# Nice Modular Varieties

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**CONTENTS**



We consider some more or less classical hypersurfaces in projective space, known to be birational to a quotient of the unit ball in the corresponding dimension by an arithmetic subgroup. We are interested in understanding the intersection of each such hypersurface with its Hessian from the point of view of arithmetic groups. In addition to unifying certain results found previously in the literature, we compute for four of these hypersurfaces the Hessian as well as its intersection with the hypersurface.

## INTRODUCTION

An arithmetic quotient $X_\Gamma = \Gamma\backslash\mathcal{D}$, where $\mathcal{D}$ is a bounded symmetric domain and $\Gamma$ is an arithmetic group, is an algebraic object: a fundamental theorem of Baily–Borel states that a special type of automorphic forms embeds it in projective space, i.e., it is quasiprojective, and there is a normal algebraic projective variety $X_\Gamma^*$ (called the Baily–Borel embedding of the Satake compactification) containing $X_\Gamma$. In special cases there is even such an embedding $X_\Gamma \subset X_\Gamma^* \subset \mathbb{P}^n$, displaying $X_\Gamma^*$ as a *hypersurface*. Often these hypersurfaces turn out to be very special, sometimes even *spectacular*. In several known examples, $X_\Gamma^*$ has a large automorphism group and is "exceptionally singular", meaning it has a large number of singularities compared with the degree of $X_\Gamma^*$. For example, the so-called Varchenko bound gives an upper bound $\mu(d, n{-}1)$ on the number of ordinary double points which a hypersurface of degree $d$ and dimension $n{-}1$ can have. For $(d,\, n{-}1) = (3,3)$ and $(4,3)$ (threefolds of degrees 3 and 4, respectively), these numbers are 10 and 45, and this maximum is achieved by such arithmetic quotients (these are discussed in Examples 6 and 8 below). In these two cases, the double points are in fact just the compactification locus $X_\Gamma^* - X_\Gamma$, as these are ball quotients whose boundary components are zero-dimensional. In this paper we will define a no-


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tion which we call *nice modular varieties*, a notion which makes the exhibited features of the examples discussed below explicit and precise, in the case that $X^*_\Gamma$ is a hypersurface.

## 1. NICE AND EXCEPTIONAL MODULAR VARIETIES

Let $X \subset \mathbb{P}^n$ be a projective hypersurface defined by the vanishing of the homogeneous polynomial $f$, $X = \{x \in \mathbb{P}^n : f(x) = 0\}$; recall the Hessian matrix of $X$, is the matrix

$$\operatorname{Hess} X := \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{ij},$$

a $(n+1) \times (n+1)$ matrix, the entries of which are homogenous polynomials of degree $d-2$ (where $d = \deg(f)$). The *Hessian variety* is

$$\mathbf{Hess}\, X := \{x \in \mathbb{P}^n : \det(\operatorname{Hess} X)(x) = 0\},$$

and it is again a hypersurface, of degree $(n+1)(d-2)$. It describes the following geometric properties of $X$: the points $x \in \mathbf{Hess}\, X$ are those points for which the polar quadric of $x$ with respect to $X$ is *singular* (for precise definitions of the polar quadric, see [Hunt 1996, Definition B.1.6] and the discussion there). It follows in particular that the intersection $\mathbf{Hess}\, X \cap X$ consists of $x \in X$ for which the tangent hyperplane section $X_x := T_x X \cap X$ of $X$ has a singularity at $x$ worse than an ordinary double point. This intersection $D_X := \mathbf{Hess}\, X \cap X$ is a divisor known classically as the *parabolic divisor* of $X$.

The Hessian variety is a *covariant* of $X$; an example of a contravariant is the *dual variety* [Hunt 1996, Appendix B]. Let $d_X : \mathbb{P}^n \to (\mathbb{P}^n)^\wedge$ be the duality mapping

$$(x_0 : \ldots : x_n) \mapsto \left(\frac{\partial f}{\partial x_0}(x) : \ldots : \frac{\partial f}{\partial x_n}(x)\right);$$

the image $d_X(X) = X^\wedge$ is the dual variety. (More precisely, take the closure of $d_X(X^0)$, where $X^0$ is the smooth locus of $X$.) It follows immediately that $D_X$ maps under $d_X$ to part of the *singular locus* of $X^\wedge$. For instance, for curves $C \in \mathbb{P}^2$, the intersection $D_C = \mathbf{Hess}\, C \cap C$ consists of the set of *inflection points*, hence the image under $d_X$ has a cusp as indicated in Figure 1. In general however the image $d_X(D_X)$ is no longer a divisor.

The notion we now formulate is based on the parabolic divisor $D_X$.

**Definition 1.1.** Let $X \subset \mathbb{P}^n$ be a projective hypersurface in $\mathbb{P}^n$. We call $X$ a *nice modular variety* if it satisfies these conditions:

(i) There is an algebraic subvariety (reducible) $D$, which is contained in the parabolic divisor, $D \subset D_X$, such that $X - D = \Gamma\backslash\mathcal{D} =: X_\Gamma$ is an arithmetic quotient.

(ii) The complement of $D$ in the parabolic divisor, $D_X - D$ is a "modular subvariety" (this notion will be defined more precisely in a moment) of the quotient $X_\Gamma = X - D$.

(iii) The automorphism group $\operatorname{Aut}(X)$ acts as "modular transformations", i.e., there is another arithmetic subgroup $\tilde\Gamma$, such that $\Gamma \triangleleft \tilde\Gamma$ is a normal subgroup and the quotient $\tilde\Gamma/\Gamma \cong \operatorname{Aut}(X)$.

Point (iii) in this definition amounts to requiring that the quotient

$$X/\operatorname{Aut}(X) = \tilde\Gamma\backslash\mathcal{D}$$

is itself a (in general singular) arithmetic quotient. We now define what we mean by "modular subvariety". We are assuming that $X - D = X_\Gamma = \Gamma\backslash\mathcal{D}$, where $\mathcal{D}$ is a bounded symmetric domain of dimension $n-1$. This means that there is a $\mathbb{Q}$-group $G_\mathbb{Q}$, such that $\mathrm{PG}_\mathbb{Q}(\mathbb{R}) \cong \operatorname{Aut}(\mathcal{D})$ (the group is the projective group of $\mathbb{R}$ points), and $\Gamma \subset G_\mathbb{Q}(\mathbb{Q})$ is commensurable to $G_\mathbb{Q}(\mathbb{Z})$. A *modular subgroup* $H_\mathbb{Q} \subset G_\mathbb{Q}$ is a $\mathbb{Q}$-subgroup, such that $PH_\mathbb{Q}(\mathbb{R}) \cong \operatorname{Aut}(\mathcal{D}')$ for a bounded symmetric domain $\mathcal{D}'$; it then follows that we have an embedding $\mathcal{D}' \hookrightarrow \mathcal{D}$. Finally, we assume that $\Gamma \cap H_\mathbb{Q}(\mathbb{Q})$ is an arithmetic group $\Gamma'$ in $H_\mathbb{Q}(\mathbb{Q})$. It then follows that the image in $X_\Gamma$ of $\Gamma'\backslash\mathcal{D}'$ is a (possibly singular) arithmetic subquotient; this is what we will refer to as a *modular subvariety*. Since we are discussing the case here where $D_X$ is a divisor, hence the modular subvariety should be a divisor, it follows that the domain $\mathcal{D}$ is either a complex ball $\mathbb{B}_{n-1}$ or the non-compact dual of a quadric. In these cases it is known that the subdomain $\mathcal{D}' \hookrightarrow \mathcal{D}$ is the fixed point set of an involution $\iota \subset \operatorname{Aut}(\mathcal{D})$, and the condition that $\Gamma \cap H_\mathbb{Q}(\mathbb{Q})$ is arithmetic is implied by requiring this involution to lie in the arithmetic group $\tilde\Gamma$.

Since (ii) implies that the Hessian variety contains modular subvarieties, one can ask whether it itself is a modular variety. This motivates the following definition.

**Definition 1.2.** Let $X \subset \mathbb{P}^n$ be a hypersurface, and assume that (i) of the previous definition holds, i.e., $X - D = X_\Gamma$. We call $X$ an *exceptional modular variety* if in addition $\mathbf{Hess}\, X$ is again an arithmetic quotient.

In a sense it seems surprising that such varieties exist at all. However, as we shall see, it turns out to be common, or even somewhat universal (for hypersurfaces). We will give examples of both nice and exceptional modular varieties, and a number of these examples turn out to be both.

## 2. BALL QUOTIENTS

Conditions (ii) and (iii) of the definition of nice modular varieties are related. Suppose that $X$ is the compactification of a ball quotient. If $X$ is the Satake compactification, then the locus $D$ consists of isolated points. Suppose that $X$ is smooth outside of this locus. Then we can show:

**Theorem 2.1.** *Assume that $X$ is a Satake compactification, and that the isolated singularities $D$ are rational. Moreover, assume that every component of $D_X$ contains components of $D$. Then condition* (iii) *in Definition* 1.1 *implies condition* (ii).

*Proof.* Condition (iii) in the definition implies that the automorphisms of $X$ can be lifted to automorphisms of the ball, i.e., for each $\gamma \in \mathrm{Aut}(X)$, there is a $\tilde{\gamma} \in \mathrm{Aut}(\mathbb{B}_{n-1})$, such that $\tilde{\gamma}$ induces the action of $\gamma$ on $X = X_\Gamma$. In particular, the element $\tilde{\gamma}$ is a *linear* automorphism, and its fixed point set (modulo $\Gamma$ it has finite order) consists of a *linearly* embedded subball. The image of this subball is then in the branch locus of $X \to X/\mathrm{Aut}(X)$. The Hessian is a covariant of $X$, in other words $\mathrm{Aut}(\mathbf{Hess}\, X) = \mathrm{Aut}(X)$, hence the intersection is fixed (not necessarily pointwise) under the action of $\mathrm{Aut}(X)$. Now we invoke the assumption that $D \subset D_X$ consists of rational singularities. In this case, there are elements of finite order in $\mathrm{Aut}(X)$ which fix the components of $D$. This is because a Satake compactification of a ball quotient has singularities at the cusps, which, when resolved, give rise to abelian varieties (see [Hunt 1996, Section 5.6.2] for a discussion), and the fact that the singularities are rational implies that the group has torsion "at the cusps", i.e., the corresponding parabolic subgroups have torsion. These torsion elements fix the singularities of $D$ pointwise. By assumption $D \subset D_X$ and every component of $D_X$ contains a component of $D$. The corresponding elements which fix that component of $D$ then also fix (a neighborhood of) the component of $D_X$ containing it, and hence there are elements fixing all components of $D_X$ pointwise, as claimed. From this it follows immediately that $D_X$ consists of modular subvarieties. □

## 3. EXAMPLES

### Example 1: The Klein Quartic

Consider the quartic curve

$$\mathcal{K}_4 := \{x_0x_1^3 + x_1x_2^3 + x_3x_0^3 = 0\} \subset \mathbb{P}^2.$$

As is well-known, this is the compactification of the modular curve $(\Gamma(7)\backslash\mathbb{S}_1)^*$, where $\mathbb{S}_1$ is the upper half-plane and $\Gamma(7) \subset \mathrm{SL}(2,\mathbb{Z})$ denotes the principal congruence subgroup, so (i) is satisfied. The automorphism group is the simple group $G_{168} = \Gamma(1)/\Gamma(7)$ of order 168, so (iii) is satisfied. The Hessian variety is the curve of degree 6 $\mathbf{Hess}\,\mathcal{K}_4 = \{x_0^5x_1 + x_1^5x_2 + x_2^5x_0 - 5x_0^2x_1^2x_2^2 = 0\}$. The intersection $\mathbf{Hess}\,\mathcal{K}_4 \cap \mathcal{K}_4$ consists of the 24 inflection points of the curve. Note that 24 is also the number of cusps of the modular curve. Under the action of $G_{168}$, there is a single orbit of order 24, and the set of cusps is one such, hence

$$D_{\mathcal{K}_4} = \mathbf{Hess}\,\mathcal{K}_4 \cap \mathcal{K}_4 = (\Gamma(7)\backslash\mathbb{S}_1)^* - \Gamma(7)\backslash\mathbb{S}_1,$$

so the condition (ii) is satisfied (actually, the condition is vacant in this case), and *the Klein quartic $\mathcal{K}_4$ is a nice modular variety.*

**Remark.** The complement $\mathcal{K}_4 - D_{\mathcal{K}_4}$ is (a connected component of) the moduli space of elliptic curves with a full level 7 structure, so we have a moduli interpretation of this complement.

### Example 2: The Fermat Cubic Curve

Consider the Fermat cubic curve $\mathcal{F}_3 := \{x_0^3 + x_1^3 + x_2^3 = 0\} \subset \mathbb{P}^2$. It is elliptic, and since it has an automorphism of order 3, it is the elliptic curve $\mathbb{C}/(\mathbb{Z} \oplus \varrho\mathbb{Z})$, where $\varrho = e^{\frac{2\pi i}{3}}$ is a primitive third root of unity. On the other hand, the modular curve $\Gamma(6)\backslash\mathbb{S}_1$ is also elliptic, and (fixing a point of order 3 on that elliptic curve as the origin) has an automorphism (fixing the given point of order 3) of order

3. It follows that $\mathcal{F}_3 = (\Gamma(6)\backslash\mathbb{S}_1)^*$. The intersection of the Hessian (which is $x_0x_1x_2 = 0$) consists of the nine inflection points on that curve (which are just the points of order three). The group $\Gamma(6)$ has 12 cusps, and it is easy to see that

$$(\Gamma(6)\backslash\mathbb{S}_1)^* - \Gamma(6)\backslash\mathbb{S}_1 = 12 \text{ 2- and 3- torsion points.}$$

Hence, $\mathcal{F}_3$ is almost, but not quite, a nice modular variety ($D \not\subset D_{\mathcal{F}_3}$).

### Example 3: The Cayley Cubic Surface

There is a unique cubic surface in $\mathbb{P}^3$ with four ordinary double points; it is called the Cayley cubic and can be given by the equation

$$\mathfrak{C} := \{\sigma_3(x_0, x_1, x_2, x_3) = 0\} \subset \mathbb{P}^3,$$

where $\sigma_3$ denotes the elementary symmetric polynomial in the four variables $x_0, \ldots, x_3$: $x_1x_2x_3 + x_0x_2x_3 + x_0x_1x_3 + x_0x_1x_2 = 0$. The four hyperplanes $x_0 = 0, \ldots, x_3 = 0$ define the faces of the coordinate tetrahedron, the vertices of which are the nodes of $\mathfrak{C}$. There are 9 lines on $\mathfrak{C}$, the six edges of the coordinate tetrahedron, and three additional lines. The former six lines are the exceptional $\mathbb{P}^1$'s, viewing $\mathfrak{C}$ as the blow-up of $\mathbb{P}^2$ at the six points depicted in Figure 1 of [Hunt 1998], the latter three are the three diagonals which are denoted in that figure by $N_{0124}$, $N_{0125}$ and $N_{0123}$ (see [Hunt 1996, Section 4.1.3] for details about and pictures of this and other cubic surfaces).

It has been shown in [Hunt 1998, Theorem 1.1] that the Cayley cubic surface is the compactification of a ball quotient; i.e., if we set $D = 4$ nodes, then the condition (i) is satisfied, $\mathfrak{C} - D = \Gamma\backslash\mathbb{B}_3$. (The dual variety, the unique quartic surface singular in three lines which meet at a point, is in fact also a special case of a Humbert surface on the Siegel modular variety of level 2 in dimension 3, but we will not use this.) The Hessian variety of $\mathfrak{C}$ is the quartic surface given by the equation $\{\sigma_1(x_0 : \ldots : x_3)\sigma_3(x_0 : \ldots : x_3) - 4\sigma_4(x_0 : \ldots : x_3) = 0\}$. It has 14 double points, 10 of which are the vertices of the Sylvester polyhedron and the other four at the four singular points of $\mathfrak{C}$. Clearly the intersection of $\mathfrak{C}$ and its Hessian is given by $\{\sigma_3 = \sigma_4 = 0\}$, and as $\{\sigma_4 = 0\}$ is just the coordinate tetrahedron, this intersection consists of the six lines which are the edges of that tetrahedron. On the other hand, from [Hunt 1998, Section 1] we know that the 9 lines on $\mathfrak{C}$ are modular curves, and condition (ii) is satisfied.

As to condition (iii), the surface is itself a modular subvariety on a three-dimensional ball quotient with symmetry group $\Sigma_6$ (the Segre cubic, discussed below), and the symmetry group of $\mathfrak{C}$, $\Sigma_4$ is a subgroup of this. It is known from [Hunt 1996, Chapter 3] that the quotient of the Segre cubic by $\Sigma_6$ is a ball quotient, hence it follows that $\mathfrak{C}/\Sigma_4$ is a (sub-)ball quotient, and (iii) is satisfied. Hence, *the Cayley cubic $\mathfrak{C}$ is a nice modular variety.*

**Remark.** The surface $\mathfrak{C}$ parameterizes certain abelian fourfolds with complex multiplication by the Eisenstein numbers $K = \mathbb{Q}(\sqrt{-3})$, and the modular curves correspond to loci where these fourfolds *split*, i.e., are *reducible*. Hence we have

$$\mathfrak{C} - D_{\mathfrak{C}} \subset \left\{\begin{matrix}\text{the moduli space of } \textit{irreducible} \\ \text{abelian fourfolds with the men-} \\ \text{tioned complex multiplication}\end{matrix}\right\} = \mathfrak{C} - 9 \text{ lines.}$$

This justifies the notion *modular* in our nomenclature. A similar remark applies to other examples.

In the remaining examples we will need some more work to verify the properties.

### Example 4: The Hessian of the Clebsch Cubic Surface

The Clebsch cubic surface $\mathcal{C}$, also known as the Clebsch diagonal surface, is given in $\mathbb{P}^4$ by the equations

$$\mathcal{C} = \left\{\sum_{i=1}^{5} y_i = \sum_{i=1}^{5} y_i^3 = 0\right\} \subset \mathbb{P}^4, \qquad (3\text{–}1)$$

and since the first equation is linear, this is in fact a cubic hypersurface in a $\mathbb{P}^3$. The automorphism group of $\mathcal{C}$ is the symmetric group on 5 letters (clear from the defining equation), and as the Hessian variety is a covariant, it also has the same symmetry group $\Sigma_5$. The Hessian is the quartic with equation

$$\mathbf{Hess}\,\mathcal{C} = \{\sigma_1\sigma_3 - \sigma_4 = 0\},$$

where $\sigma_i$ stands for $\sigma_i(x_0 : \ldots : x_3)$; it has, as does any Hessian of a smooth cubic, 10 nodes. Let $\mathcal{H} := \mathbf{Hess}\,\mathcal{C}$.

**Proposition 3.1.** *$\mathcal{H}$ is a nice modular variety.*

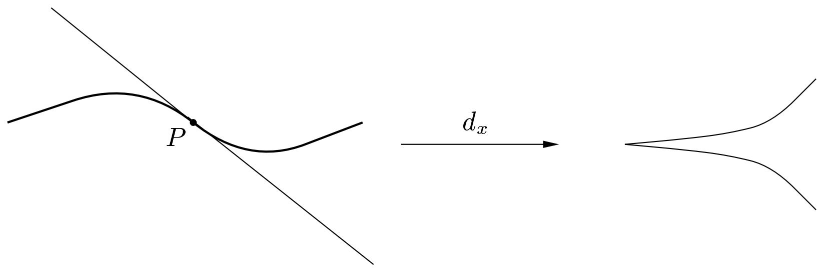


**FIGURE 1.** The dual of an inflection point is a cusp.

*Proof.* (i) $\mathcal{H}$ is a ball quotient: It has been shown in [Hunt 1996], that the Burkhardt quartic $\mathcal{B}_4$ (this example will be discussed in detail below) is the Baily–Borel embedding of the Satake compactification of a ball quotient. Also, in [Hunt 1996, Section 5.2.5], it is shown that $\mathcal{H}$ is a *hyperplane section* (the hyperplanes are called $n$-primes there) of $\mathcal{B}_4$ (there are 216 such hyperplane sections). The automorphism group of $\mathcal{B}_4$ is the simple group of order 25,920, and this group reduces to the $\Sigma_5$ on these hyperplane sections. Unfortunately, there are no elements of $\mathrm{Aut}(\mathcal{B}_4)$ which fix these hyperplanes point-wise (from which one could immediately conclude that they are subball quotients). Instead, an easy calculation as in [Hunt 1996, Lemma 5.6.1.4] shows that they fulfill *relative proportionality*, from which it follows that they are subball quotients. More precisely, letting $D \subset \mathcal{H}$ denote the union of the ten nodes, we have $\mathcal{H} - D = \Gamma\backslash\mathbb{B}_2$, and condition (i) is satisfied. Now, from [Hunt 1996, Theorem 5.6.1], $\mathcal{B}_4$ is the Satake compactification of the Picard modular group of level 2, and its automorphism group is $G_{25,920} \cong \mathrm{PU}(3,1;\mathcal{O}_K/2\mathcal{O}_K)$, where $K = \mathbb{Q}(\sqrt{-3})$ denotes the field of Eisenstein numbers. From this it follows that $\mathcal{B}_4/\mathrm{Aut}(\mathcal{B}_4)$ is again a ball quotient, and from this it follows that the same holds for the subball quotients $\mathcal{H}$, verifying (iii). Finally we consider condition (ii). The Hessian of $\mathcal{H}$ is a surface of degree 8, and intersects $\mathcal{H}$ in a curve of degree 32. We have, using the computer algebra package Macaulay, calculated this intersection: it consists of the 10 $\kappa$-lines of $\mathcal{B}_4$ [Hunt 1996, 5.2.1, I] lying in the $n$-prime whose section with $\mathcal{B}_4$ is $\mathcal{H}$ (these 10 lines are edges of the Sylvester pentahedron of $\mathcal{C}$), each counted twice, together with the intersection $\mathcal{H} \cap \mathcal{C}$. It follows here just as in Theorem 2.1 that this curve is also a modular subvariety on the ball quotient $\mathcal{H}$. It now follows that (ii) is also satisfied. □

### Example 5: The Clebsch Diagonal Surface

Now that we have established that $\mathcal{H}$ is a modular variety, we can prove the following.

**Theorem 3.2.** *The Clebsch diagonal surface $\mathcal{C}$ is an exceptional modular variety.*

*Proof.* $\mathcal{C}$ is a modular variety: this is slightly different from the cases above in that $\mathcal{C}$ is a *compact* quotient. First, Hirzebruch [1976] has shown that $\mathcal{C}$ is the compactification of the Hilbert modular surface of level 2 and the field $\mathbb{Q}(\sqrt{5})$. Thus $\mathcal{C}$ is modular. There is yet another interpretation. Recall from [Hunt 1996, Proposition 4.1.10] that the Clebsch cubic is the result of embedding the blow-up of $\mathbb{P}^2$ at the six points which are the vertices and center of a pentahedron, or what amounts to the same, the six five-fold points of the icosahedral arrangement, by the system of cubics through those six points. It was shown in [Barthel et al. 1987] that there are two ball quotients which are branched covers of the plane, branched along the lines of the icosahedral arrangement. Of these, there is one for which the threefold points of the arrangement (which are just the 10 Eckard points of $\mathcal{C}$) are *not* blown up. This shows that $\mathcal{C}$ is a (compact) ball quotient.

$\mathcal{C}$ is exceptional, since by the previous result its Hessian is modular. □

**Remark.** This surface also occurs as a hyperplane section of the Segre cubic, the example considered next. Since the Segre cubic is itself a ball quotient, it seems natural that $\mathcal{C}$ is also. In this case it would be a compact quotient on the non-compact quotient $\mathcal{S}_3$. This could be shown to be the case if one could show relative proportionality for $\mathcal{C}$. We sketch what happens when one tries to do this. The Segre cubic threefold (the next example) is known to be a ball quotient, more precisely [Hunt 1996, Theorem 3.2.5] it is the Picard modular threefold of level $\sqrt{-3}$ for the field $K$ of Eisenstein numbers. From its equation (3–2) below, one can see that there are five hyperplane sections, given by $x_i = 0$, which are copies of the Clebsch surface $\mathcal{C}$ (cf. (3–1)). The symmetry group of the Segre cubic $\mathcal{S}_3$ is the symmetric group $\Sigma_6$ on six letters, which acts by permutations of the $x_i$. Moreover, the symmetry group $\Sigma_6$ of $\mathcal{S}_3$ consists of ball automorphisms, i.e.,

$$\Sigma_6 = \mathrm{PU}(3,1;\mathcal{O}_K/\sqrt{-3}\mathcal{O}_K),$$

and as above it follows that $\mathcal{S}_3/\operatorname{Aut}(\mathcal{S}_3)$ is again a ball quotient. Once again, there are no elements of this symmetry group fixing the hyperplane, so we can not immediately conclude the intersection with $\mathcal{S}_3$ is a subball quotient. To show it is a subball quotient, we must verify relative proportionality for $\mathcal{C}$. Without going into detail, we sketch this. First, we need to verify relative proportionality for a *torsion-free* quotient, i.e., for a certain cover. In [Hunt 1996] we have constructed a cover $\tilde{Y} \to \tilde{\mathbb{P}}^3$ which is a smooth compactification of a ball quotient with a Zariski open subset cover the complement in $\mathcal{S}_3$ of the ten nodes. We use this cover; let $S$ be one of the copies of $\mathcal{C}$ on $\mathcal{S}_3$, and denote also its proper transform on $\tilde{\mathbb{P}}^3$ by the same symbol. Finally, let $T \to S$ be the cover given by one of the inverse images in $\tilde{Y}$ of $S$. Then for this subvariety of the compactification of ball quotient $\tilde{Y}$, we must verify

$$c_1^2(\tilde{Y})_{|T} = \left(\tfrac{4}{3}\right)^2 c_1^2(T).$$

(Here we need not use the logarithmic Chern classes, as the surface does not meet the compactification locus). To calculate the right-hand side, we must precisely determine the branch locus of $T \to S$. This is by [Hunt 1996, 4.1.12] the set of 15 lines complementary to a double six. In terms of the blow up of $\mathbb{P}^2$ mentioned above, this means it is branched precisely over the 15 lines, not over the six exceptional $\mathbb{P}^1$'s. There is no such example in Höfer's list. Thus we start getting suspicious. Note that for each Eckard point of $\mathcal{C}$, there are three of the branch divisors meeting. In other words, $T$ is *singular* at the inverse images of those points. This is the reason that relative proportionality cannot be verified. It is not clear whether $\mathcal{C}$ is a subball quotient.

**Example 6: The Segre Cubic Threefold**

This variety has been studied in great detail; see in particular [Hunt 1996, Chapter 3] for this and the following examples. The Segre cubic is given by the same equations as the Clebsch diagonal surface in one dimension higher:

$$\mathcal{S}_3 = \left\{ \sum_0^5 x_i = \sum_0^5 x_i^3 = 0 \right\} \subset \mathbb{P}^5. \tag{3–2}$$

It is the unique cubic hypersurface in $\mathbb{P}^3$ which has 10 ordinary double points. It is also the unique $\Sigma_6$-invariant cubic hypersurface. As mentioned above, it was proved in [Hunt 1996, Theorem 3.2.5] that $\mathcal{S}_3$ is the Satake compactification of a ball quotient (in fact even the arithmetic group was identified there), so setting $D =$ the 10 nodes, the condition (i) is fulfilled. Furthermore, the Hessian variety of $\mathcal{S}_3$ is just the Nieto quintic $\mathcal{N}_5$ (see Example 9 below). It was shown in [Hunt 1996] that the intersection $\mathcal{S}_3 \cap \mathcal{N}_5$ consists of the union of the 15 Segre planes on both of these varieties. In the interpretation of $\mathcal{S}_3$ as a ball quotient in [Hunt 1996], these 15 Segre planes are subball quotients. In fact, $\mathcal{S}_3$ is the Picard modular threefold of level $\sqrt{-3}$ for the field $K$ of Eisenstein numbers, and each of the 15 Segre planes is a copy of the Picard modular surface of level $\sqrt{-3}$ for the field $K$. Furthermore all of the 10 nodes are contained in the union of the 15 Segre planes (there are four of the nodes in each of the planes, and six of the planes pass through each node, forming a $(6)_4$ configuration). It follows that (ii) is satisfied. Finally, as mentioned above, the symmetry group $\operatorname{Aut}(\mathcal{S}_3) \cong \Sigma_6 \cong \mathrm{PU}(3,1;\mathcal{O}_K/\sqrt{-3}\mathcal{O}_K)$, and condition (iii) is immediate for $\mathcal{S}_3$. Thus *The Segre cubic is a nice modular variety.* This case was quite easy to verify.

**Remark.** We have the following moduli interpretation of the complement of the parabolic divisor on $\mathcal{S}_3$:

$$\mathcal{S}_3 - D_{\mathcal{S}_3} = \left\{ \begin{array}{l} \text{the moduli space of } \textit{irreduci-} \\ \textit{ble} \text{ abelian fourfolds with the} \\ \text{given complex multiplication} \end{array} \right\}.$$

See [Hunt 1996, 3.2.6].

**Example 7: The Igusa Quartic**

Here we are dealing with the variety in $\mathbb{P}^4$ which is the *dual variety* to the Segre cubic; it is a quartic $\mathcal{I}_4$ which was classically known and studied in modern times again by Igusa and other mathematicians. The singular locus of this quartic consists of 15 lines, which meet in 15 points. This is the most singular in the pencil of $\Sigma_6$-invariant quartics. It is well-known that this is just the Satake compactification of the Siegel modular variety $\Gamma(2)\backslash\mathbb{S}_2$, where $\mathbb{S}_2$ is the Siegel space of degree 2 (and dimension 3) and $\Gamma(2) \subset Sp(4, \mathbb{Z})$ is the principal congruence subgroup of level 2, so condition (i) in the definition is satisfied. The automorphism group is $\Sigma_6$, which follows on the one hand since $\mathcal{I}_4$ is dual to $\mathcal{S}_3$ and the dual variety is a covariant. But here it arises from the isomorphism $\Sigma_6 = \mathrm{PSp}(4, \mathbb{F}_2)$, hence condition (iii) is satisfied. We remark that from the fact that $\mathcal{I}_4$ is dual to $\mathcal{S}_3$, we have some information on its geometry. For example, the 15 Segre planes correspond to the 15 singular lines, which meet in 15 points, corresponding to the 15 hyperplane sections of $\mathcal{S}_3$ which cut out three of the 15 Segre planes. Dual to the 10 nodes of $\mathcal{S}_3$ are 10 *bitangent surfaces*, i.e., quadrics which are tangent hyperplane sections of $\mathcal{I}_4$. Finally, the Hessian variety has degree 10, and we have again using Macaulay calculated the intersection with $\mathcal{I}_4$: it consists of the 10 bitangents, the quadric surfaces just mentioned, each counted twice. An equation of the Igusa quartic is

$$\mathcal{I}_4 = \left\{(y_0y_1+y_0y_2+y_1y_2-y_3y_4)^2-4y_0y_1y_2 \textstyle\sum y_i = 0\right\}$$

and the intersection $\mathcal{I}_4 \cap \{y_0 = 0\}$ is such a double quadric (bitangent). These bitangents are modular subvarieties of the Siegel space quotient, hence condition (ii) is satisfied, and *the Igusa quartic is a nice modular variety*. Again this example was quite easy to check.

**Remark.** It is well-known that under the moduli interpretation of $\mathcal{I}_4$ as the moduli space of principally polarized abelian surfaces with a level 2 structure, the 10 bitangents correspond to those abelian surfaces which *split* as a product. Hence, we have

$$\mathcal{I}_4 - D_{\mathcal{I}_4} = \left\{\begin{matrix}\text{the (rough) moduli space of}\\ \textit{irreducible} \text{ abelian surfaces}\\ \text{with a level-two structure}\end{matrix}\right\}.$$

#### Example 8: The Burkhardt Quartic

The Burkhardt quartic is another variety which had been studied classically. Details from a more modern point of view can be found in [Hunt 1996, Chapter 5]. The Burkhardt quartic $\mathcal{B}_4$ is the unique invariant of degree 4 of the simple group $G_{25,920}$ of order 25,920, which has a representation in $\mathbb{C}^5$ which is generated by unitary reflections. There are 45 reflections (of order two), defining a set of 45 $\mathbb{P}^3$'s (called Jordan primes) in $\mathbb{P}^4$. (These 45 correspond in a well-defined manner to the 45 tritangents of a smooth cubic surface, having to do with the fact that the group of automorphisms of the 27 lines (equivalently, the Galois group of the equation defining the 27 lines) is the nontrivial extension by $\mathbb{Z}/2\mathbb{Z}$ of $G_{25,920}$ (equivalently, the Weyl group of $E_6$). The Burkhardt quartic is also the unique quartic hypersurface in $\mathbb{P}^4$ which has 45 nodes, and each of these nodes is the polar point of one of the 45 Jordan primes. The equation of the Burkhardt quartic is

$$\mathcal{B}_4 = \{y_0^4-y_0(y_1^3+y_2^3+y_3^3+y_4^3)+3y_1y_2y_3y_4 = 0\} \subset \mathbb{P}^4. \tag{3–3}$$

From the equation one sees that the hyperplane $y_0 = 0$ intersects $\mathcal{B}_4$ in the union of 4 $\mathbb{P}^2$'s. There are in fact 40 such special hyperplanes (called Steiner primes), and the stabilizer of each is one of the subgroups of order 648 in $G_{25,920}$. There are also 40 $\mathbb{P}^2$'s altogether (called Jacobi planes) cut out by the Steiner primes, and the stabilizer of each of these is also a group of order 648, but these two types of subgroups of order 648 are *not* conjugate in $G_{25,920}$.

It was proved in [Hunt 1996, Theorem 5.6.1] that $\mathcal{B}_4$ is the Baily–Borel embedding of a ball quotient (in fact the group was also determined). More precisely, letting $D$ denote the union of the 45 nodes, it holds that $\mathcal{B}_4 - D = \Gamma_{\mathcal{B}}\backslash\mathbb{B}_3$, and condition (i) is satisfied. The symmetry group as already mentioned is $G_{25,920} \cong \mathrm{PU}(3,1;\mathcal{O}_K/2\mathcal{O}_K)$, and as the group $\Gamma_{\mathcal{B}}$ is the Picard modular group of level 2 (proved in the theorem mentioned above), also $X/\operatorname{Aut}(X)$ is a ball quotient, and (iii) is satisfied. The Hessian of $\mathcal{B}_4$ is, as was the case for $\mathcal{I}_4$, of degree 10, so its intersection with $\mathcal{B}_4$ is of degree 40. Its equation is

$$\begin{aligned}&32y_0^6y_1y_2y_3y_4 - 8y_0^4(y_1^3y_2^3 + \cdots + y_3^3y_4^3)\\ &\quad - 4y_0^3y_1y_2y_3y_4(y_1^3 + \cdots + y_4^3)\\ &\quad + y_0\big(6(y_1^3y_2^3y_3^3 + \cdots + y_2^3y_3^3y_4^3)\\ &\qquad\quad - (y_1^6y_2^3 + \cdots + y_3^3y_4^6)\big)\\ &\quad + 18y_0^2(y_1^2y_2^2y_3^2y_4^2)\\ &\quad + y_1y_2y_3y_4(y_1^3y_2^3 + \cdots + y_3^3y_4^3 - y_1^6 - \cdots - y_4^6).\end{aligned}$$

From this is it easy to see that for $y_0 = 0$, both of these varieties contain the four planes

$$y_1 = 0, \quad \dots, \quad y_4 = 0.$$

Since the Hessian is a covariant, it has the same symmetry group, hence contains all 40 planes, and the union of these 40 planes is of degree 40, and is consequently the entire intersection. These planes, called $j$-planes by Burkhardt, are themselves Picard modular surface of level 2, by [Hunt 1996, Lemma 5.6.1.1]. It follows that condition (ii) is also fulfilled, and $\mathcal{B}_4$ *is a nice modular variety.*

**Remark.** We again can give a moduli interpretation of the complement of the parabolic divisor. The moduli interpretation of the $j$-planes was determined in [Hunt 1996, Lemma 5.6.1.1] and consists of those abelian fourfolds with complex multiplication and level structure which *split*, i.e., we have the equality

$$\mathcal{B}_4 - D_{\mathcal{B}_4} = \left\{ \begin{array}{l} \text{moduli space of } \textit{irreducible} \\ \text{principally polarized abelian} \\ \text{fourfolds given by the } \textsc{Pel}\text{-} \\ \text{data of [Hunt 1996, 5.7.4]} \end{array} \right\}.$$

In addition, it is known that the Hessian of $\mathcal{B}_4$ is also a modular variety. Hence :

**Corollary 3.3.** $\mathcal{B}_4$ *is both a nice and an exceptional modular variety.*

*Proof.* We have already shown it to be nice. To see that it is exceptional, we need to show that the Hessian is also a modular variety. Recall that the Burkhardt quartic $\mathcal{B}_4$ has the peculiar property of being self-Steinerian, meaning it is its own Steinerian variety [Hunt 1996, Section 5.3]. Next, recall that the Hessian variety is the locus of points $x \in \mathbb{P}^4$, such that the polar quadric of $x$ with respect to $\mathcal{B}_4$ is singular (generically a cone), and the Steinerian is the (closure of the) locus of the vertices of these cones. It then follows that quite generally the Steinerian and the Hessian are birational to each other. Since $\mathcal{B}_4$ is a modular variety, it immediately follows that the Hessian is also. Hence, by definition, $\mathcal{B}_4$ is exceptional. □

### Example 9: The Nieto Quintic

This beautiful hypersurface is, as mentioned above, just the Hessian of the Segre cubic and is discussed in detail in [Barth and Nieto 1994; Hunt 1996; Hunt 1998]. Its defining equation is

$$\mathcal{N}_5 = \{\sigma_1(x_0 : \dots : x_5) = \sigma_5(x_0 : \dots : x_5) = 0\} \subset \mathbb{P}^5, \tag{3–4}$$

in terms of the elementary symmetric functions; it is clearly invariant under $\Sigma_6$, which it must be as the Hessian is a covariant and $\mathcal{S}_3$ is also invariant. The singular locus of $\mathcal{N}_5$ consists of 20 lines which meet in 15 points, and in addition the 10 nodes of $\mathcal{S}_3$. It is the most singular quintic in the pencil of $\Sigma_6$-invariant quintics. On $\mathcal{N}_5$ there are five different loci of interest: three in the singular locus already mentioned, consisting of 20 lines, 15 intersection points and 10 isolated points. There are in addition two types of $\mathbb{P}^2$'s on $\mathcal{N}_5$, 15 Segre planes and 15 other planes.

**Proposition 3.4.** $\mathcal{N}_5$ *is a nice modular variety.*

*Proof.* It was proved in [Barth and Nieto 1994] that $\mathcal{N}_5$ is birational to a compactification of a Siegel space quotient, and in [Hunt 1998] it was shown that it is birational to the compactification of a ball quotient. For the second we have: blow up $\mathcal{N}_5$ by blowing up the 10 isolated points (with $\mathbb{P}^1 \times \mathbb{P}^1$ as exceptional divisors), the 15 intersection points (with copies of the Cayley cubic surface as exceptional divisors), and the 20 lines (with again $\mathbb{P}^1 \times \mathbb{P}^1$ as exceptional divisors). Let $D$ denote the union of the 20 copies of $\mathbb{P}^1 \times \mathbb{P}^1$ which are the exceptional divisors over the 20 singular lines of $\mathcal{N}_5$ and the 10 copies of $\mathbb{P}^1 \times \mathbb{P}^1$ which are the exceptional divisors over the 10 nodes of $\mathcal{S}_3$. Then

$$\mathcal{N}_5 - D = \Gamma_{\mathcal{N}} \backslash \mathbb{B}_3,$$

and condition (i) is satisfied. Actually, in [Hunt 1998] it is only proved that this is a ball quotient. The fact that the group $\Gamma_{\mathcal{N}}$ is *arithmetic* is deeper, and was shown in [Hunt 1999] as a consequence of a general criterion for arithmeticity.

The Hessian of $\mathcal{N}_5$ has degree 15, and intersects $\mathcal{N}_5$ in a surface of degree 75. We have, again using Macaulay, checked that this intersection consists of the 15 Segre planes (each once) as well as the 15 other planes (each with multiplicity 4). In [Hunt 1998, Section 6] it is verified that each of these planes is a subball quotient (relative proportionality for them is verified). Hence (ii) is fulfilled.

We now check (iii), that the automorphism group can be lifted to automorphisms of the ball. This follows from the following fact.

**Lemma** [Hunt 2000, Corollary 5.4]. Let $X_\Gamma$ be a compactification of a ball quotient, $G$ a group of symmetries acting on $X_\Gamma$ and $\pi : X_{\Gamma'} \to X_\Gamma$ a branched cover with $X_{\Gamma''}$ a torsion-free ball quotient. If the branch locus of $\pi$ is $G$-invariant, then there is a discrete group $\Gamma''$, normal inclusions $\Gamma' \triangleleft \Gamma \triangleleft \Gamma''$, and (at least birationally) $X_{\Gamma''} = X_\Gamma / G$.

Such a covering of (a birational model of) the Nieto quintic has been constructed in [Hunt 1998]; it is branched along the union of the the 30 planes, along the 15 exceptional divisors which are the resolutions of the 15 intersection points of the 20 lines (each of these divisors is also a subball quotient), and along the components of the compactification locus. The lemma can be applied, and $\mathcal{N}_5/\operatorname{Aut}(\mathcal{N}_5)$ is again a ball quotient, verifying (iii). This completes the proof that $\mathcal{N}_5$ is a nice modular variety. □

We now get:

**Corollary 3.5.** *The Segre cubic is a nice and an exceptional modular variety.*

### Example 10: The Invariant Quintic Fourfold

The beautiful geometry of this variety is discussed in detail in [Hunt 1996, Chapter 6]. In this section we only recall the loci we require in the sequel. The quintic, to be denoted $\mathcal{I}_5$ in what follows, has the equation

$$\mathcal{I}_5 = \bigl\{x_6^5 - 6x_6^3\sigma_1(x) - 27x_6\bigl(\sigma_1^2(x) - 4\sigma_2(x)\bigr) - 648x_1x_2x_3x_4x_5 = 0\bigr\}. \quad (3\text{–}5)$$

It is the unique degree 5 invariant of the Weyl group of $E_6$, acting on $\mathbb{P}^5$ as a reflection group; there are 36 reflection hyperplanes in that $\mathbb{P}^5$. The singular locus of $\mathcal{I}_5$ consists of 120 lines which meet in 36 points; the 36 points correspond to the 36 positive roots, while the 120 lines correspond to the 120 subroot systems of type $A_2$. So, for example, each of these lines contains three of the 36 points ($A_2$ contains three positive roots). It is easy to desingularize $\mathcal{I}_5$: first blow up $\mathbb{P}^5$ in the 36 points; this resolves the singular points on $\mathcal{I}_5$ and replaces each with a copy of the Segre cubic. Then blow up along the proper transforms of the 120 lines; this replaces each line by a divisor isomorphic to $(\mathbb{P}^1)^3$. We let $\tilde{\mathcal{I}}_5$ denote this resolved fourfold.

From the equation (3–5) one sees that the intersection $\mathcal{I}_5 \cap \{x_6 = 0\}$ consists of the union of 5 $\mathbb{P}^3$'s which lie on $\mathcal{I}_5$. That hyperplane is one of the 27 hyperplanes defined by *primitive weight vectors*, so there are 27 such hyperplane sections. There are 45 such $\mathbb{P}^3$'s which lie on $\mathcal{I}_5$, and these 27 and 45 are in covariant relation with the 27 lines and 45 tritangents of a cubic surface.

It was shown in [Hunt 2000, Theorem 5.15] that $\mathcal{I}_5$ is ($W(E_6)$-equivariantly) birational to the compactification of a four-dimensional arithmetic ball quotient. More precisely, let $D$ denote the union on $\tilde{\mathcal{I}}_5$ of the 120 exceptional $(\mathbb{P}^1)^3$'s. Then $\tilde{\mathcal{I}}_5 - D = \Gamma_{\mathcal{J}} \backslash \mathbb{B}_4$, and condition (i) is satisfied. The Hessian variety of $\mathcal{I}_5$ has degree 18, hence the intersection with $\mathcal{I}_5$ has degree 90. We have, again using Macaulay, checked that this intersection consists of the 45 $\mathbb{P}^3$'s on $\mathcal{I}_5$, each being counted twice. On $\tilde{\mathcal{I}}_5$, we also have the 36 exceptional divisors resolving the triple points in the proper transform of that intersection. The 45 $\mathbb{P}^3$'s, as well as the 36 exceptional copies of the Segre cubic over the 36 points, are modular subvarieties of the ball quotient, which is shown in [Hunt 2000], Theorem 3.3. Hence condition (ii) is satisfied. Finally, in considering (iii), the question is again as to whether the automorphisms of $\mathcal{I}_5$ lift to the ball. They do, as was proved in [Hunt 2000, Lemma 5.1] (which there is a corollary of the lemma above). Hence (iii) is satisfied, and $\mathcal{I}_5$ *is a nice modular variety.*

We formulate these results as a theorem.

**Theorem 3.6.** 1. *The following varieties are examples of nice modular varieties: the Klein quartic $\mathcal{K}_4$, the Cayley cubic $\mathfrak{C}$, the Hessian of the Clebsch cubic $\mathcal{H}$, the Segre cubic $\mathcal{S}_3$, the Igusa quartic $\mathcal{I}_4$, the Burkhardt quartic $\mathcal{B}_4$, the Nieto quintic $\mathcal{N}_5$ and the invariant quintic fourfold $\mathcal{I}_5$.*

2. *The following varieties are examples of exceptional modular varieties: The Clebsch cubic $\mathcal{C}$, the Segre cubic $\mathcal{S}_3$ and the Burkhardt quartic $\mathcal{B}_4$.*

**Remark.** The notion of Janus-like algebraic varieties was introduced in [Hunt and Weintraub 1994]. Many of the varieties discussed above — at least $\mathcal{F}_3$, $\mathfrak{C}$, $\mathcal{S}_3$, $\mathcal{I}_4$, $\mathcal{B}_4$, $\mathcal{N}_5$, and $\mathcal{C}$ — turn out to be Janus-like [Hunt and Weintraub 1994; Hunt 1996; Hunt 1998]. We do not know whether this is coincidental or the notion of Janus-like and nice modular varieties are related. Moreover, the dual of the invariant quintic, a $W(E_6)$-invariant of degree 32, is (almost) a quotient of a type IV domain, in the sense that a twofold cover of it is such a quotient; hence $\mathcal{I}_5$ is (almost) Janus-like [Freitag and Hunt 1999].

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