Hausdorff Convergence and the Limit Shape of the Unicorns

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We discuss the Hausdorff convergence of hyperbolic components in parameter space as a one-parameter family of transcendental functions is dynamically approximated by polynomials. This convergence is strongly suggested by computer experiments and is proved in a weaker form, which is illustrated with exponential, sine and cosine families. Furthermore, we consider the convergence of subhyperbolic components. Our result also applies to the antiholomorphic exponentials, which allows us to investigate the limit shape of the unicorns.

1. INTRODUCTION

Consider a family of entire functions depending on a complex parameter. By iteration it gives rise to a family of dynamical systems. The question is what can be said about the dynamical systems of the family and of their dependence on the parameter. Of particular interest are how the Fatou or stable set, where the dynamics is "well-behaved", and the Julia set, where the dynamics is "chaotic", depend on the parameter. Suppose now that an entire *transcendental* family is approximated by a sequence of families of *polynomials* (of increasing degree) locally uniformly on compact sets. The singular values, that is, values where one cannot define a local inverse, are of paramount importance for the dynamics. To make "dynamical sense" we ask all families to have the same finite number of (free) singular values, in which case we speak of *dynamical approximation*. Since it is generally easier to study polynomials rather than transcendental functions, the question is: What can be said about the transcendental family as a dynamical system

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by looking at the polynomial families as the degree tends to infinity?

This type of convergence question has first been addressed in [Devaney et al. 1986], which studied the approximation of the exponentials $E(\lambda, z) = \lambda\, e^z$ by

$$P_d(\lambda, z) = \lambda \left(1 + \frac{z}{d}\right)^d.$$

This example gives a nice connection between the connectedness locus of P_2, the well-known Mandelbrot set, and the parameter plane of E. It is shown there that certain external rays of the connectedness loci of the polynomials P_d converge to "hairs" in the parameter plane of the exponentials E, and that hyperbolic components converge pointwise. Similar results are established in [Fagella 1995] for the family $\mathcal{G}(\lambda, z) = \lambda\, z\, e^z$ approximated by

$$\mathcal{Q}_d(\lambda, z) = \lambda\, z \left(1 + \frac{z}{d}\right)^d,$$

which shows up in the study of the complex standard family. The strongest result up to now in the parameter plane is the convergence of hyperbolic components as kernels in the sense of Carathéodory in [Krauskopf and Kriete 1995b]. In the *dynamical plane* the Hausdorff convergence of the Julia sets, for suitable values of the parameter, is proved in [Krauskopf 1993] for the example from [Devaney et al. 1986]. This result has been extended to the dynamical approximation of entire transcendental functions in [Kisaka 1995; Krauskopf and Kriete 1996] and of meromorphic functions in [Krauskopf and Kriete 1995a; Kriete 1995].

In this paper we are concerned with the convergence of hyperbolic components *in the parameter plane* in the strongest sense, that is, with respect to the Hausdorff metric, which is strongly suggested by computer experiments for several examples. For families with one free singular value we consider the m-layers, defined as the set of hyperbolic components for which the free singular value converges to a cycle of minimal period m. Under a certain compactness condition we prove Hausdorff convergence of m-layers. In fact we conjecture that the compactness condition can be dropped. Furthermore we conjecture that for some of our examples the hyperbolic components themselves converge in the Hausdorff metric, which is false in general.

For illustration we use the example from [Devaney et al. 1986] and complex sine and cosine families, dynamically approximated using Chebyshev polynomials. We adapt our results to subhyperbolic components and discuss the example in [Fagella 1995]. Finally we investigate the family of antiholomorphic exponentials $\bar{E}(\lambda, z) = \lambda e^{\bar{z}}$ approximated by

$$\bar{P}_d(\lambda, z) = \lambda\left(1 + \frac{\bar{z}}{d}\right)^d.$$

The connectedness loci of the $\bar{P}_d$ are called *unicorns* due to their shapes and their limit shape is provided by the parameter plane of $\bar{E}$.

This paper is organized as follows. In the next section we recall some basic notions from iteration theory. Hyperbolic components and m-layers are introduced in Section 3. The Main Theorem is stated in Section 4 and illustrated with a number of examples in Sections 5–7. The proof can be found in Section 8. The compactness condition that appears in the statement of the Main Theorem is discussed in Section 9 for our examples. Section 10 gives information on the algorithms used to compute the figures.

2. FACTS FROM ITERATION THEORY

In this section we briefly recall some notations and basic facts from iteration theory as they can be found for example in the surveys [Bergweiler 1993; Baker 1988; Erëmenko and Lyubich 1992; Milnor 1990] or the monographs [Beardon 1991; Carleson and Gamelin 1993; Steinmetz 1993]. Consider an entire function $f : \mathbb{C} \to \mathbb{C}$, that is, a function holomorphic on the whole of $\mathbb{C}$. Then f is either a polynomial or an entire transcendental function with an essential singularity at ∞. In the sequel we consider f as a map from the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to itself, which is perfectly justified for polynomials and a convention for transcendental functions. Let $f^{\circ n}$ denote the n-th iterate of f. For a given point $\zeta \in \widehat{\mathbb{C}}$ we call

$$O^+(\zeta) := \{f^{\circ n}(\zeta) : n \in \mathbb{N}\}$$

the (*forward*) *orbit* of ζ. The point ζ is called *m-periodic* if $f^{\circ m}(\zeta) = \zeta$, and m is called the *minimal period* if it is minimal with this property. If ζ is m-periodic its orbit $C := O^+(\zeta)$ is also called a *cycle*. Of importance is the *multiplier* of C, defined as $M(C) = (f^{\circ m})'(\zeta)$. A cycle C is called attracting (respectively repelling) if its multiplier is smaller (respectively larger) than one in absolute value. It is called indifferent if $|M(C)| = 1$. Every attracting cycle C comes with a *basin* $A(C)$, defined as the set of points $z \in \widehat{\mathbb{C}}$ whose orbits accumulate on C under iteration. In other words, $A(C) = \{z \in \mathbb{C} : \omega(z) = C\}$, where $\omega(z)$ is the set of all accumulation points of the sequence of iterates $\{f^{\circ n}(z)\}$.

The key problem is to get a global picture of the dynamics of f on $\widehat{\mathbb{C}}$, which is why one studies the following sets. The *Fatou set* $F(f)$ is defined as the set of points $\zeta \in \widehat{\mathbb{C}}$, such that on some neighborhood $V \subset \widehat{\mathbb{C}}$ of ζ the iterates $f^{\circ n}|_V$ are holomorphic and form a normal family. This means that in any sequence of these iterates one can find a subsequence that converges uniformly on compact sets to a limit, so that the Fatou set is the set of "well-behaved" points. Clearly any basin $A(C)$ is a subset of the Fatou set. The *Julia set* $J(f)$ is the complement of $F(f)$ in $\widehat{\mathbb{C}}$, and it contains the "chaotic" points. It is a well-known theorem that the repelling periodic points of f are dense in $J(f)$, which provides an alternative way of defining these two sets. Note that we regard both $F(f)$ and $J(f)$ as subsets of $\widehat{\mathbb{C}}$ even if f is transcendental.

Of great importance are the *singular values* of f, where one cannot define a local inverse of f. The set $\text{sing}(f)$ of all *finite* singular values of f consists of the *finite critical values* of f and, if f is transcendental, also of the *finite asymptotic values*. An important fact we use much in the sequel is that each basin of an attracting periodic orbit contains at least one singular value. We call f *hyperbolic* if $O^+(\text{sing}(f)) \Subset F(f)$, that is, if the complete forward orbit of all singular values is relatively compact in the Fatou set. Note that in particular the singular values cannot accumulate on the Julia set if f is hyperbolic.

3. HYPERBOLIC COMPONENTS AND m-LAYERS

From now on we study families of entire maps depending holomorphically on a parameter λ from $\mathbb{C}^L$, or an arbitrary complex manifold of any finite dimension for that matter. That is, we consider a holomorphic function $\mathcal{F} : \mathbb{C}^L \times \mathbb{C}\,; (\lambda, z) \mapsto \mathcal{F}(\lambda, z)$, viewed as a family of entire functions $\mathcal{F}(\lambda, \cdot)$. We say that a family $\mathcal{F}$ has *one free singular value* if there is a singular value $c(\lambda)$ depending holomorphically on λ and such that $\mathcal{F}(\lambda, \cdot)$ is hyperbolic if and only if $O^+(c(\lambda)) \Subset F(\mathcal{F}(\lambda, \cdot))$. This means that all singular values other than $c(\lambda)$ are not important as far as hyperbolicity goes in the following sense. Singular values other than $c(\lambda)$ are either absorbed by attracting cycles or have the same dynamics as the free singular values, for example for symmetry reasons.

In this paper we restrict our attention to families of finite type with one free singular value. In other words, each family has only finitely many singular values, of which only one needs to be considered to check for hyperbolicity; see also the examples below. In fact a family with one free singular value is essentially a one-parameter family of entire maps, which is why we take the parameter space to be $\mathbb{C}$ in the sequel. We now define the set

$$\mathcal{H}(\mathcal{F}) := \{\lambda \in \mathbb{C} : \mathcal{F}(\lambda, \cdot) \text{ is hyperbolic}\},$$

which is open because attracting cycles persist under changes of λ; compare [Krauskopf and Kriete 1995b]. A connected component of $\mathcal{H}(\mathcal{F})$ is called a *hyperbolic component*. (We assume that $\mathcal{H}(\mathcal{F})$ is nonempty for all families considered here.)

Because there is only one free singular value $c(\lambda)$, we can define the sets

$$\mathcal{H}_m(\mathcal{F}) := \left\{ \begin{aligned} \lambda \in \mathbb{C} : c(\lambda) \text{ is in the basin of a} \\ \text{minimal } m\text{-cycle of } \mathcal{F}(\lambda,\cdot) \end{aligned} \right\},$$

which we call *m-layers*. We get a natural partition of $\mathcal{H}$ into m-layers, that is, $\mathcal{H}_i(\mathcal{F}) \cap \mathcal{H}_j(\mathcal{F}) = \varnothing$ for $i \neq j$ and $\mathcal{H}(\mathcal{F}) = \dot{\bigcup}_{m\in\mathbb{N}} \mathcal{H}_m(\mathcal{F})$. The m-layers themselves consist typically of infinitely many hyperbolic components.

4. MAIN RESULT

The setting of the Main Theorem is the following. We consider a limit family $\mathcal{F}_\infty$ of entire transcendental functions of *finite type* and a family of polynomials $\mathcal{F}_d$, converging to $\mathcal{F}_\infty$ uniformly on compact sets. Clearly the $\mathcal{F}_d$ must be of increasing degree in d. To get control over the dynamics we ask that $\mathcal{F}_\infty$ and $\mathcal{F}_d$ have the same (finite) number of singular values, independently of d and λ, which is what we call *dynamical approximation.* Furthermore, we suppose that $\mathcal{F}_\infty$ and $\mathcal{F}_d$ have one free singular value $c_\infty = c_\infty(\lambda)$ and $c_d = c_d(\lambda)$, respectively.

We are interested in convergence with respect to the Hausdorff metric. Recall that for two closed sets $A, B \in \widehat{\mathbb{C}}$ the Hausdorff distance is defined as

$$h(A,B) := \inf\{\varepsilon > 0 : A \subset V_\varepsilon(B) \text{ and } B \subset V_\varepsilon(A)\},$$

where $V_\varepsilon(X)$ denotes the ε-neighborhood of a closed set $X \in \widehat{\mathbb{C}}$ with respect to the chordal metric on $\widehat{\mathbb{C}}$. When we talk of the Hausdorff convergence of hyperbolic components or m-layers, which are open sets, we always mean the convergence of the respective boundaries with respect to $\widehat{\mathbb{C}}$. For the statement of our result we need the following technical **C**ompactness property.

Definition 4.1. We say that the approximating families $\mathcal{F}_d$ have *Property C* if the following holds.

Consider a λ from a given compact set $K \subset \mathbb{C}$, such that $c_d(\lambda)$ is attracted by some m-cycle $C_d(\lambda)$ of $\mathcal{F}_d(\lambda,\cdot)$ for almost all $d \in \mathbb{N}$. Then the limit $C_\infty(\lambda)$ of the $C_d(\lambda)$ exists and lies in a disk around the origin with radius R, where R depends only on K and m.

Theorem 4.2 (Main Theorem). *Let $\mathcal{F}_\infty$ be a family of entire transcendental functions that is dynamically approximated by families of polynomials $\mathcal{F}_d$. Suppose that all families have one free singular value and that the approximating families have Property C. Then for every m-layer $\mathcal{H}_m(\mathcal{F}_\infty)$ of $\mathcal{F}_\infty$ there exists a sequence $\{\mathcal{H}_m(\mathcal{F}_d)\}$ of m-layers of $\mathcal{F}_d$ converging to $\mathcal{H}_m(\mathcal{F}_\infty)$ with respect to the Hausdorff metric as d tends to infinity.*

The purpose of Property C is to avoid the following problem. Clearly each attracting cycle $C_\infty(\lambda)$ of $\mathcal{F}_\infty(\lambda,\cdot)$ is approximated by attracting cycles $C_d(\lambda)$ of $\mathcal{F}_d(\lambda,\cdot)$. However, it is not at all clear that the limit $C_\infty(\lambda)$ of attracting cycles $C_d(\lambda)$ of $\mathcal{F}_d(\lambda,\cdot)$ is finite. It may happen that $\infty \in C_\infty(\lambda)$. This would cause great difficulties since $\mathcal{F}_\infty$ is transcendental and has ∞ as an essential singularity. However, we do not know of an example of this. In Section 9 we discuss a method of checking Property C. We conjecture that dynamical convergence is such a strong notion that Property C can be dropped from the assumptions of the Main Theorem.

We have stated the Main Theorem for entire functions of *finite type* for two reasons. First, in the proof we use the result in [Krauskopf and Kriete 1995b] on the kernel convergence of hyperbolic components, which is stated and proved for functions of finite type. Second, all the examples we present are of finite type. However, we remark that the proof in [Krauskopf and Kriete 1995b] can be generalized to the case of a limit family with infinitely many singular values that is dynamically approximated. Here this means that the limit family and the approximating families have a constant finite number of *free* singular values. As a consequence, the Main Theorem applies in this more general setting. The full proof of this is beyond the scope of this paper.

The next section illustrates the Main Theorem with a number of examples.

5. THREE EXAMPLES

Example 5.1 (Exponential Family). As mentioned in the introduction, the classic example of dynamical approximation is from [Devaney et al. 1986], the polynomials

$$P_d(\lambda, z) = \lambda \left(1 + \frac{z}{d}\right)^d$$

converging locally uniformly to

$$E(\lambda, z) = \lambda\, e^z.$$

The families E and P_d have the unique singular value 0 and the P_d have Property C; see Section 9. As a consequence of the Main Theorem, the m-layers $\mathcal{H}_m(P_d)$ converge in the Hausdorff metric to the m-layer $\mathcal{H}_m(E)$, as is illustrated in Figure 1.

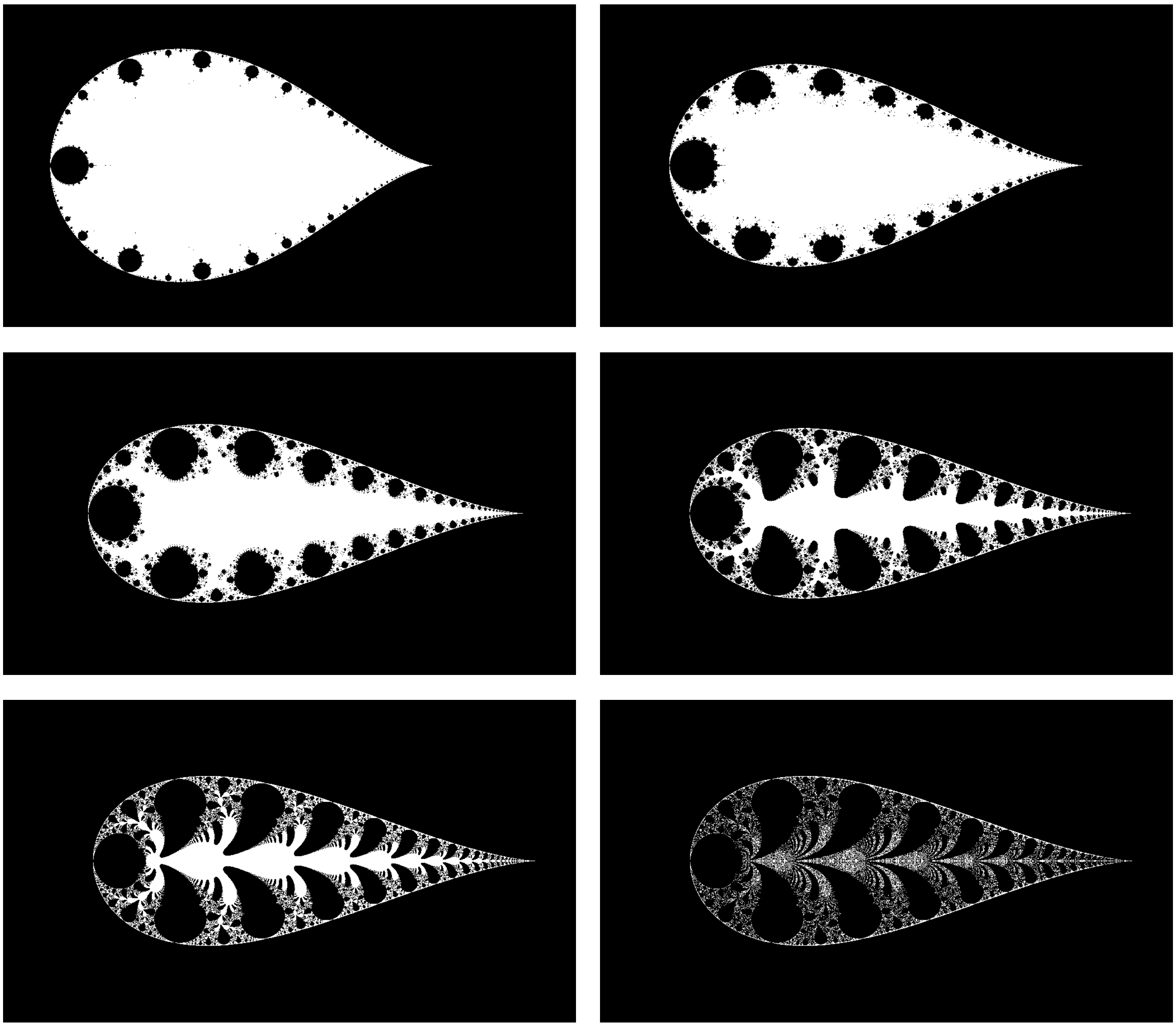

FIGURE 1. The hyperbolic components of P_2, P_4, P_{16}, P_{256}, P_{65536} and E from Example 1. All figures in this paper depict a chart in the neighborhood of the point ∞, which can be found in the middle of the pictures. For more information on how the figures were computed, see Section 10.

Consider the *Mandelbrot sets*

$$M_d := \{c \in \mathbb{C} : \{P_d^{\circ n}(0)\} \text{ stays bounded}\},$$

which are in fact the connectedness loci of the P_d, that is, the sets for which $J(P_d(\lambda, \cdot))$ is connected. The Mandelbrot sets M_d are compact and, consequently, they contain the closure of all hyperbolic components inside them. Our result implies that this closure converges to the closure of $\mathcal{H}(E)$. This is a statement about the limit of the Mandelbrot sets M_d as the degree d goes to infinity.

It is generally believed that each Mandelbrot set M_d is the closure of the hyperbolic components it contains; this is known as the Generic Hyperbolicity Conjecture. For the limit, the conjecture is that $\mathcal{H}(E)$ is dense in

$$M_\infty := \{c \in \mathbb{C} : \{E^{\circ n}(0)\} \text{ stays bounded}\},$$

which in turn is conjecturally dense in $\widehat{\mathbb{C}}$. If this is indeed true, then the closure of the limit of the Mandelbrot sets M_d, as d goes to infinity, is $\widehat{\mathbb{C}}$.

Example 5.2 (Approximation by Chebyshev polynomials). It is well-known that the Chebyshev polynomials provide an excellent approximation of the cosine and sine functions; see for example [Rivlin 1974]. Recall that these polynomials can be defined recursively by

$$\begin{aligned} T_0(z) &= 1, \\ T_1(z) &= z, \\ T_d(z) &= 2\,z\,T_{d-1}(z) - T_{d-2}(z). \end{aligned}$$

For our purposes it is important that $T_{4d}(z/(4d))$ and $T_{4d+1}(z/(4d+1))$ converge locally uniformly and dynamically to $\cos(z)$ and $\sin(z)$, respectively. Consider now the families

$$\mathcal{S}_d(\lambda, z) = \lambda T_{4d+1}\left(\frac{z}{4d+1}\right)$$

that dynamically approximate the family

$$\mathcal{S}_\infty(\lambda, z) = \lambda \sin(z).$$

These families have the singular values $\pm\lambda$. Since the functions are odd the orbit of $-\lambda$ is the negative of the orbit of λ, so that both singular values behave in the same way. Consequently, to check for hyperbolicity it is sufficient to consider the orbit of, say, the singular value λ, which we call the free singular value. The convergence of the $\mathcal{H}_m(\mathcal{S}_d)$ to $\mathcal{H}_m(\mathcal{S}_\infty)$ appears to be very rapid and is illustrated in Figure 2.

In the same way $\lambda\cos(z)$ can be approximated, but here we consider a slightly different example, namely the family

$$\mathcal{C}_\infty(\lambda, z) = \lambda(\cos(z) - 1),$$

which is dynamically approximated by

$$\mathcal{C}_d(\lambda, z) = \lambda\left(T_{4d}\left(\frac{z}{4d}\right) - 1\right).$$

For these families the critical value 0 is an attracting fixed point for any λ, so that the second critical value -2λ is the free singular value. Again, the convergence of $\mathcal{H}(\mathcal{C}_d)$ to $\mathcal{H}(\mathcal{C}_\infty)$ seems to be very rapid, as shown in Figure 3.

We have not been able to show that our examples involving the Chebyshev polynomials have Property C; compare Section 9. Nonetheless, Figures 2 and 3 suggest that even the hyperbolic components themselves converge.

Figures 1–3 give the impression that the hyperbolic components themselves converge in the Hausdorff metric. We conjecture that this is true for the above families and for many other examples, but in general this is false, as we show now.

Counterexample 5.3. The polynomial counterexample in [Krauskopf and Kriete 1995b, Example 2] can be adapted to this setting. Here we present an (even worse) example, where a single hyperbolic component is pinched into infinitely many hyperbolic components in the limit, with ∞ as their common boundary point.

Consider the families

$$\mathcal{F}_d(\lambda, z) = \left(1 - \frac{1}{d} + \frac{1}{2}\left(1 + \frac{\lambda}{d}\right)^d\right)\left(\left(1 + \frac{z}{d}\right)^d - 1\right)$$

and

$$\mathcal{F}_\infty(\lambda, z) = \left(1 + \tfrac{1}{2}e^\lambda\right)(e^z - 1).$$

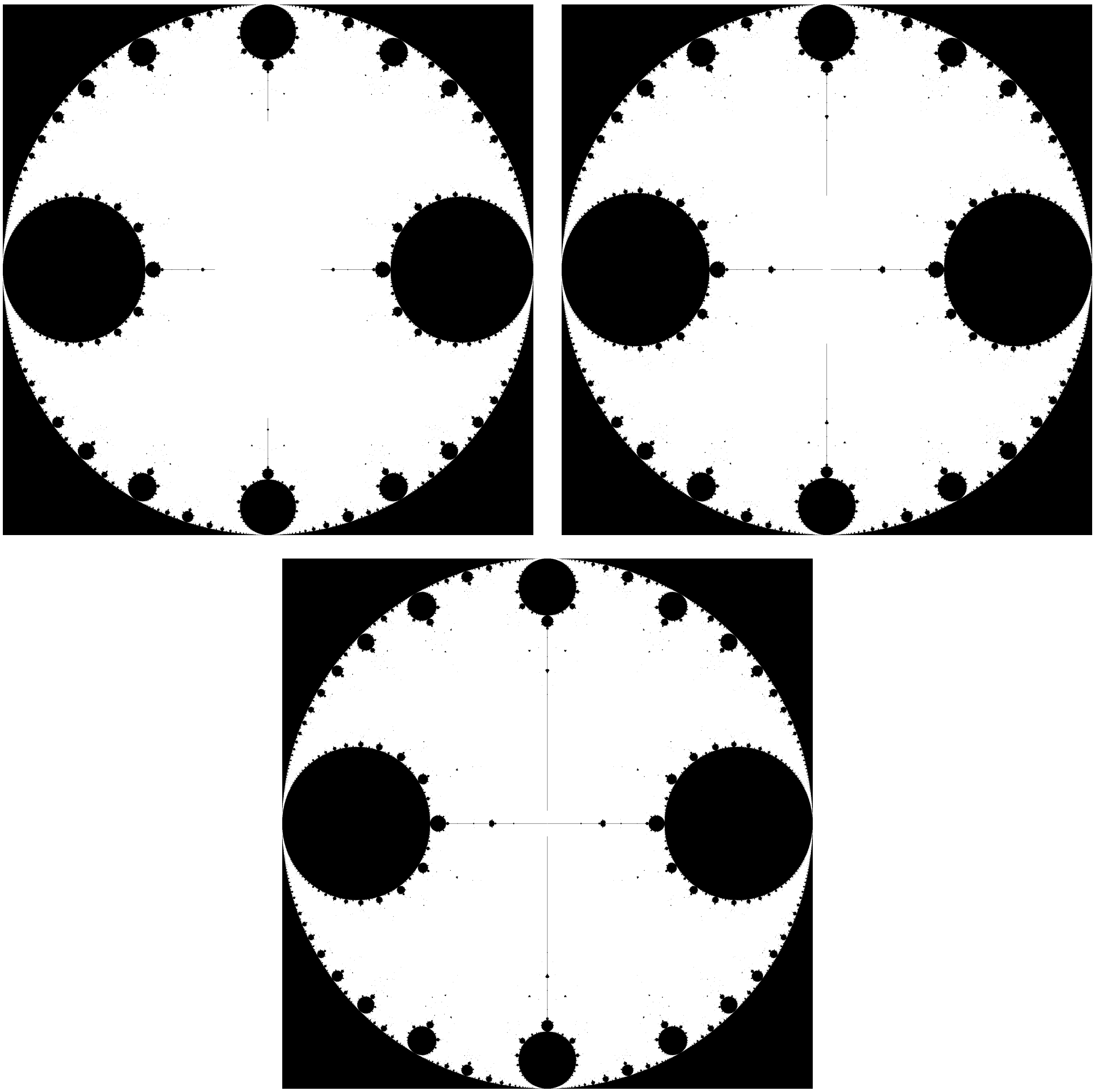

FIGURE 2. The hyperbolic components of $\mathcal{S}_1$, $\mathcal{S}_{16}$ and $\mathcal{S}_\infty$ from Example 5.2.

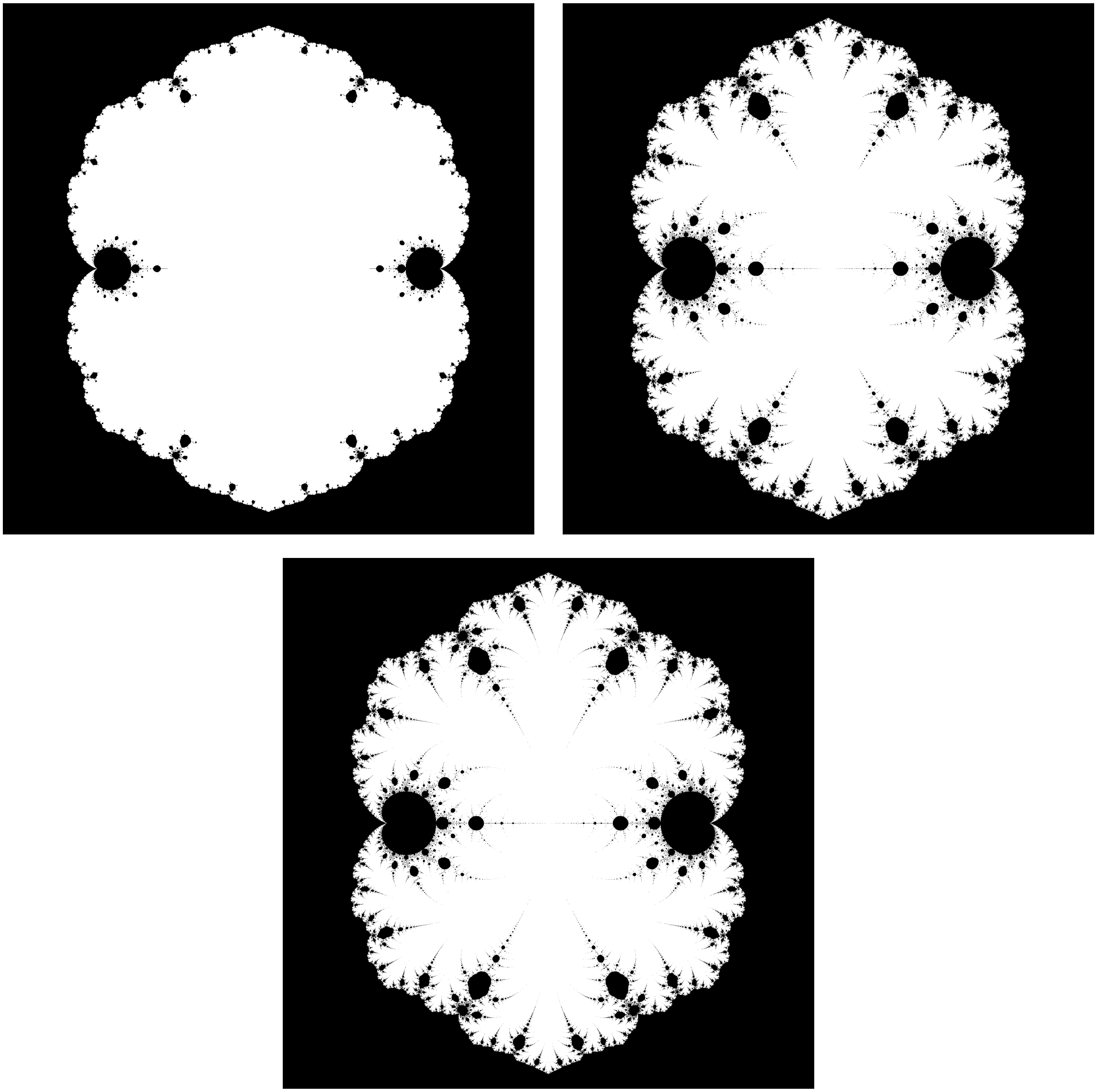

FIGURE 3. The hyperbolic components of $\mathcal{C}_1$, $\mathcal{C}_{16}$ and $\mathcal{C}_\infty$ from Example 5.2.

It follows from the convergence of $(1+z/d)^d$ to e^z that the $\mathcal{F}_d$ converge locally uniformly and dynamically to $\mathcal{F}_\infty$. In fact, the function $\mathcal{F}_\infty(\lambda,\cdot)$ has no critical value and the single asymptotic value $-\left(1+\frac{1}{2}e^\lambda\right)$. The function $\mathcal{F}_d(\lambda,\cdot)$ has the single critical value

$$-\left(1-\frac{1}{d}+\frac{1}{2}\left(1+\lambda/d\right)^d\right).$$

The origin is a fixed point for the $\mathcal{F}_d$ and it is attracting if its multiplier is smaller than one in absolute value, in which case the unique singular value is attracted to the origin. The curve given by

$$|M_d(0)| = \left|\left(1-\frac{1}{d}+\frac{1}{2}\left(1+\frac{\lambda}{d}\right)^d\right)\right| = 1$$

is the boundary of a component of $\mathcal{H}_1(\mathcal{F}_d)$. This component gets pinched near the critical point $-d$ as the degree d is increased; see Figure 4. In a similar fashion the curve defined by

$$|M_\infty(0)| = \left|\left(1+\tfrac{1}{2}e^\lambda\right)\right| = 1$$

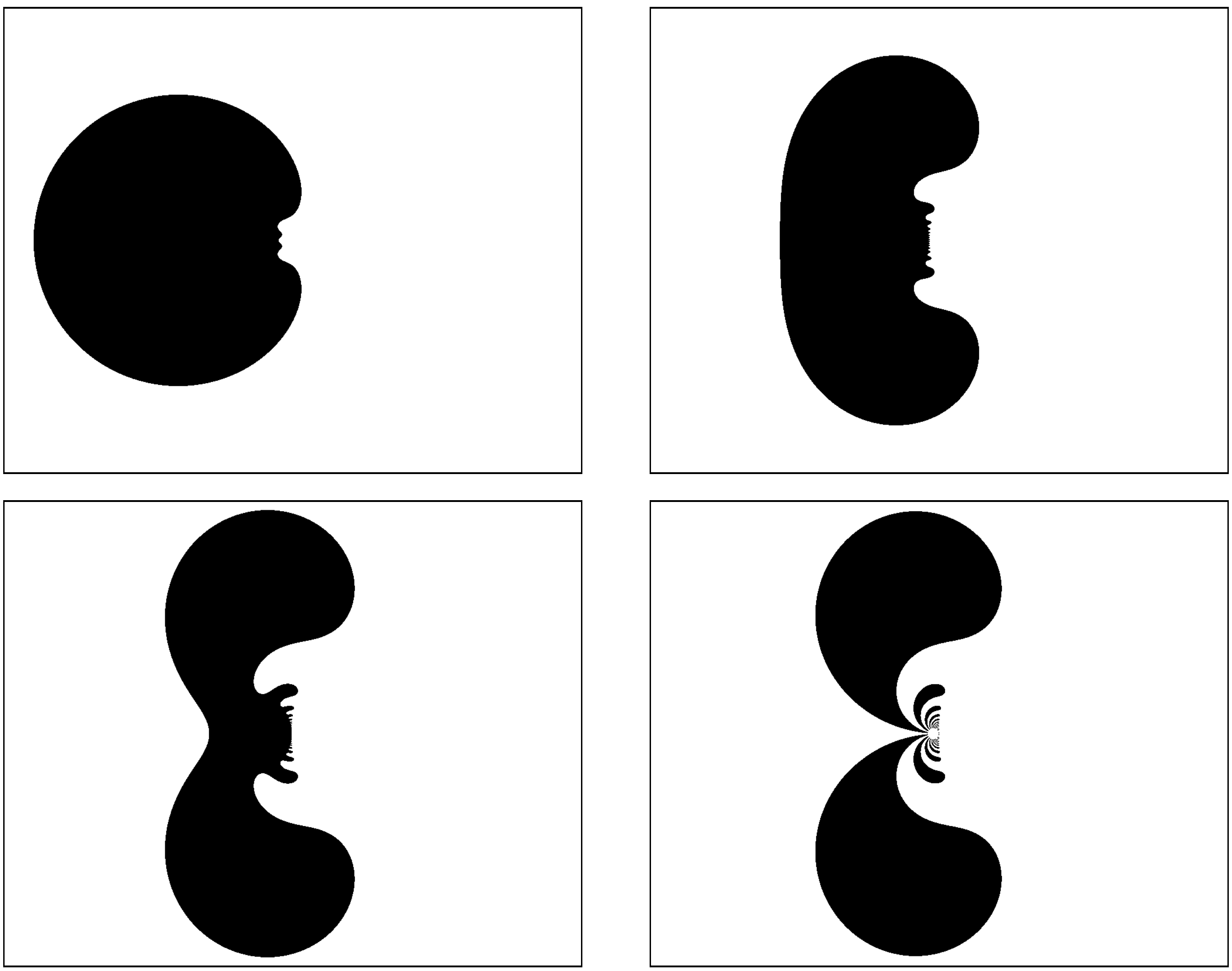

FIGURE 4. The hyperbolic components of $\mathcal{F}_4$, $\mathcal{F}_{16}$, $\mathcal{F}_{256}$ and $\mathcal{F}_\infty$ from Example 5.3.

is the boundary of components of $\mathcal{H}_1(\mathcal{F}_\infty)$. But this curve bounds infinitely many hyperbolic components that have ∞ as a common point on their boundary; see Figure 4 (lower right).

Consequently the above hyperbolic component of $\mathcal{H}_1(\mathcal{F}_d)$ will never lie in an ε-neighborhood of any of the infinitely many components of $\mathcal{H}_1(\mathcal{F}_\infty)$, no matter how big d is. This shows that hyperbolic components do *not* converge in the Hausdorff metric in general. However, the approximating family has Property C (see Section 9), and by the Main Theorem the 1-layers $\mathcal{H}_1(\mathcal{F}_d)$ converge to the 1-layer $\mathcal{H}_1(\mathcal{F}_\infty)$. Furthermore, the hyperbolic components do converge as kernels, according to Theorem 8.1 in Section 8.

In this counterexample the dependence on the parameter is the key. We conjecture that, if the families depend on the parameter "in a natural way", like it is the case for all our other examples, the hyperbolic components themselves converge with respect to the Hausdorff metric.

6. EXAMPLE: SUBHYPERBOLIC COMPONENTS

In this section we consider the example from [Fagella 1995], which shows up in the study of the complex standard family. The family

$$\mathcal{G}_\infty(\lambda, z) = \lambda z e^z$$

is approximated dynamically by the polynomial families

$$\mathcal{Q}_d(\lambda, z) = \lambda z \left(1 + \frac{z}{d}\right)^d .$$

The limit family $\mathcal{G}_\infty$ has two singular values: the asymptotic value 0 and the critical value $-\lambda/e$. For all $\lambda \in \mathbb{C}$ the singular value 0 is a fixed point with multiplier λ. As a consequence, $\mathcal{Q}_d(\lambda, z)$ can only have hyperbolic components for $|\lambda| < 1$. By definition, $\mathcal{G}_\infty(\lambda, \cdot)$ is *subhyperbolic* if each singular value is preperiodic or absorbed by an attracting cycle. We define the set

$$\mathcal{H}^s(\mathcal{G}_\infty) := \operatorname{Int}\left(\{\lambda \in \mathbb{C} : \mathcal{G}_\infty(\lambda, \cdot) \text{ is subhyperbolic}\}\right),$$

and see that $\lambda \in \mathcal{H}^s(\mathcal{G}_\infty)$ if and only if $c(\lambda) = -\lambda/e$ is attracted to an attracting cycle, so that we can consider $c(\lambda)$ as the free singular value. We call a component of $\mathcal{H}^s(\mathcal{G}_\infty)$ a *subhyperbolic component*. Like for hyperbolic components we can define the m-layers

$$\mathcal{H}^s_m(\mathcal{G}_\infty) := \left\{ \begin{array}{r} \lambda \in \mathbb{C} : c(\lambda) \text{ is in the basin of a} \\ \text{minimal } m\text{-cycle of } G_\infty(\lambda, \cdot) \end{array} \right\},$$

which gives the partition

$$\mathcal{H}^s(\mathcal{G}_\infty) = \dot{\bigcup_{m \in \mathbb{N}}} \mathcal{H}^s_m(\mathcal{G}_\infty) .$$

The functions $\mathcal{Q}_d(\lambda, \cdot)$ also have two singular values, namely the critical value 0, which is again a fixed point, and the (free) critical value

$$c_d(\lambda) := -\lambda\left(1 - \frac{1}{d+1}\right)^{(d+1)} .$$

In complete analogy we define subhyperbolic components and the m-layers $\mathcal{H}^s_m(\mathcal{Q}_d)$ that partition the set $\mathcal{H}^s(\mathcal{Q}_d)$.

With these slight modifications in the definitions we can use the same arguments as given in Section 8 to obtain the following general result for m-layers of subhyperbolic components.

Theorem 6.1. *If the approximating families have Property C, then every m-layer $\mathcal{H}^s_m(\mathcal{F}_\infty)$ is the limit of a sequence of m-layers $\{\mathcal{H}^s_m(\mathcal{F}_d)\}$ as d tends to infinity.*

The estimate used in [Fagella 1995] to prove pointwise convergence of the hyperbolic components can be used to show Property C; compare Section 9. As a consequence each m-layer $\mathcal{H}^s(\mathcal{Q}_d)$ converges in the Hausdorff metric to $\mathcal{H}^s(\mathcal{G}_\infty)$, which is illustrated in Figures 5 and 6.

7. EXAMPLE: THE LIMIT SHAPE OF THE UNICORNS

In this section we apply our results to the families of anti-holomorphic maps

$$\bar{P}_d(\lambda, z) = \lambda \left(1 + \frac{\bar{z}}{d}\right)^d .$$

This family converges to

$$\bar{E}(\lambda, z) = \lambda e^{\bar{z}}.$$

We begin with some background information. Consider the anti-holomorphic families $A_d(z) := \bar{z}^d + c$, where c is a complex parameter, and define the sets

$$M_d^* := \{c \in \mathbb{C} : \{A_d^{\circ n}(0)\} \text{ stays bounded}\}.$$

These sets are the connectedness loci of the maps A_n and are called *multicorns* due to their shapes. Best known is the *tricorn* M_2^*, the anti-holomorphic counterpart of the Mandelbrot set M_2. The name tricorn was introduced by Milnor, who found it in the parameter space of cubic holomorphic maps; see [Milnor 1992]. It is known to be locally disconnected and has been studied in [Crowe et al. 1989; Winters 1990]. In [Nakane 1993] it is shown that the tricorn is in fact connected, so that external angles can be introduced in the same way as for the Mandelbrot set. In fact all multicorns are connected (but not locally connected) as is shown in [Nakane and Schleicher ≥ 1997]. The multicorn M_d^* is symmetric with respect to the group D_{d+1}, generated by the rotation over $2\pi/(d+1)$ around 0, and by the reflection in the real axes. By means of defining $\lambda := d\, c^{(d-1)}$ one can divide out the rotational symmetry, that is, the group $\mathbb{Z}_{d+1}$. As a result one gets the *unicorn*, the connectedness locus of the map $\bar{P}_d$, which can be defined as

$$U_d := \left\{c \in \mathbb{C} : \{\bar{P}_d^{\circ n}(0)\} \text{ stays bounded}\right\};$$

compare [Lau and Schleicher 1996].

We want to make some sense of the question: what is the limit shape of the unicorns? The answer is provided by the convergence of the $\bar{P}_d$ to $\bar{E}$, so that we need to consider the set

$$U_\infty := \left\{c \in \mathbb{C} : \{\bar{E}^{\circ n}(0)\} \text{ stays bounded}\right\}.$$

A first study of the parameter space of $\bar{E}$ can be found in [Baker and Rippon 1989]. In complete analogy to the holomorphic case of the functions P_d and E from Example 5.1 we can define the m-levels $\mathcal{H}_m(\bar{P}_d)$ of $\bar{P}_d$ and investigate how they converge

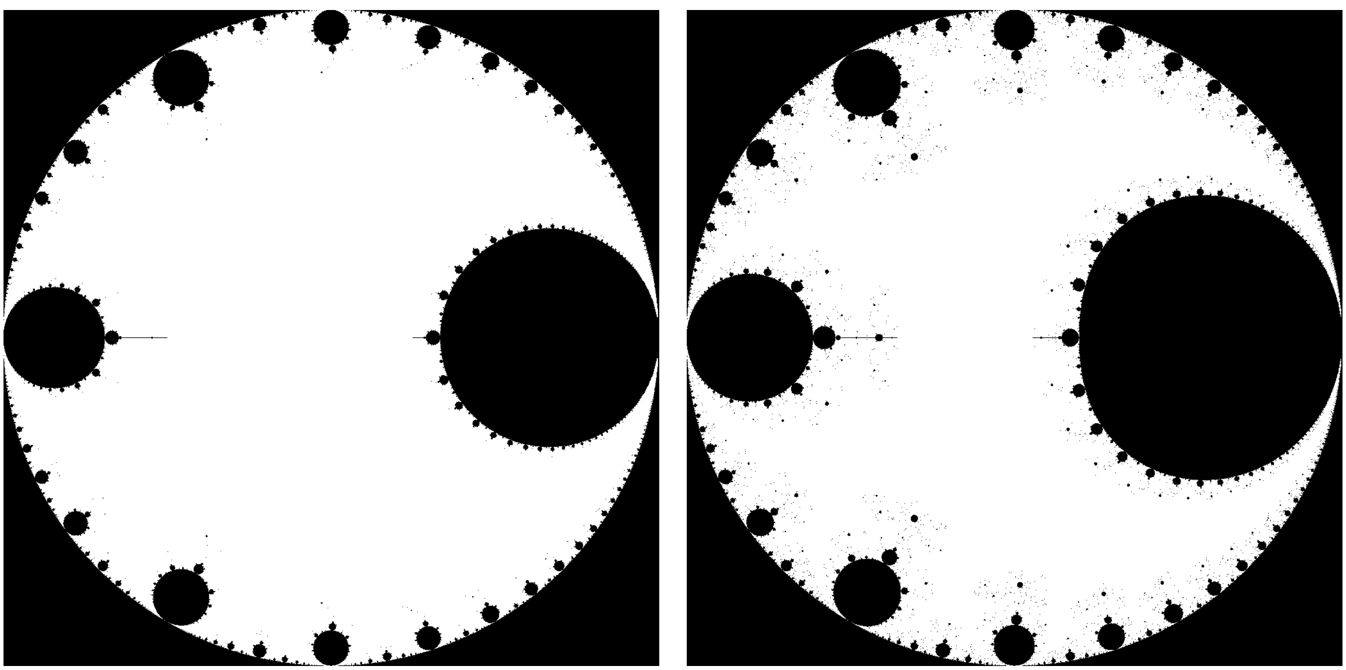

FIGURE 5. The hyperbolic components of $\mathcal{Q}_1$ and $\mathcal{Q}_4$ of Section 6. Continued in Figure 6.

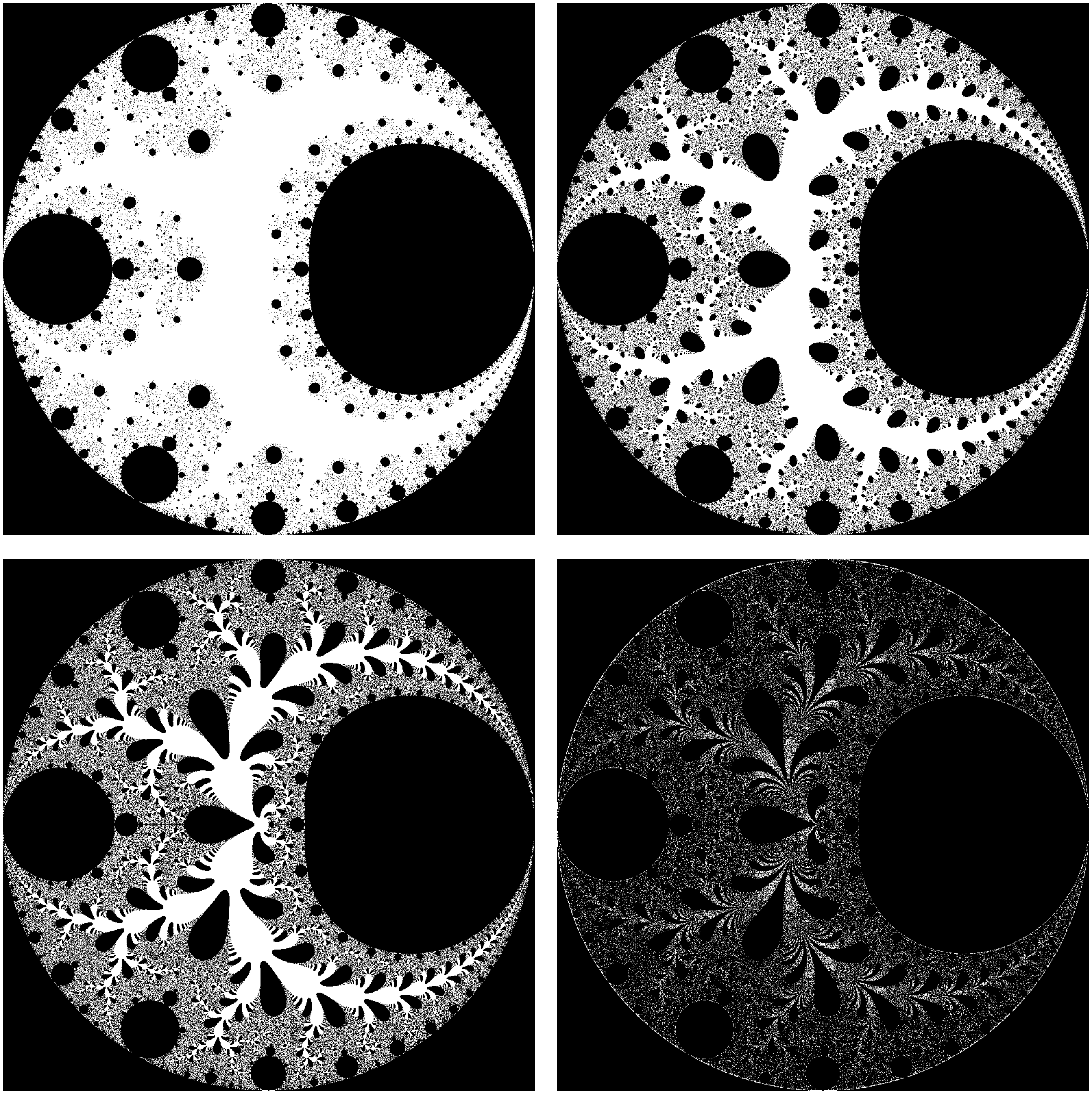

FIGURE 6. The hyperbolic components of $\mathcal{Q}_{16}$, $\mathcal{Q}_{256}$, $\mathcal{Q}_{65536}$ and $\mathcal{G}_\infty$ (see Section 6 and Figure 5).

to the m-levels $\mathcal{H}_m(\bar{E})$ of $\bar{E}$. Again, the Generic Hyperbolicity Conjecture asserts that

$$\mathcal{H}(\bar{P}_d) = \bigcup_{m\in\mathbb{N}} \mathcal{H}_m(\bar{P}_d)$$

is dense in U_d, and that $\mathcal{H}(\bar{E}) = \bigcup_{m\in\mathbb{N}} \mathcal{H}_m(\bar{E})$ is dense in U_∞. Furthermore, it is a conjecture that U_∞ is dense in $\mathbb{C}$. Hence, it makes sense to investigate the limit shape of the unicorns by looking at the convergence of m-layers; see Figures 7 and 8.

Theorem 7.1. *For each $m \in \mathbb{N}$ the m-levels of $\bar{P}_d$ converge to the m-levels of $\bar{E}$ in the Hausdorff metric as the degree d goes to infinity.*

Proof. The key lies in the well-known fact that the second iterate of an anti-holomorphic map is holomorphic. We study the holomorphic families

$$\begin{aligned}\mathcal{F}_d((\lambda,\mu),z) &= \left(P_d(\lambda,\cdot)\circ P_d(\mu,\cdot)\right)(z)\\ &= \lambda\left(1+\frac{\mu}{d}\left(1+\frac{z}{d}\right)^d\right)^d,\end{aligned}$$

converging uniformly on compact sets to the holomorphic family

$$\begin{aligned}\mathcal{F}_\infty((\lambda,\mu),z) &= \left(E(\lambda,\cdot)\circ E(\mu,\cdot)\right)(z)\\ &= \lambda\exp\left(\mu\exp(z)\right),\end{aligned}$$

where $(\lambda,\mu)\in\mathbb{C}^2$. We find the second iterate of the original functions after choosing $\mu=\bar{\lambda}$. Note that we get the second iterates of the families P_d and E from Example 5.1 by choosing $\mu=\lambda$. In other words, these two choices define one dimensional cross-sections through the two-dimensional (λ,μ)-spaces of the above families. The two sections intersect in the line of real λ-values, so that the unicorn U_d and the Mandelbrot set M_d coincide there. We restrict to the section

$$\mathcal{M} = \left\{(\lambda,\mu)\in\mathbb{C}^2 : \mu=\bar{\lambda}\right\},$$

which clearly is a complex manifold biholomorphically equivalent to $\mathbb{C}$. Note that, under iteration of $\mathcal{F}_\infty((\lambda,\bar{\lambda}),\cdot)$, the singular value $e^{\bar{\lambda}}$ converges to an attracting cycle exactly when λ does, even though the two orbits are disjoint. Consequently, we may

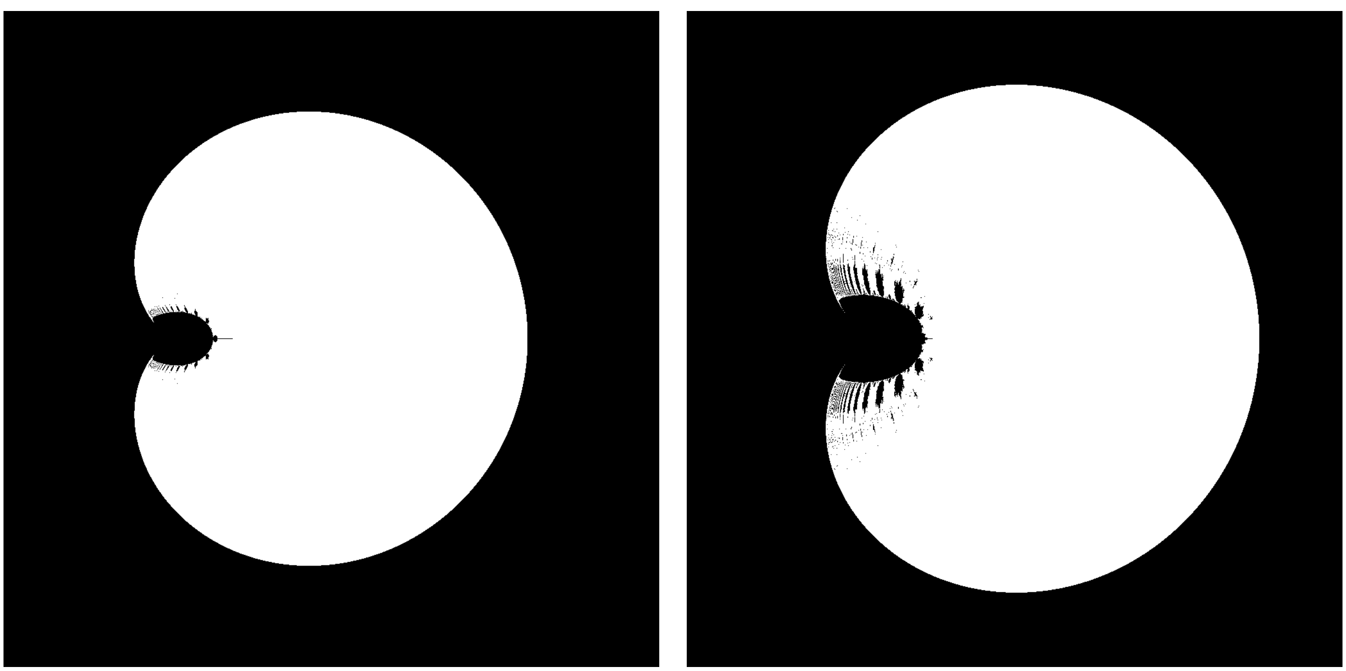

FIGURE 7. The hyperbolic components of $\bar{P}_2$ and $\bar{P}_4$ (see Section 7). Continued in Figure 8.

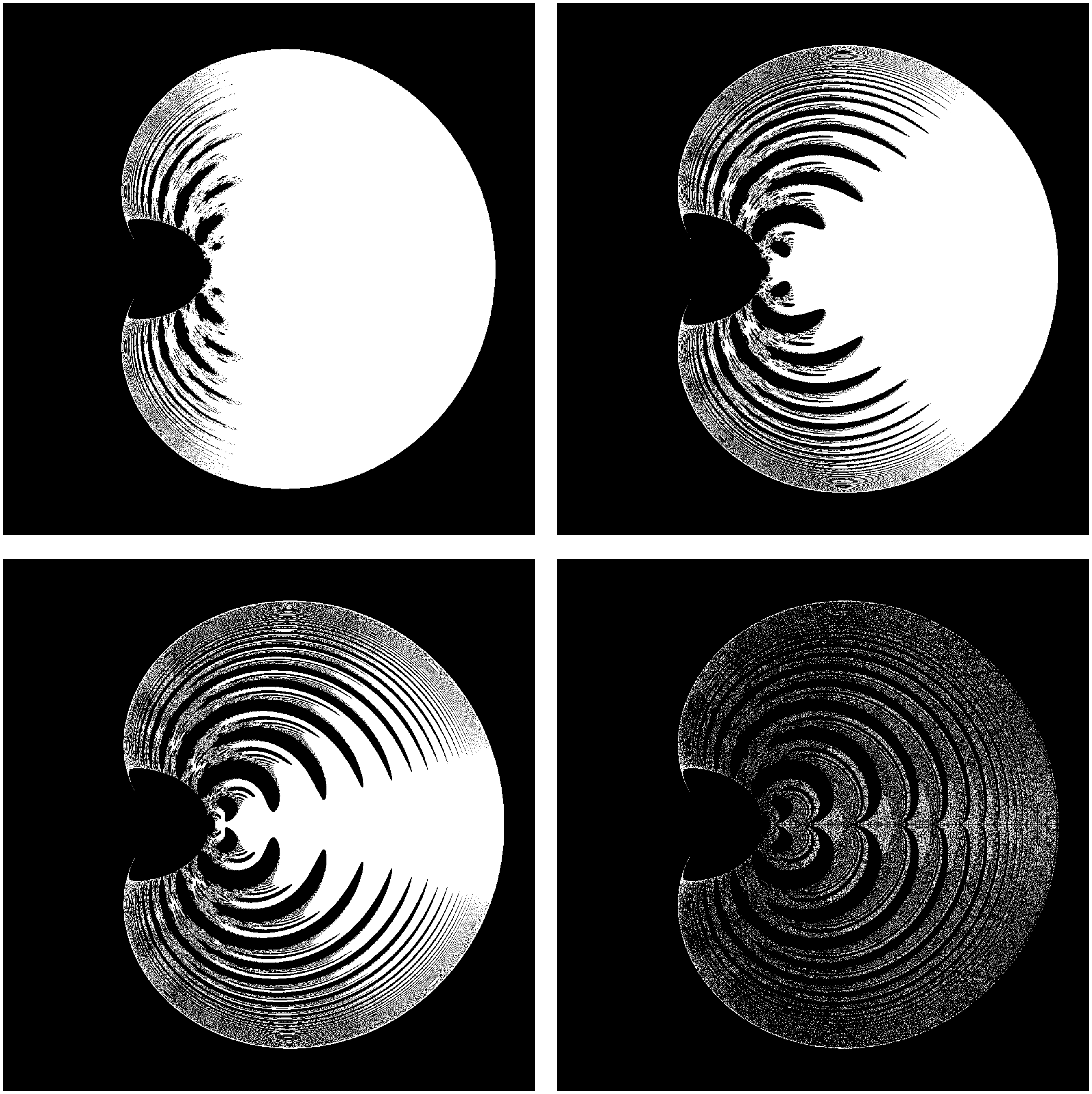

FIGURE 8. The hyperbolic components of $\bar{P}_{16}$, $\bar{P}_{256}$, $\bar{P}_{65536}$ and $\bar{E}$, illustrating the convergence of the unicorns (see Section 7 and Figure 7.)

consider λ as the free singular value of $\mathcal{F}_\infty((\lambda, \bar{\lambda}), \cdot)$. Because the P_d have Property C also the $\mathcal{F}_d$ do as is shown in Section 9. Consequently, m-layers of $\mathcal{F}_d$ converge to m-layers of $\mathcal{F}_\infty$ with respect to the Hausdorff metric according to the Main Theorem, which gives the result on the considered cross section. In the same fashion one can get the convergence of m-layers for the anti-holomorphic counterparts of other examples. □

8. PROOF OF THE MAIN THEOREM

The proof of the Main Theorem makes use of the kernel convergence of hyperbolic components. Recall that a connected open set H_∞ is the *kernel* (in the sense of Carathéodory) of the sequence of connected open sets $\{H_d\}$, if every compact set $K \in H_\infty$ lies in almost every H_d, and H_∞ is maximal with this property.

Theorem 8.1. *Let $\mathcal{F}_\infty$ be a family of entire transcendental functions that is dynamically approximated by families of polynomials $\mathcal{F}_d$. Then for each hyperbolic component $H_\infty \subset \mathcal{H}(\mathcal{F}_\infty)$ there exists a sequence $\{H_d\}$ of hyperbolic components $H_d \subset \mathcal{H}(\mathcal{F}_d)$ with H_∞ as kernel.*

This theorem is the main result in [Krauskopf and Kriete 1995b]. As remarked earlier, Theorem 8.1 can be extended to the case that $\mathcal{F}_\infty$ has infinitely many singular values, but is still dynamically approximated by the $\mathcal{F}_d$, in the sense that all families have the same constant number of *free* singular values. This is so because in the proof we use the parametrizations for a finite number of singular values that determine hyperbolicity. This allows to extend the Main Theorem to this more general setting. Here we state and prove it for functions of *finite type* to keep the argument simple. Note that all present examples are from this class of functions.

Proof of the Main Theorem. This proof uses the kernel convergence combined with the crucial fact that the families $\mathcal{F}_d$ have Property C. We fix $m \in \mathbb{N}$ and choose a component G_∞ of $\mathcal{H}_m(\mathcal{F}_\infty)$. According to Theorem 8.1 the component G_∞ is kernel of a sequence $\{G_d\}$ of components G_d of $\mathcal{H}(\mathcal{F}_d)$. The persistence of attracting cycles shows that $G_d \subset \mathcal{H}_m(\mathcal{F}_d)$ for almost every $d \in \mathbb{N}$.

Now define G as the set of all $\lambda_\infty \in \mathbb{C}$ such that there exists a sequence $\{\lambda_d\}$ with $\lambda_d \in \partial\mathcal{H}_m(\mathcal{F}_d)$, having λ_∞ as an accumulation point. We fix $\lambda_\infty \in G$ and have to show that $\lambda_\infty \in \partial\mathcal{H}_m(\mathcal{F}_\infty)$. Clearly $\lambda_\infty \notin \mathcal{H}_m(\mathcal{F}_\infty)$.

First, we assume that $\lambda_\infty \notin \overline{\mathcal{H}_m(\mathcal{F}_\infty)}$ and that λ_∞ is an isolated point of G. Then there exists a sequence of components G_d of $\mathcal{H}_m(\mathcal{F}_d)$ converging to λ_∞ in the Hausdorff metric. Let a be the multiplier of the limit cycle $\omega(c_\infty(\lambda_\infty))$ (with respect to $\mathcal{F}_\infty(\lambda_\infty, \cdot)$). Clearly, $a \in S^1$ and we choose a root of unity $b \in S^1$ such that $b \neq a$. For each $d \in \mathbb{N}$ the mapping

$$\varphi_d : \partial G_d \to S^1$$

that takes λ to $M(\omega(c_d(\lambda))$ is a covering. Therefore, for every $d \in \mathbb{N}$ there is some λ_d such that $M(\omega(c_d(\lambda_d)) = b$. Property C allows passing to the limit and we obtain $b = a$, a contradiction.

We now assume that λ_∞ is not an isolated point of G and want to show that $\lambda_\infty \in \partial\mathcal{H}_m(\mathcal{F}_\infty)$. By construction we find a sequence $\{\lambda_d\}$ converging to λ_∞ such that for each λ_d the function $\mathcal{F}_d(\lambda_d, \cdot)$ has an attracting periodic point z_d of minimal period m. By Property C we have $z_d \in D_R(0)$. After selecting a subsequence if necessary, passing to the limit proves that $\mathcal{F}_\infty(\lambda_\infty, \cdot)$ has $z_\infty = \lim_{d\to\infty} z_d$ as an indifferent periodic point. In addition, we have $z_\infty \in \overline{D_R(0)}$. It is possible to parametrize this periodic point on some neighborhood U of λ_∞ and thus, we find a hyperbolic component B of $\mathcal{F}_\infty$ with $\lambda_\infty \in \partial B$. Recall that z_∞ is an *indifferent* cycle and, hence, a bifurcation may occur. This means that for $\lambda \in B$ the cycle $\omega(c_\infty(\lambda))$ is an attracting cycle of period m and that its minimal period $\widetilde{m}$ divides m.

If $m = \widetilde{m}$ then clearly $B \subset \mathcal{H}_m(\mathcal{F}_\infty)$, therefore $\lambda_\infty \in \partial\mathcal{H}_m(\mathcal{F}_\infty)$. To complete the proof we show that the period does not drop, that is, we show that always $\widetilde{m} = m$. We assume now that $\widetilde{m} < m$ and use the following lemma to get a contradiction.

Lemma 8.2. *Let $C_\infty := \omega(c_\infty(\lambda_\infty))$ be a cycle of minimal period $\widetilde{m}$ and limit of attracting cycles $C_d := \omega(c_d(\mu_d))$ of $\mathcal{F}_d(\mu_d, \cdot)$ of minimal period m, where $\lim_{d\to\infty} \mu_d = \lambda_\infty$ and $\widetilde{m}|m$. If $\widetilde{m} < m$ then $M(\omega(c_\infty(\lambda_\infty))) = 1$.*

Proof of Lemma 8.2. By Property C there is some periodic point $z_d \in C_d \cap D_R(0)$ for every d. Passing to the limit we find a point $z_\infty \in C_\infty \cap \overline{D_R(0)}$. Since $\widetilde{m} < m$ the point z_∞ is a multiple root of $(\mathcal{F}^{\circ m}_{\infty,\lambda_\infty}(z) - z)$. Differentiation shows that z_∞ is a root of $((\mathcal{F}^{\circ m}_{\infty,\lambda_\infty})'(z) - 1)$. □

Since λ_∞ is not an isolated point of G we find, under the assumption that $\lambda_\infty \notin \overline{\mathcal{H}_m(\mathcal{F}_\infty)}$, a non-discrete set $S \subset G \setminus \overline{\mathcal{H}_m(\mathcal{F}_\infty)}$ with the following property. For all $\lambda \in S$ the function $\mathcal{F}_\infty(\lambda, \cdot)$ has an indifferent cycle $C_\lambda := \omega(c_\infty(\lambda))$ with $C_\lambda \cap \overline{D_R(0)} \neq \varnothing$. Furthermore, the minimal period of C_λ is strictly smaller than m for all $\lambda \in S$. By Lemma 8.2 we conclude that $\left(\mathcal{F}^{\circ m}_{\infty,\lambda}\right)'(z_\lambda) = 1$ for all $\lambda \in S$. The Identity Theorem shows that $\mathcal{F}_\infty(\lambda, \cdot)$ has an indifferent cycle of period m and with multiplier 1 for every $\lambda \in \mathbb{C}$, which in turn implies $\mathcal{H}(\mathcal{F}_\infty) = \varnothing$. This is a contradiction and the proof is complete. □

9. PROPERTY C

Let the functions F_d converge uniformly on compact sets to the non-constant function F_∞ and suppose that F_∞ has a cycle C_∞ of minimal period m. By a Rouché type argument one gets persistence of periodic cycles, meaning that almost all F_d have a cycle C_d of period m. A difficult open question concerns the reverse statement.

Question 9.1. *If the F_d have a cycle C_d of minimal period m, can one conclude that F_∞ has a* finite *cycle C_∞ of period m? Under which conditions on the functions is this true?*

Property C ensures that for the families under consideration limits of cycles are finite. In fact it is a big problem to prove this for a given family. One idea of showing Property C is the following. For any $\lambda \in \mathcal{H}_m(\mathcal{F}_d)$ the function $\mathcal{F}_d(\lambda, \cdot)$ has an attracting cycle C_d of minimal period m. This means that the multiplier is bounded by one in absolute value, that is,

$$|M(C_d)| = \left| \prod_{z \in C_d} \mathcal{F}'_d(\lambda, z) \right| < 1.$$

If one knows a nice relation between the derivative $\mathcal{F}'_d$ and the function $\mathcal{F}_d$ one gets an estimate on the product of the points in C_d. This may allow one to show the finiteness of one point in the limit, which suffices to get Property C.

This approach was used in [Devaney et al. 1986] to show the finiteness of the limits of attracting cycles for the families $P_{d,\lambda}$ from Example 5.1, which is necessary even for the pointwise convergence of hyperbolic components; see also [Krauskopf 1993]. Clearly one has $P'_{d,\lambda}(z) = P_{d,\lambda}(z)/(1 + z/d)$, which gives

$$\left| \prod_{z \in C_d} P'_{d,\lambda}(z) \right| < \left| \prod_{z \in C_d} \frac{d\,z}{z + d} \right| < 1.$$

It is now straightforward to show that for at least one $\hat{z} \in C_d$ one has $|\hat{z}| \leq 2$ for $d > 2$. Consequently, one gets a finite cycle in the limit. In much the same way one can show Property C for the families from Example 5.3, using the relation $\mathcal{F}'_d(\lambda, z) = (\mathcal{F}_d(\lambda, z) + 1)/(1 + z/d)$. In [Fagella 1995] the finiteness of the limits of attracting cycles is shown for the families $\mathcal{G}_d$ from Section 6 with a similar argument, using the relation $\mathcal{G}'_d(\lambda, z) = (d + z + d\,z)\mathcal{G}_d(\lambda, z)/(d\,z + z^2)$.

In order to prove Property C for the families $\mathcal{F}_{d,(\lambda,\mu)}$ from Section 7 we use the fact that these families are defined by composition. The estimate on the derivative of each constituent, which we know from Example 5.1, is sufficient to show Property C. Let C_d be an attracting m-cycle of $\mathcal{F}_{d,(\lambda,\mu)}$

for d sufficiently large and $\lambda \in K$. The multiplier can then be calculated as

$$\begin{aligned} M(C_d) &= \prod_{z \in C_d} \mathcal{F}'_d((\lambda,\mu),z) \\ &= \prod_{z \in C_d} [E(\lambda,\cdot) \circ E(\mu,\cdot)]'(z) \\ &= \prod_{z \in C_d} \left(\frac{E(\lambda,E(\mu,z))}{1+E(\mu,z)/d}\right)\left(\frac{E(\mu,z)}{1+z/d}\right) \\ &= \prod_{z \in C_d} \left(\frac{E(\mu,z)}{1+E(\mu,z)/d}\right)\left(\frac{z}{1+z/d}\right) \end{aligned}$$

As above, if $|M(C_d)| < 1$ we have either $\hat{z} \leq 2$ or $E(\mu,\hat{z}) \leq 2$ for some $\hat{z}$ and $d > 2$, which gives Property C. It is clear from this calculation that one can extend the idea of using a relationship between the derivative and the function to arbitrary compositions. However, the following problem remains.

Question 9.2. *If two sequences of families $\widehat{\mathcal{F}}_d$ and $\widetilde{\mathcal{F}}_d$ have Property C, is it true that the composed sequence $\mathcal{F}_d := \widehat{\mathcal{F}}_d \circ \widetilde{\mathcal{F}}_d$ has Property C?*

Finally we return to the Chebyshev polynomials from Example 5.2. The problem is that there is no nice relation between $\mathcal{T}'_d$ and $\mathcal{T}_d$ so that the above method does not work. We believe that the families $\mathcal{S}_d$ and $\mathcal{C}_d$ have Property C, but to our disappointment we were unable to find a different method of proof.

10. ALGORITHMS

All pictures in this paper show hyperbolic components in parameter space of some family $\mathcal{F}(\lambda,\cdot)$ with one free singular value $c(\lambda)$. The basic idea is rather simple. Choose a grid, each point representing a parameter value λ in the complex plane. Choose a maximal number of iterations `itermax` and a maximal period `permax`. For each grid point iterate the singular value $c(\lambda)$ under the map $\mathcal{F}(\lambda,\cdot)$ and check if it is attracted to some cycle of period $m <$ `itermax`. If this is the case before the maximal number of iterations `itermax` is reached, color the point black, if not, color it white. In this fashion one gets an approximation of $\bigcup_{j=1}^{\texttt{permax}} \mathcal{H}_j$, which can be thought of as a collection of level sets of the respective multiplier maps. We typically used `itermax` $= 5000$ and `permax` $= 60$, which gives a good approximation of $\mathcal{H}(\mathcal{F})$. This procedure is time-consuming, but ideally suited for parallelization.

We note here that we underestimate $\mathcal{H}(\mathcal{F})$ since a point is only colored black if the orbit of the singular value is in a basin of some cycle. This is why our figures may appear emptier than figures in other publications, where a grid point is usually colored black if the orbit of the singular value does not escape off to infinity. Another difference is that Figures 1–8 show a chart near infinity, which is most interesting for the convergence addressed here. An additional advantage is that possible artifacts, typically occurring near infinity, are dramatically reduced in size. Such a chart can conveniently be computed by applying the transformation $\lambda \mapsto \bar{\lambda}^{-1}$ to each grid point before iterating the respective map.

In the course of the iteration one will encounter orbits that escape off to infinity. If the family $\mathcal{F}(\lambda,\cdot)$ is polynomial this can be checked by testing whether the iterates enter the basin of infinity, that is, become bigger than an a priori known bound. Clearly one can run into numerical problems for polynomials of very high degree, the evaluation of which is also more expensive. Note that the polynomials from Examples 5.1 and 5.3, as well as those of Sections 6 and 7, can be evaluated by essentially $\log_2(d)$ multiplications. If the family $\mathcal{F}(\lambda,\cdot)$ is transcendental the situation is fundamentally more difficult since infinity is an essential singularity and not an attractor. As a consequence an orbit does not necessarily go off to infinity if an iterate is larger than a prescribed bound. To make things worse, transcendental functions grow extremely fast. In our examples one needs to evaluate $\exp(x)$ for a real variable x, which gives an overflow error in double precision if $|x|$ is bigger than about 710. We use the convention that in this case the associated point is colored black if $x > 0$ and white otherwise, which works well in practice; compare [Krauskopf 1993]. Note that, as a consequence of this convention, in the pictures for transcendental functions some points are colored black even though they were not shown to converge to an attracting cycle. However, since any algorithm has to cope with this problem this does not contradict the fact that our figures appear 'emptier'. These artifacts are kept to a minimum since we consider a chart near infinity.

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