Invariants of Orthogonal G-Modules from the Character Table

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CONTENTS

Character-theoretic methods using Clifford algebras are developed to describe the quadratic forms on a vector space V that are invariant under a finite subgroup G of GL(V) such that the order of the commutator factor group of G is odd.

1. INTRODUCTION

Let G be a finite group. Any $\mathbb{Q}G$-module V is uniquely determined by its character χ_V. So in principle χ_V also determines the G-invariant quadratic forms on V. However, there is not much known how to calculate rational invariants of these quadratic forms without describing the action of G on V explicitly. Of course, the G-invariant form q on V is not unique, since for all $a \in \mathbb{Q}$, the form aq is also G-invariant, and there are more G-invariant forms, if V is not irreducible over the reals. So one can only hope to calculate say the determinant of q, which is defined as the determinant of a Gram matrix of the corresponding bilinear form modulo squares, if $\dim V$ is even.

Invariants for a nondegenerate quadratic space $\varphi := (V, q)$ over an arbitrary field K can be read off from the Clifford algebra $C(\varphi)$. This is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra functorially attached to φ and it determines the two most important invariants of the quadratic space φ: its determinant and its Clifford invariant (see Theorem 2.1). According to [Hasse 1924], if V is a vector space over a number field K, then these two invariants together with the dimension of V and the signatures of φ at all real places of K describe the K-isometry class of the nondegenerate quadratic space φ over K completely.

If G acts on $\varphi = (V, q)$ as isometries, then G acts on $C(\varphi)$ as algebra automorphisms respecting the grading. If $\dim V$ is even, $C(\varphi) =: c(\varphi)$ is central

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simple; if $\dim V$ is odd, the even part $c(\varphi) := C_0(\varphi)$ of the Clifford algebra is a central simple algebra. So the simple $c(\varphi)$-module W becomes a module for some projective representation of G (in the sense of Schur) of which the character χ_W (of a covering group of G) is a certain square root of a character constructed from χ_V. An irreducible constituent that occurs in χ_W with odd multiplicity gives information on either the determinant of φ, if $\dim V$ is even, or the Clifford invariant of φ, if $\dim V$ is odd (see Corollary 3.6). Applications are given in the examples in Section 4.

Related ideas have been used by A. Turull [1992], where he determines the Schur index of χ_W using explicit knowledge about (V, q) for the $(n-1)$-dimensional S_n-module V.

This paper summarizes one idea of my Habilitationsschrift [Nebe 1999]. There I develop also other methods to calculate the isometry class of φ using constructions of V as a constituent of a tensor product or of an induced module. See also [Nebe 2000].

2. CLIFFORD ALGEBRAS

Let $\varphi := (V, q)$ be a quadratic space of dimension $n := \dim V$ over a field K. Then the *Clifford algebra* $C(\varphi) := T(V)/I(\varphi)$ is the quotient of the tensor algebra $T(V) := \bigoplus_{i=0}^{\infty} \otimes^i V$ by the two-sided ideal $I(\varphi)$ generated by $v \otimes v - q(v) \cdot 1$ with $v \in V$ (see [Scharlau 1985, Chapter 9]). $C(\varphi)$ is a 2^n-dimensional K-algebra. It contains the *even Clifford algebra*

$$C_0(\varphi) := \langle v_1 \dots v_s \mid v_i \in V, s \text{ even } \rangle \le C(\varphi)$$

as a subalgebra of dimension $2^{n-1} = \dim_K(C_0(\varphi))$. Let

$$C_1(\varphi) := \langle v_1 \dots v_s \mid v_i \in V, s \text{ odd}\rangle.$$

If $(v_1, \dots, v_n)$ is a basis of V, then $(v_{i_1} \dots v_{i_s} \mid 1 \le i_1 < \dots < i_s \le n)$ is a basis of $C(\varphi)$. In particular, V is embedded in $C(\varphi)$.

If $\operatorname{char} K \neq 2$, then let B_q be the bilinear form associated to q, defined by

$$B_q(v, w) := \tfrac{1}{2}(q(v+w) - q(v) - q(w))$$

for all $v, w \in V$. Then the *determinant* $\det(\varphi)$ is the determinant of a Gram matrix of B_q modulo squares

$$\det(\varphi) := \det(B_q(v_i, v_j))_{i,j=1}^n (K^*)^2 \in K/(K^*)^2$$

and the *discriminant* of φ is

$$d_\pm(\varphi) = (-1)^{\binom{n}{2}} \det(\varphi) \in K/(K^*)^2.$$

Assume that $\operatorname{char} K \neq 2$ and φ is *nondegenerate*, that is, $d_\pm(\varphi) \neq 0$. Then:

Theorem 2.1 [Scharlau 1985, Theorem 9.2.10]. (i) *If n is even, then $C(\varphi)$ is a central simple K-algebra and $Z(C_0(\varphi))$ is isomorphic to*

$$K[X]/(X^2 - d_\pm(\varphi)).$$

(ii) *If n is odd, then $C_0(\varphi)$ is a central simple K-algebra and $Z(C(\varphi))$ is isomorphic to*

$$K[X]/(X^2 - d_\pm(\varphi)).$$

An analogous theorem holds if $\operatorname{char} K = 2$; see [Knus 1991, Theorem (2.2.3)].

Let

$$c(\varphi) := \begin{cases} C(\varphi) & \text{if } \dim V \text{ is even,} \\ C_0(\varphi) & \text{if } \dim V \text{ is odd.} \end{cases}$$

The class $[c(\varphi)] \in \operatorname{Br} K$ of the central simple K-algebra $c(\varphi)$ in the Brauer group $\operatorname{Br} K$ of K is called the *Clifford invariant* of φ.

Remarks 2.2. (i) If W is the simple right $c(\varphi)$-module, $D := \operatorname{End}_{c(\varphi)}(W)$, and $W^* := \operatorname{Hom}_D(W, D)$ is the simple left $c(\varphi)$-module, then $c(\varphi) \cong W^* \otimes_D W$ as $c(\varphi)$-bimodule.

(ii) Because of the universal property of $C(\varphi)$, the identity embedding of $V \subset C(\varphi)$ extends to a unique algebra antiautomorphism of $C(\varphi)$, the canonical involution $^-$. The involution satisfies $\overline{v_1 \dots v_s} = v_s \dots v_1$ for all $v_i \in V$ and induces an algebra antiautomorphism on the central simple K-algebra $c(\varphi)$. Via this involution W^* becomes a right $c(\varphi)$-module isomorphic to W.

The Orthogonal Group

For a nondegenerate quadratic space $\varphi = (V, q)$ the orthogonal group $O(\varphi)$ is defined as

$$O(\varphi) := \{f \in \mathrm{GL}(V) \mid q(f(v)) = q(v) \text{ for all } v \in V\}.$$

Using the universal property of $C(\varphi)$ one easily sees that the linear action of $O(\varphi)$ on V extends uniquely to K-algebra automorphisms of the Clifford algebra $C(\varphi)$. This action of $O(\varphi)$ on $C(\varphi)$ respects the grading.

The *Clifford group* is

$$\Gamma(\varphi) := \{s \in C_0(\varphi)^* \cup (C(\varphi)^* \cap C_1(\varphi)) \mid sVs^{-1} = V\}$$

[Scharlau 1985, Section 9.3; Knus 1991, Section IV.6]. For $s \in \Gamma(\varphi)$ let $\gamma(s) = 1$ if $s \in C_0(\varphi)$ and $\gamma(s) = -1$ if $s \in C_1(\varphi)$. Define $\alpha(s) : V \to V$ so that

$$\alpha(s)(v)\ \gamma(s)s^{-1}vs.$$

Then $\alpha(\Gamma(\varphi)) \subset O(\varphi)$. If $s \in V \subset C_1(\varphi)$ satisfies $q(s) \neq 0$, then $s \in \Gamma(\varphi)$ and $\alpha(s) \in O(\varphi)$ is the reflection along the anisotropic vector s. If $\operatorname{char} K \neq 2$, then the reflections along anisotropic vectors in V generate $O(\varphi)$. Therefore $\alpha(\Gamma(\varphi)) = O(\varphi)$. If $\operatorname{char} K = 2$ then α is also surjective [Knus 1991, p. 228]. The kernel of α is $K^* \subset C_0(\varphi)^*$ and one has the following exact sequence

$$1 \to K^* \to \Gamma(\varphi) \to O(\varphi) \to 1$$

which gives rise to a projective representation

$$P : O(\varphi) \to \Gamma(\varphi)$$

mapping the reflection along the anisotropic vector $s \in V$ to $s \in \Gamma(\varphi)$. By [Scharlau 1985, Lemma 9.3.2], $s\bar{s} \in K^* \subseteq C_0(\varphi)$ for all $s \in \Gamma(\varphi)$, where $^-$ is the canonical involution of $c(\varphi)$ (see Remark 2.2). This gives rise to a group homomorphism

$$\begin{aligned}\operatorname{norm} : O(\varphi) &\to K^*/(K^*)^2,\\ g &\mapsto P(g) \cdot \overline{P(g)} \cdot (K^*)^2,\end{aligned}$$

called the *Spinor norm* [Scharlau 1985, Definition 9.3.4]. Let

$$S\Gamma(\varphi) := \{s \in \Gamma(\varphi) \cap C_0(\varphi) \mid \bar{s} = s^{-1}\}.$$

Proposition 2.3. *The group $S\Gamma(\varphi)$ acts on $c(\varphi)$ by conjugation: $c(\varphi) \times S\Gamma(\varphi) \to c(\varphi), (x, s) \mapsto s^{-1}xs$. With the notations of Remark 2.2 one has*

$$c(\varphi) \cong W \otimes_D W$$

as $S\Gamma(\varphi)$-modules.

3. CLIFFORD ALGEBRAS AS G-ALGEBRAS.

Let $\varphi = (V, q)$ be a nondegenerate quadratic space and G be a subgroup of the orthogonal group $O(\varphi)$. Then φ is also called an *orthogonal KG-module.* Theoretical concepts for Clifford algebras as G-algebras are given in [Fröhlich 1972]. Here practical methods to obtain information on $C(\varphi)$ from the character $\chi = \chi_V$ of the G-module V are developed.

Since $G \leq O(\varphi)$, the action of $O(\varphi)$ on $C(\varphi)$ restricts to a linear representation $\Delta_{C(\varphi)}$ of G on the Clifford algebra that respects the grading:

Remark 3.1. The character of the KG-module $C(\varphi)$ respectively $C_0(\varphi)$ is

$$\tilde{\chi} := \sum_{i=0}^{n} \Lambda^i(\chi) \text{ respectively } \tilde{\chi}_0 := \sum_{i=0, i\ even}^{n} \Lambda^i(\chi)$$

where $\Lambda^i(\chi)$ is the i-th exterior power of χ.

Note that

$$\tilde{\chi}(g) = (-1)^n p_g(-1) \text{ for all } g \in G$$

where p_g is the characteristic polynomial of g on V. With this trick one can calculate $\tilde{\chi}(g)$ (and $\tilde{\chi}_0(g)$) with the help of GAP [Schönert et al. 1994], by restricting χ to the subgroup $\langle g \rangle \leq G$, for any group G whose character table and powermap is known.

Assumption. From now on we assume that the order of the commutator factor group G/G' is odd.

Lemma 3.2. *There are $a_g \in K^*$ (for all $g \in G$) such that $P_0(g) := a_g P(g)$ satisfies*

$$P_0(g) \subseteq C_0(\varphi), \quad P_0(g)\overline{P_0(g)} = 1,$$

and $P_0 \otimes P_0 : G \to \mathrm{GL}(c(\varphi))$ is a linear representation equivalent to $\Delta_{c(\varphi)}$.

Proof. The mapping $P : G \to C(\varphi)^*, g \mapsto P(g)$ is a projective representation of G. Hence $P(g)P(h) = a(g, h)P(gh)$ for some $a(g, h) \in K^*$ and for all $h, g \in G$. Since $K^* \subseteq C_0(\varphi)$, the grading of the Clifford algebra defines a group homomorphism $G \to \mathbb{Z}/2\mathbb{Z}$, $g \mapsto \deg(P(g))$. Since $|G/G'|$ is odd, this group homomorphism is trivial, hence $P(G) \subseteq C_0(\varphi)$. In particular $P(G) \subseteq c(\varphi)$ and P is a projective representation of G on the simple $c(\varphi)$-module W. Also the Spinor norm, $\operatorname{norm} : G \to K^*/(K^*)^2$, is a homomorphism from G to an abelian 2-group and hence trivial. Rescaling $P(g)$ with elements in K^* we may therefore assume that $P(g)\overline{P(g)} = 1$, that is, $P(g) \in S\Gamma(\varphi)$ for all $g \in G$.

Therefore, by Proposition 2.3, the G-module $c(\varphi)$ is isomorphic to the tensor square $W \otimes_D W$ of the projective G-module W. In particular the projective representation $P \otimes_D P$ is equivalent (as a projective representation) to the linear representation $\Delta_{c(\varphi)}$ of G on $c(\varphi)$. This means that there is $T \in \mathrm{GL}(c(\varphi))$ and a mapping $\alpha : G \to K^*$ such that $TP(g) \otimes P(g)T^{-1} = \alpha(g)\Delta_{c(\varphi)}(g)$ for all $g \in G$. Since P is a projective representation and $\Delta_{c(\varphi)}$ is linear, $\alpha : G \to K^*/(K^*)^2$ is a homomorphism. Again, since

$|G/G'|$ is odd, we have $\alpha(g) = \beta(g)^2 \in (K^*)^2$ for all $g \in G$ and $P_0(g) := \beta(g)^{-1}P(g)$ has the desired properties. $\square$

Corollary 3.3 (compare [Gagola and Garrison 1982, Theorem 1.2, Corollary 4.3]). *Assume that* $\operatorname{char} K \neq 2$ *and let* $g \in G$ *be an element of order* 2 *and* e *the dimension of the* -1*-eigenspace of* g *in* V. *Then* $P_0(g)^2 = (-1)^{\binom{e}{2}}$ id.

Proof. Let $v_1, \dots, v_e$ be an orthogonal basis of the -1-eigenspace of g on V. Then

$$P_0(g) = a_g v_1 \dots v_e,$$
$$\overline{P_0(g)} = a_g v_e \dots v_1 = (-1)^{\binom{e}{2}} P_0(g).$$

Since $P_0(g)\overline{P_0(g)} = \mathrm{id}$, one has $P_0(g)^2 = (-1)^{\binom{e}{2}}$ id. $\square$

If $c(\varphi) \cong D^{a\times a}$ for some central K-division algebra D, then the simple $c(\varphi)$-module W is isomorphic to D^a. Over the algebraic closure of K, the $c(\varphi)$-module W is isomorphic to the sum of m copies of a simple module, where m is the *index* of D ($\dim_K(D) = m^2$).

We now fix a covering group $u : \tilde{G} \to G$ of G such that P_0 is equivalent to a linear representation of $\tilde{G}$. Let W be the simple $c(\varphi)$-module and m the index of $\operatorname{End}_{c(\varphi)}(W)$.

Corollary 3.4. *Let* $m\chi_W$ *be the character of a linear* $K\tilde{G}$*-module that is equivalent to* W *over the algebraic closure of* K. *Regarding* χ *as a character of* $\tilde{G}$ *one has*

$$\chi_W \otimes \chi_W = \begin{cases} \tilde{\chi} & \text{if } n \text{ is even,} \\ \tilde{\chi}_0 & \text{if } n \text{ is odd.} \end{cases}$$

For the next theorem we additionally assume that K is a number field. By [Scharlau 1985, Lemma 9.2.8] $c(\varphi)$ is a tensor product of quaternion algebras. Since K is a number field, this implies that $c(\varphi)$ is a matrix ring over a quaternion algebra (see [Reiner 1975, Theorem (32.9)], for instance) and

$$C_0(\varphi) \cong D^{a\times a},$$

where $D = L := Z(C_0(\varphi))$ or D is a quaternion division algebra over L.

Theorem 3.5. *With the notations above let* m *be the Schur index of* D, W *the simple* $C_0(\varphi)$*-module and* $m\chi_W$ *the corresponding character of* $\tilde{G}$. *Assume that there is an absolutely irreducible character* ψ *of* $\tilde{G}$ *occurring with odd multiplicity in* χ_W.

(a) *If* n *is even and* $L = Z(C_0(\varphi))$ *is a field then* L *is a subfield of the character field* $K(\psi)$.

(b) *Assume that* n *is odd. If the Schur index of* ψ *is odd, then* $K(\psi)$ *splits* D. *Otherwise let* U *be the irreducible* $K\tilde{G}$*-module whose character contains* ψ. *Then* $D \subset \operatorname{End}_{K\tilde{G}}(U)$.

Proof. By Lemma 3.2 $P(G)$ is already contained in $C_0(\varphi)$ and therefore $\operatorname{End}_{C_0(\varphi)}(W) \subseteq \operatorname{End}_{\tilde{G}}(W)$.

In both cases $C_0(\varphi) \cong D^{a\times a}$ and $\dim_K(C_0(\varphi)) = a^2m^2[L:K] = 2^{n-1}$ is a power of 2. Let x be the multiplicity of ψ in χ_W, U the irreducible K-module whose character contains ψ and $D_U := \operatorname{End}_{K\tilde{G}}(U)$. Then D_U is a skew field with center $K(\psi)$ and of index, say, m_U. Let U' be the U-homogeneous component in $W_{|\tilde{G}}$. Then $\operatorname{End}_{K\tilde{G}}(U') \cong D_U^{y\times y}$ for some $y \in \mathbb{N}$. Since the multiplicity of ψ in $m\chi_W$ and the multiplicity of ψ in $\chi_{U'}$ are equal, one has

$$mx = ym_U. \tag{3–1}$$

Since D has no zero divisors, D embeds into

$$\operatorname{End}_{K\tilde{G}}(U')$$

and hence

$$\begin{aligned} A &:= D_U^{op} \otimes_{K(\psi)} (D \otimes_K K(\psi)) \\ &\to D_U^{op} \otimes_{K(\psi)} D_U^{y\times y} \cong K(\psi)^{m_U^2 y\times m_U^2 y} =: B, \end{aligned}$$

where D_U^{op} denotes the opposite algebra of D_U. If $K(\psi) \otimes_K L$ is a field then let $\varepsilon := [L:K] \in \{1,2\}$. Then A is a central simple $K(\psi)\otimes_K L$-algebra isomorphic to $\tilde{D}^{k\times k}$ for some central $K(\psi)\otimes_K L$-division algebra $\tilde{D}$ of index, say, $\tilde{m}$. If $K(\psi)\otimes_K L$ is not a field then let $\varepsilon := 1$. Then A is a direct sum of two isomorphic central simple $K(\psi)$-algebras isomorphic to $\tilde{D}^{k\times k}$ for some central $K(\psi)$-division algebra $\tilde{D}$ of index, say, $\tilde{m}$.

In both cases the dimension of A over its center is

$$m_U^2 \cdot m^2 = \tilde{m}^2 \cdot k^2 \tag{3–2}$$

and the $K(\psi)$-dimension of a simple A-module is $\varepsilon \cdot \tilde{m}^2 \cdot k$ and divides the $K(\psi)$-dimension of the simple B-module, which is $m_U^2 \cdot y$

$$\varepsilon \cdot \tilde{m}^2 \cdot k \text{ divides } m_U^2 \cdot y.$$

Claim. $\tilde{m}$ *is odd and* $\varepsilon = 1$.

Proof. Since K is a number field, m is either 1 or 2. If $m=1$, then m_U and y are odd by (3–1) (recall that x is odd) and hence also $\tilde{m}$ and ε are odd. Assume that $m=2$. Then (3–1) implies that either m_U is even and $y \cdot m_U/2$ is odd, or y is even and $m_U \cdot y/2$ is odd. Assume $2 \mid \tilde{m}$. If m_U is even, 2^3 divides $\tilde{m}^2 \cdot k$; if m_U is odd, 2^2 divides $\tilde{m}^2 \cdot k$. But this power of 2 does not divide $m_U^2 \cdot y$ in both cases, which is a contradiction. Therefore $\tilde{m}$ is odd and k is even. If m_U is even, then 4 divides k by (3–2) and therefore ε is odd. If m_U is odd, then also $\varepsilon = 1$ since k is even and $y/2$ is odd. The claim follows. □

In particular $\varepsilon = 1$ and hence L is a subfield of $K(\psi)$ which proves (a).

Now we prove (b). Since n is odd, $L = K$ and $([D_U]^{-1} \cdot [D \otimes_K K(\psi)])$ has odd order in the Brauer group of $K(\psi)$ because

$$\left([D_U]^{-1} \cdot [D \otimes_K K(\psi)]\right)^{\tilde{m}} = 1 \in \mathrm{Br}(K(\psi))$$

Therefore the local index $m_\wp(D_U)$ is even, if and only if the local index $m_\wp(D \otimes_K K(\psi))$ is 2, for all (infinite and finite) places $\wp$ of $K(\psi)$. Hence $D \otimes_K K(\psi)$ embeds into D_U (compare [Reiner 1975, Exercise 29.7]). □

In the applications absolutely irreducible orthogonal G-modules (V, q) over totally real number fields K are of special interest. Then q is (positive or negative) definite. If n is even, then the discriminant of q is negative, if $n \equiv 2 \pmod 4$ and positive, if $4 \mid n$.

Corollary 3.6. *In addition to the assumptions of the theorem let K be a totally real number field and assume that φ is definite.*

(a) *Let n be even. If $[K(\psi) : K]$ is odd or $n \equiv 0 \pmod 4$ and all intermediate fields $K(\psi) \supset L \supset K$ of degree $[L : K] = 2$ are complex fields, then the discriminant $d_\pm(\varphi) = 1$.*
If $n \equiv 2 \pmod 4$, then $K(\psi)/K$ has a totally complex intermediate field L with $[L : K] = 2$. One of these fields is isomorphic $K\left[\sqrt{d_\pm(\varphi)}\right]$.
(b) *Assume that n is odd. If ψ has Schur index 1, then the Clifford invariant $[c(\varphi)]$ satisfies*

$$[c(\varphi) \otimes_K K(\psi)] = [K(\psi)] \in \mathrm{Br}_2(K(\psi)).$$

If ψ has Schur index 2 then $[K(\psi) \otimes_K c(\varphi)] = [\mathrm{End}_{K(\psi)\tilde{G}}(U)] \in \mathrm{Br}_2(K(\psi))$ for the irreducible $K(\psi)G$-module U with character 2ψ.

4. EXAMPLES

We will now apply the methods presented before to some irreducible representations of finite quasi simple groups. The notations are taken from [Conway et al. 1985].

Example 1

Let $G \cong 2.O_8^+(2)$. Then G is perfect and its universal covering group is $\tilde{G} \cong 2^2.O_8^+(2)$. Let V be the 8-dimensional faithful $\mathbb{Q}G$-module with character χ and q a non zero G-invariant quadratic form on V. If $\varphi := (V, q)$, then $\dim(c(\varphi)) = 2^8$ and $\tilde{\chi} = \chi_W \otimes \chi_W$ for a 16-dimensional $\tilde{G}$-module W. One calculates that $\chi_W = \chi_8 + \chi_8'$ is the sum of the two irreducible characters $\chi_8, \chi_8' \neq \chi$ which belong to absolutely irreducible rational modules of degree 8 of $\tilde{G}$. Therefore $d_\pm(\varphi) = 1$ and also $[c(\varphi)] = [\mathbb{Q}]$.

Of course, that $d_\pm(\varphi) = 1$ is well known and can also be seen by inspection of the modular constituents of V [Jansen et al. 1995].

Example 2

Let $G \cong M^cL$ and $\varphi := (V, q)$ a 22-dimensional orthogonal $\mathbb{Q}G$-module with character χ. The universal covering group of G is $3.G$. Therefore $P_0 : G \to c(\varphi)$ can be chosen to be linear. There is a unique character χ_W of G satisfying $\chi_W \otimes \chi_W = \tilde{\chi}$. In the notation of [Conway et al. 1985] one has $\chi_W = 2(\chi_1 + \chi_2 + \chi_3) + \chi_5 + \chi_6$. Now the character field $\mathbb{Q}[\chi_5] = \mathbb{Q}[\chi_6] = \mathbb{Q}[\sqrt{-15}]$. Since $\dim(V) \equiv 2 \pmod 4$, Corollary 3.6 yields $d_\pm(\varphi) = -15$.

Example 3

Let $G \cong S_6(3)$ and χ the irreducible character of degree 78 with orthogonal $\mathbb{Q}G$-module $\varphi := (V, q)$. The universal covering group of G is $\tilde{G} \cong 2.S_6(3)$. Let χ_W be the character of $\tilde{G}$ on the simple $c(\varphi)$-module. If $g \in G$ is an element of order 2 in class $2B$ in the notation of [Conway et al. 1985], then $-g$ has a 42-dimensional fixed space on V. Therefore χ_W is a faithful character of $\tilde{G}$ by Corollary 3.3. With GAP one finds that there is only one faithful character χ_W of $\tilde{G}$ satisfying $\chi_W \otimes \chi_W = \tilde{\chi}$. The character χ_W contains the two complex conjugate irreducible characters ψ_1 and ψ_2 of degree 13 with multiplicity 1683. Since $\dim(V) \equiv 2 \pmod 4$ and $\mathbb{Q}[\psi_1] = \mathbb{Q}[\sqrt{-3}]$ Corollary 3.6 yields $d_\pm(\varphi) = -3$.

Example 4

The applications are not restricted to the characteristic 0 case. Let V be the 4-dimensional $\mathbb{F}_2A_6$-module. If V admits a nondegenerate A_6-invariant quadratic form q, then there is a projective representation

$$A_6 \to C_0((V,q))^*$$

yielding an irreducible $\overline{\mathbb{F}_2}A_6$-module of dimension 2. Since there is no such module, one concludes that V is not of quadratic type.

Now let $\varphi := (V,q)$ be a 4-dimensional simple orthogonal $\mathbb{F}_3A_5$-module. Then there is a linear representation $2.A_5 \to C_0(\varphi)^*$ giving rise to a nontrivial action of $\bar{\mathbb{F}}_3 2.A_5$ on the 2-dimensional simple $C_0(\varphi)$-module. Since the two irreducible $\bar{\mathbb{F}}_3 2.A_5$-modules of dimension 2 are only realisable over $\mathbb{F}_9$, the determinant $d_\pm(\varphi) = -1$ is not a square in $\mathbb{F}_3^*$.

Example 5

Let $G = U_3(5)$ and $\varphi := (V,q)$ be a 21-dimensional simple orthogonal $\mathbb{Q}G$-module. The universal covering group of G is $3.G$. Therefore $P_0 : G \to c(\varphi)$ can be chosen to be linear. There is a unique character χ_W of G satisfying $\chi_W \otimes \chi_W = \tilde{\chi}$. In the notation of [Conway et al. 1985] one has $\chi_W = 2\chi_1 + \chi_2 + 2\chi_3 + 2\chi_7 + 2\chi_{10} + \chi_{11} + \chi_{12} + \chi_{13} + \chi_{14}$. The character field of χ_2 (of degree 20) is $\mathbb{Q}$ and its rational Schur index is 2. If U is the irreducible $\mathbb{Q}G$-module with character $2\chi_2$ then $\mathrm{End}_{\mathbb{Q}G}(U) = \mathcal{Q}_{\infty,5}$ the rational quaternion algebra ramified only at 5 and the infinite place. Now Corollary 3.6 yields $[c(\varphi)] = [\mathcal{Q}_{\infty,5}]$.

Let q be a power of an odd prime p. As noted by a referee, the group $U_3(q)$ has a absolutely irreducible rational representation V of degree $q^2 - q + 1$ (see [Simpson and Frame 1973]) and a rational character χ of degree $q(q-1)$ with Schur index 2 at ∞ and p ([Gross 1990, Section 14]). So one might hope to generalize this calculation to arbitrary prime powers q. For $q = 7$ the character χ occurs with odd multiplicity in χ_W, so here $[c(\varphi)] = [\mathcal{Q}_{\infty,7}]$. If $q = 3$ or $q = 11$, then $\dim(V) \equiv \pm 1 \pmod 8$. Therefore $\mathbb{R}$ splits $c(\varphi)$ for any positive definite quadratic form on V and χ can not occur with odd multiplicity in χ_W. But here one finds that the trivial character has odd multiplicity in χ_W, hence $[c(\varphi)] = 1$ in these cases. For $q = 9$ the calculations do not allow to determine $[c(\varphi)]$ uniquely. It seems to be very difficult to calculate the candidates for χ_W (for $q = 7, 9, 11$ one has two possibilities for the character χ_W) generically.

As a referee pointed out, some of the examples above can also be considered from the integral lattice point of view, replacing the determinant by the determinant module $L^\#/L$, where L is a G-invariant lattice in the $\mathbb{Q}G$-module V and L is integral, that is, contained in its dual lattice

$$L^\# := \{v \in V \mid B_q(v,L) \subseteq \mathbb{Z}\}.$$

If L is a maximal integral G-invariant lattice in V, then $L^\#/L$ is a direct sum of simple selfdual $\mathbb{F}_pG$-modules, where p runs through the primes dividing $|L^\#/L|$. If all selfdual p-modular constituents of V have even degree, then p does not divide the determinant of V. This observation immediately yields that $\det(\varphi) = 1$ in Example 1, since the character is absolutely irreducible modulo every prime p. Similarly the only primes p that can divide $\det(\varphi)$ are 3 and 5 in Example 2 and 3 in Example 3. In Example 3 one even can conclude that $\det(\varphi) = 3$, because otherwise G would fix an even unimodular lattice $L \subset V$, and hence $8 \mid \dim V$.

Conclusion

This method is better called a trick, because it can not be applied too often. For instance, for the finite simple groups only the first two or three characters χ_V usually have a real chance that there is such a constituent ψ of χ_W with odd multiplicity. But when this trick can be applied, the calculations are easy and it is surprising to see the determinant of a G-invariant quadratic form appear in the character table.

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