The One-Equator Property

Mitja Lakner and Peter Petek

CONTENTS

We show the existence of points in the Mandelbrot cardioid that have the one-equator property, a property useful for the study of quaternionic dynamics. The question whether the Julia set is homeomorphic to a codimension-one sphere becomes a good deal more subtle in quaternionic dynamics.

1. INTRODUCTION: QUATERNIONIC JULIA SETS

The quaternions $\mathbb{H}$ can be represented as a direct sum $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3$. A quaternion will be denoted by

$$X = (\xi, \vec{x}),$$

where $\xi \in \mathbb{R}$ is the real part and $\vec{x} \in \mathbb{R}^3$ is the vector part. Multiplication, given by

$$AB = (\alpha, \vec{a})(\beta, \vec{b}) = (\alpha\beta - \vec{a}\vec{b},\ \alpha\vec{b} + \beta\vec{a} + \vec{a} \times \vec{b}),$$

is associative, distributive, yet not commutative. In particular, squaring in $\mathbb{H}$ reduces to

$$X^2 = (\xi^2 - \vec{x}^2,\ 2\xi\vec{x}).$$

The complex field $\mathbb{C}$ can be imbedded by

$$a + bi \mapsto (a, b\vec{\imath}),$$

where a and b are real and $\vec{\imath}$ is the first vector of the canonical basis $(\vec{\imath}, \vec{\jmath}, \vec{k})$.

In the complex plane the Julia set is defined by normal families. In the skew field of quaternions there are no nontrivial analytic functions, but we can define the *quaternionic Julia set* $J_c^{\mathbb{H}}$ of a quadratic function

$$x \mapsto x^2 + c$$

as the boundary of the basin of attraction of the point at infinity. In the complex plane it is known

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that for every parameter c from the big Mandelbrot cardioid

$$M_1 = \{\lambda/2 - \lambda^2/4 : |\lambda| < 1\}$$

the corresponding Julia set J_c is a homeomorphic image of the circle [Carleson and Gamelin 1993].

In the analysis of the quaternionic Julia sets, we can take the parameter c to be complex, with no loss in generality. But even if c is in the Mandelbrot cardioid, the corresponding Julia set need not be homeomorphic to the three-sphere S^3. Holbrook [1987] has shown that $J_c^{\mathbb{H}}$ is multiply connected, and therefore topologically not a sphere, if the complex Julia set J_c crosses the imaginary axis more than twice.

Suppose that $\pm ai \in J_c$, where $a > 0$. The whole sphere

$$S_a^2 = \{(0, \vec{x}) : \|\vec{x}\| = a\}$$

belongs to $J_c^{\mathbb{H}}$, because any point in S_a^2 is mapped to the complex point $(ai)^2 + c$. We shall call this sphere an *equator* of $J_c^{\mathbb{H}}$. If there is only one equator it is possible to divide the quaternionic Julia set into two *hemispheres* [Kozak and Petek 1994]. Only in this situation can the quaternionic Julia set be homeomorphic to the sphere.

Of course, if we take c real and in the Mandelbrot cardioid M_1—that is, $c \in (-\frac{3}{4}, \frac{1}{4}) \subset \mathbb{R}$—the complex Julia set J_c intersects the imaginary axis only twice. This follows from the symmetry of the Julia set J_c with respect to the imaginary axis for real c. For these values of c the corresponding quaternionic Julia set is obtained by rotating the complex Julia set J_c around the real axis:

$$J_c^{\mathbb{H}} = \{(\xi, \vec{x}) : \xi + \|\vec{x}\|\, i \in J_c\},$$

so $J_c^{\mathbb{H}}$ is homeomorphic to the three-sphere.

2. DEFINITION OF THE ONE-EQUATOR PROPERTY

Let $f_c : \mathbb{C} \to \mathbb{C}$ be the quadratic function

$$f_c(z) = z^2 + c.$$

Definition 2.1. The complex number c from the Mandelbrot cardioid M_1 has the *one-equator property* if the Julia set J_c intersects the imaginary axis exactly twice.

It is easy to find points that do not have the one-equator property.

Example 2.2. Set $c = -0.7 + 0.1i \in M_1$ and take the points $w_1 = 0.818i$ and $w_2 = 0.822i$ on the imaginary axis. The sequence $(f_c^n(w_1))_{n\in\mathbb{N}}$ diverges to infinity, whereas the sequence $(f_c^n(w_2))_{n\in\mathbb{N}}$ converges to the attracting fixed point of f_c. Since w_1 is below w_2, the imaginary axis intersects the Julia set J_c more than twice. See Figure 1.

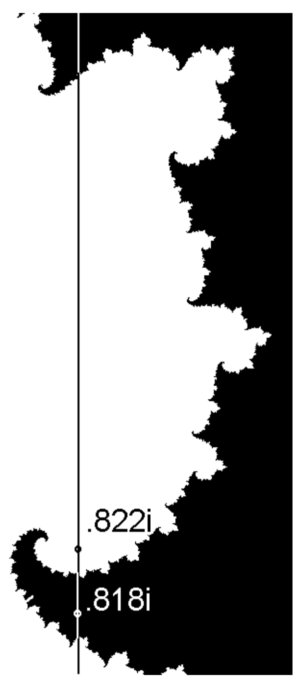

FIGURE 1. Part of Julia set J_c for $c = -0.7 + 0.1i$, which does not have one-equator property.

The same idea allows one to find many other points that don't have the one-equator property, as illustrated in Figure 2. It is much harder to show the existence of points off the real axis that do have the one-equator property.

FIGURE 2. Black dots represent points in the upper half of the Mandelbrot cardioid that are known not to have the one-equator property. The appearance of the picture is a consequence of the numerical method, which is the following. For a given c, we look for positive numbers $t_1 < t_2$ such that it_1 goes to infinity under iteration and it_2 goes to the finite attracting point. If there is such a pair we know that c does not have the one-equator property; see Example 2.2.

3. EXISTENCE OF THE ONE-EQUATOR PROPERTY

The idea in proving that there are points with the one-equator property is to find a parameter c in M_1 such that the Julia set J_c (which is the closure of repelling periodic points) intersects the imaginary axis in a periodic point z_0 of order n in which the derivative of the iterate f_c^n is real and greater than 1, so at least locally there will be only one intersection. If the derivative is not real, the Julia set will look locally like a spiral [Carleson and Gamelin 1993], because locally, near a periodic point z_0, the iterate behaves like $z - z_0 \mapsto (f_c^n)'(z_0)(z - z_0)$.

We first solve the equation $f_c^n(it) = it$ for the periodic point it as a function of a real parameter t. We get 2^{n-1} curves $c(t)$ parametrized by the real parameter t.

Example 3.1. For $n = 1$, the equation $(it)^2 + c = it$ has the solution $c(t) = it + t^2$. We have $c(t) \in M_1$ only for $|t| < \frac{1}{2}$, and thus it is the attractive fixed point, not on J_c.

For $n = 2$, the equation $(-t^2 + c)^2 + c = it$ has solutions $c_1(t) = it + t^2$ and $c_2(t) = -it + t^2 - 1$. The curve c_1 gives attracting fixed points, and c_2 misses M_1.

For higher n we can't get analytical solutions. Let's take one of these curves $t \mapsto c(t)$ and look at the real function

$$\beta_n(t) = \mathrm{Im}(f_{c(t)}^n)'(it).$$

We find numerically that $n = 4$ is the smallest integer for which this function changes sign inside the cardioid off the real axis. Let t_0 be the zero of $\beta = \beta_4$ lying in the interval $(1.04, 1.06)$, and let c_0 be the solution of the equation $f_{c_0}^4(it_0) = it_0$. This c_0 is the point we are looking for. See Figure 3.

Let it_0 be the repelling periodic point of order 4 of the function $f_{c_0}(z) = z^2 + c_0$. We denote by $z_j = f_{c_0}^j(it_0)$, for $j = 0, 1, 2, 3$, the periodic points of the 4-cycle. We interpret indices j cyclically modulo 4.

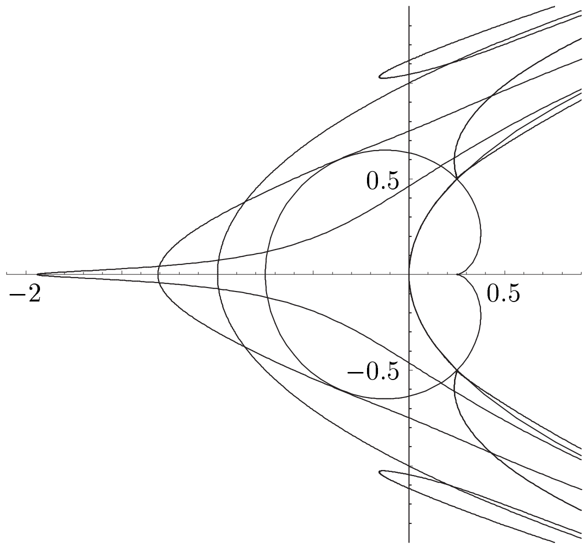

FIGURE 3. Solution curves of $f_c^4(it) = it$ in the c-plane. One of the curves goes near the point $c = -0.067 - 0.419i$, for $t = 1.05$.

The numerical values are

$$t_0 = 1.0493404831,$$
$$c_0 = -0.0669671577 - 0.4194471409i,$$
$$z_0 = it_0,$$
$$z_1 = -1.1680826073 - 0.4194471409i,$$
$$z_2 = 1.1215139157 + 0.560450679i,$$
$$z_3 = 0.8767213418 + 0.8376593302i.$$

The attracting fixed point is $z^* = -0.149118101 - 0.3230900049i$. Therefore the circle $K(z^*, \rho)$ with radius $\rho = 1-2|z^*| \approx 0.288$ and center z^* lies inside the basin of attraction of z^*. This is a consequence of the inequality

$$|f_{c_0}(z^* + h) - z^*| \leq (2\,|z^*| + |h|)|h| < |h|,$$

which holds if $h| < 1 - 2\,|z^*|$.

We now construct two open domains $D^+_{r,j}$ and $D^-_{r,j}$ as follows. At the point z_j take the two circles of radius $R = 1.5$ tangent to the path $t \mapsto f^j_{c_0}(it_0+it)$. Take also the circle with center z_j and radius r. These three circles bound two wedges with vertex z_j; we define $D^+_{r,j}$ as the wedge that opens away from 0 and $D^-_{r,j}$ as the wedge that opens toward 0.

For $r_1 = 0.3$ and $r_2 = 0.9$ we will prove the following facts.

Proposition 3.2. (i) $f_{c_0}(D^{\pm}_{r_1,j}) \subset D^{\pm}_{r_2,j+1}$ *for* $j = 0, 1, 2, 3$.
(ii) *For all points z from $D^{\pm}_{r_1,j}$ we have*

$$|f_{c_0}(z) - z_{j+1}| > 1.5\,|z - z_j|$$

for $j = 0, 1, 2, 3$.
(iii) *For $z \in D^+_{r_2,j} \backslash D^+_{r_1,j}$ we have $|f_{c_0}(z)| > 1.5$, and therefore z is in the basin of attraction of infinity.* (See Figure 4.)
(iv) *For $z \in D^-_{r_2,j} \backslash D^-_{r_1,j}$ we have $|f^3_{c_0}(z) - z^*| < \rho$, and the point z is in the basin of attraction of the attracting fixed point z^* of f_{c_0}.* (See Figure 5.)

(The 1.5 in (ii) and (iii) is unrelated to the constant R in the definition of $D^+_{r,j}$ and $D^-_{r,j}$.)

Proof. Properties (i) and (ii) are consequences of the following two lemmas:

Lemma 3.3. *Let $f(z) = z^2 + c_0$, and let a nonzero complex point a and a direction $e^{i\alpha}$, for $\alpha \in \mathbb{R}$, be given. In local coordinates at a, in which the direction is preserved, the function f is given by*

$$z \mapsto g(z) = 2\,|a|\,z + \frac{|a|}{a}\,e^{i\alpha} z^2.$$

Proof. Let $L_{a,\alpha}(z) = a + e^{i\alpha} z$, $b = f(a)$; the direction $e^{i\beta}$ is determined by the image of direction $e^{i\alpha}$ in point b, $\beta = \arg(ae^{i\alpha})$. Then our normalized function is the composition $L^{-1}_{b,\beta} \circ f \circ L_{a,\alpha} = g$. ☐

Lemma 3.4. *Let $g(z) = kz + e^{i\gamma} z^2$, where $k > 1$, and let D_r be the wedge*

$$\{z : |z \pm iR| > R,\ |z| < r,\ \mathrm{Re}\ z > 0\}.$$

Then for pairs (k, γ) from Lemma 3.3 we have, for all periodic points z_j,

$$g(D_{r_1}) \subset D_{r_2} \tag{3.1}$$

and

$$|g(z)| > 1.5\,|z| \quad \textit{for all } z \in D_{r_1}. \tag{3.2}$$

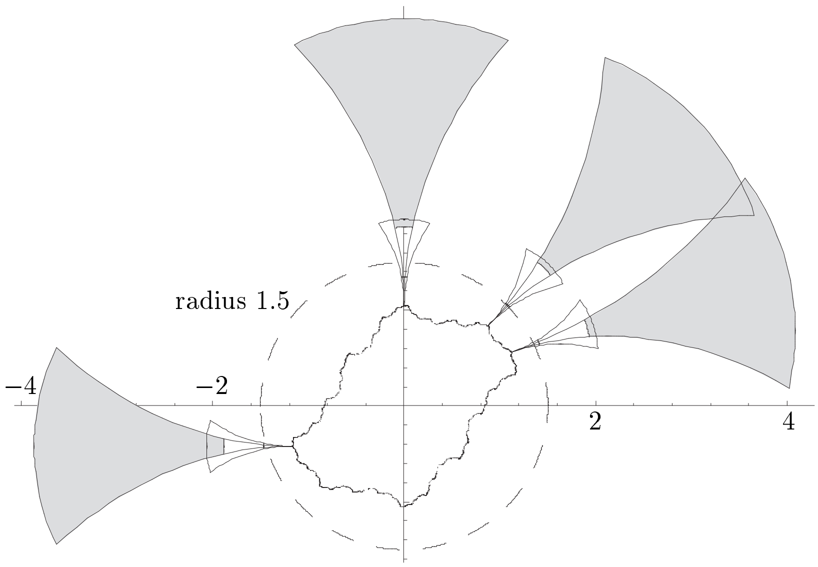

FIGURE 4. Convergence to infinity. The images under f_{c_0} of the regions $D^+_{r_2,j} \backslash D^+_{r_1,j}$ are shaded and lie outside the circle of escape radius 1.5.

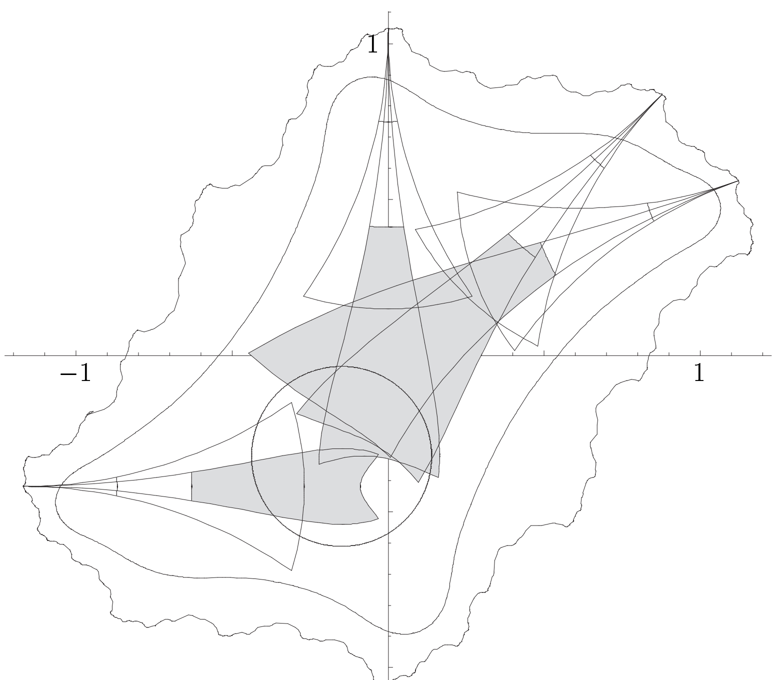

FIGURE 5. Convergence to to the finite attracting point. The images under f_{c_0} of the regions $D^-_{r_2,j} \backslash D^-_{r_1,j}$ are shaded. The smooth curve is the third preimage of the attracting circle. The regions $D^-_{r_2,j} \backslash D^-_{r_1,j}$ lie within this smooth curve.

Proof. The function g is analytic, so it is enough to show the inclusion $g(\partial D_{r_1}\setminus\{0\}) \subset D_{r_2}$. If we parametrize circles $S(\pm iR, R)$ by

$$\kappa(t) = R(\sin t \pm i(1-\cos t)),$$

the expression $|g\circ\kappa(t)\mp iR|^2 - R^2$ can be simplified to

$$8R^2\sin^2\frac{t}{2}\Big(\tfrac{1}{2}k(k-1) + R^2(1-\cos t) \\ +R(k-1)\sin(t\pm s)\mp Rk\sin s\Big).$$

It is a strictly positive function of t on the interval $(0,\pi/2)$ for all four periodic points z_j. This follows from elementary reasoning, because the pairs (k,γ) in question are $(2.099, 0)$, $(2.482, -0.345)$, $(2.508, -0.119)$, $(2.425, 0.046)$. Since, for all $|z| = r_1$ and all j, we have $|g(z)| \le kr_1 + r_1^2 < r_2$, we have verified (3.1). Inequality (3.2) is trivial since

$$|g(z)| \ge (k-|z|)|z| \ge (2-0.3)\,|z| \ge 1.5\,|z|. \quad \square$$

This concludes the proof of parts (i) and (ii) of Proposition 3.2. For the proof of (iii) and (iv), it is enough to look at the shaded domains $D^{\pm}_{r_2,j}\setminus D^{\pm}_{r_1,j}$. More formally, to prove (iii), we should look at the boundaries of domains $D^{+}_{r_2,j}\setminus D^{+}_{r_1,j}$, because the function f_{c_0} is analytic. We can choose points $w_1,\dots,w_n$ from the boundaries of our domains in such a way that the union of the disks with center w_k and radius 0.05 covers the boundaries. For all these finitely many points (approximately 80) we can computationally verify that

$$|f_{c_0}(w_k)| > 1.8.$$

Each point from our boundaries can be written in the form $w_k + h$ for some $|h| < 0.05$. From the inequalities

$$|f_{c_0}(w_k)| - |f_{c_0}(w_k+h)| \le |f_{c_0}(w_k+h) - f_{c_0}(w_k)| \\ \le (2\,|w_k| + |h|)|h| \le M|h|$$

we get

$$|f_{c_0}(w_k+h)| \ge |f_{c_0}(w_k)| - M|h| \\ > 1.8 - 6\times 0.05 = 1.5,$$

since all domains lie in the circle of radius 2.5 centered at the origin.

The proof of (iv) is similar, with $|h| < 0.001$,

$$|f^3_{c_0}(w_k+h) - f^3_{c_0}(w_k)| \le M|h|$$

for $M = 50$, and finally

$$|f^3_{c_0}(w_k+h) - z^*| \le |f^3_{c_0}(w_k) - z^*| + M|h| \\ < 0.23 + 50\times 0.001 < \rho.$$

The verification that $|f^3_{c_0}(w_k) - z^*| < 0.23$ must be carried out for approximately 4000 points w_k. $\square$

Theorem 3.5. *There is a point c_0 in the Mandelbrot cardioid, off the real axis, that has the one-equator property.*

Proof. We have to show that the sequence

$$w_n = f^n_{c_0}(it),$$

where $t \ge 0$, diverges to infinity if $t > t_0$ and converges to the attracting fixed point z^* of f_{c_0} for $0 \le t < t_0$.

We consider first the case $t > t_0$. If $t \ge t_0 + r_2$, then $|w_0| > 1.5$ and therefore we are in the attraction basin of infinity. If $r_1 < t - t_0 < r_2$, part (iii) of Proposition 3.2 gives $|w_1| > 1.5$, so again go to infinity.

If $0 < t - t_0 \le r_1$, parts (i) and (ii) of Proposition 3.2 say that some w_n is in $D^{+}_{r_2,j}\setminus D^{+}_{r_1,j}$; by part (iii) we have $|w_{n+1}| > 1.5$.

Now assume instead that $0 \le t < t_0$.

If $0 < t \le t_0 - r_2 = 0.14$, then w_0 is in the basin of attraction of finite attracting point z^*. If $r_1 < t_0 - t < r_2$, then by part (iv) of the proposition we have $w_3 = f^3_{c_0}(it) \in K(z^*, \rho)$. Finally, If $0 < t_0 - t \le r_1$, parts (i) and (ii) say that some w_n is in $D^{-}_{r_2,j}\setminus D^{-}_{r_1,j}$; by part (iv) we have $w_{n+3} \in K(z^*, \rho)$. $\square$

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Mitja Lakner, University of Ljubljana, FAGG, Jamova 2, 61000 Ljubljana, Slovenija (mlakner@fagg.uni-lj.si)

Peter Petek, Institute of Mathematics, Physics and Mechanics, Jadranska 19, 61111 Ljubljana, Slovenija (peter.petek@uni-lj.si)

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