)$$

where the summation on the second line is over all $n_0 \in \mathbb{N}$ and all $(n_1, \ldots, n_6) \in \mathbb{N}^6$ such that $n_i > 0$ if $\alpha_i \in w(R^-)$, and where

$$\xi = \xi_0^{\deg(f)n_0} \prod_{i=1}^{6} (\xi_0^{b_i} \xi_i)^{n_i},$$

$$\tau(n_0, \ldots, n_6) = -(49 + \deg(f)s)n_0 + \sum_{i=1}^{6} \Big(a_i - b_i\Big(s + \frac{\dim G}{\deg f}\Big)\Big)n_i.$$

3. EXPLICIT FORM OF $Z_{\mathbb{Q}_p}(s)$

The explicit form for $Z_{\mathbb{Q}_p}(s)$ as a rational function may now be obtained. These first few computations are straightforward: $\dim\rho = 3402$, $\dim G = 49$, $\deg f = 7$, $(a_1, \ldots, a_6) = (6, 10, 12, 12, 10, 6)$, and

$$\mu(G_{\mathbb{Q}_p} \cap \mathrm{GL}_{3402}(\mathbb{Z}_p)) = (1 - p^{-1}) \prod_{i=1}^{6} (1 - p^{-(i+1)}).$$

There is also a formula for $\Phi(\xi(\pi))$, which will now be exhibited. It is a result of Élie Cartan [1913, pp. 3–5] that the weights ω^* of G satisfy

$$\omega^* = \varpi^{-1} \prod_{i=1}^{6} \alpha_i^{c_i},$$

where $c = (c_1, \ldots, c_6) \in \Gamma$, a finite subset of $\mathbb{N}^6$. With these conventions, and with $n = (n_1, \ldots, n_6)$, we then have

$$\Phi(\xi(\pi)) = p^{-(49n_0 + (\sum_{c \in \Delta_\xi} c + \sum_{\alpha \in R} c_{\alpha,\xi}) \cdot n)},$$

where Δ_ξ is the subset of Γ that forms a basis for $\mathbb{Q}^6$ such that $c \cdot n$ for all $c \in \Delta_\xi$ takes the smallest value, and $c_{\alpha,\xi}$ is the $c \in \Gamma$ associated to a row $r = r_{\alpha,\xi}$ such that

$$\frac{d}{du}(\theta_\alpha(u))|_{u=0} \neq 0$$

and $c \cdot n$ takes the smallest value (see [Igusa 1989, p. 713]). The proof of this fact can be found in [Martin 1992b]. Thus, the computation of $\Phi(\xi(\pi))$ involves a detailed understanding of the weights of ρ and the homomorphisms θ_α, for $\alpha \in R$.

The weights of ρ are easily described, and are exactly all unique products $\omega^1 \cdot \omega^3 \cdot \omega^5$, where ω^1, ω^3 and ω^5 denote arbitrary weights of Λ_1, Λ_3 and Λ_5, respectively [Cartan 1913, pp. 11–13]. In particular, define $\omega_i : T \to \mathbb{G}_m$ by $\omega_i(t_1, \ldots, t_7) = t_i$, so that ω_i is the weight of the basis element v_i. In this notation, then,

$$\begin{aligned}
\omega^1 &= \omega_i && \text{for } 1 \le i \le 7,\\
\omega^3 &= \omega_{j_1} \cdot \omega_{j_2} \cdot \omega_{j_3} && \text{for } 1 \le j_1 < j_2 < j_3 \le 7,\\
\omega^5 &= \omega_{k_1} \ldots \omega_{k_5} && \text{for } 1 \le k_1 < \cdots < k_5 \le 7,\\
R &= \{\alpha_{ij} = \omega_i\omega_j^{-1} : 1 \le i, j \le 7 \text{ and } i \ne j\},\\
S &= \{\alpha_i = \omega_i\omega_{i+1}^{-1} : 1 \le i \le 6\}.
\end{aligned}$$

The highest weight ϖ of the irreducible representation of G in $\mathbb{A}^{3402}$ defined by $g \mapsto {}^tg^{-1}$ is the product $(\omega_7 \cdot \omega_5\omega_6\omega_7 \cdot \omega_3 \ldots \omega_7)^{-1}$. It follows that $(b_1, \ldots, b_6) = (9, 18, 20, 22, 17, 12)$ and that, if we set $U = \{(0,1,0,0,0,0), (0,0,0,1,0,0), (0,0,0,0,0,1), (1,1,0,0,0,0)\}$, the value of Δ_ξ is given by the following table:

condition	Δ_ξ
$n_2 < n_4 < n_6$	$U \cup \{(0,1,1,0,0,0), (0,0,0,1,1,0)\}$
$n_2, n_6 < n_4$	$U \cup \{(0,1,1,0,0,0), (0,0,0,0,1,1)\}$
$n_2, n_6 > n_4$	$U \cup \{(0,0,1,1,0,0), (0,0,0,1,1,0)\}$
$n_2 > n_4 > n_6$	$U \cup \{(0,0,1,1,0,0), (0,0,0,0,1,1)\}$

Information about the $\theta_{\alpha_{ij}}$, for $1 \le i, j \le 7$ with $i \ne j$, is obtained in a more subtle fashion. Let $\theta'_{\alpha_{ij}}(\mathbb{G}_a)$ denote the unipotent subgroup of SL_7 composed of unipotent matrices $\theta'_{\alpha_{ij}}(u)$ with non-identically zero (i,j)-th entry u. We mean to exploit the fact that $\theta_{\alpha_{ij}}(u) = \rho(\theta'_{\alpha_{ij}}(u))$, for $u \in \mathbb{G}_a$. But first we consider $\Lambda_1 \otimes \Lambda_3 \otimes \Lambda_5(\theta'_{\alpha_{ij}}(\mathbb{G}_a))$, which, by construction, is defined by the action of a $\theta'_{\alpha_{ij}}(u)$ on a basis element

$$v_{i_1} \otimes v_{j_1} \wedge v_{j_2} \wedge v_{j_3} \otimes v_{k_1} \wedge \cdots \wedge v_{k_5}.$$

This action is given by

$$\begin{aligned}
&\theta'_{\alpha_{ij}}(u)(v_{i_1} \otimes v_{j_1} \wedge v_{j_2} \wedge v_{j_3} \otimes v_{k_1} \wedge \cdots \wedge v_{k_5})\\
&= (\theta'_{\alpha_{ij}}(u)v_{i_1})(\theta'_{\alpha_{ij}}(u)v_{j_1} \wedge v_{j_2} \wedge v_{j_3})(\theta'_{\alpha_{ij}}(u)v_{k_1} \wedge \cdots \wedge v_{k_5}).
\end{aligned}$$

Therefore,

$$\frac{d}{du}(\Lambda_1 \otimes \Lambda_3 \otimes \Lambda_5(\theta'_{\alpha_{ij}}(u)))|_{u=0}$$

has nonzero entries in the rows indexed by these basis elements:

$$\begin{aligned}
&v_i \otimes v_{j_1} \wedge v_{j_2} \wedge v_{j_3} \otimes v_{k_1} \wedge \cdots \wedge v_{k_5},\\
&v_{i_1} \otimes v_{l_1} \wedge v_{l_2} \wedge v_{l_3} \otimes v_{k_1} \wedge \cdots \wedge v_{k_5} \text{ for } l_1, l_2, l_3\\
&\qquad \text{with } 1 \le l_1 < l_2 < l_3 \le 7 \text{ and } i \in \{l_1, l_2, l_3\},\\
&v_{i_1} \otimes v_{j_1} \wedge v_{j_2} \wedge v_{j_3} \otimes v_{m_1} \wedge \cdots \wedge v_{m_5} \text{ for } m_1, \dots, m_5\\
&\qquad \text{with } 1 \le m_1 < \cdots < m_5 \le 7 \text{ and } i \in \{m_1, \dots, m_5\}.
\end{aligned}$$

Consider the case $(i,j) = (1,2)$. In this example,

$$\frac{d}{du}(\Lambda_1 \otimes \Lambda_3 \otimes \Lambda_5(\theta'_{\alpha_{12}}(u)))|_{u=0}$$

has nonzero entries in the rows

$$\begin{aligned}
&v_1 \otimes v_{j_1} \wedge v_{j_2} \wedge v_{j_3} \otimes v_{k_1} \wedge \cdots \wedge v_{k_5},\\
&v_{i_1} \otimes v_1 \wedge v_{l_2} \wedge v_{l_3} \otimes v_{k_1} \wedge \cdots \wedge v_{k_5}\\
&\qquad\qquad \text{for } l_2, l_3 \text{ with } 1 < l_2 < l_3 \le 7,\\
&v_{i_1} \otimes v_{j_1} \wedge v_{j_2} \wedge v_{j_3} \otimes v_1 \wedge v_{m_2} \wedge \cdots \wedge v_{m_5}\\
&\qquad \text{for } m_2, \dots, m_5 \text{ with } 1 < m_2 < \cdots < m_5 \le 7.
\end{aligned}$$

Since $v_{i_1} \otimes v_{j_1} \wedge v_{j_2} \wedge v_{j_3} \otimes v_{k_1} \wedge \cdots \wedge v_{k_5}$ has weight $\omega_{i_1} \cdot \omega_{j_1}\omega_{j_2}\omega_{j_3} \cdot \omega_{k_1} \cdots \omega_{k_5}$, and since ϖ is known to be $(\omega_7 \cdot \omega_5\omega_6\omega_7 \cdot \omega_3 \cdots \omega_7)^{-1}$, the row where

$$\frac{d}{du}(\Lambda_1 \otimes \Lambda_3 \otimes \Lambda_5(\theta'_{\alpha_{12}}(u)))|_{u=0}$$

has a nonzero entry and weight, with associated $c \in \Gamma$ making $c \cdot n$ minimal (where $n = (n_1, \dots, n_6)$), is $v_7 \otimes v_5 \wedge v_6 \wedge v_7 \otimes v_1 \wedge v_4 \wedge \cdots \wedge v_7$. The particular c here is $(1,1,0,0,0,0)$. Note that this choice for c is independent of n, and so also independent of ξ. It is not always possible to arrange this, as the table on the next column will illustrate. Again from Cartan, ρ has a basis of elements of the form

$$v = \sum_l c_{i,l} v_{i_1,l} \otimes v_{j_1,l} \wedge v_{j_2,l} \wedge v_{j_3,l} \otimes v_{k_1,l} \wedge \cdots \wedge v_{k_5,l},$$

where $c_{i,l} \in \mathbb{Q}_p$ and the vectors

$$v_{i_1,l} \otimes v_{j_1,l} \wedge v_{j_2,l} \wedge v_{j_3,l} \otimes v_{k_1,l} \wedge \cdots \wedge v_{k_5,l}$$

are all of the same weight [Cartan 1913, pp. 11–13]. This, together with the fact that the weight $\omega_7 \cdot \omega_5\omega_6\omega_7 \cdot \omega_1\omega_4 \dots \omega_7$ associated to $v_7 \otimes v_5 \wedge v_6 \wedge v_7 \otimes v_1 \wedge v_4 \wedge \cdots \wedge v_7$ does occur in ρ with multiplicity one and

$$\frac{d}{du}\rho(\theta'_{\alpha_{ij}}(u))|_{u=0}$$

has a nonzero entry in the row indexed by v only if

$$\frac{d}{du}(\Lambda_1 \otimes \Lambda_3 \otimes \Lambda_5(\theta'_{\alpha_{ij}}(u)))|_{u=0}$$

has a nonzero entry in a row indexed by one of the $v_{i_1,l} \otimes v_{j_1,l} \wedge v_{j_2,l} \wedge v_{j_3,l} \otimes v_{k_1,l} \wedge \cdots \wedge v_{k_5,l}$, allows one to conclude that $v_7 \otimes v_5 \wedge v_6 \wedge v_7 \otimes v_1 \wedge v_4 \wedge \cdots \wedge v_7$ is a basis element for ρ. Therefore, $c_{\alpha_{12},\xi} = c = (1,1,0,0,0,0)$.

Following the same line of reasoning for the other $\alpha_{ij} \in R$, we get these values for $c_{\alpha_{ij},\xi}$:

i	$j=1$	2	3	4	5	6	7
1		110000	110000	111000	111100	111110	111111
2	010000		010000	011000	011100	011110	011111
3	000000	000000		011000 001100	001100	001110	001111
4	000000	000000	010000 000100		000100	000110	000111
5	000000	000000	000000	000000		000110 000011	000011
6	000000	000000	000000	000000	000100 000001		000001
7	000000	000000	000000	000000	000000	000000	

We have omitted the commas and parentheses from the vectors due to typesetting limitations. Moreover, when two vectors are given for the same (i,j), the correct answer depends on the n_i. Thus,

$$\begin{aligned}
c_{\alpha_{12},\xi} &= (1,1,0,0,0,0) \quad \text{for all } \xi,\\
c_{\alpha_{13},\xi} &= (1,1,0,0,0,0) \quad \text{for all } \xi,\\
c_{\alpha_{34},\xi} &= \begin{cases} (0,1,1,0,0,0) & \text{if } n_2 < n_4,\\ (0,0,1,1,0,0) & \text{if } n_2 > n_4,\end{cases}\\
c_{\alpha_{43},\xi} &= \begin{cases} (0,1,0,0,0,0) & \text{if } n_2 < n_4,\\ (0,0,0,1,0,0) & \text{if } n_2 > n_4,\end{cases}\\
c_{\alpha_{56},\xi} &= \begin{cases} (0,0,0,1,1,0) & \text{if } n_4 < n_6,\\ (0,0,0,0,1,1) & \text{if } n_4 > n_6,\end{cases}\\
c_{\alpha_{65},\xi} &= \begin{cases} (0,0,0,1,0,0) & \text{if } n_4 < n_6,\\ (0,0,0,0,0,1) & \text{if } n_4 > n_6,\end{cases}
\end{aligned}$$

and so on. The author used the Maple package crystal [Joyner and Martin 1994] to help check these computations.

The choices in Δ_ξ and $c_{\alpha_{ij},\xi}$ require that the sum in parentheses in (2.1) be split into eleven pieces, reflecting the possible expressions for $\Phi(\xi(\pi))$ in terms of the relative values of n_2, n_4 and n_6:

condition	$\Phi(\xi(\pi))$
$n_2 > n_4 > n_6$	$p^{-(49n_0+7n_1+14n_2+13n_3+16n_4+11n_5+10n_6)}$
$n_2 > n_4 = n_6$	$p^{-(49n_0+7n_1+14n_2+13n_3+16n_4+11n_5+10n_6)}$
$n_2, n_6 > n_4$	$p^{-(49n_0+7n_1+14n_2+13n_3+19n_4+11n_5+7n_6)}$
$n_2 = n_4 > n_6$	$p^{-(49n_0+7n_1+14n_2+13n_3+16n_4+11n_5+10n_6)}$
$n_2 = n_4 = n_6$	$p^{-(49n_0+7n_1+14n_2+13n_3+16n_4+11n_5+10n_6)}$
$n_2 = n_4 < n_6$	$p^{-(49n_0+7n_1+14n_2+13n_3+19n_4+11n_5+7n_6)}$
$n_4 > n_6 > n_2$	$p^{-(49n_0+7n_1+17n_2+13n_3+13n_4+11n_5+10n_6)}$
$n_4 > n_6 = n_2$	$p^{-(49n_0+7n_1+17n_2+13n_3+13n_4+11n_5+10n_6)}$
$n_4 > n_2 > n_6$	$p^{-(49n_0+7n_1+17n_2+13n_3+13n_4+11n_5+10n_6)}$
$n_2 < n_4 = n_6$	$p^{-(49n_0+7n_1+17n_2+13n_3+13n_4+11n_5+10n_6)}$
$n_2 < n_4 < n_6$	$p^{-(49n_0+7n_1+17n_2+13n_3+16n_4+11n_5+7n_6)}$

As a result, we have the explicit expression for $Z_{\mathbb{Q}_p}(s)$ given in the sidebar below.

4. TESTING THE FUNCTIONAL EQUATION

It remains to check whether the functional equation holds. For notational convenience, denote by X_w the rational function within the outer parentheses of the expression for $Z_{\mathbb{Q}_p}(s)$ in the sidebar.

The first step is to express $Z_{\mathbb{Q}_p}(s)$ with respect to the basis $\nu(S)$, where $\nu \in \mathrm{Sym}(7)$ is the unique element such that $\nu(R^-) = R^+$:

$$Z_{\mathbb{Q}_p}(s) = \frac{(1-p^{-1})\prod_{i=1}^{6}(1-p^{-(i+1)})}{1-p^{-(49+7s)}} \cdot \sum_{w\in\mathrm{Sym}(7)} p^{-\lambda(\nu w)} \cdot \frac{p^{-Q_{\nu w}(1)(1+9s)}}{1-p^{-(1+9s)}} \cdot \frac{p^{-Q_{\nu w}(3)(1+20s)}}{1-p^{-(1+20s)}} \cdot \frac{p^{-Q_{\nu w}(5)(1+17s)}}{1-p^{-(1+17s)}} \cdot X_{\nu w},$$

where the Q-notation is defined in the sidebar. Sending p to p^{-1} and using the relations $\lambda(\nu w) = \operatorname{card} R^+ - \lambda(w)$ and

$$\left.\frac{1}{1-p^A}\right|_{p\mapsto p^{-1}} = \frac{-p^A}{1-p^A},$$

$Z_{\mathbb{Q}_p}(s)$ AS A RATIONAL FUNCTION IN p^{-1} AND p^{-s}

Define $Q_w(i_1, i_2, \ldots)$ to be equal to 1 if there exists $j \in \{i_1, i_2, \ldots\}$ such that $\alpha_j \in w(R^-)$, and equal to 0 otherwise. Then

$$\begin{aligned} Z_{\mathbb{Q}_p}(s) = {} & \frac{(1-p^{-1})\prod_{i=1}^{6}(1-p^{-(i+1)})}{1-p^{-(49+7s)}} \cdot \sum_{w\in\mathrm{Sym}(7)} p^{-\lambda(w)} \cdot \frac{p^{-Q_w(1)(1+9s)}}{1-p^{-(1+9s)}} \cdot \frac{p^{-Q_w(3)(1+20s)}}{1-p^{-(1+20s)}} \cdot \frac{p^{-Q_w(5)(1+17s)}}{1-p^{-(1+17s)}} \\ & \cdot \Biggl(\frac{p^{-(4+18s)}}{1-p^{-(4+18s)}} \biggl(\frac{p^{-(8+40s)}}{1-p^{-(8+40s)}} \frac{p^{-Q_w(6)(12+52s)}}{1-p^{-(12+52s)}} + \frac{p^{-Q_w(4,6)(12+52s)}}{1-p^{-(12+52s)}} + \frac{p^{-(1+12s)}}{1-p^{-(1+12s)}} \frac{p^{-Q_w(4)(12+52s)}}{1-p^{-(12+52s)}} \biggr) \\ & + \biggl(\frac{p^{-(8+40s)}}{1-p^{-(8+40s)}} \frac{p^{-Q_w(6)(12+52s)}}{1-p^{-(12+52s)}} + \frac{p^{-Q_w(2,4,6)(12+52s)}}{1-p^{-(12+52s)}} + \frac{p^{-(1+12s)}}{1-p^{-(1+12s)}} \frac{p^{-Q_w(2,4)(12+52s)}}{1-p^{-(12+52s)}} \biggr) \\ & + \frac{p^{-(1+22s)}}{1-p^{-(1+22s)}} \biggl(\frac{p^{-(5+34s)}}{1-p^{-(5+34s)}} \frac{p^{-Q_w(6)(12+52s)}}{1-p^{-(12+52s)}} + \frac{p^{-Q_w(2,6)(12+52s)}}{1-p^{-(12+52s)}} + \frac{p^{-(5+34s)}}{1-p^{-(5+34s)}} \frac{p^{-Q_w(2)(12+52s)}}{1-p^{-(12+52s)}} \biggr) \\ & + \frac{p^{-(5+34s)}}{1-p^{-(5+34s)}} \frac{p^{-Q_w(2)(12+52s)}}{1-p^{-(12+52s)}} + \frac{p^{-(1+12s)}}{1-p^{-(1+12s)}} \frac{p^{-(5+34s)}}{1-p^{-(5+34s)}} \frac{p^{-Q_w(2)(12+52s)}}{1-p^{-(12+52s)}} \Biggr). \end{aligned}$$

we obtain

$$Z_{\mathbb{Q}_p}(s) = p^{-7s}\frac{(1-p^{-1})\prod_{i=1}^{6}(1-p^{-(i+1)})}{1-p^{-(49+7s)}} \cdot \sum_{w\in \mathrm{Sym}(7)} p^{-\lambda(w)} \cdot \frac{p^{-Q_w(1)(1+9s)}}{1-p^{-(1+9s)}} \cdot \frac{p^{-Q_w(3)(1+20s)}}{1-p^{-(1+20s)}} \cdot \frac{p^{-Q_w(5)(1+17s)}}{1-p^{-(1+17s)}} \cdot (-X_{\nu w})|_{p\mapsto p^{-1}}.$$

Lemma. *The equality* $Z_{\mathbb{Q}_p}(s)|_{p\mapsto p^{-1}} = p^{-7s}Z_{\mathbb{Q}_p}(s)$ *implies* $-X_\nu|_{p\mapsto p^{-1}} = X_1$.

Proof. Recall that $\lambda(w) = \mathrm{card}(R^+ \cap w(R^-))$. Write

$$p^{-\lambda(w)}\frac{p^{-Q_w(1)(1+9s)}}{1-p^{-(1+9s)}}\frac{p^{-Q_w(3)(1+20s)}}{1-p^{-(1+20s)}}\frac{p^{-Q_w(5)(1+17s)}}{1-p^{-(1+17s)}}X_w$$

as the Laurent series

$$c(w)_1 p^{-N(w)} + c(w)_2 p^{-(N(w)+1)} + \cdots,$$

and

$$p^{-\lambda(w)} \cdot \frac{p^{-Q_w(1)(1+9s)}}{1-p^{-(1+9s)}} \cdot \frac{p^{-Q_w(3)(1+20s)}}{1-p^{-(1+20s)}} \cdot \frac{p^{-Q_w(5)(1+17s)}}{1-p^{-(1+17s)}} \cdot (-X_{\nu w})\Big|_{p\mapsto p^{-1}}$$

as

$$c(\nu w)_1 p^{-N(\nu w)} + c(\nu w)_2 p^{-(N(\nu w)+1)} + \cdots,$$

where the $c(w)_i$ and $c(\nu w)_i$ are constants in $\mathbb{C}$ and the $N(w)$ and $N(\nu w)$ are natural numbers. We claim that

$$\min\{N(w) : w \in \mathrm{Sym}(7)\} = N(1) = 0$$

and $N(w) > 0$ if $w \neq 1$. If $w = 1$, the leading term of

$$p^{-\lambda(1)} \cdot \frac{p^{-Q_1(1)(1+9s)}}{1-p^{-(1+9s)}} \cdot \frac{p^{-Q_1(3)(1+20s)}}{1-p^{-(1+20s)}} \cdot \frac{p^{-Q_1(5)(1+17s)}}{1-p^{-(1+17s)}} \cdot X_1$$

is 1, due to the term

$$\frac{p^{-Q_1(2,4,6)(12+52s)}}{1-p^{-(12+52s)}} = \frac{1}{1-p^{-(12+52s)}}$$

and the fact that $\lambda(1) = 0$. Hence,

$$c(1)_1 p^{-N(1)} + c(1)_2 p^{-(N(1)+1)} + \cdots$$

equals 1 plus lower-order terms, so $N(1) = 0$. Furthermore, for any other $w \in \mathrm{Sym}(7)$, we have $p^{-\lambda(w)} \neq 1$. It follows that $N(w) > N(1)$, establishing the claim. A similar argument proves that

$$\min\{N(\nu w) : w \in \mathrm{Sym}(7)\} = N(\nu) = 0$$

and $N(\nu w) > 0$ if $w \neq 1$. These two results imply $-X_\nu|_{p\mapsto p^{-1}}$ must equal X_1, completing the proof of the lemma. □

But $-X_\nu|_{p\mapsto p^{-1}} = X_1$ if and only if $p^{-(8+40s)} = p^{-(5+34s)}$. Therefore, the expected functional equation $Z_{\mathbb{Q}_p}(s)|_{p\mapsto p^{-1}} = p^{-7s}Z_{\mathbb{Q}_p}(s)$ does not hold.

5. REMARKS

The function Φ is not residual in the sense of [Denef and Meuser 1991]. Further, $Z_{\mathbb{Q}_p}(s)$ is not simple in the sense of [Igusa 1989, p. 713], or [Martin 1996]. Thus, the fact that $Z_{\mathbb{Q}_p}(s)$ does not satisfy the expected functional equation does not contradict prior work. Typical questions, such as whether a modified form of the functional equation holds for $Z_{\mathbb{Q}_p}(s)$ and, more generally, what other representations ρ have associated generalized Igusa local zeta function $Z_{\mathbb{Q}_p}(s)$ (for certain p) such that $Z_{\mathbb{Q}_p}(s)$ has a finite form that expresses $Z_{\mathbb{Q}_p}(s)$ as a rational function $Z(p^{-1}, p^{-s})$ satisfying

$$Z(p^{-1}, p^{-s})|_{p\mapsto p^{-1}} = p^{-\deg(f)s} Z(p^{-1}, p^{-s}),$$

are being examined. Partials answers can be found in [Martin 1992a; 1996; a]. The obstruction to the expected functional equation holding for the above $Z_{\mathbb{Q}_p}(s)$ is due to the failure of $X_w = -(X_{\nu w})|_{p\mapsto p^{-1}}$ to hold for all $w \in \mathrm{Sym}(7)$. This failure may be traced back to the vectors

$$(a_1, \ldots, a_6) = (6, 10, 12, 12, 10, 6),$$
$$(b_1, \ldots, b_6) = (9, 18, 20, 22, 17, 12),$$

$c \in \Delta_\xi$, and $c_{\alpha_{ij,\xi}}$ for $1 \le i, j \le 7$ and $i \neq j$. However, experimental evidence suggests that the $c_{\alpha_{ij,\xi}}$ are the key to this obstruction. Therefore, we make the following conjecture:

Conjecture. *Given a finite algebraic extension K of $\mathbb{Q}_p$ with ring of integers O_K and ideal of nonunits πO_K satisfying*

$$\operatorname{card}(O_K/\pi O_K) = q,$$

the generalized Igusa local zeta function $Z_K(s)$ associated to a simply connected simple Chevalley K-group G' and finite dimensional K-representation ρ of G' satisfies the expected functional equation

$$Z(q^{-1}, q^{-s})|_{q\mapsto q^{-1}} = q^{-\deg(f)s} Z(q^{-1}, q^{-s})$$

if and only if the matrix of vectors $c_{\alpha_{ij,\xi}}$, where $1 \le i, j \le \operatorname{rank} G'$ and $i \ne j$, has the property

$$c_{\alpha_{ij,\xi}} = c^{\bullet}_{\alpha_{ji,\xi}},$$

where $(x_1, x_2, \dots, x_n)^{\bullet} = (x_n, x_{n-1}, \dots, x_1)$.

Igusa has suggested that a matrix of vectors satisfying this property be called Hermitian symmetric.

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