The Volume Spectrum of Hyperbolic 4-Manifolds

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CONTENTS

Keywords: hyperbolic manifolds, 4-manifolds, volume, 24-cell
AMS Classification: Primary 30F40, 51M10, 53C25

We construct complete, open, hyperbolic 4-manifolds of smallest volume by gluing together the sides of a regular ideal 24-cell in hyperbolic 4-space. We also show that the volume spectrum of hyperbolic 4-manifolds is the set of all positive integral multiples of $4\pi^2/3$.

1. INTRODUCTION

A *hyperbolic manifold* is a Riemannian manifold of constant sectional curvature -1. The set of all volumes of complete hyperbolic n-manifolds of finite volume is called the *volume spectrum* of hyperbolic n-manifolds. It has been known for over a hundred years that the volume spectrum of hyperbolic 2-manifolds is the set of all positive integral multiples of 2π. In contrast to dimension two, Jørgensen and Thurston [Thurston 1979] have shown that the volume spectrum of hyperbolic 3-manifolds is a closed, non-discrete, well-ordered subset of the positive real numbers, having order type ω^ω. In particular, there is a smallest positive number that is the volume of a complete hyperbolic 3-manifold. This number is at present unknown.

In this paper, we geometrically construct examples of complete, open, hyperbolic 4-manifolds of smallest volume and show that the volume spectrum of complete, open, hyperbolic 4-manifolds is the set of all positive integral multiples of $4\pi^2/3$. This implies that the volume spectrum of hyperbolic 4-manifolds is also the set of all positive integral multiples of $4\pi^2/3$, since the volume of a closed hyperbolic 4-manifold is also a multiple of $4\pi^2/3$. All the manifolds constructed in this paper are open, so this paper sheds no light on the volume spectrum of closed hyperbolic 4-manifolds.

The first explicit example of a hyperbolic 4-manifold of finite volume in the literature is the closed hyperbolic 4-manifold constructed by Davis [1985] by gluing together the opposite sides of a regular 120-

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cell in hyperbolic 4-space. Ratcliffe and Tschantz [1994] constructed an explicit example of a complete, open, hyperbolic 4-manifold of finite volume by gluing together the sides of a regular ideal 24-cell in hyperbolic 4-space. The examples of hyperbolic 4-manifolds of smallest volume in this paper are also obtained by gluing together the sides of a regular ideal 24-cell in hyperbolic 4-space.

Hyperbolic 4-manifolds of small volume are currently of interest in cosmology in the theory of quantum gravity. See [Gibbons 1996] where the examples in this paper are considered in the theory of quantum gravity. More examples of hyperbolic 4-manifolds of small volume that are considered in the theory of quantum gravity are given in [Ratcliffe and Tschantz 1998].

We now set up notation in order to describe our examples and further results. A real $(n+1)\times(n+1)$ matrix A is said to be *Lorentzian* if A preserves the *Lorentzian inner product*

$$x \circ y \;=\; x_1y_1 + x_2y_2 + \cdots + x_ny_n - x_{n+1}y_{n+1}.$$

The *hyperboloid model* of hyperbolic n-space is the metric space

$$H^n \;=\; \{x \in \mathbb{R}^{n+1} : x \circ x = -1 \text{ and } x_{n+1} > 0\}$$

with metric d defined by

$$\cosh d(x,y) = -x \circ y.$$

A Lorentzian $(n+1)\times(n+1)$ matrix A is said to be either *positive* or *negative* according as A maps H^n to H^n or $-H^n$. The isometries of H^n correspond to the positive Lorentzian $(n+1)\times(n+1)$ matrices.

Let Γ^n be the group of positive Lorentzian $(n+1)\times(n+1)$ matrices with integer entries. The group Γ^n is an infinite discrete subgroup of the group $O(n,1)$ of Lorentzian $(n+1)\times(n+1)$ matrices. The *principal congruence two subgroup* of Γ^n is the group Γ^n_2 of all matrices in Γ^n that are congruent to the identity matrix modulo two. The congruence two subgroup Γ^n_2 is not torsion-free, but it only has 2-torsion [Newman 1972, Theorem IX.7].

In this paper, we construct and classify all the hyperbolic space-forms H^n/Γ where Γ is a torsion-free subgroup of minimal index in the congruence two subgroup Γ^n_2 for $n = 2,3,4$. We call such a space-form an *integral, congruence two, hyperbolic n-manifold of minimum volume*. We show that there are 2, 13, 1171 isometry classes of integral, congruence two, hyperbolic n-manifolds of minimum volume for $n = 2,3,4$, respectively. These hyperbolic manifolds have smallest volume among all complete hyperbolic n-manifolds for $n = 2,4$. Thus there are at least 1171 different complete hyperbolic 4-manifolds of smallest volume. By a theorem of Wang [1972], there are only finitely many complete hyperbolic 4-manifolds, up to isometry, with the same finite volume.

The 1171 integral, congruence two, hyperbolic 4-manifolds of minimum volume are the simplest complete hyperbolic 4-manifolds of finite volume. They are all constructed by gluing together the sides of a regular ideal 24-cell in hyperbolic 4-space in a particularly simple way. A complete hyperbolic 4-manifold that is obtained by gluing together the sides of a regular ideal 24-cell is called a 24-*cell manifold*. We shall call an integral, congruence two, hyperbolic 4-manifold of minimum volume a *congruence two* 24-*cell manifold*. All but 22 of the 1171 congruence two 24-cell manifolds are nonorientable.

The nonorientable congruence two 24-cell manifolds are far more interesting than the few orientable ones. The first 24-cell manifold we constructed is the nonorientable manifold referred to as the *hyperbolic* 24-*cell space* in [Ratcliffe 1994]. It has a symmetry group of order 128 all of whose elements are induced by symmetries of the 24-cell. Of all the congruence two 24-cell manifolds, the hyperbolic 24-cell space is constructed by the most symmetric side-pairing of the 24-cell.

Quite surprisingly, there is a congruence two 24-cell manifold with an even larger symmetry group of order 320. This manifold has the largest symmetry group among all the congruence two 24-cell manifolds. If one equates beauty with symmetry, then this manifold is the most beautiful congruence two 24-cell manifold. It has a symmetry of order 5 that cyclically permutes its 5 cusps. This manifold is one of only two congruence two 24-cell manifolds with the property that all of their cusps have the same type. Both of these manifolds are nonorientable.

A nonorientable manifold is double covered by an orientable manifold, and one should think of a nonorientable manifold as an orientable manifold together with an orientation reversing fixed point free involution. In fact, the orientable double covers of our nonorientable manifolds are of interest in cosmology [Ratcliffe and Tschantz 1998].

By the Gauss–Bonnet theorem (see [Gromov 1982; Hopf 1926]), the volume of a complete hyperbolic 4-manifold M of finite volume is given by

$$\mathrm{Vol}(M) = \frac{4\pi^2}{3}\chi(M),$$

where $\chi(M)$ is the Euler characteristic of M. We prove that there are complete, open, orientable, hyperbolic 4-manifolds of finite volume whose Euler characteristic is any given positive integer. Therefore the volume spectrum of hyperbolic 4-manifolds is the set of all positive integral multiples of $4\pi^2/3$.

We also determine the structure of the congruence two subgroup Γ_2^n for $n = 2, 3, 4$. In particular, we show that Γ_2^n is a reflection group with respect to a noncompact right-angled polytope P^n in hyperbolic n-space for $n = 2, 3, 4$. This implies that H^n/Γ_2^n is isometric to P^n for $n = 2, 3, 4$. We prove that Γ_2^n has a torsion-free subgroup of finite index i if and only if i is divisible by 2^n for $n = 2, 3, 4$. We classify, up to isomorphism, all the torsion-free subgroups of Γ_2^n of index 2^n for $n = 2, 3, 4$. These are the groups whose orbit spaces are the integral, congruence two, hyperbolic n-manifolds of minimum volume for $n = 2, 3, 4$. Thus the integral, congruence two, hyperbolic n-manifolds of minimum volume are the minimal, nonsingular, covering spaces of the orbifold P^n for $n = 2, 3, 4$. The classification of these manifolds in dimension n will play a role in the classification in dimension $n + 1$ for $n = 2, 3$.

This paper is organized as follows: In Sections 2, 3, and 4, we determine the structure of the congruence two subgroup Γ_2^n and construct and classify all the congruence two hyperbolic n-manifolds of minimum volume for $n = 2, 3, 4$, respectively. In Section 5, we describe our coding for the side-pairings of a fundamental domain for all these manifolds. In Section 6, we give tables that list side-pairings and isometry invariants of all 1171 congruence two 24-cell manifolds.

2. INTEGRAL, CONGRUENCE TWO, HYPERBOLIC SURFACES

In this section, we determine the structure of the congruence two subgroup Γ_2^2 of the group Γ^2 of integral, positive, Lorentzian 3×3 matrices and classify all the congruence two hyperbolic surfaces of minimum area.

According to Fricke [1891, § 3], the group Γ^2 is a reflection group with respect to a noncompact triangle Δ^2 in H^2 whose Coxeter diagram is

Vertices for Δ^2 are $(0, 0, 1), (\sqrt{2}/2, \sqrt{2}/2, \sqrt{2})$, and $(1, 0, 1)$ (at infinity).

The group Γ^2 is generated by the matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -2 & 2 \\ -2 & -1 & 2 \\ -2 & -2 & 3 \end{pmatrix},$$

which represent the reflections in the sides of Δ^2. By mapping these matrices into $\mathrm{GL}(3, \mathbb{Z}/2\mathbb{Z})$, we see that the index of Γ_2^2 in Γ^2 is two.

Let Σ^2 be the group of order two generated by the first matrix in the above list of matrices. Then Σ^2 is a set of coset representatives for Γ_2^2 in Γ^2. We therefore have a natural, split, short, exact sequence of groups

$$1 \to \Gamma_2^2 \to \Gamma^2 \to \Sigma^2 \to 1.$$

We now pass to the conformal disk model B^2 of the hyperbolic plane. The vertices of Δ^2 are now $(0, 0)$, $(1 - \sqrt{2}/2, 1 - \sqrt{2}/2)$, $(1, 0)$. The triangle Δ^2 is a triangle of the barycentric subdivision of the ideal square Q^2 whose vertices are $(\pm 1, 0)$, $(0, \pm 1)$. See Figure 1. Let P^2 be the intersection of Q^2 with the first quadrant of $\mathbb{R}^2$. Then P^2 is a noncompact right-triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$.

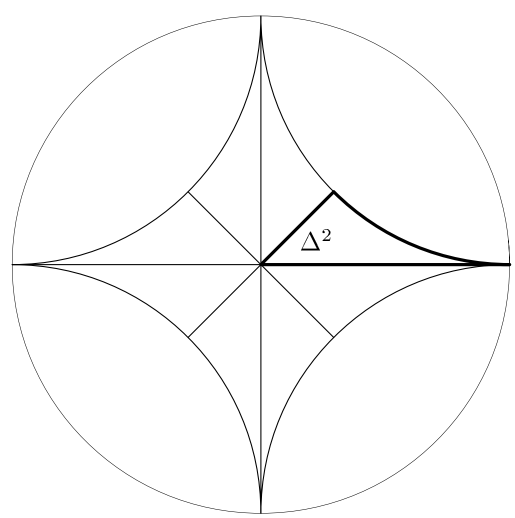


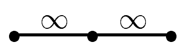
FIGURE 1. The triangle Δ^2.

Observe that $P^2 = \Sigma^2 \Delta^2$ and Σ^2 is the group of symmetries of P^2. The Lorentzian matrices that represent the reflections in the sides of P^2 are

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -2 & 2 \\ -2 & -1 & 2 \\ -2 & -2 & 3 \end{pmatrix}.$$

These matrices are all in Γ^2_2. Now since Σ^2 is a set of coset representatives for Γ^2_2 in Γ^2, we have that $P^2 = \Sigma^2 \Delta^2$ is a fundamental polygon for Γ^2_2. We therefore have the following theorem.

Theorem 1. *The congruence two subgroup Γ^2_2 of the group Γ^2 of integral, positive, Lorentzian 3×3 matrices is a reflection group with respect to a noncompact triangle P^2 whose Coxeter diagram is*



Let K^2 be the Klein four group generated by the reflections in the vertical and horizontal sides of P^2. The next corollary follows immediately from Theorem 1.

Corollary 1. *Every nonidentity element of Γ^2_2 of finite order has order two and every finite subgroup of Γ^2_2 is conjugate in Γ^2_2 to a subgroup of the Klein four group K^2.*

Let Γ be a torsion-free subgroup of Γ^2_2 of finite index i. Then $M = H^2/\Gamma$ is a hyperbolic surface of finite area. By the Gauss–Bonnet theorem, we have

$$\text{Area}(M) = -2\pi\chi(M),$$

where $\chi(M)$ is the Euler characteristic of M. Therefore the area of M is a multiple of 2π. As the area of P^2 is $\pi/2$, we deduce that i is divisible by 4.

Now suppose that $i = 4$. Then K^2 forms a set of coset representatives of Γ in Γ^2_2 and so the ideal square $Q^2 = K^2 P^2$ is a fundamental polygon for Γ. Let S be a side of Q^2. Then there is a nonidentity element g of Γ that pairs a side S' of Q^2 to S. Let k be the element of K^2 that maps S' to S, and let r be the reflection in the side S. Then gkr leaves S invariant. The only elements of Γ^2_2 that leave S invariant are the identity and r. We cannot have $gkr = r$, since g has infinite order. Therefore $gkr = 1$. Thus the side-pairing transformations of Q^2 are of the form rk where k is in K^2 and r is the reflection in a side of Q^2.

Now Γ is generated by the side-pairing transformations of Q^2, and so Γ is determined by the side-pairing of Q^2. There are only three side-pairings for Q^2 of the above form. Two of these pairings are equivalent by a symmetry of Q^2 and yield the hyperbolic thrice-punctured sphere M^2_1 and the third pairing yields the symmetric, hyperbolic, twice-punctured, projective plane M^2_2. See Figure 2. Thus we have the following theorem.

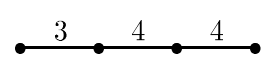
Theorem 2. *There are, up to isometry, exactly two hyperbolic space-forms H^2/Γ where Γ is a torsion-free subgroup of minimal index in the congruence two subgroup Γ^2_2 of the group Γ^2 of integral, positive, Lorentzian 3×3 matrices.*

Theorem 3. *The congruence two group Γ^2_2 has a torsion-free subgroup of index i if and only if i is divisible by* 4.

Proof. We have already shown that every torsion-free subgroup of Γ^2_2 has index divisible by 4. Let Γ be a torsion-free subgroup of Γ^2_2 of index four. Then Γ is a free group of rank two. Therefore Γ maps homomorphically onto $\mathbb{Z}$. Hence Γ has a subgroup of index i for each positive integer i. Therefore Γ^2_2 has a torsion-free subgroup of index $4j$ for each positive integer j. □

3. INTEGRAL, CONGRUENCE TWO, HYPERBOLIC 3-MANIFOLDS

In this section, we determine the structure of the congruence two subgroup Γ^3_2 of the group Γ^3 of integral, positive, Lorentzian 4×4 matrices and classify all integral, congruence two, hyperbolic 3-manifolds of minimum volume. Coxeter [1950] proved that the group Γ^3 is a reflection group with respect to a noncompact tetrahedron Δ^3 in H^3 (see Figure 3) whose Coxeter diagram is



Vertices for Δ^3 are

$$(0,0,0,1),\ (\sqrt{6}/6, \sqrt{6}/6, \sqrt{6}/6, \sqrt{6}/2),$$
$$(\sqrt{2}/2, \sqrt{2}/2, 0, \sqrt{2}),\ \text{and } (1,0,0,1) \text{ (at infinity)}.$$

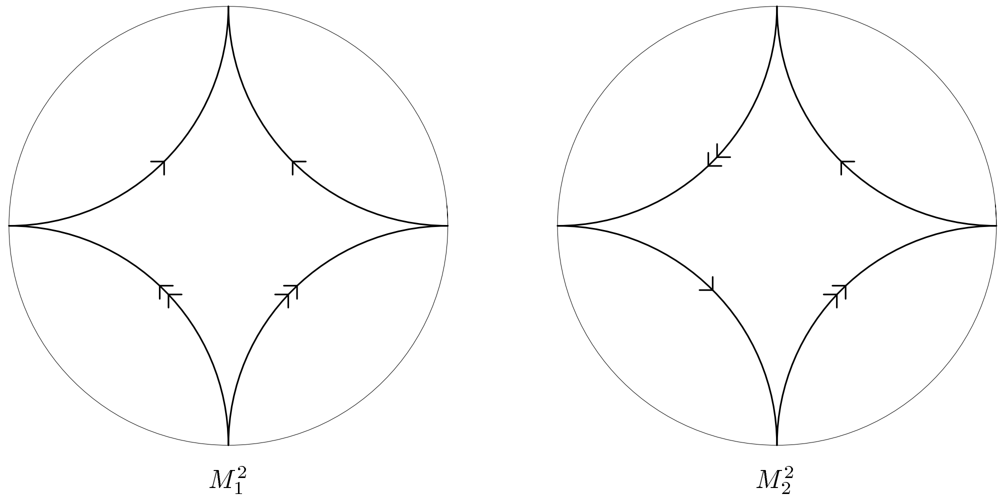


FIGURE 2. The 3-punctured sphere and the 2-punctured projective plane.

The group Γ^3 is generated by the four matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 & -1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which represent the reflections in the sides of Δ^3. By mapping these matrices into $\mathrm{GL}(4, \mathbb{Z}/2\mathbb{Z})$ and computing the order of the group that their images generate, we deduce that the index of Γ_2^3 in Γ^3 is 12.

Let Σ^3 be the group generated by the first three matrices in the above list of matrices. These are the generators of Γ^3 that project to nonzero elements of $\mathrm{GL}(4, \mathbb{Z}/2\mathbb{Z})$. Then Σ^3 is the group generated by the reflections in the three sides of Δ^3 incident with the vertex $(\sqrt{6}/6, \sqrt{6}/6, \sqrt{6}/6, \sqrt{6}/2)$. Therefore Σ^3 is isomorphic to a spherical triangle reflection group whose Coxeter diagram is obtained from the Coxeter diagram of Γ_2^3 by deleting its third vertex and its adjoining edges. Hence, the Coxeter diagram of Σ^3 is the disjoint union of an edge labeled by 3 and a vertex. Therefore Σ^3 is the direct product of the dihedral group D^3 of order six generated by the first two matrices and the group of order two generated by the third matrix in the above list of matrices. Thus Σ^3 has order 12.

Now Σ^3 injects into $\mathrm{GL}(4, \mathbb{Z}/2\mathbb{Z})$, since it has the same order as its image. Hence Σ^3 is a set of coset representatives for Γ_2^3 in Γ^3. We therefore have a natural, split, short, exact sequence of groups

$$1 \to \Gamma_2^3 \to \Gamma^3 \to \Sigma^3 \to 1.$$

We now pass to the conformal ball model B^3 of hyperbolic 3-space. The vertices of Δ^3 are now

$$(0,0,0), \quad (1-\sqrt{6}/3, 1-\sqrt{6}/3, 1-\sqrt{6}/3),$$
$$(1-\sqrt{2}/2, 1-\sqrt{2}/2, 0), \quad (1,0,0),$$

and Δ^3 is a tetrahedron of the barycentric subdivision of the ideal octahedron O^3 whose vertices are $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$.

Let T^3 be the intersection of O^3 with the positive octant of $\mathbb{R}^3$. Then T^3 is a noncompact tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. The dihedral angles of T^3 along the edges joining the ideal vertices of T^3 are all $\pi/4$. The barycentric subdivision of O^3 subdivides T^3 into six copies of Δ^3 that are the images of Δ^3 under the elements of the dihedral group D^3. See Figure 3.

Let P^3 be the union of T^3 and the tetrahedron $(T^3)'$ obtained by reflecting T^3 in the side of T^3 spanned by its ideal vertices, that is, the front face of T^3 in Figure 3. Then P^3 is a noncompact polyhedron with six sides and five vertices, two actual, $(0,0,0)$ and $(1/3, 1/3, 1/3)$, and three ideal. See Figure 4. The dihedral angles of P^3 are all $\pi/2$. Observe that $P^3 = \Sigma^3 \Delta^3$ and Σ^3 is the group of symmetries of P^3.

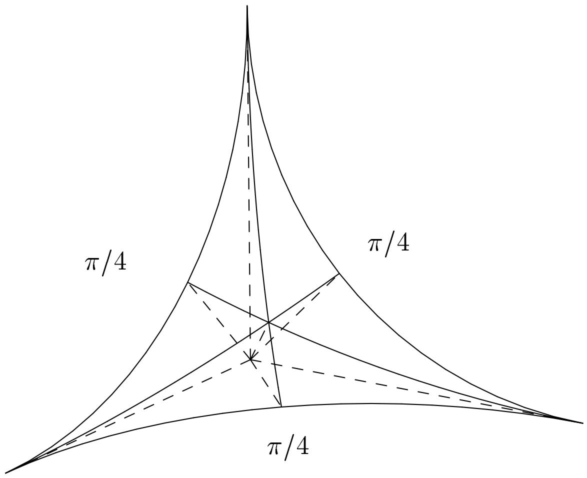


FIGURE 3. The subdivision of T^3 into six copies of Δ^3.

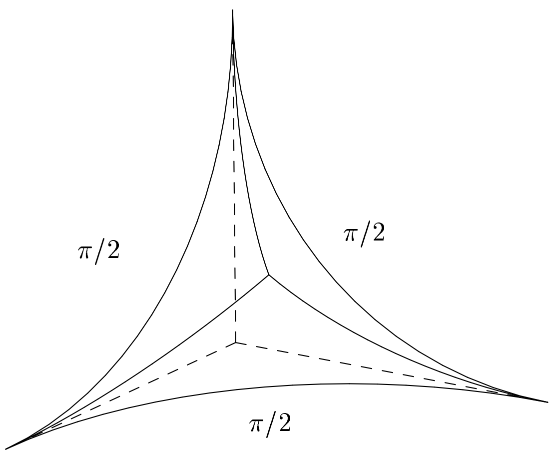


FIGURE 4. The polyhedron P^3.

The Lorentzian matrices that represent the reflections in the sides of P^3 are

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & -2 & -2 & 3 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 & -2 & 2 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & -1 & 2 \\ -2 & 0 & -2 & 3 \end{pmatrix}, \quad \begin{pmatrix} -1 & -2 & 0 & 2 \\ -2 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ -2 & -2 & 0 & 3 \end{pmatrix}.$$

All are in Γ_2^3. Since Σ^3 is a set of coset representatives for Γ_2^3 in Γ^3, we see that $P^3 = \Sigma^3\Delta^3$ is a fundamental polyhedron for Γ_2^3. Therefore:

Theorem 4. *The congruence two subgroup Γ_2^3 of the group Γ^3 of integral, positive, Lorentzian 4×4 matrices is a reflection group with respect to the noncompact polyhedron P^3.*

Let K^3 be the elementary 2-group of order 8 generated by the three reflections in the coordinate planes of $\mathbb{R}^3$. We shall identify K^3 with the corresponding subgroup of Γ^3 generated by the first three matrices in last displayed list of matrices. The next corollary follows immediately from Theorem 4.

Corollary 2. *Every nonidentity element of Γ_2^3 of finite order has order two, every finite subgroup of Γ_2^3 is conjugate in Γ^3 to a subgroup of the elementary 2-group K^3, and there are two conjugacy classes of maximal finite subgroups of Γ_2^3 in Γ_2^3 corresponding to the two actual vertices of P^3.*

Let Γ be a torsion-free subgroup of Γ_2^3 of finite index. Then K^3 acts freely on the set of cosets of Γ in Γ_2^3 by $g\Gamma \mapsto kg\Gamma$, since Γ is torsion-free. Therefore $|K^3| = 8$ divides $[\Gamma_2^3 : \Gamma]$. Now suppose that $[\Gamma_2^3 : \Gamma] = 8$. Then the set $Q^3 = K^3P^3$ is a fundamental polyhedron for Γ (Figure 5). It is a rhombic dodecahedron with 14 vertices, 8 actual $(\pm\frac{1}{3}, \pm\frac{1}{3}, \pm\frac{1}{3})$, and 6 ideal $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$. It has the same group of symmetries as the cube with vertices $(\pm\frac{1}{3}, \pm\frac{1}{3}, \pm\frac{1}{3})$. All its dihedral angles are $\pi/2$.

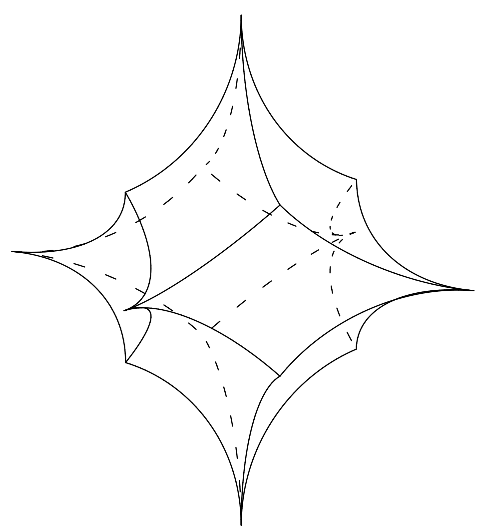


FIGURE 5. The polyhedron Q^3.

Let S be a side of Q^3. Then there is a nonidentity element g of Γ that pairs a side S' of Q^3 to S. Let k be the element of K^3 that maps S' to S, and let r be the reflection in the side S. Then gkr leaves S invariant. The only elements of Γ_2^3 that leave S invariant are the identity, the reflection r, the reflection s in the coordinate plane P perpendicular to S, and the 180° rotation $rs = sr$ about the line $P \cap S$. Note that s is in K^3. We cannot have $gkr = r$ or $gkr = sr$, since g has infinite order. Therefore $gkr = 1$ or $gkr = s$. Hence, the side-pairing transformations of Q^3 are of the form rk where k is in K^3 and r is the reflection in a side of Q^3.

Now Γ is generated by the side-pairing transformations for Q^3, and so Γ is determined by the side-pairing of Q^3. The side-pairing for Q^3 restricts to a side-pairing for each of the three copies of Q^2 obtained from Q^3 by intersecting Q^3 with a coordinate plane. Extending the 3 possible side-pairings for Q^2 that yield an integral, congruence two, hyperbolic surface on each of the 3 copies of Q^2 in Q^3 yields 1728 side-pairings for Q^3 of the above form. Only 107 of these satisfy the conditions of Poincaré's fundamental polyhedron theorem [Ratcliffe 1994] for the gluing of a complete hyperbolic 3-manifold. These 107 side-pairings for Q^3 fall into 20 equivalence classes under equivalence by a symmetry of Q^3. The classification of the hyperbolic 3-manifolds that correspond to these side-pairings of Q^3 is summarized in our next theorem.

Theorem 5. *There are, up to isometry, exactly* 13 *hyperbolic space-forms* H^3/Γ *where* Γ *is a torsion-free subgroup of minimal index in the congruence two subgroup* Γ_2^3 *of the group* Γ^3 *of integral, positive, Lorentzian* 4×4 *matrices. Only three of these manifolds are orientable.*

Proof. All the 107 side-pairings for Q^3 identify the 8 vertices $(\pm 1/3, \pm 1/3, \pm 1/3)$ of Q^3 to one point. Notice that each of the vertices $(\pm 1/3, \pm 1/3, \pm 1/3)$ is the corner vertex of a right-angled corner (like the corner of a room). This suggests a cut and paste operation on Q^3. Cut Q^3 along the three coordinate planes into 8 copies of the polyhedron P^3, turn around each of the 8 copies of P^3, and reassemble the polyhedron Q^3 according to the gluing pattern of the side-pairing of Q^3 so that the vertices $(\pm 1/3, \pm 1/3, \pm 1/3)$ are all glued together at $(0,0,0)$. We call this an *inside-out operation* on Q^3. Each of the 107 side-pairings of Q^3 induces a new side-pairing on Q^3 after an inside-out operation on Q^3 that yields the same manifold. After comparing the new side-pairings with the old ones, up to symmetry of Q^3, the number of manifolds is reduced to 13. Table 1 lists side-pairings and isometric invariants for the 13 manifolds.

We denote the manifolds in Table 1 by $M_1^3, \ldots, M_{13}^3$ indexed by the row number in the column that says N. The column headed by SP describes the side-pairing of Q^3 in a coded form that will be explained in Section 5. The column headed by O indicates the orientability of the manifolds with 1 for orientable and 0 for nonorientable. A manifold in Table 1 is orientable if and only if all the side-pairing transformations of the corresponding side-pairing of Q^3 are orientation preserving.

The column of Table 1 headed by C lists the number of cusps of the manifolds. The link of each cusp is either a torus or a Klein bottle. The column headed by LT indicates the link type of each cusp with T representing a torus and K a Klein bottle. The column headed by S lists the number of symmetries of the manifold. The column headed by H_1 lists the first homology groups of the manifolds with the 3 digit number abc representing $\mathbb{Z}^a \oplus \mathbb{Z}_2^b \oplus \mathbb{Z}_4^c$. The column headed by H_2 lists the second homology groups of the manifolds with the entry a representing $\mathbb{Z}^a$. Notice that the 4-cusped manifolds are classified by their first homology groups.

N	SP	O	C	S	H_1	H_2	LT
1	142	1	3	48	300	2	TTT
2	147	1	3	16	300	2	TTT
3	143	0	3	8	300	2	KTT
4	156	0	3	8	300	2	KTT
5	357	0	3	16	220	1	KKT
6	136	0	3	8	220	1	KKT
7	153	0	3	16	201	1	KKT
8	157	0	3	8	201	1	KKT
9	367	0	3	8	102	0	KKK
10	174	1	4	64	400	3	TTTT
11	134	0	4	16	310	2	KKTT
12	165	0	4	8	220	1	KKKT
13	135	0	4	16	121	0	KKKK

TABLE 1. Minimal volume, integral, congruence 2, hyperbolic 3-manifolds.

The side-pairings of Q^3 for the 3-cusped manifolds have the property that opposite ideal vertices of Q^3 are identified. Consequently, each boundary of a maximum open cusp of a 3-cusped manifold is tangent to itself in the two points represented by $(0,0,0)$ and $(\pm 1/3, \pm 1/3, \pm 1/3)$. Therefore these two points are canonical points of the manifold.

The ball centered at $(0,0,0)$ inscribed in Q^3 meets the boundary of Q^3 in the centers of the 12 sides of Q^3. Let M be one of the 3-cusped manifolds and let c be the point of M represented by $(0,0,0)$. Then the boundary of the maximum open ball centered at c in M is tangent to itself in the six point $c_1, \dots, c_6$ represented by the centers of the sides of Q^3.

Let $\varphi : M \to M'$ be an isometry from M to the 3-cusped manifold M'. Let c' be the point of M' represented by $(0,0,0)$. By applying an inside-out operation to Q^3 if $\varphi(c) \neq c'$, we may assume that $\varphi(c) = c'$. Let $c'_1, \dots, c'_6$ be the six points of M' represented by the centers of the sides of Q^3. Then φ must map the set $\{c_1, \dots, c_6\}$ to the set $\{c'_1, \dots, c'_6\}$. Consequently φ lifts to a symmetry of Q^3. Thus, an isometry between two 3-cusped manifolds is induced either by a symmetry of Q^3 or by an inside-out operation on Q^3 followed by a symmetry of Q^3. As the 13 manifolds are already classified up to such an isometry, the classification is complete. □

We next show that the three orientable manifolds M_1^3, M_2^3, M_{10}^3 are homeomorphic to complements of links in the 3-sphere. When we pass to the upper half space model U^3 of hyperbolic 3-space, the vertices of Q^3 become $(0,0,0)$, $(\pm 1,0,0)$, $(0,\pm 1,0)$, $(\pm 1/3, \pm 1/3, 1/3)$, $(\pm 1, \pm 1, 1)$, and ∞. We identify the boundary plane of U^3 with the complex plane $\mathbb{C}$. Then the orientation preserving isometries of U^3 correspond to the elements of the group $\mathrm{PSL}(2,\mathbb{C})$ and the side-pairing transformations for the manifolds M_1^3, M_2^3, M_{10}^3 in Table 1 correspond to elements of the Picard group $\mathrm{PSL}(2,\mathbb{Z}[i])$. Consequently, the manifolds M_1^3, M_2^3, M_{10}^3 correspond to torsion-free subgroups of the Picard group.

The hyperbolic manifolds $M_1^3, \dots, M_{13}^3$ all have the same volume as Q^3. The polyhedron Q^3 is built up from 96 copies of the tetrahedron Δ^3, so its volume equals $96 \operatorname{Vol}(\Delta^3)$. According to Bianchi [1892, § 12], the Picard group has a natural extension, by the Klein four group $\{\pm 1, \pm i\}$, that is isomorphic to Γ^3. Therefore, we have

$$\operatorname{Vol}(U^3/\mathrm{PSL}(2,\mathbb{Z}[i])) = 4 \operatorname{Vol}(\Delta^3).$$

Hence, we have

$$\operatorname{Vol}(Q^3) = 24 \operatorname{Vol}(U^3/\mathrm{PSL}(2,\mathbb{Z}[i])).$$

Therefore, the manifolds M_1^3, M_2^3, M_{10}^3 correspond to torsion-free subgroups of the Picard group of index 24. Milnor [1982] has shown that

$$\operatorname{Vol}(U^3/\mathrm{PSL}(2,\mathbb{Z}[i])) = \tfrac{2}{3}\mathcal{L}(\pi/4),$$

where $\mathcal{L}$ is the Lobachevsky function [Milnor 1982]. Therefore, we have

$$\operatorname{Vol}(Q^3) = 16\mathcal{L}(\pi/4) = 7.3277247\dots.$$

From the classification of all the index 24 torsion-free subgroups of the Picard group given by Brunner, Frame, Lee, and Wielenberg [Brunner et al. 1984], we deduce that the 4-cusped manifold M_{10}^3 is homeomorphic to the link complement 8_2^4. One can also derive a presentation for the fundamental group of M_{10}^3 from the side-pairing for M_{10}^3 in Table 1 and transform the presentation into the presentation for the group of the link complement 8_2^4 given by Wielenberg [1978]. It then follows from Mostow's rigidity theorem that M_{10}^3 is homeomorphic to the link complement 8_2^4.

Cutting off the top half $(T^3)'$ of each of the 8 copies of P^3 in Q^3 leaves a regular ideal octahedron. The eight copies of the tetrahedron $(T^3)'$ can be assembled around their corner points to form a regular ideal octahedron. Therefore each of the manifolds $M_1^3, \dots, M_{13}^3$ can be obtained by gluing together two regular ideal octahedrons along their sides. The manifold M_2^3 has an inside-out symmetry that interchanges the two octahedrons and has no fixed points. The quotient space under the action of this inside-out symmetry of M_2^3 is the Whitehead link complement obtained by gluing together the sides of a regular octahedron as in [Thurston 1997]. From the classification of all the index 24 torsion-free subgroups of the Picard group [Brunner et al. 1984], we deduce that M_2^3 is homeomorphic to the link complement 8_9^3. One can also derive a presentation for the fundamental group of M_2^3 from the side-pairing for M_2^3 in Table 1 and transform the presentation into the presentation for the group of the link complement 8_9^3 given by Wielenberg [1978].

It then follows from Mostow's rigidity theorem that M_2^3 is homeomorphic to the link complement 8_9^3.

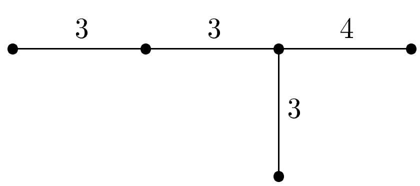
The manifold M_1^3 is homeomorphic to the link complement 6_2^3 (Borromean rings), since the corresponding side-pairing of two regular ideal octahedrons is the one given by Thurston [1997]. The side-pairing of Q^3 in Table 1 for the Borromean rings has also been described by Hilden, Lozano, and Montesinos [Hilden et al. 1992].

Theorem 6. *The congruence two group Γ_2^3 has a torsion-free subgroup of index i if and only if i is divisible by* 8.

Proof. We have already shown that every torsion-free subgroup of Γ_2^3 has index divisible by 8. From Table 2 we see that Γ_2^3 has a torsion-free subgroup Γ whose first homology group has an infinite cyclic summand. Therefore Γ maps homomorphically onto $\mathbb{Z}$. Hence Γ has a subgroup of index i for each positive integer i. Therefore Γ_2^3 has a torsion-free subgroup of index $8j$ for each positive integer j. □

4. INTEGRAL, CONGRUENCE TWO, HYPERBOLIC 4-MANIFOLDS

In this section, we determine the structure of the congruence two subgroup Γ_2^4 of the group Γ^4 of integral, positive, Lorentzian 5×5 matrices and classify all the integral, congruence two, hyperbolic 4-manifolds of minimum volume. Vinberg [1967] has proved that the group Γ^4 is a reflection group with respect to a noncompact 4-simplex Δ^4 in H^4 whose Coxeter diagram is



Vertices for Δ^4 are

$$(0,0,0,0,1),$$
$$(\sqrt{6}/6, \sqrt{6}/6, \sqrt{6}/6, 0, \sqrt{6}/2),$$
$$(\sqrt{2}/2, \sqrt{2}/2, 0, 0, \sqrt{2}),$$
$$(\sqrt{5}/5, \sqrt{5}/5, \sqrt{5}/5, \sqrt{5}/5, 3\sqrt{5}/5),$$
$$(1,0,0,0,1) \quad \text{(at infinity)}.$$

The group Γ^4 is generated by the five matrices

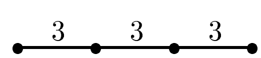
$$\begin{pmatrix} 0&1&0&0&0\\ 1&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0&0\\ 0&0&1&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&0&0&0&1 \end{pmatrix},$$

$$\begin{pmatrix} 0&-1&-1&0&1\\ -1&0&-1&0&1\\ -1&-1&0&0&1\\ 0&0&0&1&0\\ -1&-1&-1&0&2 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&-1&0\\ 0&0&0&0&1 \end{pmatrix},$$

which represent the reflections in the sides of Δ^4. By mapping these matrices into $\mathrm{GL}(5, \mathbb{Z}/2\mathbb{Z})$ and computing the order of the group that their images generate, we deduce that the index of Γ_2^4 in Γ^4 is 120.

Let Σ^4 be the group generated the first four matrices in the above list of matrices. These are the generators of Γ^4 that project to nonzero elements of $\mathrm{GL}(5, \mathbb{Z}/2\mathbb{Z})$. Then Σ^4 is the group generated by the reflections in the sides of Δ^4 incident with the vertex $(\sqrt{5}/5, \sqrt{5}/5, \sqrt{5}/5, \sqrt{5}/5, 3\sqrt{5}/5)$. Thus Σ^4 is isomorphic to a spherical tetrahedral reflection group whose Coxeter diagram is obtained from the Coxeter diagram of Γ_2^4 by deleting its right most vertex and its adjoining edge. Hence Σ^4 has the Coxeter diagram



Therefore Σ^4 is isomorphic to the group of symmetries of a regular 4-simplex and so Σ^4 is isomorphic to the symmetric group S_5. Thus Σ^4 has order 120.

Now Σ^4 injects into $\mathrm{GL}(5, \mathbb{Z}/2\mathbb{Z})$, since it has the same order as its image. Hence Σ^4 is a set of coset representatives for Γ_2^4 in Γ^4. We therefore have a natural, split, short, exact sequence of groups

$$1 \to \Gamma_2^4 \to \Gamma^4 \to \Sigma^4 \to 1.$$

We now pass to the conformal ball model B^4 of hyperbolic 4-space. The vertices of the simplex Δ^4 are now (0,0,0,0), $(1-\sqrt{6}/3, 1-\sqrt{6}/3, 1-\sqrt{6}/3, 0)$, $(1-\sqrt{2}/2, 1-\sqrt{2}/2, 0, 0)$, $((3-\sqrt{5})/4, (3-\sqrt{5})/4, (3-\sqrt{5})/4, (3-\sqrt{5})/4)$, and $(1,0,0,0)$.

Let Q^4 be the regular ideal 24-cell in B^4 with vertices $(\pm1,0,0,0)$, $(0,\pm1,0,0)$, $(0,0,\pm1,0)$, $(0,0,0,\pm1)$, and $(\pm1/2, \pm1/2, \pm1/2, \pm1/2)$. The dihedral angles of Q^4 are all $\pi/2$. Let P^4 be the intersection of Q^4 with the positive hexadecant of $\mathbb{R}^4$. Then P^4 is a noncompact convex polytope with actual vertices $(0,0,0,0)$, $(0,1/3,1/3,1/3)$, $(1/3,0,1/3,1/3)$, $(1/3,1/3,0,1/3)$, $(1/3,1/3,1/3,0)$ and ideal vertices $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, and $(1/2,1/2,1/2,1/2)$. Let T^4 be the 4-simplex with vertices $(0,0,0,0)$, $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$. Call T^4 a 4-dimensional corner simplex, with (0,0,0,0) the corner vertex of T^4, and each side of T^4 incident with $(0,0,0,0)$ a corner side of T^4.

The ideal vertices of P^4 are the vertices of a regular 4-simplex S^4 in B^4. The polytope P^4 is obtained from S^4 by gluing on to each side of S^4 a 4-dimensional corner simplex whose corner vertex is an actual vertex of P^4. Each corner simplex has 4 corner sides and each corner side matches up with a corner side of an adjacent corner simplex to give a total of $4\cdot 5/2 = 10$ sides of P^4. The dihedral angles of P^4 are all $\pi/2$. Observe that $P^4 = \Sigma^4\Delta^4$ and Σ^4 is the group of symmetries of S^4 and of P^4.

The Lorentzian matrices that represent the reflections in the sides of P^4 are

$$\begin{pmatrix} -1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0&0\\ 0&-1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&-1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&-1&0\\ 0&0&0&0&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&-1&-2&2\\ 0&0&-2&-1&2\\ 0&0&-2&-2&3 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0&0\\ 0&-1&0&-2&2\\ 0&0&1&0&0\\ 0&-2&0&-1&2\\ 0&-2&0&-2&3 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&0&0\\ 0&-1&-2&0&2\\ 0&-2&-1&0&2\\ 0&0&0&1&0\\ 0&-2&-2&0&3 \end{pmatrix}, \quad \begin{pmatrix} -1&0&0&-2&2\\ 0&1&0&0&0\\ 0&0&1&0&0\\ -2&0&0&-1&2\\ -2&0&0&-2&3 \end{pmatrix},$$

$$\begin{pmatrix} -1&0&-2&0&2\\ 0&1&0&0&0\\ -2&0&-1&0&2\\ 0&0&0&1&0\\ -2&0&-2&0&3 \end{pmatrix}, \quad \begin{pmatrix} -1&-2&0&0&2\\ -2&-1&0&0&2\\ 0&0&1&0&0\\ 0&0&0&1&0\\ -2&-2&0&0&3 \end{pmatrix}.$$

These matrices are all in Γ^4_2. Now since Σ^4 is a set of coset representatives for Γ^4_2 in Γ^4, we have that $P^4 = \Sigma^4\Delta^4$ is a fundamental polytope for Γ^4_2. We therefore have the following theorem.

Theorem 7. *The congruence two subgroup Γ^4_2 of the group Γ^4 of integral, positive, Lorentzian 5×5 matrices is a reflection group with respect to the polytope P^4.*

Let K^4 be the elementary 2-group of order 16 generated by the 4 reflections in the coordinate hyperplanes of $\mathbb{R}^4$. We shall identify K^4 with the corresponding subgroup of Γ^4 generated by the first 4 matrices in last displayed list of matrices. The next corollary follows immediately from Theorem 7.

Corollary 3. *Every nonidentity element of Γ^4_2 of finite order has order two, every finite subgroup of Γ^4_2 is conjugate in Γ^4 to a subgroup of the elementary 2-group K^4, and there are 5 conjugacy classes of maximal finite subgroups of Γ^4_2 in Γ^4_2 corresponding to the 5 actual vertices of P^4.*

Let Γ be a torsion-free subgroup of Γ^4_2 of finite index. Then the group K^4 acts freely on the set of cosets of Γ in Γ^4_2 by $g\Gamma \mapsto kg\Gamma$, since Γ is torsion-free. Therefore $|K^4| = 16$ divides $[\Gamma^4_2 : \Gamma]$.

Now suppose that $[\Gamma^4_2 : \Gamma] = 16$. Then the regular ideal 24-cell $Q^4 = K^4P^3$ is a fundamental polytope for Γ. The polytope Q^4 has 24 sides each of which is a regular ideal octahedron.

Let S be a side of Q^4. Then there is a nonidentity element g of Γ that pairs a side S' of Q^4 to S. Let k be the element of K^4 that maps S' to S, and let r be the reflection in the side S. Then gkr leaves S invariant. The elements of Γ^4_2 that leave S invariant are the identity, the reflection r, the reflections s and t in the two coordinate hyperplanes perpendicular to

S, the 180° rotations $rs = sr$, $rt = tr$, and $st = ts$, and the involution rst. Note that both s and t are in K^4. We cannot have gkr equal to r or sr or tr or str, since g has infinite order. Therefore gkr equals 1 or s or t or st. Hence, the side-pairing transformations of Q^4 in Γ are of the form rk where k is in K^4 and r is the reflection in a side of Q^4.

Now Γ is generated by the side-pairing transformations for Q^4, and so Γ is determined by the side-pairing of Q^4. The side-pairing for Q^4 restricts to a side-pairing for each of the 4 copies of Q^3 obtained from Q^4 by intersecting Q^4 with a coordinate hyperplane. Extending the 107 possible side-pairings for Q^3 that yield an integral, congruence two, hyperbolic 3-manifold on each of the 4 copies of Q^3 in Q^4, in a consistent fashion, yields 179625 side-pairings for Q^4 of the above form. Exactly 137075 of these satisfy the conditions of Poincaré's fundamental polyhedron theorem [Ratcliffe 1994] for the gluing of a complete hyperbolic 4-manifold. These 137075 side-pairings for Q^4 fall into 5757 equivalence classes under equivalence by a symmetry of Q^4. The classification of these hyperbolic 4-manifolds is summarized in our next theorem.

Theorem 8. *There are, up to isometry, exactly* 1171 *hyperbolic space-forms* H^4/Γ *where* Γ *is a torsion-free subgroup of minimal index in the congruence two subgroup* Γ_2^4 *of the group* Γ^4 *of integral, positive, Lorentzian* 5×5 *matrices. Only* 22 *of these manifolds are orientable.*

Proof. We pass to the conformal ball model B^4 of hyperbolic 4-space and consider the 137075 side-pairings of the 24-cell Q^4 that yield an integral, congruence two, hyperbolic 4-manifold. Let e_1, e_2, e_3, e_4 be the standard basis vectors of $\mathbb{R}^4$. Each side-pairing of Q^4 induces an equivalence relation on the 24 ideal vertices of Q^4. The equivalence classes are called cycles. The cycle of an ideal vertex $v = \pm e_i$ of Q^4 is either just itself or itself and its antipodal vertex, since an element of K^4 either fixes v or maps v to $-v$. It turns out that the eight vertices $\pm e_1, \pm e_2, \pm e_3, \pm e_4$ fall into either four or five cycles of the form 2, 2, 2, 2 or 1, 1, 2, 2, 2. The remaining 16 ideal vertices of Q^4 either form one cycle or divide into two cycles of 8 vertices. The possible vertex cycle structures are 2, 2, 2, 2, 16 or 1, 1, 2, 2, 2, 16 or 2, 2, 2, 2, 8, 8. Thus all the integral, congruence two, hyperbolic 4-manifolds of minimum volume have 5 or 6 cusps. All the orientable manifolds have 5 cusps.

Let M^4 be a 5-cusped manifold. Its side-pairing has a vertex cycle structure 2, 2, 2, 2, 16. Consider a maximum open cusp of M^4 of vertex cycle order 2. Its boundary is tangent to itself at the point represented by the origin. Let $\pm e_i$ be the corresponding two vertices of Q^4. The horospheres based at $\pm e_i$ passing through the origin are also tangent to 24 edges of Q^4 at their Euclidean midpoints, and these points represent 3 more self-tangency points of the boundary of the cusp. Thus, the boundary of a maximum open cusp of M^4 of vertex cycle order 2 is tangent to itself at 4 points. The boundary of a maximum open cusp of M^4 of vertex cycle order 16 is also tangent to itself at 4 points. It turns out that all the self-tangency points of the maximal cusps of M^4 consist of only 5 points. Each of these 5 points is a self-tangency point of the boundary of only 4 of the maximal cusps. Thus M^4 has a set of 5 canonical points. The 5 canonical points of M^4 are represented by the 5 actual vertices of the polytope P^4.

Each actual vertex of P^4 is the vertex of a right-angled corner. This suggests a cut and paste operation on Q^4. Cut Q^4 along the 4 coordinate hyperplanes into 16 copies of the polytope P^4. By reassembling a 24-cell around a different corner of P^4 than the origin, we get possibly 5 different ways to glue up M^4. We call such a cut and paste operation an *inside-out operation* on M^4.

The 5757 equivalence classes of side-pairings of Q^4 under equivalence by symmetry of Q^4 split up into 5378 classes with 5 vertex cycles and 379 classes with 6 vertex cycles. By considering canonical points and inside-out operations as in the classification of the 3-cusped 3-manifolds in the proof of Theorem 5, the 5378 classes of side-pairings, with 5 vertex cycles, represent exactly 1090 isometry classes of 5-cusped hyperbolic 4-manifolds. Table 2 on page 117 lists side-pairings and isometric invariants for the 22 orientable 5-cusped manifolds. Table 3, starting on page 117, lists side-pairings and isometric invariants for the 1068 nonorientable 5-cusped manifolds.

Now let M^4 be a 6-cusped manifold. Assume first that it is glued up by a side-pairing of Q^4 with vertex cycle structure 1, 1, 2, 2, 2, 16. Take a maximum open cusp of M^4 of vertex cycle order 1. Its boundary is tangent to itself at 12 points. The maximum open cusps of vertex cycle order 2 or 16 are tangent to themselves in 4 points as before. Thus, the cusps of vertex cycle order 1 are intrinsically different from the cusps of vertex cycle order 2 or 16. In fact, the volume of a maximum cusp of vertex cycle order 1 is $16\sqrt{2}/3$ whereas the volume of a maximum cusp of vertex cycle order 2 or 16 is $16/3$.

Only one of the five self-tangency points of the boundaries of the maximum open cusps of vertex cycle order 2 or 16 is a self-tangency point of the boundaries of all 4 maximum open cusps of vertex cycle order 2 or 16. Thus M^4 has a single canonical point represented by one of the actual vertices of P^4 other than the origin. It turns out that by performing an inside-out operation, M^4 can also be glued up by a side-pairing of Q^4 that has the vertex cycle structure 2, 2, 2, 2, 8, 8. Here the cusps of vertex cycle order 1 correspond to the cusps of vertex cycle order 8 and the canonical point of M^4 is represented by the origin. The 12 self-tangency points of a maximum open cusp of vertex cycle order 8 are represented by the centers of the 24 sides of Q^4.

The 379 equivalence classes of side-pairings of Q^4 with 6 vertex cycles split up into 298 classes with vertex cycle structure 1, 1, 2, 2, 2, 16 and 81 classes with vertex cycle structure 2, 2, 2, 2, 8, 8. These 379 classes of side-pairings of Q^4 represent exactly 81 isometry classes of 6-cusped hyperbolic 4-manifolds corresponding to the 81 classes of side-pairings with vertex cycle structure 2, 2, 2, 2, 8, 8 by the same argument as before. This completes the classification of the integral, congruence two, hyperbolic 4-manifolds of minimum volume. Table 4 on page 124 lists side-pairings and isometric invariants for the 81 nonorientable 6-cusped manifolds. □

Theorem 9. *The congruence two group* Γ_2^4 *has a torsion-free subgroup of index* i *if and only if* i *is divisible by* 16.

Proof. We have already shown that every torsion-free subgroup of Γ_2^4 has index divisible by 16. From Table 2 we see that Γ_2^3 has a torsion-free subgroup Γ whose first homology group has an infinite cyclic summand. Therefore Γ maps homomorphically onto $\mathbb{Z}$. Hence Γ has a subgroup of index i for each positive integer i. Therefore Γ_2^4 has a torsion-free subgroup of index $16j$ for each positive integer j. □

By the Gauss–Bonnet theorem [Gromov 1982; Hopf 1926], the volume of a complete hyperbolic 4-manifold M of finite volume is given by

$$\mathrm{Vol}(M) = \frac{4\pi^2}{3}\chi(M),$$

where $\chi(M)$ is the Euler characteristic of M. All the 4-manifolds in Theorem 8 have Euler characteristic 1. Therefore all the hyperbolic 4-manifolds in Theorem 8 have minimum volume among all complete hyperbolic 4-manifolds. More generally, the volume spectrum of complete hyperbolic 4-manifolds is given by our last theorem.

Theorem 10. *There are complete, orientable, arithmetic, hyperbolic* 4*-manifolds of finite volume whose Euler characteristic is any given positive integer. Therefore, the volume spectrum of complete hyperbolic* 4*-manifolds of finite volume is the set of all positive integral multiples of* $4\pi^2/3$.

Proof. The first manifold in Table 2 has a positive first Betti number. Therefore it has an m-fold covering for each positive integer m. Thus, there are complete, orientable, arithmetic, hyperbolic 4-manifolds of finite volume whose Euler characteristic is any given positive integer m. □

All the hyperbolic 4-manifolds used to prove Theorem 10 are open. The volume spectrum of closed hyperbolic 4-manifolds is unknown. The closed orientable hyperbolic 4-manifold of smallest known volume is the Davis 120-cell space [Davis 1985] whose Euler characteristic is 26. We have constructed a closed nonorientable hyperbolic 4-manifold whose Euler characteristic is 17. This example will be discussed in a future paper.

5. SIDE-PAIRING CODING

In this section, we describe the coding that we use to list all the side-pairings of Q^n for the integral, congruence two, hyperbolic n-manifolds of minimum volume for $n = 2, 3, 4$. Reading this section is necessary only if the reader wants to reconstruct the manifolds in Tables 1, 2, 3, and 4.

We know that a minimal index torsion-free subgroup of Γ_2^n has as a fundamental domain Q^n which is an ideal square in dimension 2, a semi-ideal rhombic dodecahedron in dimension 3, or an ideal 24-cell in dimension 4. In each case, the side-pairing maps must be of the form rk, where k is in the group K^n generated by the reflections in the coordinate hyperplanes, and r is a reflection in a side of Q^n. Let $r_1, r_2, \ldots, r_m$ be the reflections in the sides of Q^n (in a fixed ordering to be specified below). Then to specify one of our manifolds it suffices to list the corresponding sequence $k_1, k_2, \ldots, k_m$ of elements of K^n such that the side-pairing maps are $r_i k_i$ for $i = 1, 2, \ldots, m$. It will turn out that we don't have to specify quite this much, and that in fact to get a manifold we must have

$$k_{4j+1} = k_{4j+2} = k_{4j+3} = k_{4j+4} \quad \text{for } j = 0, \ldots, (m/4) - 1$$

for a particular ordering of the sides. Thus in the end we will write out only every fourth k_i but internally we compute with the complete list of the k_i.

The elements of K^n are given by diagonal $(n+1) \times (n+1)$ matrices of the form

$$k = \operatorname{diag}(a_1, \ldots, a_n, 1) \quad \text{for } a_i = \pm 1.$$

We encode this k as a single (binary) number

$$\sum_{i=1}^{n} \frac{(1 - a_i)}{2}\, 2^{i-1}$$

so that a 1 entry in the matrix is a 0 bit and a -1 entry in the matrix is a 1 bit in the corresponding position in the binary number. Matrix multiplication then corresponds to bitwise mod 2 addition and the binary representation of the number corresponds to an element of $\mathbb{Z}_2^n$. The action of k in the first coordinate thus corresponds to the least significant bit of the code for k. We write the code numbers as a single hexadecimal digit with A = 10, B = 11, ..., F = 15 in dimension 4.

In each dimension, in the conformal ball model, each side of the fundamental domain Q^n will have ideal vertices on just two of the coordinate axes. Conversely, every pair of ideal vertices of the form $\pm e_i$ that are not antipodal to each other will determine a side of Q^n. We can express this in another way in $\mathbb{R}^{n,1}$ by noting that the sides of Q^n correspond to the vectors of the form $s = (a_1, \ldots, a_n, 1)$ such that exactly two of the a_i are ± 1 and the others (if any) are 0. Such a vector s is a unit normal vector of the corresponding side with respect to the Lorentzian inner product in $\mathbb{R}^{n,1}$. Note that two sides with unit normal vectors s and s' are adjacent if and only if $s \circ s' = 0$, since the dihedral angle between adjacent sides is $\pi/2$. Moreover, two sides with unit normal vectors s and s' are tangent at infinity if and only if $s \circ s' = -1$.

If r is the reflection in the side S, and rk is a side-pairing map, then the side mapped to side S by rk is kS. One of the conditions for a valid side-pairing must be that if rk is part of the side-pairing, r being the reflection in side S and r' is the reflection in the side $S' = kS$, then $S' \neq S$ and $r'k$ is also in the side-pairing, since it is the inverse of rk. Each side can only be paired to one of three others (those given by a unit normal vector s with the same zero coordinates) and, whichever one it is paired to, the other two of these sides will then be paired to each other, since the ideal vertices $\pm e_i$ can only be mapped to $\pm e_i$ by an element of K^n. This implies that each 2-dimensional coordinate cross-section of the 3-dimensional manifolds will be one of the 2-dimensional manifolds and each 3-dimensional coordinate cross-section of the 4-dimensional manifolds will be one of the 3-dimensional manifolds.

It remains to choose a particular sequence of the sides. We list on the next page (bottom left) the vectors $s_1, s_2, \ldots, s_m$ corresponding to the sides of Q^n. Then the reflections $r_1, r_2, \ldots, r_m$ in these sides are determined and a sequence $k_1, k_2, \ldots, k_m$ of elements of K^n (encoded as hexadecimal digits) will define side-pairing maps $r_i k_i$ giving a manifold. Each of the sets of four sides that must be paired with each other will be taken as a block of consecutive sides in our ordering. The blocks are determined by the nonzero coordinates in the s_i and we proceed starting with the first two coordinates nonzero and end with those pairs involving the last 3 coordinates nonzero in a lexicographic fashion.

The reflection r_1 in the side corresponding to s_1 is given by

$$\begin{pmatrix} -1 & -2 & 2 \\ -2 & -1 & 2 \\ -2 & -2 & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & -2 & 0 & 2 \\ -2 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ -2 & -2 & 0 & 3 \end{pmatrix}$$

in dimensions 2 and 3, respectively, and by

$$\begin{pmatrix} -1 & -2 & 0 & 0 & 2 \\ -2 & -1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 & 3 \end{pmatrix} \quad \text{in dimension 4.}$$

The other r_i can be obtained as $g^{-1}r_1g$ for some matrix $g \in \bar{K}^n$ with $gs_i = s_1$, where $\bar{K}^n$ is the group of order $2^n n!$ generated by K^n and all the permutation matrices that fix the last coordinate.

Starting with dimension 2 (and using the codes for the elements of K^2), we have $s_1 = (1,1,1)$, k_1 is not the identity (coded by 0) but can be any of 1, 2, or 3. If $k_1 = 1$, then $k_1s_1 = (-1,1,1) = s_2$, and $k_2 = 1$, leaving s_3 and s_4 to be paired so $k_3 = k_4 = 1$ also. If $k_1 = 2$, then $k_1s_1 = (1,-1,1) = s_3$, and $k_3 = 2$, leaving s_2 and s_4 to be paired so $k_2 = k_4 = 2$ as well. If $k_1 = 3$, then $k_1s_1 = (-1,-1,1) = s_4$, and $k_4 = 3$, and as before $k_2 = k_3 = 3$ also. Thus the complete codes for the three possible side-pairings of the ideal square Q^2 are 1111, 2222, and 3333, which we can abbreviate to just 1, 2, and 3. Now it turns out that side-pairings 1 and 2 are equivalent under a symmetry of Q^2 and we have two manifolds, 1 being the thrice-punctured sphere and 3 being the twice-punctured projective plane.

In dimension 3, each two-dimensional coordinate plane cross-section reduces to the two-dimensional case. In the xy-plane, $s_1 = (1,1,0,1)$ cannot be fixed by k_1 so $k_1 \neq 0,4$ but k_1 can be 1, 2, 3, 5, 6, or 7. Restricting to the 2-dimensional cross-section, the above reasoning tells us that if k_1 is 1 or 5 then $k_2 = k_1$ and $k_3 = k_4$ are also either 1 or 5. If k_1 is 2 or 6 then $k_1 = k_3$ and $k_2 = k_4$ are also 2 or 6, and if k_1 is 3 or 7 then $k_1 = k_4$ and $k_2 = k_3$ is 3 or 7. Note that maps 5, 6, and 7 reflect the z coordinate whereas the corresponding maps 1, 2, and 3 do not. Similar reasoning applies in the xz- and yz-planes. We build a table of the possible side-pairing maps for each block of four sides. In the xz case s_5 would be fixed by $k_5 = 0$ or 2, and in the yz case s_9 would be fixed by $k_9 = 0$ or 1.

k_1–k_4	k_5–k_8	k_9–k_{12}
1111	1111	2222
1155	1133	2233
2222	3311	3322
2626	3333	3333
3333	4444	4444
3773	4646	4545
5511	5555	5454
5555	5775	5555
6262	6464	6666
6666	6666	6776
7337	7557	7667
7777	7777	7777

There are 3 choices to be made with 12 alternatives each. We need to check which of the $12^3 = 1728$ combinations give manifolds. Since sides of the rhombic dodecahedron Q^3 are at right angles along edges, we need a side-pairing to give cycles of 4 edges. There are 8 actual vertices and these have to form a single cycle under the side-pairing. It turns out that these conditions imply that

$$k_1 = k_2 = k_3 = k_4,\; k_5 = k_6 = k_7 = k_8,\; k_9 = k_{10} = k_{11} = k_{12},$$

Dimension 2

$$\begin{aligned} s_1 &= (\;\;1,\;\;1,1) \\ s_2 &= (-1,\;\;1,1) \\ s_3 &= (\;\;1,-1,1) \\ s_4 &= (-1,-1,1) \end{aligned}$$

Dimension 3

$$\begin{aligned} s_1 &= (\;\;1,\;\;1,\;\;0,1) \\ s_2 &= (-1,\;\;1,\;\;0,1) \\ s_3 &= (\;\;1,-1,\;\;0,1) \\ s_4 &= (-1,-1,\;\;0,1) \\ s_5 &= (\;\;1,\;\;0,\;\;1,1) \\ s_6 &= (-1,\;\;0,\;\;1,1) \\ s_7 &= (\;\;1,\;\;0,-1,1) \\ s_8 &= (-1,\;\;0,-1,1) \\ s_9 &= (\;\;0,\;\;1,\;\;1,1) \\ s_{10} &= (\;\;0,-1,\;\;1,1) \\ s_{11} &= (\;\;0,\;\;1,-1,1) \\ s_{12} &= (\;\;0,-1,-1,1) \end{aligned}$$

Dimension 4

$$\begin{aligned} s_1 &= (\;\;1,\;\;1,\;\;0,\;\;0,1) \\ s_2 &= (-1,\;\;1,\;\;0,\;\;0,1) \\ s_3 &= (\;\;1,-1,\;\;0,\;\;0,1) \\ s_4 &= (-1,-1,\;\;0,\;\;0,1) \\ s_5 &= (\;\;1,\;\;0,\;\;1,\;\;0,1) \\ s_6 &= (-1,\;\;0,\;\;1,\;\;0,1) \\ s_7 &= (\;\;1,\;\;0,-1,\;\;0,1) \\ s_8 &= (-1,\;\;0,-1,\;\;0,1) \\ s_9 &= (\;\;0,\;\;1,\;\;1,\;\;0,1) \\ s_{10} &= (\;\;0,-1,\;\;1,\;\;0,1) \\ s_{11} &= (\;\;0,\;\;1,-1,\;\;0,1) \\ s_{12} &= (\;\;0,-1,-1,\;\;0,1) \\ s_{13} &= (\;\;1,\;\;0,\;\;0,\;\;1,1) \\ s_{14} &= (-1,\;\;0,\;\;0,\;\;1,1) \\ s_{15} &= (\;\;1,\;\;0,\;\;0,-1,1) \\ s_{16} &= (-1,\;\;0,\;\;0,-1,1) \\ s_{17} &= (\;\;0,\;\;1,\;\;0,\;\;1,1) \\ s_{18} &= (\;\;0,-1,\;\;0,\;\;1,1) \\ s_{19} &= (\;\;0,\;\;1,\;\;0,-1,1) \\ s_{20} &= (\;\;0,-1,\;\;0,-1,1) \\ s_{21} &= (\;\;0,\;\;0,\;\;1,\;\;1,1) \\ s_{22} &= (\;\;0,\;\;0,-1,\;\;1,1) \\ s_{23} &= (\;\;0,\;\;0,\;\;1,-1,1) \\ s_{24} &= (\;\;0,\;\;0,-1,-1,1) \end{aligned}$$

The vectors $s_1, s_2, \ldots, s_m$ corresponding to the sides of Q^n (see preceding page).

and that k_1, k_5, k_9 are linearly independent in the $\mathbb{Z}_2$-vector space K^3. Thus a list of only k_1, k_5, k_9 will suffice to determine the manifold.

There are then 107 possibilities left for k_1, k_5, k_9:

134	153	174	237	267	345	375	542	576	645	712	746
135	156	175	243	273	346	376	543	612	647	713	753
136	157	214	245	274	352	512	546	613	652	714	754
137	162	215	247	276	354	513	547	614	654	715	756
142	163	216	253	314	357	516	562	615	657	732	762
143	164	217	254	315	362	517	564	632	672	735	763
146	165	234	256	316	364	532	567	634	673	736	764
147	172	235	263	317	367	534	573	637	674	742	765
152	173	236	265	342	372	537	574	643	675	745	

This list includes side-pairings that are equivalent under symmetries of Q^3 as well as different side-pairings yielding isometric manifolds.

The 4-dimensional case is simplified by looking at the four coordinate hyperplane cross-sections. Each cross-section must reduce to one of the above 107 cases. We also have that the choices of side-pairings for the 3-dimensional cross-sections must give the same pairs of sides, that is, the common 2-dimensional cross-sections must be the same. Once we have chosen consistent cross-section side-pairings, the side-pairings of the ideal 24-cell will be determined. Then it becomes a matter of filtering the possible side-pairings according to the 3-face, 2-face, and 1-face cycle conditions of Poincaré's fundamental polyhedron theorem, and then reducing the list by symmetries of the 24-cell and hidden isometries between the resulting manifolds. Since each three-dimensional cross-section has equal k_i for each of the sides s_i intersecting a 2-dimensional cross-section, the blocks of 4 sides intersecting each 2-dimensional cross-section will also have equal k_i in the 4-dimensional case. Thus a list of only

$$k_1, k_5, k_9, k_{13}, k_{17}, k_{21}$$

will suffice to determine the manifold, since

$$k_{4i+1} = k_{4i+2} = k_{4i+3} = k_{4i+4} \quad \text{for } i = 0, 1, 2, 3, 4, 5.$$

We now describe in detail how to extract a side-pairing of our fundamental domain Q^n from a coded side-pairing. Consider the coded side-pairing for the Borromean rings complement 142 on line 1 of Table 1. It represents the side-pairing

$$111144442222$$

for the 12 sides of the rhombic dodecahedron Q^3. Here

$$\begin{aligned} 1 &\text{ represents } \operatorname{diag}(-1,1,1,1),\\ 2 &\text{ represents } \operatorname{diag}(1,-1,1,1),\\ 4 &\text{ represents } \operatorname{diag}(1,1,-1,1). \end{aligned}$$

The set S_i of vertices of the side of Q^3 in $\mathbb{R}^3$ whose normal vector in $\mathbb{R}^{3,1}$ is s_i is given by

$$\begin{aligned}
S_1 &= \{(1,0,0), (\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}), (\tfrac{1}{3},\tfrac{1}{3},-\tfrac{1}{3}), (0,1,0)\}\\
S_2 &= \{(-1,0,0), (-\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}), (-\tfrac{1}{3},\tfrac{1}{3},-\tfrac{1}{3}), (0,1,0)\}\\
S_3 &= \{(1,0,0), (\tfrac{1}{3},-\tfrac{1}{3},\tfrac{1}{3}), (\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3}), (0,-1,0)\}\\
S_4 &= \{(-1,0,0), (-\tfrac{1}{3},-\tfrac{1}{3},\tfrac{1}{3}), (-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3}), (0,-1,0)\}\\
S_5 &= \{(1,0,0), (\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}), (\tfrac{1}{3},-\tfrac{1}{3},\tfrac{1}{3}), (0,0,1)\}\\
S_6 &= \{(-1,0,0), (-\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}), (-\tfrac{1}{3},-\tfrac{1}{3},\tfrac{1}{3}), (0,0,1)\}\\
S_7 &= \{(1,0,0), (\tfrac{1}{3},\tfrac{1}{3},-\tfrac{1}{3}), (\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3}), (0,0,-1)\}\\
S_8 &= \{(-1,0,0), (-\tfrac{1}{3},\tfrac{1}{3},-\tfrac{1}{3}), (-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3}), (0,0,-1)\}\\
S_9 &= \{(0,1,0), (\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}), (-\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}), (0,0,1)\}\\
S_{10} &= \{(0,-1,0), (\tfrac{1}{3},-\tfrac{1}{3},\tfrac{1}{3}), (-\tfrac{1}{3},-\tfrac{1}{3},\tfrac{1}{3}), (0,0,1)\}\\
S_{11} &= \{(0,1,0), (\tfrac{1}{3},\tfrac{1}{3},-\tfrac{1}{3}), (-\tfrac{1}{3},\tfrac{1}{3},-\tfrac{1}{3}), (0,0,-1)\}\\
S_{12} &= \{(0,-1,0), (\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3}), (-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3}), (0,0,-1)\}
\end{aligned}$$

The corresponding ordered sets of vertices of the paired sides are given by

$$\begin{aligned}
S'_1 &= \{(-1,0,0), (-\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}), (-\tfrac{1}{3},\tfrac{1}{3},-\tfrac{1}{3}), (0,1,0)\}\\
S'_2 &= \{(1,0,0), (\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}), (\tfrac{1}{3},\tfrac{1}{3},-\tfrac{1}{3}), (0,1,0)\}\\
S'_3 &= \{(-1,0,0), (-\tfrac{1}{3},-\tfrac{1}{3},\tfrac{1}{3}), (-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3}), (0,-1,0)\}\\
S'_4 &= \{(1,0,0), (\tfrac{1}{3},-\tfrac{1}{3},\tfrac{1}{3}), (\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3}), (0,-1,0)\}\\
S'_5 &= \{(1,0,0), (\tfrac{1}{3},\tfrac{1}{3},-\tfrac{1}{3}), (\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3}), (0,0,-1)\}\\
S'_6 &= \{(-1,0,0), (-\tfrac{1}{3},\tfrac{1}{3},-\tfrac{1}{3}), (-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3}), (0,0,-1)\}\\
S'_7 &= \{(1,0,0), (\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}), (\tfrac{1}{3},-\tfrac{1}{3},\tfrac{1}{3}), (0,0,1)\}\\
S'_8 &= \{(-1,0,0), (-\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}), (-\tfrac{1}{3},-\tfrac{1}{3},\tfrac{1}{3}), (0,0,1)\}\\
S'_9 &= \{(0,-1,0), (\tfrac{1}{3},-\tfrac{1}{3},\tfrac{1}{3}), (-\tfrac{1}{3},-\tfrac{1}{3},\tfrac{1}{3}), (0,0,1)\}\\
S'_{10} &= \{(0,1,0), (\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}), (-\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}), (0,0,1)\}\\
S'_{11} &= \{(0,-1,0), (\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3}), (-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3}), (0,0,-1)\}\\
S'_{12} &= \{(0,1,0), (\tfrac{1}{3},\tfrac{1}{3},-\tfrac{1}{3}), (-\tfrac{1}{3},\tfrac{1}{3},-\tfrac{1}{3}), (0,0,-1)\}
\end{aligned}$$

Here we have $S'_i = k_i S_i$ where $k_i = \operatorname{diag}(a_1, a_2, a_3, 1)$ acts on S_i as a 3×3 matrix $\operatorname{diag}(a_1, a_2, a_3)$.

$$\begin{aligned}
S_1 &= \{(1,0,0,0),(1,1,1,1)/2,(1,1,-1,1)/2,(1,1,1,-1)/2,(1,1,-1,-1)/2,(0,1,0,0)\}\\
S_2 &= \{(-1,0,0,0),(-1,1,1,1)/2,(-1,1,-1,1)/2,(-1,1,1,-1)/2,(-1,1,-1,-1)/2,(0,1,0,0)\}\\
S_3 &= \{(1,0,0,0),(1,-1,1,1)/2,(1,-1,-1,1)/2,(1,-1,1,-1)/2,(1,-1,-1,-1)/2,(0,-1,0,0)\}\\
S_4 &= \{(-1,0,0,0),(-1,-1,1,1)/2,(-1,-1,-1,1)/2,(-1,-1,1,-1)/2,(-1,-1,-1,-1)/2,(0,-1,0,0)\}\\
S_5 &= \{(1,0,0,0),(1,1,1,1)/2,(1,-1,1,1)/2,(1,1,1,-1)/2,(1,-1,1,-1)/2,(0,0,1,0)\}\\
S_6 &= \{(-1,0,0,0),(-1,1,1,1)/2,(-1,-1,1,1)/2,(-1,1,1,-1)/2,(-1,-1,1,-1)/2,(0,0,1,0)\}\\
S_7 &= \{(1,0,0,0),(1,1,-1,1)/2,(1,-1,-1,1)/2,(1,1,-1,-1)/2,(1,-1,-1,-1)/2,(0,0,-1,0)\}\\
S_8 &= \{(-1,0,0,0),(-1,1,-1,1)/2,(-1,-1,-1,1)/2,(-1,1,-1,-1)/2,(-1,-1,-1,-1)/2,(0,0,-1,0)\}\\
S_9 &= \{(0,1,0,0),(1,1,1,1)/2,(-1,1,1,1)/2,(1,1,1,-1)/2,(-1,1,1,-1)/2,(0,0,1,0)\}\\
S_{10} &= \{(0,-1,0,0),(1,-1,1,1)/2,(-1,-1,1,1)/2,(1,-1,1,-1)/2,(-1,-1,1,-1)/2,(0,0,1,0)\}\\
S_{11} &= \{(0,1,0,0),(1,1,-1,1)/2,(-1,1,-1,1)/2,(1,1,-1,-1)/2,(-1,1,-1,-1)/2,(0,0,-1,0)\}\\
S_{12} &= \{(0,-1,0,0),(1,-1,-1,1)/2,(-1,-1,-1,1)/2,(1,-1,-1,-1)/2,(-1,-1,-1,-1)/2,(0,0,-1,0)\}\\
S_{13} &= \{(1,0,0,0),(1,1,1,1)/2,(1,-1,1,1)/2,(1,1,-1,1)/2,(1,-1,-1,1)/2,(0,0,0,1)\}\\
S_{14} &= \{(-1,0,0,0),(-1,1,1,1)/2,(-1,-1,1,1)/2,(-1,1,-1,1)/2,(-1,-1,-1,1)/2,(0,0,0,1)\}\\
S_{15} &= \{(1,0,0,0),(1,1,1,-1)/2,(1,-1,1,-1)/2,(1,1,-1,-1)/2,(1,-1,-1,-1)/2,(0,0,0,-1)\}\\
S_{16} &= \{(-1,0,0,0),(-1,1,1,-1)/2,(-1,-1,1,-1)/2,(-1,1,-1,-1)/2,(-1,-1,-1,-1)/2,(0,0,0,-1)\}\\
S_{17} &= \{(0,1,0,0),(1,1,1,1)/2,(-1,1,1,1)/2,(1,1,-1,1)/2,(-1,1,-1,1)/2,(0,0,0,1)\}\\
S_{18} &= \{(0,-1,0,0),(1,-1,1,1)/2,(-1,-1,1,1)/2,(1,-1,-1,1)/2,(-1,-1,-1,1)/2,(0,0,0,1)\}\\
S_{19} &= \{(0,1,0,0),(1,1,1,-1)/2,(-1,1,1,-1)/2,(1,1,-1,-1)/2,(-1,1,-1,-1)/2,(0,0,0,-1)\}\\
S_{20} &= \{(0,-1,0,0),(1,-1,1,-1)/2,(-1,-1,1,-1)/2,(1,-1,-1,-1)/2,(-1,-1,-1,-1)/2,(0,0,0,-1)\}\\
S_{21} &= \{(0,0,1,0),(1,1,1,1)/2,(-1,1,1,1)/2,(1,-1,1,1)/2,(-1,-1,1,1)/2,(0,0,0,1)\}\\
S_{22} &= \{(0,0,-1,0),(1,1,-1,1)/2,(-1,1,-1,1)/2,(1,-1,-1,1)/2,(-1,-1,-1,1)/2,(0,0,0,1)\}\\
S_{23} &= \{(0,0,1,0),(1,1,1,-1)/2,(-1,1,1,-1)/2,(1,-1,1,-1)/2,(-1,-1,1,-1)/2,(0,0,0,-1)\}\\
S_{24} &= \{(0,0,-1,0),(1,1,-1,-1)/2,(-1,1,-1,-1)/2,(1,-1,-1,-1)/2,(-1,-1,-1,-1)/2,(0,0,0,-1)\}
\end{aligned}$$

$$\begin{aligned}
S'_1 &= \{(-1,0,0,0),(-1,1,1,1)/2,(-1,1,-1,1)/2,(-1,1,1,-1)/2,(-1,1,-1,-1)/2,(0,1,0,0)\}\\
S'_2 &= \{(1,0,0,0),(1,1,1,1)/2,(1,1,-1,1)/2,(1,1,1,-1)/2,(1,1,-1,-1)/2,(0,1,0,0)\}\\
S'_3 &= \{(-1,0,0,0),(-1,-1,1,1)/2,(-1,-1,-1,1)/2,(-1,-1,1,-1)/2,(-1,-1,-1,-1)/2,(0,-1,0,0)\}\\
S'_4 &= \{(1,0,0,0),(1,-1,1,1)/2,(1,-1,-1,1)/2,(1,-1,1,-1)/2,(1,-1,-1,-1)/2,(0,-1,0,0)\}\\
S'_5 &= \{(1,0,0,0),(1,1,-1,1)/2,(1,-1,-1,1)/2,(1,1,-1,-1)/2,(1,-1,-1,-1)/2,(0,0,-1,0)\}\\
S'_6 &= \{(-1,0,0,0),(-1,1,-1,1)/2,(-1,-1,-1,1)/2,(-1,1,-1,-1)/2,(-1,-1,-1,-1)/2,(0,0,-1,0)\}\\
S'_7 &= \{(1,0,0,0),(1,1,1,1)/2,(1,-1,1,1)/2,(1,1,1,-1)/2,(1,-1,1,-1)/2,(0,0,1,0)\}\\
S'_8 &= \{(-1,0,0,0),(-1,1,1,1)/2,(-1,-1,1,1)/2,(-1,1,1,-1)/2,(-1,-1,1,-1)/2,(0,0,1,0)\}\\
S'_9 &= \{(0,-1,0,0),(1,-1,1,1)/2,(-1,-1,1,1)/2,(1,-1,1,-1)/2,(-1,-1,1,-1)/2,(0,0,1,0)\}\\
S'_{10} &= \{(0,1,0,0),(1,1,1,1)/2,(-1,1,1,1)/2,(1,1,1,-1)/2,(-1,1,1,-1)/2,(0,0,1,0)\}\\
S'_{11} &= \{(0,-1,0,0),(1,-1,-1,1)/2,(-1,-1,-1,1)/2,(1,-1,-1,-1)/2,(-1,-1,-1,-1)/2,(0,0,-1,0)\}\\
S'_{12} &= \{(0,1,0,0),(1,1,-1,1)/2,(-1,1,-1,1)/2,(1,1,-1,-1)/2,(-1,1,-1,-1)/2,(0,0,-1,0)\}\\
S'_{13} &= \{(1,0,0,0),(1,1,1,-1)/2,(1,-1,1,-1)/2,(1,1,-1,-1)/2,(1,-1,-1,-1)/2,(0,0,0,-1)\}\\
S'_{14} &= \{(-1,0,0,0),(-1,1,1,-1)/2,(-1,-1,1,-1)/2,(-1,1,-1,-1)/2,(-1,-1,-1,-1)/2,(0,0,0,-1)\}\\
S'_{15} &= \{(1,0,0,0),(1,1,1,1)/2,(1,-1,1,1)/2,(1,1,-1,1)/2,(1,-1,-1,1)/2,(0,0,0,1)\}\\
S'_{16} &= \{(-1,0,0,0),(-1,1,1,1)/2,(-1,-1,1,1)/2,(-1,1,-1,1)/2,(-1,-1,-1,1)/2,(0,0,0,1)\}\\
S'_{17} &= \{(0,-1,0,0),(-1,-1,1,-1)/2,(1,-1,1,-1)/2,(-1,-1,-1,-1)/2,(1,-1,-1,-1)/2,(0,0,0,-1)\}\\
S'_{18} &= \{(0,1,0,0),(-1,1,1,-1)/2,(1,1,1,-1)/2,(-1,1,-1,-1)/2,(1,1,-1,-1)/2,(0,0,0,-1)\}\\
S'_{19} &= \{(0,-1,0,0),(-1,-1,1,1)/2,(1,-1,1,1)/2,(-1,-1,-1,1)/2,(1,-1,-1,1)/2,(0,0,0,1)\}\\
S'_{20} &= \{(0,1,0,0),(-1,1,1,1)/2,(1,1,1,1)/2,(-1,1,-1,1)/2,(1,1,-1,1)/2,(0,0,0,1)\}\\
S'_{21} &= \{(0,0,-1,0),(-1,1,-1,-1)/2,(1,1,-1,-1)/2,(-1,-1,-1,-1)/2,(1,-1,-1,-1)/2,(0,0,0,-1)\}\\
S'_{22} &= \{(0,0,1,0),(-1,1,1,-1)/2,(1,1,1,-1)/2,(-1,-1,1,-1)/2,(1,-1,1,-1)/2,(0,0,0,-1)\}\\
S'_{23} &= \{(0,0,-1,0),(-1,1,-1,1)/2,(1,1,-1,1)/2,(-1,-1,-1,1)/2,(1,-1,-1,1)/2,(0,0,0,1)\}\\
S'_{24} &= \{(0,0,1,0),(-1,1,1,1)/2,(1,1,1,1)/2,(-1,-1,1,1)/2,(1,-1,1,1)/2,(0,0,0,1)\}
\end{aligned}$$

N	SP	S	H_1	H_2	H_3	LT
1	1428BD	16	330	700	4	AAABF
2	14278D	16	240	600	4	AABBF
3	1477B8	16	240	600	4	AABBF
4	1477BE	16	240	600	4	AABBF
5	1478ED	16	240	600	4	AABBF
6	14278E	16	240	600	4	ABBBF
7	142DBE	48	150	500	4	ABBBF
8	1427BD	16	150	500	4	ABBBF
9	1477EB	16	150	500	4	ABBBF
10	1477ED	16	150	500	4	ABBBF
11	1478EB	16	150	500	4	ABBBF
12	147BDE	16	150	500	4	ABBBF
13	14B8ED	16	150	500	4	ABBBF
14	1427BE	16	150	500	4	BBBBF
15	1477DE	16	150	500	4	BBBBF
16	14B7E8	16	060	400	4	BBBBF
17	14B7ED	16	060	400	4	BBBBF
18	14BDE7	16	060	400	4	BBBBF
19	14B7DE	16	060	400	4	BBFFF
20	14B8E7	16	051	400	4	ABFFF
21	14BD7E	16	051	400	4	ABFFF
22	17BE8D	16	051	400	4	ABFFF

TABLE 2. Orientable, 5 cusped, minimal volume, integral, congruence 2, hyperbolic 4-manifolds.

Now take the first orientable 4-manifold in Table 2 above. The hexadecimal digits in the column headed SP are interpreted as follows:

1 represents $\mathrm{diag}(-1, 1, 1, 1, 1)$,
2 represents $\mathrm{diag}(1, -1, 1, 1, 1)$,
4 represents $\mathrm{diag}(1, 1, -1, 1, 1)$,
8 represents $\mathrm{diag}(1, 1, 1, -1, 1)$,
B represents $\mathrm{diag}(-1, -1, 1, -1, 1)$,
D represents $\mathrm{diag}(-1, 1, -1, -1, 1)$,

and the code 1428BD represents the side-pairing

1111444422228888BBBBDDDD

for the 24 sides of the 24-cell Q^4. The set S_i of vertices of the side of Q^4 whose normal vector is s_i is given in the upper half of page 116. The corresponding ordered sets of the vertices of the paired sides are given in the lower half of the same page. Here we have $S_i' = k_i S_i$ where $k_i = \mathrm{diag}(a_1, a_2, a_3, a_4, 1)$ acts on S_i as a 4×4 matrix $\mathrm{diag}(a_1, a_2, a_3, a_4)$.

6. TABLES

Tables 2–4 list side-pairings and isometric invariants of all the congruence two 24-cell manifolds. In each table, N is the row number. The column

N	SP	S	H_1	H_2	H_3	LT
23	1569A4	32	420	620	2	AAGGH
24	134B2E	16	420	620	2	AAGHH
25	134B3E	16	420	620	2	AAGHH
26	13483D	16	420	620	2	AAGHJ
27	1348BD	16	420	620	2	AAGHJ
28	13492C	16	420	620	2	AAHHI
29	1349AC	16	420	620	2	AAHHI
30	1429AC	48	420	620	2	AFGGG
31	13C835	16	420	530	1	AGGGJ
32	13482C	16	420	530	1	AGGHH
33	1348AC	16	420	530	1	AGGHJ
34	1348BC	16	420	530	1	AGHHI
35	146928	64	420	440	0	GGGGH
36	1468AF	32	330	610	3	AAAJJ
37	156F8C	32	330	610	3	AABJJ
38	143BD8	16	330	610	3	ABFGH
39	14378D	16	330	610	3	ABFHH
40	143CF9	16	330	520	2	AAGGJ
41	13FF8A	32	330	520	2	AAGHH
42	13482D	16	330	520	2	AAGHJ
43	1348FD	16	330	520	2	AAGHJ
44	136B84	16	330	520	2	AAGIJ
45	13ED28	16	330	520	2	AAGIJ
46	13493C	16	330	520	2	AAHHI
47	1349EC	16	330	520	2	AAHHI
48	134A2F	16	330	520	2	AAHIJ
49	134A3F	16	330	520	2	AAHIJ
50	1369A4	16	330	520	2	AAIIJ
51	156A9C	16	330	520	2	ABGGH
52	13D834	16	330	520	2	ABGGJ
53	136F8A	32	330	520	2	ABGHH
54	134B6E	16	330	520	2	ABGHH
55	134B7E	16	330	520	2	ABGHH
56	13D935	16	330	520	2	ABGHI
57	13482E	16	330	520	2	ABGHJ
58	13487D	16	330	520	2	ABGHJ
59	1348AD	16	330	520	2	ABGHJ
60	134B3C	16	330	520	2	ABGHJ
61	136DA8	16	330	520	2	ABGHJ
62	1439AC	16	330	520	2	ABGHJ
63	143B9C	16	330	520	2	ABGHJ
64	13483F	16	330	520	2	ABGJJ
65	13496C	16	330	520	2	ABHHI
66	1349BC	16	330	520	2	ABHHI
67	134A2C	16	330	520	2	ABHIJ
68	1368A4	16	330	520	2	AFGGI
69	1347B8	16	330	520	2	AFGII
70	1437C9	16	330	520	2	BFGGH
71	13EB34	16	330	430	1	AGGGH
72	13EB64	16	330	430	1	AGGGH
73	13582D	16	330	430	1	AGGHH
74	135B3F	16	330	430	1	AGGHH
75	1358BD	16	330	430	1	AGGHI
76	13C8B4	16	330	430	1	AGGHI
77	13CB36	16	330	430	1	AGGHI
78	13EFCA	16	330	430	1	AGGHI
79	157B9D	16	330	430	1	AGGHI
80	13483E	16	330	430	1	AGGHJ
81	1348FC	16	330	430	1	AGGHJ
82	134B2C	16	330	430	1	AGGHJ
83	13583C	16	330	430	1	AGGHJ
84	135B2E	16	330	430	1	AGGHJ
85	136B2E	16	330	430	1	AGGHJ
86	136F2A	16	330	430	1	AGGHJ
87	13EE64	16	330	430	1	AGGHJ
88	13C875	16	330	430	1	AGGIJ
89	13EA35	16	330	430	1	AGGIJ
90	134B3D	16	330	430	1	AGGJJ
91	136D28	16	330	430	1	AGGJJ
92	1348EC	16	330	430	1	AGHHI
93	13593D	16	330	430	1	AGHHI
94	135A2F	16	330	430	1	AGHHI
95	136CA8	16	330	430	1	AGHHI
96	13EFC4	16	330	430	1	AGHHI
97	147B9C	16	330	430	1	AGHHJ
98	1347A8	16	330	430	1	AGHIJ
99	13486C	16	330	430	1	AGHIJ
100	13487C	16	330	430	1	AGHIJ

TABLE 3 (start). Non-orientable, 5 cusped, minimal volume, integral, congruence 2, hyperbolic 4-manifolds.

N	SP	S	H_1	H_2	H_3	LT
101	13492E	16	330	430	1	AGHIJ
102	13592C	16	330	430	1	AGHIJ
103	135A3E	16	330	430	1	AGHIJ
104	1367A8	16	330	430	1	AGHIJ
105	1368AE	16	330	430	1	AGHIJ
106	13692C	16	330	430	1	AGHIJ
107	1359AC	16	330	430	1	AGIIJ
108	1349BD	16	330	430	1	AGIJJ
109	1369AE	16	330	430	1	AHHHI
110	1347B9	16	330	430	1	AHIIJ
111	1349AD	16	330	430	1	AHIIJ
112	13C874	16	330	430	1	BGGGH
113	1358AC	16	330	430	1	BGGGJ
114	13EECA	16	330	430	1	BGGGJ
115	13682E	16	330	430	1	BGGHH
116	136C2A	16	330	430	1	BGGHH
117	1359BD	16	330	430	1	BGGHI
118	136E28	16	330	430	1	BGGHJ
119	13EEC4	16	330	430	1	BGGHJ
120	13C837	16	330	430	1	BGGIJ
121	157A9C	16	330	430	1	BGHHH
122	136A2C	16	330	430	1	BGHHI
123	153A9C	16	330	430	1	BGHHI
124	1539AD	16	330	430	1	BGHHJ
125	143AC9	16	330	430	1	FGGGH
126	1539BD	16	330	430	1	FGGGH
127	1439BC	16	330	430	1	FGGHH
128	136B8C	16	330	421	1	AGHHH
129	13678A	16	330	421	1	AGHIJ
130	13CB35	16	330	340	0	GGGGJ
131	13583D	16	330	340	0	GGGHH
132	136F28	16	330	340	0	GGGHH
133	13C836	16	330	340	0	GGGHI
134	135B3E	16	330	340	0	GGGHJ
135	136C28	16	330	340	0	GGGHJ
136	136D2A	16	330	340	0	GGGHJ
137	13C935	16	330	340	0	GGGIJ
138	136E2A	16	330	340	0	GGGJJ
139	13EF64	16	330	340	0	GGHHH
140	135A3F	16	330	340	0	GGHHI
141	13EA64	16	330	340	0	GGHHI
142	1358AD	16	330	340	0	GGHHJ
143	13593C	16	330	340	0	GGHIJ
144	156B9D	16	330	340	0	GHHHI
145	1359BC	16	330	340	0	GHIJJ
146	135B2F	16	330	331	0	GGGHH
147	136B2C	16	330	331	0	GGGHH
148	13582C	16	330	331	0	GGGHJ
149	13682C	16	330	331	0	GGGHJ
150	13592D	16	330	331	0	GGHHI
151	13692E	16	330	331	0	GGHHI
152	1358BC	16	330	331	0	GGHHJ
153	13C8B5	16	330	331	0	GGHHJ
154	135A2E	16	330	331	0	GGHIJ
155	136A2E	16	330	331	0	GGHIJ
156	1359AD	16	330	331	0	GHHHI
157	3579BF	32	250	430	2	ABGII
158	1437F9	16	240	510	3	ABBGJ
159	14378E	16	240	510	3	ABBJJ
160	143DF8	16	240	510	3	ABFGH
161	14779C	16	240	510	3	ABFGH
162	14F7C9	16	240	510	3	ABFGJ
163	14386D	16	240	510	3	ABFHH
164	143E8D	16	240	510	3	ABFHH
165	1437FA	16	240	510	3	ABFHI
166	14279C	16	240	510	3	BBFGG
167	1427AC	16	240	510	3	BBFGG
168	1437D8	16	240	510	3	BBFGH
169	143D68	16	240	510	3	BBFGH
170	1CFF38	32	240	420	2	AAGJJ
171	153CF4	32	240	420	2	AAHII
172	13D864	16	240	420	2	ABGGJ
173	143DE8	16	240	420	2	ABGGJ
174	13DB37	16	240	420	2	ABGHI
175	13486D	16	240	420	2	ABGHJ
176	1348ED	16	240	420	2	ABGHJ
177	136B3C	16	240	420	2	ABGHJ
178	136F38	16	240	420	2	ABGHJ
179	13D8B4	16	240	420	2	ABGHJ
180	13EE84	16	240	420	2	ABGHJ
181	13EED4	16	240	420	2	ABGHJ
182	1477B9	16	240	420	2	ABGHJ
183	1479BE	16	240	420	2	ABGHJ
184	1347E8	16	240	420	2	ABGIJ
185	136B8D	16	240	420	2	ABGIJ
186	13EEDB	16	240	420	2	ABGIJ
187	14BC79	16	240	420	2	ABGIJ
188	134B2D	16	240	420	2	ABGJJ
189	13683F	16	240	420	2	ABGJJ
190	1368AF	16	240	420	2	ABGJJ
191	136D3A	16	240	420	2	ABGJJ
192	13DD64	16	240	420	2	ABGJJ
193	143B68	16	240	420	2	ABGJJ
194	13497C	16	240	420	2	ABHHI
195	1349FC	16	240	420	2	ABHHI
196	136CA9	16	240	420	2	ABHII
197	13D975	16	240	420	2	ABHII
198	13493E	16	240	420	2	ABHIJ
199	134A7F	16	240	420	2	ABHIJ
200	13693E	16	240	420	2	ABHIJ
201	136C3B	16	240	420	2	ABHIJ
202	136CA4	16	240	420	2	ABHIJ
203	13DCF4	16	240	420	2	ABHIJ
204	156A3C	16	240	420	2	ABHIJ
205	156DAE	16	240	420	2	ABHIJ
206	1347F9	16	240	420	2	ABIIJ
207	13D9A4	16	240	420	2	ABIIJ
208	13DA36	16	240	420	2	ABIIJ
209	13EA75	16	240	420	2	ABIIJ
210	13492F	16	240	420	2	ABIJJ
211	134A3D	16	240	420	2	ABIJJ
212	136A3D	16	240	420	2	ABIJJ
213	136E39	16	240	420	2	ABIJJ
214	136E84	16	240	420	2	ABIJJ
215	13EE75	16	240	420	2	ABIJJ
216	1429FA	16	240	420	2	AFGGG
217	136F84	16	240	420	2	AFGHI
218	1439FA	16	240	420	2	AFGHI
219	13487E	16	240	420	2	AFGHJ
220	134B6C	16	240	420	2	AFGHJ
221	143B9E	16	240	420	2	AFGHJ
222	136DA4	16	240	420	2	AFGIJ
223	156F9C	16	240	420	2	AFGIJ
224	134B7D	16	240	420	2	AFGJJ
225	1347FC	16	240	420	2	AFHII
226	1367A9	16	240	420	2	AFHII
227	157DB9	16	240	420	2	AFHII
228	13496E	16	240	420	2	AFHIJ
229	156DA9	16	240	420	2	AFHIJ
230	1347A9	16	240	420	2	AFIII
231	1569F4	32	240	420	2	BBGGH
232	13D8A4	16	240	420	2	BBGGJ
233	143EC9	16	240	420	2	BBGGJ
234	143FD8	16	240	420	2	BBGGJ
235	5EFF7A	32	240	420	2	BBGHH
236	136B3E	16	240	420	2	BBGHH
237	136F3A	16	240	420	2	BBGHH
238	13EB74	16	240	420	2	BBGHH
239	13EA74	16	240	420	2	BBGHI
240	15FA9C	16	240	420	2	BBGHI
241	13683D	16	240	420	2	BBGHJ
242	136D38	16	240	420	2	BBGHJ
243	13D874	16	240	420	2	BBGHJ
244	13EE74	16	240	420	2	BBGHJ
245	1437BD	16	240	420	2	BBGHJ
246	13D836	16	240	420	2	BBGIJ
247	13D974	16	240	420	2	BBGIJ
248	1437CA	16	240	420	2	BBGIJ
249	1539AF	16	240	420	2	BBGIJ
250	13DD74	16	240	420	2	BBGJJ
251	13EF74	16	240	420	2	BBHHH
252	13693C	16	240	420	2	BBHHI
253	136C39	16	240	420	2	BBHHI
254	13D9B5	16	240	420	2	BBHHI
255	13DC74	16	240	420	2	BBHHJ
256	156AD9	16	240	420	2	BBHHJ
257	153CA4	32	240	420	2	BBHII
258	136A3F	16	240	420	2	BBHIJ
259	136E3B	16	240	420	2	BBHIJ
260	143C69	16	240	420	2	BFGGH
261	147CA9	16	240	420	2	BFGGH
262	14379C	16	240	420	2	BFGHH
263	14396C	16	240	420	2	BFGHH
264	14B7C9	16	240	420	2	BFGHI
265	1437BE	16	240	420	2	BFGHJ

TABLE 3 (continued).

N	SP	S	H_1	H_2	H_3	LT
266	1358EC	16	240	330	1	AGGIJ
267	1368B4	16	240	330	1	AGGIJ
268	13E837	16	240	330	1	AGGIJ
269	143FCA	16	240	330	1	AGGIJ
270	13482F	16	240	330	1	AGGJJ
271	143A9F	16	240	330	1	AGGJJ
272	13EF84	16	240	330	1	AGHHJ
273	1359FD	16	240	330	1	AGHII
274	13E936	16	240	330	1	AGHII
275	13EA94	16	240	330	1	AGHII
276	134A3C	16	240	330	1	AGHIJ
277	136C29	16	240	330	1	AGHIJ
278	13EE9B	16	240	330	1	AGHIJ
279	14AB9C	16	240	330	1	AGHIJ
280	14FB9C	16	240	330	1	AGHJJ
281	136A94	16	240	330	1	AGIIJ
282	13CA37	16	240	330	1	AGIIJ
283	13493F	16	240	330	1	AGIJJ
284	134A2D	16	240	330	1	AGIJJ
285	136E2B	16	240	330	1	AGIJJ
286	1347EC	16	240	330	1	AHHIJ
287	1349FD	16	240	330	1	AHIIJ
288	1347FD	16	240	330	1	AHIJJ
289	13678C	16	240	330	1	AHIJJ
290	13496D	16	240	330	1	AIIJJ
291	136DA9	16	240	330	1	AIJJJ
292	1358ED	16	240	330	1	BGGGH
293	13EB94	16	240	330	1	BGGGH
294	13EBC4	16	240	330	1	BGGGH
295	13DB34	16	240	330	1	BGGGJ
296	13683C	16	240	330	1	BGGHH
297	136BCE	16	240	330	1	BGGHH
298	136C38	16	240	330	1	BGGHH
299	13EC64	16	240	330	1	BGGHH
300	13EF9A	16	240	330	1	BGGHH
301	1368BE	16	240	330	1	BGGHI
302	136F9A	16	240	330	1	BGGHI
303	13C9B4	16	240	330	1	BGGHI
304	13CB74	16	240	330	1	BGGHI
305	13D837	16	240	330	1	BGGHI
306	13DA35	16	240	330	1	BGGHI
307	13E964	16	240	330	1	BGGHI
308	1569BF	16	240	330	1	BGGHI
309	13587D	16	240	330	1	BGGHJ
310	135B2D	16	240	330	1	BGGHJ
311	135B3D	16	240	330	1	BGGHJ
312	135B7F	16	240	330	1	BGGHJ
313	13683E	16	240	330	1	BGGHJ
314	136B2D	16	240	330	1	BGGHJ
315	136B3F	16	240	330	1	BGGHJ
316	136B9C	16	240	330	1	BGGHJ
317	136C3A	16	240	330	1	BGGHJ
318	136E3A	16	240	330	1	BGGHJ
319	13EEDA	16	240	330	1	BGGHJ
320	1569EA	16	240	330	1	BGGHJ
321	156FC9	16	240	330	1	BGGHJ
322	1357A8	16	240	330	1	BGGIJ
323	1359FC	16	240	330	1	BGGIJ
324	1368E4	16	240	330	1	BGGIJ
325	136DB8	16	240	330	1	BGGIJ
326	13C7A8	16	240	330	1	BGGIJ
327	13CA35	16	240	330	1	BGGIJ
328	1539BF	16	240	330	1	BGGIJ
329	13582E	16	240	330	1	BGGJJ
330	13583E	16	240	330	1	BGGJJ
331	13587C	16	240	330	1	BGGJJ
332	135B7E	16	240	330	1	BGGJJ
333	136B3D	16	240	330	1	BGGJJ
334	136E38	16	240	330	1	BGGJJ
335	1439BE	16	240	330	1	BGGJJ
336	143A69	16	240	330	1	BGGJJ
337	1579ED	16	240	330	1	BGGJJ
338	1369EC	16	240	330	1	BGHHI
339	136A3E	16	240	330	1	BGHHI
340	136CB9	16	240	330	1	BGHHI
341	136F29	16	240	330	1	BGHHI
342	136F3B	16	240	330	1	BGHHI
343	13C974	16	240	330	1	BGHHI
344	13CCE6	16	240	330	1	BGHHI
345	13EFD4	16	240	330	1	BGHHI
346	357B9D	16	240	330	1	BGHHI
347	13CCE5	16	240	330	1	BGHHJ
348	13DC64	16	240	330	1	BGHHJ
349	1479EC	16	240	330	1	BGHHJ
350	157A3C	16	240	330	1	BGHHJ
351	15EA9C	16	240	330	1	BGHHJ
352	136A9C	16	240	330	1	BGHII
353	13C936	16	240	330	1	BGHII
354	15FB9D	16	240	330	1	BGHII
355	13592F	16	240	330	1	BGHIJ
356	13593F	16	240	330	1	BGHIJ
357	13597D	16	240	330	1	BGHIJ
358	135A7F	16	240	330	1	BGHIJ
359	13692F	16	240	330	1	BGHIJ
360	13693D	16	240	330	1	BGHIJ
361	1369BE	16	240	330	1	BGHIJ
362	136A3C	16	240	330	1	BGHIJ
363	136D2B	16	240	330	1	BGHIJ
364	136D39	16	240	330	1	BGHIJ
365	136E9B	16	240	330	1	BGHIJ
366	136F39	16	240	330	1	BGHIJ
367	13D964	16	240	330	1	BGHIJ
368	13DCE4	16	240	330	1	BGHIJ
369	13EAC4	16	240	330	1	BGHIJ
370	13EB75	16	240	330	1	BGHIJ
371	13EEC6	16	240	330	1	BGHIJ
372	13EECB	16	240	330	1	BGHIJ
373	14A93C	16	240	330	1	BGHIJ
374	15396D	16	240	330	1	BGHIJ
375	1569FD	16	240	330	1	BGHIJ
376	13EEC5	16	240	330	1	BGHJJ
377	13EED5	16	240	330	1	BGHJJ
378	156E9D	16	240	330	1	BGHJJ
379	1357B9	16	240	330	1	BGIIJ
380	136AC4	16	240	330	1	BGIIJ
381	13D936	16	240	330	1	BGIIJ
382	15396A	16	240	330	1	BGIIJ
383	13597C	16	240	330	1	BGIJJ
384	135A2C	16	240	330	1	BGIJJ
385	135A3C	16	240	330	1	BGIJJ
386	135A7E	16	240	330	1	BGIJJ
387	13693F	16	240	330	1	BGIJJ
388	136D3B	16	240	330	1	BGIJJ
389	13D9B4	16	240	330	1	BGIJJ
390	14396A	16	240	330	1	BGIJJ
391	13D9A5	16	240	330	1	BHHHI
392	13DC65	16	240	330	1	BHHHI
393	13DC75	16	240	330	1	BHHHI
394	156A3E	16	240	330	1	BHHHJ
395	13D965	16	240	330	1	BHHII
396	13EF75	16	240	330	1	BHHIJ
397	13EFD5	16	240	330	1	BHHIJ
398	156B3D	16	240	330	1	BHHIJ
399	156D39	16	240	330	1	BHHIJ
400	143CE9	16	240	330	1	FGGGH
401	136B94	16	240	330	1	FGGGI
402	136BC4	16	240	330	1	FGGGI
403	143F9C	16	240	330	1	FGGHH
404	13679A	16	240	330	1	FGGHI
405	1367B8	16	240	330	1	FGGHI
406	14BAC9	16	240	330	1	FGGHI
407	14B93C	16	240	330	1	FGGHJ
408	15397D	16	240	330	1	FGGHJ
409	153B9F	16	240	330	1	FGGHJ
410	1569ED	16	240	330	1	FGGHJ
411	1369B4	16	240	330	1	FGGII
412	1369E4	16	240	330	1	FGGII
413	13C7B8	16	240	330	1	FGGII
414	1357B8	16	240	330	1	FGGIJ
415	156AC9	16	240	330	1	FGHHI
416	157D39	16	240	330	1	FGHHI
417	157A3E	16	240	330	1	FGHHJ
418	157F9C	16	240	330	1	FGHJJ
419	1357A9	16	240	330	1	FGIIJ
420	13EB85	16	240	321	1	AGHIJ
421	153B8F	16	240	321	1	AHHJJ
422	157F8C	16	240	321	1	AHHJJ
423	1367AE	16	240	321	1	AHIJJ
424	13EFDA	16	240	321	1	BGHHI
425	13EFDB	16	240	321	1	BGHII
426	146F28	32	240	240	0	GGGGH
427	13FFC8	32	240	240	0	GGGHH
428	13C8F4	16	240	240	0	GGGHI
429	13583F	16	240	240	0	GGGHJ
430	13E935	16	240	240	0	GGGIJ

TABLE 3 (continued).

N	SP	S	H_1	H_2	H_3	LT
431	13EE6A	16	240	240	0	GGGIJ
432	135B3C	16	240	240	0	GGGJJ
433	13EE9A	16	240	240	0	GGGJJ
434	136FC8	32	240	240	0	GGHHH
435	1358FD	16	240	240	0	GGHHH
436	13586D	16	240	240	0	GGHHI
437	1368EC	16	240	240	0	GGHHI
438	136C2B	16	240	240	0	GGHHI
439	136DEA	16	240	240	0	GGHHJ
440	13ED64	16	240	240	0	GGHHJ
441	157DA4	32	240	240	0	GGHII
442	157DF4	32	240	240	0	GGHII
443	135A3D	16	240	240	0	GGHIJ
444	135B6E	16	240	240	0	GGHIJ
445	136E29	16	240	240	0	GGHIJ
446	136F2B	16	240	240	0	GGHIJ
447	13CB75	16	240	240	0	GGHIJ
448	13CC65	16	240	240	0	GGHIJ
449	1539EA	16	240	240	0	GGHIJ
450	136CB8	16	240	240	0	GGHJJ
451	13EB95	16	240	240	0	GGHJJ
452	157E9D	16	240	240	0	GGHJJ
453	13C937	16	240	240	0	GGIIJ
454	13593E	16	240	240	0	GGIJJ
455	136B9D	16	240	240	0	GGIJJ
456	136D29	16	240	240	0	GGIJJ
457	13EE65	16	240	240	0	GGIJJ
458	136E9A	16	240	240	0	GGJJJ
459	13EFC6	16	240	240	0	GHHHI
460	5BFFDA	32	240	240	0	GHHII
461	135A6F	16	240	240	0	GHHII
462	136ACE	16	240	240	0	GHHII
463	13EFCB	16	240	240	0	GHHII
464	156CEA	16	240	240	0	GHHII
465	1359EC	16	240	240	0	GHHIJ
466	13CDE6	16	240	240	0	GHHIJ
467	13EF65	16	240	240	0	GHHIJ
468	13EFC5	16	240	240	0	GHHIJ
469	156ACE	16	240	240	0	GHHIJ
470	15EB9D	16	240	240	0	GHHIJ
471	357AB9	16	240	240	0	GHHIJ
472	153A9E	16	240	240	0	GHHJJ
473	13596C	16	240	240	0	GHIIJ
474	13C975	16	240	240	0	GHIIJ
475	156DB9	16	240	240	0	GHIIJ
476	13C9B5	16	240	240	0	GHIJJ
477	13CD65	16	240	240	0	GHIJJ
478	13CD75	16	240	240	0	GHIJJ
479	157DA9	16	240	240	0	GHIJJ
480	1369BF	16	240	240	0	GIIJJ
481	367B9C	32	240	240	0	HHHJJ
482	13582F	16	240	231	0	GGGHJ
483	13682F	16	240	231	0	GGGHJ
484	135B2C	16	240	231	0	GGGJJ
485	135B6F	16	240	231	0	GGHHI
486	1368BF	16	240	231	0	GGHHJ
487	13EDA8	16	240	231	0	GGHHJ
488	13EF6A	16	240	231	0	GGHHJ
489	367AB5	16	240	231	0	GGHII
490	13586C	16	240	231	0	GGHIJ
491	135A2D	16	240	231	0	GGHIJ
492	136A2D	16	240	231	0	GGHIJ
493	136CEA	16	240	231	0	GGHIJ
494	13C865	16	240	231	0	GGHIJ
495	1358FC	16	240	231	0	GGHJJ
496	13C8F5	16	240	231	0	GGHJJ
497	13592E	16	240	231	0	GGIJJ
498	1367C8	16	240	231	0	GGIJJ
499	1367EA	16	240	231	0	GGIJJ
500	13E9A6	16	240	231	0	GHHHI
501	13596D	16	240	231	0	GHHII
502	136F9B	16	240	231	0	GHHII
503	13EF9B	16	240	231	0	GHHII
504	1359ED	16	240	231	0	GHHIJ
505	136A9D	16	240	231	0	GHHIJ
506	13EA95	16	240	231	0	GHHIJ
507	135A6E	16	240	231	0	GHIIJ
508	13679B	16	240	231	0	GHIIJ
509	1367B9	16	240	231	0	GHIIJ
510	136DB9	16	240	231	0	GHIIJ
511	13C7B9	16	240	231	0	GHIIJ
512	13C9A5	16	240	231	0	GHIIJ
513	13EA65	16	240	231	0	GHIIJ
514	357BAF	32	240	231	0	HHHII
515	357B9A	16	240	231	0	HHHII
516	367B95	16	240	231	0	HHHIJ
517	1569AF	32	231	510	3	ABBGG
518	1579BF	32	231	420	2	AAGII
519	1347AC	16	231	420	2	ABHIJ
520	134A6F	16	231	420	2	ABHIJ
521	1369AF	16	231	420	2	ABIIJ
522	1347BD	16	231	420	2	ABIJJ
523	1539FA	16	231	420	2	AFGGI
524	1479FC	16	231	420	2	BFGGH
525	13E7A8	16	231	330	1	AGGIJ
526	13682D	16	231	330	1	AGGJJ
527	136FC4	16	231	330	1	AGHIJ
528	13E9AC	16	231	330	1	AGIIJ
529	1349ED	16	231	330	1	AGIJJ
530	136A2F	16	231	330	1	AGIJJ
531	136DE4	16	231	330	1	AGIJJ
532	13C8A5	16	231	330	1	BGGHJ
533	13E8AC	16	231	330	1	FGGGI
534	1579FD	16	231	330	1	FGGGI
535	136CE4	16	231	330	1	FGGHI
536	136EC4	16	231	330	1	FGGIJ
537	15397A	16	231	330	1	FGGIJ
538	13EB65	16	231	231	0	GGGIJ
539	136B2F	16	231	231	0	GGGJJ
540	13C965	16	231	231	0	GGIIJ
541	13692D	16	231	231	0	GGIJJ
542	1367CE	16	231	231	0	GHHIJ
543	1367EC	16	231	231	0	GHHIJ
544	357ABE	16	231	231	0	GHHIJ
545	367ABF	32	222	321	1	BGGGG
546	13E7AC	16	160	330	2	BBGII
547	143DFA	16	150	410	3	ABBIJ
548	143EDB	16	150	410	3	ABBIJ
549	1477AF	16	150	410	3	ABFGH
550	14A79C	16	150	410	3	ABFGI
551	14A78D	16	150	410	3	ABFHH
552	143B6E	16	150	410	3	ABFHJ
553	14386F	16	150	410	3	ABFJJ
554	14F7CA	16	150	410	3	BBBGJ
555	1437AC	16	150	410	3	BBBHJ
556	1437DB	16	150	410	3	BBBIJ
557	14279F	16	150	410	3	BBFGG
558	1427AF	16	150	410	3	BBFGG
559	14779A	16	150	410	3	BBFGH
560	1477A9	16	150	410	3	BBFGH
561	14A7C9	16	150	410	3	BBFGI
562	143E69	16	150	410	3	BBFGJ
563	14F79C	16	150	410	3	BBFGJ
564	143E6B	16	150	410	3	BBFHI
565	1437AF	16	150	410	3	BBFHJ
566	143D6A	16	150	410	3	BBFIJ
567	1477A8	16	150	320	2	ABGGJ
568	153CFA	16	150	320	2	ABIII
569	157CEA	16	150	320	2	ABIII
570	14AB3C	16	150	320	2	ABIJJ
571	13EE85	16	150	320	2	ABJJJ
572	14AB78	16	150	320	2	AFGGJ
573	134A7C	16	150	320	2	AFHIJ
574	15BA79	16	150	320	2	AFHIJ
575	15AD69	16	150	320	2	AFIII
576	3579CF	16	150	320	2	AFIII
577	1347ED	16	150	320	2	AFIIJ
578	13497F	16	150	320	2	AFIJJ
579	13678D	16	150	320	2	AFIJJ
580	1367AF	16	150	320	2	AFIJJ
581	15AD6E	16	150	320	2	AFIJJ
582	13D8F4	16	150	320	2	BBGGJ
583	13EED8	16	150	320	2	BBGGJ
584	1437E8	16	150	320	2	BBGGJ
585	14779B	16	150	320	2	BBGGJ
586	13EBD4	16	150	320	2	BBGHH
587	136BDE	16	150	320	2	BBGHI
588	136CFA	16	150	320	2	BBGHI
589	13DB75	16	150	320	2	BBGHI
590	156CFA	16	150	320	2	BBGHI
591	1368FD	16	150	320	2	BBGHJ
592	136FD8	16	150	320	2	BBGHJ
593	13D8E4	16	150	320	2	BBGHJ
594	13DDE7	16	150	320	2	BBGHJ
595	13E874	16	150	320	2	BBGHJ

TABLE 3 (continued).

N	SP	S	H_1	H_2	H_3	LT
596	13E8B7	16	150	320	2	BBGHJ
597	13EF7A	16	150	320	2	BBGHJ
598	1477DF	16	150	320	2	BBGHJ
599	1477EC	16	150	320	2	BBGHJ
600	147CEB	16	150	320	2	BBGHJ
601	136BD4	16	150	320	2	BBGIJ
602	136ED8	16	150	320	2	BBGIJ
603	13DA64	16	150	320	2	BBGIJ
604	13E875	16	150	320	2	BBGIJ
605	13EDB8	16	150	320	2	BBGIJ
606	13EE7B	16	150	320	2	BBGIJ
607	14BEC9	16	150	320	2	BBGIJ
608	156CBF	16	150	320	2	BBGIJ
609	136DFA	16	150	320	2	BBGJJ
610	13DB74	16	150	320	2	BBGJJ
611	13DDA4	16	150	320	2	BBGJJ
612	13EE7A	16	150	320	2	BBGJJ
613	13EED7	16	150	320	2	BBGJJ
614	14379F	16	150	320	2	BBGJJ
615	13DCF7	16	150	320	2	BBHHI
616	13ECB9	16	150	320	2	BBHHI
617	13EC74	16	150	320	2	BBHHJ
618	1369FC	16	150	320	2	BBHII
619	13E974	16	150	320	2	BBHII
620	13E9B6	16	150	320	2	BBHII
621	156DFB	16	150	320	2	BBHII
622	136ADF	16	150	320	2	BBHIJ
623	13D9F5	16	150	320	2	BBHIJ
624	13EC75	16	150	320	2	BBHIJ
625	13ED74	16	150	320	2	BBHIJ
626	156BDF	16	150	320	2	BBHIJ
627	157DEB	16	150	320	2	BBHIJ
628	357F9B	16	150	320	2	BBHIJ
629	13DCA4	16	150	320	2	BBHJJ
630	13DDF7	16	150	320	2	BBHJJ
631	143E6D	16	150	320	2	BBHJJ
632	156F3B	16	150	320	2	BBHJJ
633	1369F4	16	150	320	2	BBIIJ
634	13D7B9	16	150	320	2	BBIIJ
635	13DA74	16	150	320	2	BBIIJ
636	156D3B	16	150	320	2	BBIIJ
637	13D9F4	16	150	320	2	BBIJJ
638	13DDF6	16	150	320	2	BBIJJ
639	143D6E	16	150	320	2	BBJJJ
640	147CAF	16	150	320	2	BFGGH
641	1357E8	16	150	320	2	BFGGI
642	1368F4	16	150	320	2	BFGGI
643	14AC79	16	150	320	2	BFGGI
644	143F68	16	150	320	2	BFGGJ
645	14BA97	16	150	320	2	BFGGJ
646	1437EB	16	150	320	2	BFGHI
647	143ECB	16	150	320	2	BFGHI
648	143F6A	16	150	320	2	BFGHI
649	143FDA	16	150	320	2	BFGHI
650	14BCE9	16	150	320	2	BFGHI
651	1569FA	16	150	320	2	BFGHI
652	143A6F	16	150	320	2	BFGHJ
653	143CAF	16	150	320	2	BFGHJ
654	143DBE	16	150	320	2	BFGHJ
655	14B79C	16	150	320	2	BFGHJ
656	14B9EC	16	150	320	2	BFGHJ
657	1357F9	16	150	320	2	BFGII
658	136AD4	16	150	320	2	BFGII
659	1367D8	16	150	320	2	BFGIJ
660	1367FA	16	150	320	2	BFGIJ
661	13D7B8	16	150	320	2	BFGIJ
662	143C6B	16	150	320	2	BFGIJ
663	153CAF	16	150	320	2	BFGIJ
664	14396E	16	150	320	2	BFGJJ
665	14B7CE	16	150	320	2	BFGJJ
666	15BF9C	16	150	320	2	BFGJJ
667	156ADE	16	150	320	2	BFHHJ
668	157BDF	16	150	320	2	BFHII
669	13D7A9	16	150	320	2	BFIII
670	135B7D	16	150	320	2	FFGGH
671	13587E	16	150	320	2	FFGGJ
672	13597F	16	150	320	2	FFGHI
673	153C7A	16	150	320	2	FFGII
674	135A7C	16	150	320	2	FFGIJ
675	14AB97	16	150	230	1	AGJJJ
676	157CBF	16	150	230	1	AHIIJ
677	13EF85	16	150	230	1	AHJJJ
678	13DB64	16	150	230	1	BGGHJ
679	13EE94	16	150	230	1	BGGHJ
680	143CEB	16	150	230	1	BGGIJ
681	13C9F4	16	150	230	1	BGHHI
682	13EED6	16	150	230	1	BGHHJ
683	13EF94	16	150	230	1	BGHHJ
684	147ADF	16	150	230	1	BGHHJ
685	136FC9	16	150	230	1	BGHII
686	13DB65	16	150	230	1	BGHII
687	13EAD4	16	150	230	1	BGHII
688	13EFC9	16	150	230	1	BGHII
689	157CFA	16	150	230	1	BGHII
690	1357FD	16	150	230	1	BGHIJ
691	136BDF	16	150	230	1	BGHIJ
692	136CEB	16	150	230	1	BGHIJ
693	136F94	16	150	230	1	BGHIJ
694	13CA74	16	150	230	1	BGHIJ
695	13D7A8	16	150	230	1	BGHIJ
696	13ECA9	16	150	230	1	BGHIJ
697	13EE6B	16	150	230	1	BGHIJ
698	13EED9	16	150	230	1	BGHIJ
699	13EFC7	16	150	230	1	BGHIJ
700	1579EA	16	150	230	1	BGHIJ
701	15ADE6	16	150	230	1	BGHIJ
702	13CCA5	16	150	230	1	BGHJJ
703	13CCE7	16	150	230	1	BGHJJ
704	13DCB4	16	150	230	1	BGHJJ
705	13DDE6	16	150	230	1	BGHJJ
706	143F6C	16	150	230	1	BGHJJ
707	14F96C	16	150	230	1	BGHJJ
708	157F39	16	150	230	1	BGHJJ
709	35AFE6	16	150	230	1	BGHJJ
710	153CEA	16	150	230	1	BGIII
711	136DEB	16	150	230	1	BGIIJ
712	13CA75	16	150	230	1	BGIIJ
713	13D9E4	16	150	230	1	BGIIJ
714	13EA9C	16	150	230	1	BGIIJ
715	15A9FD	16	150	230	1	BGIIJ
716	15AC36	16	150	230	1	BGIIJ
717	1357EC	16	150	230	1	BGIJJ
718	136DB4	16	150	230	1	BGIJJ
719	136EC9	16	150	230	1	BGIJJ
720	13CDE7	16	150	230	1	BGIJJ
721	13EBD5	16	150	230	1	BGIJJ
722	13EEC9	16	150	230	1	BGIJJ
723	156F9B	16	150	230	1	BGIJJ
724	15AE9D	16	150	230	1	BGIJJ
725	13EEC7	16	150	230	1	BGJJJ
726	143C6F	16	150	230	1	BGJJJ
727	15B9ED	16	150	230	1	BGJJJ
728	136CFB	16	150	230	1	BHHHI
729	13DA75	16	150	230	1	BHHII
730	13EFD6	16	150	230	1	BHHII
731	136ED9	16	150	230	1	BHHIJ
732	13D9E5	16	150	230	1	BHHIJ
733	13DCE7	16	150	230	1	BHHIJ
734	13DCF6	16	150	230	1	BHHIJ
735	367F9A	16	150	230	1	BHHIJ
736	13DCA5	16	150	230	1	BHHJJ
737	156E3A	16	150	230	1	BHHJJ
738	357ECF	16	150	230	1	BHIII
739	35AD69	16	150	230	1	BHIII
740	1369FD	16	150	230	1	BHIIJ
741	13DA65	16	150	230	1	BHIIJ
742	13DCE6	16	150	230	1	BHIIJ
743	13E975	16	150	230	1	BHIIJ
744	13EDB9	16	150	230	1	BHIIJ
745	156C3A	16	150	230	1	BHIIJ
746	15AD36	16	150	230	1	BHIIJ
747	357ACE	16	150	230	1	BHIIJ
748	357BDF	16	150	230	1	BHIIJ
749	367BDE	16	150	230	1	BHIIJ
750	136ADE	16	150	230	1	BHIJJ
751	13EAD5	16	150	230	1	BHIJJ
752	13ED75	16	150	230	1	BHIJJ
753	13EF7B	16	150	230	1	BHIJJ
754	13EFD7	16	150	230	1	BHIJJ
755	153A6E	16	150	230	1	BHIJJ
756	153B6F	16	150	230	1	BHIJJ
757	153DAE	16	150	230	1	BHIJJ
758	156F39	16	150	230	1	BHIJJ
759	357FDE	16	150	230	1	BHIJJ
760	157ADE	16	150	230	1	BHJJJ

TABLE 3 (continued).

N	SP	S	H_1	H_2	H_3	LT
761	1367FB	16	150	230	1	BIIJJ
762	136DFB	16	150	230	1	BIIJJ
763	153C6F	16	150	230	1	BIIJJ
764	153D6E	16	150	230	1	BIIJJ
765	15AF36	16	150	230	1	BIIJJ
766	13E9B7	16	150	230	1	BIJJJ
767	15AE36	16	150	230	1	BIJJJ
768	136CB4	16	150	230	1	FGGHI
769	14BC69	16	150	230	1	FGGHI
770	14BF9C	16	150	230	1	FGGHJ
771	13E79A	16	150	230	1	FGGII
772	136E94	16	150	230	1	FGGIJ
773	13EB9C	16	150	230	1	FGGIJ
774	35FAB6	16	150	230	1	FGHHI
775	15BA96	16	150	230	1	FGHHJ
776	156DBE	16	150	230	1	FGHII
777	15ADE9	16	150	230	1	FGHII
778	1357ED	16	150	230	1	FGHIJ
779	13679C	16	150	230	1	FGHIJ
780	1367BE	16	150	230	1	FGHIJ
781	157ACE	16	150	230	1	FGHIJ
782	157E3A	16	150	230	1	FGHIJ
783	15AF9C	16	150	230	1	FGHIJ
784	153A7E	16	150	230	1	FGHJJ
785	153B7F	16	150	230	1	FGHJJ
786	156E9A	16	150	230	1	FGHJJ
787	1367C9	16	150	230	1	FGIIJ
788	1367EB	16	150	230	1	FGIIJ
789	13C7A9	16	150	230	1	FGIIJ
790	157CAF	16	150	230	1	FGIIJ
791	15BE36	16	150	230	1	FGIIJ
792	367AFB	16	150	230	1	FGIIJ
793	1357FC	16	150	230	1	FGIJJ
794	153C7F	16	150	230	1	FGIJJ
795	153D7E	16	150	230	1	FGIJJ
796	157F9B	16	150	230	1	FGIJJ
797	15EDA6	16	150	230	1	FGIJJ
798	367C9D	16	150	230	1	FHHII
799	367BEA	16	150	230	1	FHHIJ
800	157F8B	16	150	221	1	AJJJJ
801	13EFD8	16	150	221	1	BGHIJ
802	1368FC	16	150	221	1	BGHJJ
803	13EFD9	16	150	221	1	BHIII
804	136FD9	16	150	221	1	BHIIJ
805	35AFEC	16	150	221	1	BHIIJ
806	357FAE	16	150	221	1	BHIJJ
807	1367D9	16	150	221	1	BIIJJ
808	357AC9	16	150	221	1	FHHII
809	357AFC	16	150	221	1	FHHII
810	35FB9D	16	150	221	1	FHIIJ
811	14BA69	16	150	140	0	GGIJJ
812	13EE95	16	150	140	0	GGJJJ
813	13CDF6	16	150	140	0	GHIIJ
814	13EF6B	16	150	140	0	GHIIJ
815	13EC65	16	150	140	0	GHIJJ
816	157DAE	16	150	140	0	GHIJJ
817	15AB96	16	150	140	0	GHIJJ
818	15BA69	16	150	140	0	GHIJJ
819	13EF95	16	150	140	0	GHJJJ
820	15BE9D	16	150	140	0	GHJJJ
821	15BD69	16	150	140	0	GIIIJ
822	367AC5	16	150	140	0	GIIIJ
823	367ECF	16	150	140	0	GIIIJ
824	13C9F5	16	150	140	0	GIIJJ
825	13ED65	16	150	140	0	GIIJJ
826	13EDA9	16	150	140	0	GIIJJ
827	15ACE6	16	150	140	0	GIIJJ
828	15AD79	16	150	140	0	GIIJJ
829	367DB5	16	150	140	0	GIIJJ
830	367EB9	16	150	140	0	GIIJJ
831	13CDF7	16	150	140	0	GIJJJ
832	157E9A	16	150	140	0	GIJJJ
833	13CDA5	16	150	140	0	GJJJJ
834	367EFD	16	150	140	0	HHIII
835	367DFA	16	150	140	0	HHIJJ
836	367EF5	16	150	140	0	HIIII
837	367EC5	16	150	140	0	HIIIJ
838	357AFB	16	150	140	0	HIIJJ
839	357BEA	16	150	140	0	HIIJJ
840	35ACE6	16	150	140	0	HIIJJ
841	35ACED	16	150	140	0	HIIJJ
842	367F95	16	150	140	0	HIIJJ
843	35ADCE	16	150	140	0	HIJJJ
844	367C9B	16	150	140	0	HIJJJ
845	367FD5	16	150	140	0	HIJJJ
846	367BEC	16	150	140	0	HJJJJ
847	367DFC	16	150	140	0	IIJJJ
848	367FDE	16	150	140	0	IIJJJ
849	13586F	16	150	131	0	GGHJJ
850	135B6C	16	150	131	0	GGJJJ
851	13E79B	16	150	131	0	GHIIJ
852	135A6D	16	150	131	0	GHIJJ
853	157BCF	16	150	131	0	GHIJJ
854	15EF6C	16	150	131	0	GHJJJ
855	13596E	16	150	131	0	GIJJJ
856	13679D	16	150	131	0	GIJJJ
857	1367BF	16	150	131	0	GIJJJ
858	35ADC9	16	150	131	0	HIIIJ
859	35FAB9	16	150	131	0	HIIIJ
860	367DC5	16	150	131	0	HIIIJ
861	367DCE	16	150	131	0	HIIIJ
862	367EC9	16	150	131	0	HIIIJ
863	367BD5	16	150	131	0	HIIJJ
864	367DF5	16	150	131	0	HIIJJ
865	35ABE6	16	150	131	0	HIJJJ
866	14A86F	32	141	410	3	AABJJ
867	15BF8C	32	141	410	3	ABBJJ
868	143B6C	16	141	410	3	ABFHJ
869	156CAF	16	141	410	3	ABFII
870	156F8B	16	141	410	3	ABFJJ
871	136CF4	16	141	320	2	ABHIJ
872	136ED4	16	141	320	2	ABIJJ
873	14BA79	16	141	320	2	AFGIJ
874	13486F	16	141	320	2	AFGJJ
875	13E78C	16	141	320	2	AFIIJ
876	15BD79	16	141	320	2	AFIIJ
877	134A6D	16	141	320	2	AFIJJ
878	157DBE	16	141	320	2	AFIJJ
879	156DEB	16	141	320	2	BBGGI
880	1479FA	16	141	320	2	BFGGH
881	13C7E8	16	141	320	2	BFGGI
882	13E7CA	16	141	320	2	BFGGI
883	14F97C	16	141	320	2	BFGGJ
884	1357BD	16	141	320	2	BFGHI
885	136FD4	16	141	320	2	BFGHI
886	13E7B8	16	141	320	2	BFGHI
887	143A6D	16	141	320	2	BFGHJ
888	156BCF	16	141	320	2	BFGHJ
889	1357AC	16	141	320	2	BFGIJ
890	136DF4	16	141	320	2	BFGIJ
891	1367DE	16	141	320	2	BFHHI
892	1367FC	16	141	320	2	BFHHI
893	357F9C	16	141	320	2	BFHHI
894	13E9BC	16	141	320	2	BFHII
895	14B93E	16	141	320	2	FFGGJ
896	15397F	16	141	320	2	FFGGJ
897	153A7C	16	141	320	2	FFGHH
898	157DFB	16	141	230	1	AGIIJ
899	367ACF	16	141	230	1	BGGII
900	1368ED	16	141	230	1	BGGIJ
901	13C7EA	16	141	230	1	BGGIJ
902	13C8E5	16	141	230	1	BGGJJ
903	13E8A7	16	141	230	1	BGGJJ
904	14BA3D	16	141	230	1	BGGJJ
905	13E8BC	16	141	230	1	BGHIJ
906	136ACF	16	141	230	1	BGIIJ
907	13E9A7	16	141	230	1	BGIIJ
908	157C3A	16	141	230	1	BGIIJ
909	357EBF	16	141	230	1	BGIIJ
910	367CD9	16	141	230	1	BGIIJ
911	367EFB	16	141	230	1	BGIIJ
912	13EAC5	16	141	230	1	BGIJJ
913	157D3B	16	141	230	1	BGIJJ
914	13E7B9	16	141	230	1	BHIIJ
915	1367DF	16	141	230	1	BHIJJ
916	13C7FA	16	141	230	1	FGGHI
917	13E7A9	16	141	230	1	FGGII
918	15BD36	16	141	230	1	FGGII
919	367AC9	16	141	230	1	FGGII
920	367EBF	16	141	230	1	FGGII
921	153DBE	16	141	230	1	FGGIJ
922	1579FA	16	141	230	1	FGGIJ
923	15A9ED	16	141	230	1	FGGIJ
924	15B9FD	16	141	230	1	FGGIJ
925	367DBE	16	141	230	1	FGGIJ

TABLE 3 (continued).

N	SP	S	H_1	H_2	H_3	LT
926	15A9E6	16	141	230	1	FGGJJ
927	13E96C	16	141	230	1	FGHII
928	1367CF	16	141	230	1	FGHIJ
929	1367ED	16	141	230	1	FGHIJ
930	157F3B	16	141	230	1	FGHJJ
931	367F9C	16	141	230	1	FGHJJ
932	15BF36	16	141	230	1	FGIJJ
933	13EB6C	16	141	221	1	BGGIJ
934	13C7F9	16	141	221	1	BGIIJ
935	1367FD	16	141	221	1	BHIJJ
936	35ADF9	16	141	221	1	FGGHI
937	13EBC6	16	141	221	1	FGGHJ
938	13C9E6	16	141	221	1	FGHIJ
939	357BED	16	141	221	1	FGHIJ
940	13C7AE	16	141	221	1	FGIIJ
941	13C8F6	16	141	131	0	GGHJJ
942	13E865	16	141	131	0	GGIIJ
943	13CB65	16	141	131	0	GGIJJ
944	136BCF	16	141	131	0	GGJJJ
945	13EBC5	16	141	131	0	GGJJJ
946	13ED6A	16	141	131	0	GGJJJ
947	367CDF	16	141	131	0	GHIIJ
948	153CBF	16	141	131	0	GHIJJ
949	367AFD	16	141	131	0	GHIJJ
950	13EDA6	16	141	131	0	GHJJJ
951	15EBC6	16	141	131	0	GHJJJ
952	13C7FB	16	141	131	0	GIIIJ
953	13CA65	16	141	131	0	GIIIJ
954	13E965	16	141	131	0	GIIJJ
955	1369ED	16	141	131	0	GIJJJ
956	13C9E5	16	141	131	0	GIJJJ
957	367CD5	16	141	131	0	HIIIJ
958	367CE5	16	141	131	0	HIIJJ
959	367CED	16	141	122	0	HIIJJ
960	14A7F9	16	060	400	4	BBBFI
961	14F7A9	16	060	400	4	BBBFJ
962	14A79F	16	060	310	3	BBBGJ
963	14FD6A	16	060	310	3	BBBJJ
964	15EC7A	16	060	310	3	BBFIJ
965	14F79A	16	060	310	3	BFFGJ
966	1BFFD8	32	060	220	2	BBGJJ
967	1B6FD8	32	060	220	2	BBHJJ
968	1C6F38	32	060	220	2	BBHJJ
969	14A7DF	16	060	220	2	BBJJJ
970	14B7EC	16	060	220	2	BBJJJ
971	14BE69	16	060	220	2	BFGIJ
972	14BCE7	16	060	220	2	BFGJJ
973	14BD6E	16	060	220	2	BFGJJ
974	17BE9D	16	060	220	2	BFGJJ
975	15BCE6	16	060	220	2	BFHII
976	15BDE6	16	060	220	2	BFHII
977	35AEFD	16	060	220	2	BFHII
978	15EF7C	16	060	220	2	BFHJJ
979	15FDA6	16	060	220	2	BFIJJ
980	14AC7F	16	060	220	2	FFGGJ
981	14BF97	16	060	220	2	FFGGJ
982	17AF9C	16	060	220	2	FFGHJ
983	15BC6F	16	060	220	2	FFGIJ
984	16BF9C	16	060	220	2	FFGJJ
985	57EC6D	96	060	220	2	FFHHH
986	35FA6B	16	060	220	2	FFHHI
987	36ABE7	16	060	220	2	FFHHJ
988	14BF68	16	060	130	1	BGGJJ
989	15ACF6	16	060	130	1	BHIIJ
990	15ADF6	16	060	130	1	BHIIJ
991	36CEBF	16	060	130	1	BIIIJ
992	36FEBD	16	060	130	1	BIIIJ
993	15ECB6	16	060	130	1	BIJJJ
994	15EFD6	16	060	130	1	BIJJJ
995	15AD7E	16	060	130	1	FGIJJ
996	15AF79	16	060	130	1	FGIJJ
997	16AE9D	16	060	130	1	FGIJJ
998	15AE96	16	060	130	1	FGJJJ
999	15BE96	16	060	130	1	FGJJJ
1000	36AEF9	16	060	130	1	FHIII
1001	35ABED	16	060	130	1	FHIIJ
1002	35ADF6	16	060	130	1	FHIIJ
1003	36ABD5	16	060	130	1	FHIJJ
1004	36AFD5	16	060	130	1	FHIJJ
1005	37DE6F	16	060	130	1	FHIJJ
1006	36ADF7	16	060	130	1	FHJJJ
1007	36AFE7	16	060	130	1	FHJJJ
1008	36CDB5	16	060	130	1	FIIJJ
1009	36CEB9	16	060	130	1	FIIJJ
1010	36CF97	16	060	130	1	FIIJJ
1011	14FF28	320	060	040	0	GGGGG
1012	DEFF7B	64	060	040	0	GJJJJ
1013	37DEBF	32	060	040	0	HIIJJ
1014	36ADC5	16	060	040	0	IIJJJ
1015	14FBD7	16	051	310	3	ABFJJ
1016	15BC7F	16	051	310	3	AFFIJ
1017	15BF79	16	051	310	3	AFFIJ
1018	15BF86	16	051	310	3	AFFJJ
1019	13D7F9	16	051	310	3	BBFII
1020	13EA7C	16	051	310	3	BBFII
1021	13EC7A	16	051	310	3	BFFGH
1022	15FD7B	16	051	310	3	BFFGI
1023	14A7CF	16	051	310	3	BFFGJ
1024	15EA7C	16	051	310	3	BFFHH
1025	13D9F7	16	051	310	3	BFFHI
1026	14FB6C	16	051	220	2	ABJJJ
1027	13D7E8	16	051	220	2	BBGIJ
1028	13D7FB	16	051	220	2	BBIIJ
1029	13E97C	16	051	220	2	BBIIJ
1030	14B96E	16	051	220	2	BFGGJ
1031	13E7D8	16	051	220	2	BFGHI
1032	15BCF6	16	051	220	2	BFGHI
1033	13D8E7	16	051	220	2	BFGHJ
1034	14FA6D	16	051	220	2	BFGHJ
1035	15FA7C	16	051	220	2	BFGHJ
1036	13D7FA	16	051	220	2	BFGIJ
1037	13E87C	16	051	220	2	BFGIJ
1038	13D8F6	16	051	220	2	BFGJJ
1039	13EBD7	16	051	220	2	BFGJJ
1040	15B9E6	16	051	220	2	BFGJJ
1041	1B57ED	16	051	220	2	BFGJJ
1042	1B57FD	16	051	220	2	BFGJJ
1043	13ECB6	16	051	220	2	BFHHJ
1044	13D7EB	16	051	220	2	BFHII
1045	35AEF6	16	051	220	2	BFHIJ
1046	35AFDE	16	051	220	2	BFHIJ
1047	15EAD6	16	051	220	2	BFHJJ
1048	15EB7D	16	051	220	2	BFHJJ
1049	15FAD6	16	051	220	2	BFHJJ
1050	35AFD9	16	051	220	2	BFIII
1051	13D7AF	16	051	220	2	BFIIJ
1052	35FE6D	16	051	220	2	BFIIJ
1053	13ED7B	16	051	220	2	BFIJJ
1054	13EDB7	16	051	220	2	BFJJJ
1055	15FB7D	16	051	220	2	FFGHI
1056	13ECA7	16	051	220	2	FFGHJ
1057	15FBC6	16	051	220	2	FFGHJ
1058	15FCB6	16	051	220	2	FFGIJ
1059	36FEB7	16	051	220	2	FFIII
1060	13E7DB	16	051	211	2	BBIIJ
1061	13D7EA	16	051	130	1	BGIIJ
1062	14F96A	16	051	130	1	BGJJJ
1063	13EAD6	16	051	130	1	BHIJJ
1064	13D9E6	16	051	130	1	BIJJJ
1065	13EC6B	16	051	130	1	FGHIJ
1066	36ACD9	16	051	130	1	FGIII
1067	37AFD9	16	051	130	1	FGIII
1068	13C7EB	16	051	130	1	FGIIJ
1069	13E7C9	16	051	130	1	FGIIJ
1070	15BDF6	16	051	130	1	FGIIJ
1071	36CDBE	16	051	130	1	FGIIJ
1072	13C9F7	16	051	130	1	FGIJJ
1073	15FE7D	16	051	130	1	FGIJJ
1074	37ACDF	16	051	130	1	FHIII
1075	36AEFD	16	051	130	1	FIIJJ
1076	13E7D9	16	051	121	1	BIIIJ
1077	35AFE9	16	051	121	1	BIIJJ
1078	13E79C	16	051	121	1	FGIIJ
1079	36ACDF	16	051	121	1	FHIII
1080	36CFE7	80	051	050	0	JJJJJ
1081	37EDBC	64	051	031	0	HJJJJ
1082	36CF9E	16	051	031	0	IIIJJ
1083	37ADC9	16	051	031	0	IIIJJ
1084	37AFE9	16	051	031	0	IIIJJ
1085	15B97F	32	042	400	4	ABFFG
1086	35FEBD	32	042	220	2	BBGII
1087	13C8E7	16	042	220	2	FFGGJ
1088	14F97A	16	042	220	2	FFGGJ
1089	13EAC7	16	042	220	2	FFGIJ
1090	36ACD5	16	042	121	1	FGIIJ

TABLE 3 (conclusion).

N	SP	S	H_1	H_2	H_3	LT	N	SP	S	H_1	H_2	H_3	LT	N	SP	S	H_1	H_2	H_3	LT
1091	53FF35	32	510	820	3	AGHHAA	1118	56FF95	16	240	340	1	BHIJGG	1145	56CCF5	16	141	420	3	FHIIBB
1092	53FFCA	64	430	630	2	AAGGGG	1119	56FF9A	16	240	340	1	BHIJGG	1146	36AAF9	16	141	330	2	BFGIII
1093	53AA35	16	420	630	2	GHIJAA	1120	36AC65	16	240	340	1	BIIJGG	1147	35AAF9	16	141	330	2	BFHIGG
1094	53FF3A	16	420	540	1	AGHHGG	1121	36CC65	16	240	340	1	FIIJGG	1148	53AC35	16	141	330	2	GHIJBB
1095	5CFF3A	64	420	450	0	HHHHGG	1122	53FAC5	16	240	250	0	GHHJII	1149	53AF36	16	141	240	1	AIIJII
1096	56CC65	48	330	620	3	FIIIAA	1123	53AA6F	16	240	250	0	HHHIII	1150	56CA6F	16	141	240	1	BGHIII
1097	53AA9F	16	330	440	1	AHIIGG	1124	56FF3A	16	240	250	0	HHHJII	1151	36AACF	16	141	240	1	BGHJII
1098	53FF36	16	330	440	1	AHIJGG	1125	53AA36	16	240	250	0	HIIJGG	1152	36CC6F	16	141	240	1	BIJJGG
1099	56FF35	16	330	440	1	BGHHGG	1126	56AF95	16	240	250	0	HIJJII	1153	36AF65	16	141	240	1	BIJJII
1100	56FA65	16	330	440	1	BGIJGG	1127	56AA3F	16	240	241	0	GHHJII	1154	56CF35	16	141	240	1	FGHIII
1101	56AF35	16	330	350	0	GHJJGG	1128	53AFC5	16	231	340	1	AGHIII	1155	36CF65	16	141	240	1	FIJJII
1102	53AA65	16	330	350	0	HHIIGG	1129	53FF6A	16	231	340	1	AGHJII	1156	56AC35	16	141	150	0	GIJJII
1103	53AA3F	16	330	350	0	HHIJGG	1130	56FAC5	16	231	340	1	BGGJII	1157	36AAC5	16	141	150	0	GJJJII
1104	53FA65	16	330	350	0	HHIJGG	1131	36AA65	16	231	340	1	BGIJGG	1158	53AC36	16	141	150	0	HIIJII
1105	53FA35	16	330	341	0	GHHHGG	1132	56CA65	16	231	340	1	FGIIGG	1159	36FF65	16	132	330	2	BFIJGG
1106	56FF65	16	321	620	3	BHIJAA	1133	36AAC9	16	231	340	1	FGIJGG	1160	56AAC9	16	132	240	1	FGGIII
1107	53AF35	16	321	440	1	AGIJGG	1134	56FF6A	32	231	250	0	HHJJGG	1161	36CC9F	32	070	230	2	BBIIII
1108	56AACF	16	321	440	1	BGGHGG	1135	56CC6F	16	150	420	3	BIIJBB	1162	35FFAC	128	061	410	4	BBBBGG
1109	56AA35	16	321	350	0	GGJJGG	1136	36FC65	16	150	330	2	BFIIII	1163	35AAFC	32	051	230	2	FFHHII
1110	53FF9A	16	250	430	2	ABGIII	1137	36CC69	16	150	330	2	HIJJBB	1164	39FF65	16	051	230	2	FFHIII
1111	59FF9A	32	250	430	2	BBIIGG	1138	36AC69	16	150	240	1	BHIIII	1165	36FF95	16	051	140	1	BJJJII
1112	53FF39	16	240	520	3	AHIJBB	1139	35AA69	16	150	240	1	BHIJII	1166	35AAF6	16	051	140	1	FHJJII
1113	56FFC5	16	240	520	3	BGHHBB	1140	36AC6F	16	150	240	1	BHIJII	1167	39CC6F	32	051	050	0	JJJJII
1114	53AAC5	16	240	430	2	GHIJBB	1141	35AAC9	16	150	240	1	FHIJII	1168	35FF6C	16	042	320	3	BBFJII
1115	53AF95	16	240	340	1	AIIJII	1142	53AA96	16	150	150	0	HHIIII	1169	36FF6A	16	042	320	3	FHJJBB
1116	53AACF	16	240	340	1	BGHIII	1143	53FA36	16	150	150	0	HHIJII	1170	36FF6C	32	042	230	2	FFJJGG
1117	59FF3A	16	240	340	1	BHHIII	1144	36AA6F	32	141	420	3	BGHHBB	1171	36FF9C	32	042	230	2	FFJJGG

TABLE 4. Non-orientable, 6 cusped, minimal volume, integral, congruence 2, hyperbolic 4-manifolds.

headed by SP lists the side-pairing for the manifold in a coded form that is explained in Section 5. The column headed by S lists the number of symmetries of the manifolds. All the manifolds have a subgroup of symmetries corresponding to K^4. Therefore, the number of symmetries is a multiple of 16. The possible orders are 16, 32, 48, 64, 80, 96, 128, and 320. Only manifold number 1011 has a symmetry group of order 320.

The column of Tables 2–4 headed by H_i lists the i-th homology groups of the manifolds with the 3 digit number abc representing $\mathbb{Z}^a \oplus \mathbb{Z}_2^b \oplus \mathbb{Z}_4^c$ and the single digit entry a representing $\mathbb{Z}^a$.

The column headed by LT lists the link types of the cusps of the manifolds. Here A, B, ..., J represent the 10 closed Euclidean 3-manifolds in the order given by Hantzsche and Wendt [1935]. The orientable manifolds are A,...,F with A the 3-torus and F the Hantzsche–Wendt 3-manifold [Zimmermann 1990]. Only C, D, and E do not occur as links of cusps of our manifolds. The closed Euclidean 3-manifolds are identified by their homology [Hantzsche and Wendt 1935]. Manifold 1162 is the hyperbolic 24-cell space in [Ratcliffe 1994, p. 510]. Tables 2–4 give some indication of the diversity of hyperbolic 4-manifolds of finite volume.

ELECTRONIC AVAILABILITY

Plain text files of Tables 1–4 are available at ftp://math.vanderbilt.edu/users/tschantz/mantabs. The files are 3mantab.txt and 4mantab.txt.

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Received March 4, 1999; accepted March 9, 1999