Classification of Two Genera of 32-Dimensional Lattices of Rank 8 over the Hurwitz Order

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A generalization of Kneser's neighboring method allows us to classify two interesting genera at the same time. The new method is used to determine the genus of Hermitian unimodular lattices of rank 8 over the Hurwitz order $\mathfrak{M}$ and the genus of those $\mathfrak{M}$-lattices corresponding to unimodular $\mathbb{Z}$-lattices.

1. INTRODUCTION

Kneser's neighboring method [Kneser 1957] has extensively been used to construct all lattices in a given genus. Essentially one starts with one lattice in the genus and computes its neighbors as overlattices of certain maximal sublattices. So one really classifies two genera of lattices, being only interested in one. In this paper we generalize this method, replacing the maximal sublattices by sublattices of larger index in a more interesting genus. The resulting graph in each genus, which factors over a bipartite graph connecting the two genera (compare Proposition 2.6), contains the original neighborhood graph and hence is connected. We apply this method to determine the genus of unimodular lattices of rank 8 over the Hurwitz order $\mathfrak{M}$, as well as the one consisting of $\mathfrak{P}$-modular $\mathfrak{M}$-lattices (compare Definition 2.1). The latter classification was proposed in [Quebbemann 1984], where a mass formula for this genus is developed. There are only 11 such lattices, 8 of which are indecomposable. Four of these lattices are extremal in the sense that they do not contain vectors of length 2, and give rise to 3 non-isometric extremal even unimodular $\mathbb{Z}$-lattices of dimension 32. There are 24 $\mathfrak{M}$-unimodular lattices of rank 8, 15 of which are indecomposable. All 24 $\mathfrak{M}$-unimodular lattices

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contain Hermitian roots, so no extremal 2-modular $\mathbb{Z}$-lattice in the sense of [Quebbemann 1995] of dimension 32 has a structure as an $\mathfrak{M}$-unimodular lattice, which answers a question raised in [Bachoc 1995].

The paper is organized as follows. Section 2 presents the main ideas. They also allow to compute the mass of one genus, once the one of the other genus is known. The notation concerning the lattices is fixed in Section 3. Section 4 is dedicated to the application of the new method to $\mathfrak{M}$-lattices of rank at most 7, and Section 4 presents the results for rank 8. Gram matrices and generators for the Hermitian automorphism groups of the indecomposable lattices are available at http://www.research.att.com/~njas/lattices.

2. THE MASS FORMULAS

Let $\mathfrak{Q}$ be the quaternion algebra with center $\mathbb{Q}$, ramified at 2 and ∞. Let $\mathfrak{M}$ a maximal order in $\mathfrak{Q}$ and let $\mathfrak{P} = (1+i)\mathfrak{M}$ the two-sided maximal ideal of $\mathfrak{M}$ containing $2\mathfrak{M}$.

Definition 2.1. Let V be a left $\mathfrak{Q}$-vector space, $h : V \times V \to \mathfrak{Q}$ a positive definite Hermitian form with respect to the canonical involution of $\mathfrak{Q}$, and L an $\mathfrak{M}$-lattice in V.

(i) L is called *even* if $h(x,x)$ is even for all $x \in L$.

(ii) The *Hermitian dual lattice* L^* of L is defined as

$$L^* := \{x \in V : h(x,l) \in \mathfrak{M} \text{ for all } l \in L\}.$$

(iii) L is called *unimodular* if $L^* = L$.

(iv) L is called *$\mathfrak{P}$-modular* if $\mathfrak{P}L^* = L$.

(v) L is called *almost $\mathfrak{P}$-modular* if $L \subseteq \mathfrak{P}L^*$, and $\mathfrak{P}L^*/L \cong \mathfrak{M}/\mathfrak{P}$.

(vi) The *Hermitian automorphism group of* L is the subgroup $U(L)$ of $\mathrm{GL}(V)$ consisting of g such that $Lg = L$ and $h(xg, yg) = h(x,y)$ for all $x, y \in V$.

If L is an $\mathfrak{M}$-lattice such that $L \subseteq \mathfrak{P}L^*$, then the values of the Hermitian form on L lie in $\mathfrak{P}$, that is, $h(x,y) \in \mathfrak{P}$ for all $x, y \in L$. Especially $h(x,x)$ lies in $\mathfrak{P} \cap \mathbb{Q} = 2\mathbb{Z}$ for all $x \in L$. Therefore L becomes an even integral lattice with respect to the symmetric bilinear form $(x,y) := \frac{1}{2}\operatorname{Tr}(h(x,y))$, where Tr is the reduced trace of $\mathfrak{Q}$.

In particular, if L is a $\mathfrak{P}$-modular $\mathfrak{M}$-lattice of rank n, this construction yields an even $\mathbb{Z}$-unimodular lattice of dimension $4n$. So $\mathfrak{P}$-modular lattices do not exist if n is odd. The same construction applied to an almost $\mathfrak{P}$-modular lattice yields a $\mathbb{Z}$-lattice of determinant 2^2. Since $\mathfrak{M}$ itself is an $\mathfrak{M}$-unimodular lattice, there are $\mathfrak{M}$-unimodular lattices of arbitrary rank n. Since the different $\mathfrak{P}$ of $\mathfrak{M}$ is a principal ideal, these lattices give rise to 2-modular integral lattices of dimension $4n$ with respect to $(x,y) = \operatorname{Tr}(h(x,y))$.

Proposition 2.2. *Let L be an $\mathfrak{M}$-lattice with respect to the Hermitian form h.*

(i) *If L is $\mathfrak{M}$-unimodular, then h induces a nondegenerate Hermitian form*

$$\bar{h} : L/\mathfrak{P}L \times L/\mathfrak{P}L \to \mathfrak{M}/\mathfrak{P} \cong \mathbb{F}_4$$

defined by $\bar{h}(\bar{x},\bar{y}) = \overline{h(x,y)}$ for all $x, y \in L$.

(ii) *If $2L^* \subseteq L \subseteq \mathfrak{P}L^*$, then $L/2L^*$ is a nondegenerate symplectic vector space over $\mathfrak{M}/\mathfrak{P} \cong \mathbb{F}_4$ with respect to the form*

$$\varphi : L/2L^* \times L/2L^* \to \mathfrak{M}/\mathfrak{P} \cong \mathbb{F}_4$$

given by $\varphi(\bar{x},\bar{y}) = \overline{\tfrac{1}{2}h(x,y)(1+i)}$ for all $x, y \in L$.

Proof. (i) The form $\bar{h}$ clearly inherits the property to be Hermitian from the form h. To see the nondegeneracy choose $x \in L$ with $h(x,y) \in \mathfrak{P}$ for all $y \in L$. Then $\frac{1}{1+i}x \in L^*$ and therefore $x \in \mathfrak{P}L$, because L is $\mathfrak{M}$-unimodular.

(ii) For $x, y \in L$, one has $h(x,y) \in \mathfrak{P}$, so

$$\tfrac{1}{2}h(x,y)(1+i) \in \mathfrak{M}.$$

Therefore φ is well defined. Since $h(x,x) \in 2\mathfrak{M}$, all vectors are isotropic.

To see the nondegeneracy of φ let $x \in L$ with $\frac{1}{2}h(x,y)(1+i) \in \mathcal{P}$ for all $y \in L$. Then $h(x,y) \in 2\mathfrak{M}$ and therefore $x \in 2L^*$.

The form φ is clearly linear in the first variable. To prove the linearity in the second argument let $\rho : \mathfrak{O} \to \mathfrak{O}$ denote the canonical involution and choose $x, y \in L^*$, $b \in \mathfrak{M}$. Since both, the canonical involution ρ and conjugation by $(1+i)$ induce the Frobenius automorphism on $\mathfrak{M}/\mathfrak{P}$, one gets

$$\begin{aligned}\varphi(\bar{x}, \overline{by}) &= \overline{\tfrac{1}{2}h(x,y)\rho(b)(1+i)} = \overline{\tfrac{1}{2}h(x,y)(1+i)b} \\ &= \varphi(\bar{x}, \bar{y})\bar{b} = \bar{b}\varphi(\bar{x}, \bar{y}).\end{aligned}$$

Hence φ is bilinear. □

The main idea of the method to classify both, the $\mathfrak{M}$-unimodular and the (almost) $\mathfrak{P}$-modular lattices of a given rank is the following observation.

Proposition 2.3. (i) *Let M be an $\mathfrak{M}$-unimodular lattice of rank n. If n is even, the $\mathfrak{P}$-modular lattices contained in M are the full preimages of the maximal isotropic subspaces of the Hermitian $\mathbb{F}_4$ vector space $M/\mathfrak{P}M$. If n is odd, the almost $\mathfrak{P}$-modular lattices contained in M are the full preimages of the maximal isotropic subspaces of the Hermitian $\mathbb{F}_4$ vector space $M/\mathfrak{P}M$.*

(ii) *Let L be an $\mathfrak{M}$-lattice of rank n. If n is even, assume that L is $\mathfrak{P}$-modular and if n is odd, assume that L is almost $\mathfrak{P}$-modular. The $\mathfrak{M}$-unimodular lattices containing L are of the form $\mathfrak{P}^{-1}N$, where N is the full preimage of a maximal isotropic subspace of the symplectic $\mathbb{F}_4$ vector space $L/2L^*$ of dimension $dim_{\mathbb{F}_4}(L/2L^*) = 2{\cdot}[\frac{n}{2}]$.*

Proof. (i) Let L be a $\mathfrak{M}$-lattice corresponding to a maximal isotropic subspace of $M/\mathfrak{P}M$. Then $h(L,L) \subseteq \mathfrak{P}$ shows that $L \subseteq \mathfrak{P}L^*$. The index can be seen from the dimension of this subspace. Conversely let L be a (almost) $\mathfrak{P}$-modular lattice contained in M. Then $L \subseteq L + \mathfrak{P}M \subseteq \mathfrak{P}L^*$. Now either $L = \mathfrak{P}L^*$ and clearly $\mathfrak{P}M \subseteq L$ or $\mathfrak{P}L^*/L \cong \mathfrak{M}/\mathfrak{P}$ and one has equality in one of the two inclusions above. Equality in the first inclusion directly implies $\mathfrak{P}M \subseteq L$. Equality in the second inclusion yields $\mathfrak{P}L^* \subseteq M = M^* \subseteq \mathfrak{P}^{-1}L$. Clearly the image of L in $M/\mathfrak{P}M$ is maximal isotropic.

Part (ii) is analogous. □

To prove the completeness of the lists of isometry classes of $\mathfrak{M}$-unimodular and (almost) $\mathfrak{P}$-modular lattices we use the following mass formula, developed in [Hashimoto 1980]:

Let $M_1, \ldots, M_h$ be the Hermitian isometry classes of unimodular $\mathfrak{M}$-lattices of rank n. Then

$$\sum_{i=1}^{h} \frac{1}{|U(M_i)|} = \prod_{i=1}^{n} \frac{(2^i + (-1)^i)B_i}{4i},$$

where B_i is the i-th Bernoulli number.

Using this formula, a mass formula for the genus of (almost) $\mathfrak{P}$-modular lattices can be easily derived by a counting argument, which the second author learned from B. B. Venkov:

Proposition 2.4. *Let $M_1, \ldots, M_h$ be representatives of the isometry classes of unimodular $\mathfrak{M}$-lattices of rank n. Let $L_1, \ldots, L_s$ be representatives of the isometry classes of $\mathfrak{P}$-modular (if n is even) or almost $\mathfrak{P}$-modular (if n is odd) $\mathfrak{M}$-lattices. Let c_1 denote the number of maximal isotropic subspaces of the Hermitian $\mathbb{F}_4$ vector space of dimension n and c_2 denote the number of maximal isotropic subspaces of the symplectic $\mathbb{F}_4$ vector space of dimension $2{\cdot}[\frac{n}{2}]$. Then*

$$\sum_{j=1}^{s} \frac{1}{|U(L_j)|} = \frac{c_1}{c_2} \sum_{i=1}^{h} \frac{1}{|U(M_i)|}.$$

Proof. For $1 \le i \le h$ and $1 \le j \le s$ let

$$a_{ij} := |\{L \le M_i : L \text{ is isometric to } L_j\}|$$

and

$$b_{ji} := |\{\mathfrak{P}M \le L_j : M \text{ is isometric to } M_i\}|.$$

By Proposition 2.3 one has $\sum_{j=1}^{s} a_{ij} = c_1$ and $\sum_{i=1}^{h} b_{ji} = c_2$.

Let φ be a unitary mapping with $\varphi(L_j) \le M_i$. Then $\mathfrak{P}M_i \le \varphi(L_j)$ and hence $\mathfrak{P}\varphi^{-1}(M_i) \le L_j$. So the number of unitary embeddings of L_j into M_i equals the number of unitary embeddings of $\mathfrak{P}M_i$ into L_j. Moreover, if φ' is a further unitary embedding of L_j into M_i, with $\varphi(L_j) = \varphi'(L_j)$, then $\varphi'\varphi^{-1} \in U(L_j)$. Therefore one has

$$a_{ij}\,|U(L_j)| = b_{ji}\,|U(M_i)|. \tag{2.1}$$

Hence

$$\begin{aligned} c_1 \sum_{i=1}^{h} \frac{1}{|U(M_i)|} &= \sum_{j=1}^{s}\sum_{i=1}^{h} a_{ij}\frac{1}{|U(M_i)|} \\ &= \sum_{j=1}^{s}\sum_{i=1}^{h} b_{ji}\frac{1}{|U(L_j)|} \\ &= c_2 \sum_{j=1}^{s} \frac{1}{|U(L_j)|}. \qquad \square \end{aligned}$$

Note that an analogous proof may be applied to any two genera of lattices in the same vector space to calculate the mass of one genus knowing the mass of the other. Formulas for the numbers of maximal isotropic subspaces in a symplectic or unitary space over a finite field $\mathbb{F}_q$ may be found in [Taylor 1992, exercises (8.1), (10.4)].

The values for $\mathbb{F}_q = \mathbb{F}_4$ and dimensions ≤ 8 are:

dim	1	2	3	4	5	6	7	8
c_1	1	3	9	27	297	891	38313	114939
c_2	–	5	–	85	–	5525	–	1419925

In the spirit of this proof we define a bipartite graph:

Definition 2.5. Let n, h, s, a_{ij}, b_{ji}, M_i $(1 \leq i \leq h)$, and L_j $(1 \leq j \leq s)$ be as in Proposition 2.4. Then $\Gamma_{\text{iso}}(n)$ is the labelled bipartite graph with vertices M_i and L_j and edges

$$\{(M_i, L_j) : a_{ij} > 0\} \cup \{(L_j, M_i) : b_{ji} > 0\}$$

labelled with the corresponding number a_{ij} or b_{ji}, respectively.

Proposition 2.6. (i) *$\Gamma_{\text{iso}}(n)$ is connected.*

(ii) *The valence of each of the vertices M_i is c_1 and the valence of each of the vertices L_j is c_2, where c_1 and c_2 are defined as in Proposition 2.4.*

(iii) *Every subgraph of $\Gamma_{\text{iso}}(n)$ satisfying (ii) is the full graph $\Gamma_{\text{iso}}(n)$.*

Proof. (i) Let M and M' be two $\mathfrak{M}$-unimodular lattices. By [Bachoc 1995] there is a sequence of $\mathfrak{M}$-unimodular lattices $M := M'_1, \ldots, M'_k =: M'$ with $M'_i/(M'_i \cap M'_{i+1}) \cong \mathfrak{M}/\mathfrak{P}$ $(1 \leq i < k)$. For $1 \leq i < k$ let $K_i := (M'_i \cap M'_{i+1})$. Then the orthogonal complement with respect to the Hermitian form $\bar{h}$ of Proposition 2.2 of $K_i + \mathfrak{P}M'_i$ is $K_i^{\perp} + \mathfrak{P}M'_i = \mathfrak{P}\langle M'_{i+1}, M'_i\rangle + \mathfrak{P}M'_i$ and contained in $K_i + \mathfrak{P}M'_i$. Therefore $K_i + \mathfrak{P}M'_i$ contains a maximal isotropic subspace of $M'_i/\mathfrak{P}M'_i$. For $1 \leq i < k$ let L'_i be a full preimage of a maximal isotropic subspace of $M'_i/\mathfrak{P}M'_i$ contained in M'_{i+1}. Then $L'_1, \ldots, L'_{k-1}$ is a chain of (almost) $\mathfrak{P}$-modular lattices joining M and M' in $\Gamma_{\text{iso}}(n)$.

(ii) follows from Proposition 2.4 and (iii) is an easy consequence of (i). $\square$

3. SOME NOTATION

In this section we give constructions for the occuring root lattices and notation for the classified lattices. The $\mathfrak{M}$-lattices of rank n are understood to lie in the $\mathfrak{Q}$ vector space $\mathfrak{Q}^n$ endowed with the Hermitian form $h(x,y) = \sum_{i=1}^{n} x_i\bar{y}_i$. When considered as $\mathbb{Z}$-lattices, the corresponding scalar product is as defined in Section 2.

Rational Root Lattices

Let $n \geq 1$. The standard lattice $\mathbb{Z}^n$ is the $\mathbb{Z}$-span of an orthonormal basis of an n-dimensional Euclidean vector space.

Let $n \geq 1$. The root lattice $\mathbb{A}_n$ is defined as the n-dimensional sublattice

$$\mathbb{A}_n := \{(x_0, \ldots, x_n) \in \mathbb{Z}^{n+1} : \sum_{i=0}^{n} x_i = 0\}$$

of the standard lattice $\mathbb{Z}^{n+1}$. The discriminant group $\mathbb{A}_n^{\#}/\mathbb{A}_n$ is cyclic of order $n+1$ and generated by $\overline{p(\varepsilon_0)}$, where

$$p(\varepsilon_0) = \frac{1}{n+1}(n, -1, \ldots, -1) \in \mathbb{A}_n^{\#}$$

is the projection of the first basis vector $(1, 0, \ldots, 0)$ of $\mathbb{Z}^{n+1}$ into $\mathbb{Q}\mathbb{A}_n$.

For $n \geq 4$ the root lattice $\mathbb{D}_n$ is defined as the even sublattice of $\mathbb{Z}^n$,

$$\mathbb{D}_n := \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i \in 2\mathbb{Z}\}.$$

If $n = 8$, the standard lattice $\mathbb{Z}^8$ has an even neighbor $\mathbb{E}_8$ containing $\mathbb{D}_8$:

$$\mathbb{E}_8 := \langle \mathbb{D}_8, \tfrac{1}{2}(1,1,1,1,1,1,1,1)\rangle_{\mathbb{Z}}.$$

Complex Root Lattices

There is one infinite series of quaternionic lattices, which can be uniformly described as sublattices of $\mathfrak{M}^n$ [Martinet 1996]. Let $\mathfrak{I}$ be a left ideal of $\mathfrak{M}$. For $n \geq 1$ define

$$D_n(\mathfrak{I}) := \{(x_1, \ldots, x_n) \in \mathfrak{M}^n : \sum_{i=0}^{n} x_i \in \mathfrak{I}\}.$$

These quaternionic lattices are in fact scalar extensions of complex lattices, since they may be defined over any subfield of $\mathfrak{Q}$ containing generators of the left $\mathfrak{M}$-ideal $\mathfrak{I}$.

The $\mathfrak{P}$-modular $\mathfrak{M}$-lattice $D_2((1+i))$ is as a $\mathbb{Z}$-lattice isometric to the root lattice $\mathbb{E}_8$ and therefore denoted by E_8.

We additionally need one lattice defined over $\mathbb{Z}[\omega]$ where $\omega := \frac{1}{2}(-1+\sqrt{-3})$ is a third root of unity. This root lattice is described in [Feit 1978] as an extension of $\mathbb{A}_5$:

$$U_5 := \langle \mathbb{A}_5, \frac{1-\omega}{3}(1,\omega,\omega^2,1,\omega,\omega^2)\rangle_{\mathbb{Z}[\omega]} \subseteq \mathbb{Q}[\omega] \otimes \mathbb{Z}^6.$$

The $\mathfrak{M}$-unimodular lattice R_{24} as defined in [Bachoc 1995] can be constructed as $\mathfrak{M} \otimes_{\mathbb{Z}[\omega]} U_6$, where the root lattice U_6 is defined in [Feit 1978].

Hermitian Root Lattices

For the Hermitian root lattices, we refer to the notations of the root systems in [Cohen 1980].

The root lattice

$$BW_{16} := D_4((1+i)) + \mathfrak{P}^{-1}(1,1,1,1)$$

is an $\mathfrak{M}$-unimodular lattice spanned by the root system S_3. It is denoted by BW_{16}, because the corresponding $\mathbb{Z}$-lattice is the well known Barnes–Wall lattice BW_{16} of dimension 16.

We set

$$S_1 := \{y \in BW_{16} : h(x,y) \in \mathfrak{P}\},$$

where $x \in BW_{16}$ is any vector in BW_{16} satisfying $h(x,x) = 3$. The sublattice of index $BW_{16}/S_1 \cong \mathfrak{M}/\mathfrak{P}$ in BW_{16} is spanned by the root system S_1.

The lattice R_{20} of [Bachoc 1995] is the $\mathfrak{M}$-unimodular lattice spanned by the root system U.

Remark 3.1. Let Λ be an (almost) $\mathfrak{P}$-modular $\mathfrak{M}$-lattice of rank n. If the corresponding $\mathbb{Z}$-lattice L has vectors of length 2, the $\mathfrak{M}$-lattice generated by these vectors is of the form $E_8^m \perp \mathfrak{P}^s$, where E_8^m is an orthogonal summand of Λ [Quebbemann 1984]. If $n \leq 8$ the vectors of length 2 in L turn out to determine Λ up to isometry.

In view of this remark let $L_n(\mathfrak{P}^s)$ denote the (almost) $\mathfrak{P}$-modular $\mathfrak{M}$-lattice of rank n such that the corresponding $\mathbb{Z}$-lattice has root lattice $\mathbb{D}_4^s$.

In particular $L_n(\mathfrak{P}^n)$ is the (almost) $\mathfrak{P}$-modular lattice constructed using the indecomposable code e_n, described in [MacWilliams et al. 1978], which corresponds to a maximal isotropic subspace of $(\mathfrak{M}/\mathfrak{P})^n = \mathbb{F}_4^n$.

The lattice $L_7(\mathfrak{P}^3)$ is a suitable lattice containing $BW_{16} \perp \mathfrak{P}^3$ of index $(\mathfrak{M}/\mathfrak{P})^3$.

$L_7(\mathfrak{P})$ contains $\mathfrak{P} \perp L_{24}$ with index $\mathfrak{M}/\mathfrak{P}$, where L_{24} is a maximal common $\mathfrak{M}$-sublattice of Λ_{24} and $L_6(\mathfrak{P}^6)$.

Λ_{4n} denotes (almost) $\mathfrak{P}$-modular lattices such that the corresponding $\mathbb{Z}$-lattices have no vectors of length 2. For $n = 6$ one gets the $\mathfrak{M}$-structure of the Leech lattice as described in [Tits 1980]. The uniqueness of this structure is proved in [Quebbemann 1984]. In [Quebbemann 1995] an integral $\mathbb{Z}$-lattice of dimension 28, determinant 4 and minimum 4 is described. As can be seen from the construction, this lattice has a structure over $\mathfrak{M}$ and hence is isometric to Λ_{28}. Compare [Nebe 1996], where the corresponding $\mathbb{Z}$-lattice is denoted by $[2.J_2 \overset{2(2)}{\circ} \mathrm{SL}_2(3)]_{28}$, and [Bacher and Venkov 1996].

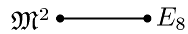
The labels a_{ij} and b_{ji} of the edges in the pictures of $\Gamma_{\text{iso}}(n)$ for $n \leq 6$ are omitted. For $n < 4$ these labels may easily be calculated from the table given just before Definition 2.5. For $n \geq 4$, the graphs $\Gamma_{\text{iso}}(n)$ are represented by tables. The columns of these tables correspond to the isomorphism classes of (almost) $\mathfrak{P}$-modular lattices $L_1, \dots, L_s$, the rows to those of $\mathfrak{M}$-unimodular lattices $M_1, \dots, M_h$. For each lattice K, a name and the order of its Hermitian automorphism group $U(K)$ is given. The entry (i,j) of the table itself consists of the numbers a_{ij} and b_{ji} in the notation of Proposition 2.4, where $1 \leq i \leq h$ and $1 \leq j \leq s$.

4. RESULTS FOR RANK 1 TO 7

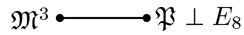
Tables 1 and 2 present the graphs $\Gamma_{\text{iso}}(n)$ for $n \leq 7$. The occuring $\mathfrak{M}$-unimodular lattices have already been determined in [Bachoc 1995], from which we also borrow some notation.

$n = 1$:

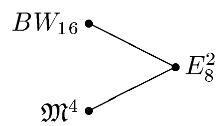
$n = 2$:



$n = 3$:

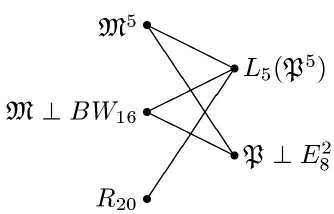


$n = 4$:



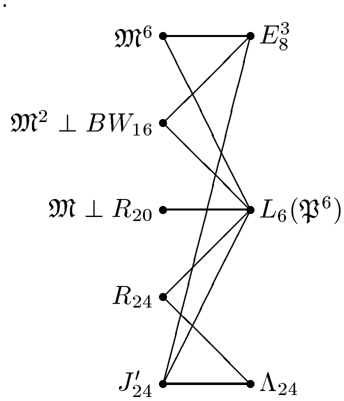
	E_8^2 $2^{15}\cdot 3^2\cdot 5^2$
$\mathfrak{M}^4$ $(2^3\cdot 3)^4\cdot(4!)$	27 25
BW_{16} $2^{13}\cdot 3^4\cdot 5$	27 60

$n = 5$:



	$L_5(\mathfrak{P}^5)$ $2^{17}\cdot 3^2\cdot 5$	$\mathfrak{P} \perp E_8^2$ $2^{18}\cdot 3^3\cdot 5^2$
$\mathfrak{M}^5$ $(2^3\cdot 3)^5\cdot(5!)$	162 1	135 25
$\mathfrak{M} \perp BW_{16}$ $2^{16}\cdot 3^5\cdot 5$	270 20	27 60
R_{20} $2^{11}\cdot 3^5\cdot 5\cdot 11$	297 64	

$n = 6$:



	E_8^3 $(2^7\cdot 3\cdot 5)^3\cdot 6$	$L_6(\mathfrak{P}^6)$ $2^{21}\cdot 3^3\cdot 5$	Λ_{24} $2^{13}\cdot 3^3\cdot 5^2\cdot 7\cdot 13$
$\mathfrak{M}^6$ $(2^3\cdot 3)^6\cdot(6!)$	405 125	486 1	
$\mathfrak{M}^2 \perp BW_{16}$ $2^{20}\cdot 3^6\cdot 5$	81 900	810 60	
$\mathfrak{M} \perp R_{20}$ $2^{14}\cdot 3^6\cdot 5\cdot 11$		891 384	
R_{24} $2^9\cdot 3^7\cdot 5\cdot 7$		567 4096	324 4160
J'_{24} $2^{16}\cdot 3\cdot(6!)$	15 4500	492 984	384 1365

TABLE 1. The graph $\Gamma_{\text{iso}}(n)$ and information about its edges, for $n = 1, \dots, 6$.

	$E_8^3 \perp \mathfrak{P}$ $2^{25}\cdot 3^5\cdot 5^3$	$L_6(\mathfrak{P}^6) \perp \mathfrak{P}$ $2^{24}\cdot 3^4\cdot 5$	$\Lambda_{24} \perp \mathfrak{P}$ $2^{16}\cdot 3^4\cdot 5^2\cdot 7\cdot 13$	$L_5(\mathfrak{P}^5) \perp E_8$ $2^{24}\cdot 3^3\cdot 5^2$	$L_7(\mathfrak{P}^7)$ $2^{24}\cdot 3^2\cdot 7$	$L_7(\mathfrak{P}^3)$ $2^{17}\cdot 3^4$	$L_7(\mathfrak{P})$ $2^{16}\cdot 3^2\cdot 5$	Λ_{28} $2^8\cdot 3^3\cdot 5^2\cdot 7$
$\mathfrak{M}^7$ $(2^3\cdot 3)^7\cdot(7!)$	2835 125	3402 1		10206 5	21870 1			
$\mathfrak{M}^3 \perp BW_{16}$ $2^{23}\cdot 3^8\cdot 5$	243 900	2430 60		2430 100	7290 28	25920 1		
$\mathfrak{M}^2 \perp R_{20}$ $2^{18}\cdot 3^7\cdot 5\cdot 11$		1782 384		891 320		35640 12		
$\mathfrak{M} \perp R_{24}$ $2^{12}\cdot 3^8\cdot 5\cdot 7$		567 4096	324 4160			17010 192	20412 64	
$\mathfrak{M} \perp J'_{24}$ $2^{19}\cdot 3^2\cdot(6!)$	15 4500	492 984	384 1365	90 300	1620 504	11520 36	24192 21	
R_{28} $2^{12}\cdot 3^5\cdot 5\cdot 7$					135 4096	1890 576	12096 1024	24192 840
R'_{28} $2^{18}\cdot 3^5$				27 4800	54 896	3672 612	6912 320	27648 525
R''_{28} $2\cdot 3^6\cdot(7!)$						2835 4096	10206 4096	25272 4160

TABLE 2. Information about the edges of $\Gamma_{\text{iso}}(7)$.

5. THE LATTICES OF RANK 8

In this section we present the main result, the classification of the $\mathfrak{M}$-unimodular and the $\mathfrak{P}$-modular lattices of rank 8. For the matrix groups, the notation is borrowed from [Nebe and Plesken 1995; Nebe $\geq$ 1997]. In particular we call a matrix group *absolutely irreducible* if the $\mathbb{Q}$-algebra generated by the matrices in the group is the full matrix algebra.

Theorem 5.1. *There are* 11 *isometry classes of* $\mathfrak{P}$-*modular lattices of rank* 8. *Seven of them yield unimodular* $\mathbb{Z}$-*lattices that contain roots*; *they may be distinguished by their root lattices*, *which are* $\mathbb{D}_4$, $\mathbb{D}_4^2$, $\mathbb{D}_4^4$, $\mathbb{D}_4^8$, $\mathbb{E}_8$, $\mathbb{E}_8 \perp \mathbb{D}_4^6$, *and* $\mathbb{E}_8^4$. *The other four lattices can be distinguished by means of their Hermitian automorphism groups* (which have been investigated using MAGMA.)

(i) $U(BW_{32}) = (Q_8 \otimes D_8 \otimes D_8 \otimes D_8).O_8^-(2)$ *is an absolutely irreducible maximal finite subgroup of* $\mathrm{GL}_8(\mathfrak{O})$. *The automorphism group of the corresponding unimodular* $\mathbb{Z}$-*lattice is* $\mathrm{Aut}(BW_{32}) = (D_8 \otimes D_8 \otimes D_8 \otimes D_8 \otimes D_8).O_{10}^+(2)$ *and an absolutely irreducible maximal finite subgroup of* $\mathrm{GL}_{32}(\mathbb{Q})$.

(ii) $U(\Lambda''_{32}) = 2_-^{1+6}.O_6^-(2) \curlywedge 2_-^{1+6} O_6^-(2)$ *is the subdirect product of two groups* $2_-^{1+6}.O_6^-(2) = (Q_8 \otimes D_8 \otimes D_8).O_6^-(2) = U(BW_{16})$ *amalgamated of the common factor group* $O_6^-(2)$. *This group is a reducible subgroup of* $\mathrm{GL}_8(\mathfrak{O})$. *The corresponding unimodular* $\mathbb{Z}$-*lattice is isometric to* BW_{32}.

(iii) $U(\Lambda_{32}) = (C_4 \otimes D_8 \otimes D_8 \otimes D_8).(U_3(3).2)$ *is an absolutely irreducible subgroup of* $U(BW_{32})$. *The automorphism group of the corresponding unimodular* $\mathbb{Z}$-*lattice is* $\mathrm{Aut}(\Lambda_{32}) = (C_4 \otimes D_8 \otimes D_8 \otimes D_8).(S_3 \times U_3(3).2)$ *and an absolutely irreducible subgroup of* $\mathrm{Aut}(BW_{32})$.

(iv) $U(\Lambda'_{32}) = (\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5) \underset{\sqrt{5}}{\otimes} \mathrm{SL}_2(5)) : S_3$ *is an absolutely irreducible maximal finite subgroup of* $\mathrm{GL}_4(\mathfrak{O} \otimes \mathbb{Q}[\sqrt{5}])$. *The automorphism group of the corresponding unimodular* $\mathbb{Z}$-*lattice is*

$$\mathrm{Aut}(\Lambda'_{32}) = ((\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5) \underset{\sqrt{5}}{\otimes} \mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)).2) : S_4$$

and an absolutely irreducible maximal finite subgroup of $\mathrm{GL}_{32}(\mathbb{Q})$.

root lattice R	construction of M	$[M:R]$	$\lvert R_4\rvert$
$D_8(1+i)$	$\frac{1+i}{2}(1,1,1,1,1,1,1,1)$	2^2	$24{\cdot}232$
$D_8(1-\omega)$	$\frac{1-\omega}{3}(1,1+\alpha,1,1+\alpha,1,1+\alpha,1,1+\alpha)$, for $\alpha=i(1-\omega)$	3^2	$24{\cdot}84$
$D_4(1+i)^2$	$\frac{1-i}{2}(1,1,1,1,0,0,0,(1+i))$ $\frac{1-i}{2}(0,0,0,(1+i),1,1,1,1)$	2^4	$24{\cdot}104$
S_1^2	$(x,-x)$ $\frac{1+i}{2}(x,x)$	2^4	$24{\cdot}72$
$\mathfrak{M}\otimes\mathbb{D}_8$	$\frac{1}{2}(1,1,1,1,1,1,1,3+i)$	2^4	$24{\cdot}56$
$D_3(1+i)\perp\mathfrak{M}\underset{\mathbb{Z}[\omega]}{\otimes}U_5$	$(\frac{1+i}{2}(1,1,1),(1-\omega)p(\varepsilon_0))$	2^4	$24{\cdot}72$
$D_2(1+i)\perp\mathfrak{M}\otimes\mathbb{D}_6$	$((1,0),\frac{1}{2}(\omega-\omega^2,1,1,1,1,1))$ $(\frac{1+i}{2}(1,1),(0,1+i,0,0,0,0))$	2^6	$24{\cdot}40$
$D_4(1-\omega)^2$	$((1,1,1,1),(\frac{1-\omega}{3}+i)(1,1,1,1))$ $((\frac{-2-\omega}{3}+i)(1,1,1,1),(1,1,1,1))$	3^4	$24{\cdot}36$
$\mathfrak{M}\otimes\mathbb{A}_8$	$(-2+3i-4\omega)p(\varepsilon_0)$	9^2	$24{\cdot}36$
$D_2(1+i)^4$	$\frac{1-i}{2}(1,1,1,1,0,0,0,(1+i))$ $\frac{1-i}{2}(0,0,1,1,1,1+(1+i)\omega,0,(1+i)\omega)$ $\frac{1-i}{2}((1+i),0,0,0,1,1,1,1)$ $(0,1,0,1,0,1,0,1,0,1)$	2^8	$24{\cdot}40$
$\mathfrak{M}\otimes\mathbb{D}_4^2$	$\frac{1}{2}(2i,0,0,0,1+2i,1,1,1)$ $\frac{1}{2}(1+2j,1,1,1,2i,0,0,0)$	2^8	$24{\cdot}24$
$D_3(1-\omega)\perp\mathfrak{M}\otimes\mathbb{A}_5$	$((1,0,0),\frac{1-\omega}{3}(1,\omega,\omega^2,1,\omega,\omega^2))$ $(1+i)(i+2j-\omega)(\frac{1-\omega}{3}(1,1,1),p(\varepsilon_0))$	$3^4\cdot 2^2$	$24{\cdot}24$
$\mathfrak{M}\otimes\mathbb{A}_4^2$	$(1+2i)(p(\varepsilon_0),p(\varepsilon_0))$ $(1+2i)\omega(p(\varepsilon_0),-p(\varepsilon_0))$	5^4	$24{\cdot}20$
$D_1(1+i)^8$	$\frac{1-i}{2}(1,1,1,1,1+i,0,0,0)$ $\frac{1-i}{2}(1+i,0,0,0,1,1,1,1)$ $\frac{1-i}{2}(0,1,\omega,\bar\omega,0,1,\omega,\bar\omega)$ $(0,0,1,1,0,0,1,1)$ $(0,0,0,0,0,1,\bar\omega,\omega)$	2^{16}	$24{\cdot}8$

TABLE 3. Information about rank-8 lattices. The first column contains the root lattice R as described in Section 3. The $\mathfrak{M}$-unimodular lattice M is generated by R and the vectors given in the second column. The last column displays the number of roots in M. The graph $\Gamma_{\text{iso}}(8)$ is encoded in Table 4. The indecomposable $\mathfrak{M}$-unimodular lattices are denoted by their Hermitian root systems. We follow the notations of Section 3. Note that the last lattice in the table is the lattice $\tilde J_4^8$ of [Bachoc 1995].

		Λ_{32} $2^{14}\cdot 3^3\cdot 7$		Λ'_{32} $2^8\cdot 3^4\cdot 5^3$		Λ''_{32} $2^{20}\cdot 3^4\cdot 5$		BW_{32} $2^{21}\cdot 3^4\cdot 5\cdot 7\cdot 17$		$L_8(\mathfrak{P})$ $2^{11}\cdot 3^3\cdot 5^2\cdot 7$	
$\mathfrak{M}^8$	$(2^3\cdot 3)^8\cdot(8!)$										
$\mathfrak{M}^4\perp BW_{16}$	$2^{28}\cdot 3^9\cdot 5$										
BW_{16}^2	$2^{27}\cdot 3^8\cdot 5^2$					51840	1	25920	119		
$\mathfrak{M}^3\perp R_{20}$	$2^{20}\cdot 3^9\cdot 5\cdot 11$										
$\mathfrak{M}^2\perp R_{24}$	$2^{16}\cdot 3^9\cdot 5\cdot 7$										
$\mathfrak{M}^2\perp J'_{24}$	$2^{27}\cdot 3^5\cdot 5$										
$\mathfrak{M}\perp R_{28}$	$2^{15}\cdot 3^6\cdot 5\cdot 7$									72576	840
$\mathfrak{M}\perp R'_{28}$	$2^{21}\cdot 3^6$									82944	525
$\mathfrak{M}\perp R''_{28}$	$2^8\cdot 3^9\cdot 5\cdot 7$									75816	4160
$\mathfrak{M}\otimes\mathbb{E}_8$	$2^{16}\cdot 3^6\cdot 5^2\cdot 7$			48384	15	18900	960	270	3264		
$D_8(1+i)$	$2^{29}\cdot 3^3\cdot 5\cdot 7$					53760	45	3840	765		
$D_8(1-\omega)$	$2^8\cdot 3^9\cdot 5\cdot 7$	65610	1152							40824	2240
$D_4(1+i)^2$	$2^{27}\cdot 3^3$	73728	63			10368	1215	576	16065		
S_1^2	$2^{19}\cdot 3^7$	20736	56	55296	125	7344	2720	216	19040		
$\mathfrak{M}\otimes\mathbb{D}_8$	$2^{16}\cdot 3^2\cdot 5\cdot 7$	30720	4608	48384	6075	2100	43200	30	146880	21504	10080
$D_3(1+i)\perp\mathfrak{M}\underset{\mathbb{Z}[\omega]}{\otimes}U_5$	$2^{15}\cdot 3^6\cdot 5$	77760	2016							25920	2100
$D_2(1+i)\perp\mathfrak{M}\otimes\mathbb{D}_6$	$2^{16}\cdot 3^3\cdot 5$	40320	14112	49152	14400	960	46080			17280	18900
$D_4(1-\omega)^2$	$2^8\cdot 3^8$	38394	70784	51840	80000	648	163840			17496	100800
$\mathfrak{M}\otimes\mathbb{A}_8$	$2^8\cdot 3^4\cdot 5\cdot 7$	41850	178560	48384	172800					19224	256320
$D_2(1+i)^4$	$2^{24}\cdot 3^2\cdot 5$	30720	126	65536	225	3712	2088	256	34272		
$\mathfrak{M}\otimes\mathbb{D}_4^2$	$2^{15}\cdot 3^3$	42624	149184	54528	159750	420	201600	6	685440	13824	151200
$D_3(1-\omega)\perp\mathfrak{M}\otimes\mathbb{A}_5$	$2^8\cdot 3^6\cdot 5$	45360	150528	51840	144000					14904	154560
$\mathfrak{M}\otimes\mathbb{A}_4^2$	$2^8\cdot 3^3\cdot 5^2$	45450	814464	54144	812160	360	884736			12600	705600
$D_1(1+i)^8$	$2^{19}\cdot 3^2$	52224	34272	55296	30375	816	73440	24	514080	6144	12600

TABLE 4. Information about the graph $\Gamma_{\mathrm{iso}}(8)$. The organization is as in Tables 1 and 2, except that the numbers a_{ij} and b_{ij} in each cell are given side by side. The table is continued on the next page.

Remark 5.2. Up to isometry there are three extremal even unimodular $\mathbb{Z}$-lattices having a structure over $\mathfrak{M}$. The $\mathbb{Z}$-lattices corresponding to Λ''_{32} and BW_{32} are both isometric to the Barnes–Wall lattice of dimension 32. In [Koch and Venkov 1989] an invariant called the "Nachbardefekt" of an even unimodular lattice without roots is defined as the minimal corank of the root systems of its neighbor lattices. The 15 extremal even unimodular lattices of Nachbardefekt ≤ 8 are classified in [Koch and Venkov 1989; [1991]; Koch and Nebe 1993; Nebe 1990]. The $\mathbb{Z}$-lattice BW_{32} is isometric to one of the 5 lattices of Nachbardefekt 0. A comparison of the orders of the automorphism groups shows that the other two lattices Λ_{32} and Λ'_{32} are not isometric to one of these 15 lattices.

	$L_8(\mathfrak{P}^2)$ $2^{20}\cdot 3^2\cdot 5$	$L_8(\mathfrak{P}^4)$ $2^{22}\cdot 3^4$	$L_8(\mathfrak{P}^8)$ $2^{30}\cdot 3^2\cdot 7$	$E_8\perp\Lambda_{24}$ $2^{20}\cdot 3^4\cdot 5^3\cdot 7\cdot 13$	$E_8\perp L_6(\mathfrak{P}^6)$ $2^{28}\cdot 3^4\cdot 5^2$	E_8^4 $2^{31}\cdot 3^5\cdot 5^4$
$\mathfrak{M}^8$			65610 1		40824 5	8505 625
$\mathfrak{M}^4\perp BW_{16}$		77760 1	21870 56		14580 300	729 9000
BW_{16}^2			36450 112			729 10800
$\mathfrak{M}^3\perp R_{20}$		106920 16			8019 1920	
$\mathfrak{M}^2\perp R_{24}$	61236 64	51030 384		972 20800	1701 20480	
$\mathfrak{M}^2\perp J'_{24}$	72576 21	34560 72	4860 2016	1152 6825	1746 5820	45 90000
$\mathfrak{M}\perp R_{28}$	36288 2048	5670 2304	405 32768			
$\mathfrak{M}\perp R'_{28}$	20736 640	11016 2448	162 7168		81 28800	
$\mathfrak{M}\perp R''_{28}$	30618 8192	8505 16384				
$\mathfrak{M}\otimes\mathbb{E}_8$	45360 256		2025 16384			
$D_8(1+i)$	43008 4		13890 1852		336 360	105 67500
$D_8(1-\omega)$		8505 16384				
$D_4(1+i)^2$	18432 240	10368 972	1170 21840		288 43200	9 810000
S_1^2	31104 1280		243 14336			
$\mathfrak{M}\otimes\mathbb{D}_8$	10416 23808	1680 27648	105 344064			
$D_3(1+i)\perp\mathfrak{M}\underset{\mathbb{Z}[\omega]}{\otimes}U_5$	3888 1536	7290 20736			81 368640	
$D_2(1+i)\perp\mathfrak{M}\otimes\mathbb{D}_6$	5460 29120	1710 65664	30 229376	12 1310400	15 921600	
$D_4(1-\omega)^2$	5832 163840	729 147456				
$\mathfrak{M}\otimes\mathbb{A}_8$	4536 294912	945 442368				
$D_2(1+i)^4$	12480 780	1920 864	210 18816	64 81900	40 28800	1 432000
$\mathfrak{M}\otimes\mathbb{D}_4^2$	3312 176640	216 82944	9 688128			
$D_3(1-\omega)\perp\mathfrak{M}\otimes\mathbb{A}_5$	2430 122880	405 147456				
$\mathfrak{M}\otimes\mathbb{A}_4^2$	2160 589824	225 442368				
$D_1(1+i)^8$	384 3840	48 3456	3 43008			

TABLE 4 (continued). Information about the graph $\Gamma_{\mathrm{iso}}(8)$.

Theorem 5.3. *There are* 24 *isometry classes of* $\mathfrak{M}$*-unimodular lattices of rank* 8, *fifteen of which consist of indecomposable lattices. These fifteen lattices may be distinguished via their Hermitian root system which is in all cases of full rank. In particular, there is no extremal* 2*-modular integral lattice of dimension* 32 *having a structure as an* $\mathfrak{M}$*-unimodular lattice.*

A description of the $\mathfrak{M}$-unimodular lattices of dimension 32 may be obtained using their Hermitian root systems as given in Section 3, and is encoded in Tables 3 and 4.

The method used to find representatives for the isometry classes of the lattices in the two genera of $\mathfrak{M}$-unimodular and $\mathfrak{P}$-modular lattices can be described as follows:

(1) Starting with decomposable $\mathfrak{M}$-unimodular lattices M, we calculate the orbits of $U(M)$ on the 114939 maximal isotropic subspaces of the Hermitian $\mathbb{F}_4$ vector space $M/\mathfrak{P}M$ and the corresponding $\mathfrak{P}$-modular lattices L as full preimages of representatives of the orbits.
(2) For the lattices L found in (1) we check whether L is already known. If not, we determine $U(L)$ with a computer program described in [Plesken and Souvignier $\geq$ 1997].
(3) For the lattices L found in (1) we compute the number of sublattices $M' \leq L^*$, which are isometric to M using Equation (2.1) on page 154.
(4) When all known $\mathfrak{M}$-unimodular lattices M are processed in this way, we look for new $\mathfrak{M}$-unimodular lattices as full preimages of maximal isotropic subspaces of L^*/L, where L is one of the known $\mathfrak{P}$-modular lattices.

Remark 5.4. For the computation of $U(L)$ in (2) it is helpful to know a subgroup of $U(L)$ which is obtained computing some elements of $U(M)$ stabilizing the maximal isotropic subspace $L/\mathfrak{P}M$. An analogous remark applies to (4).

Remark 5.5. To check whether L is already known, it suffices in most cases to compute the number of roots in L. To prove the completeness of the list of $\mathfrak{P}$-modular and $\mathfrak{M}$-unimodular lattices we use the mass formula. An analogous remark applies to (4).

Remark 5.6. Since one knows a priori the number of maximal isotropic subspaces of L^*/L yielding known $\mathfrak{M}$-unimodular lattices by step (3), one can choose L such that one has a good chance to find new $\mathfrak{M}$-unimodular lattices.

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