A Test for Identifying Fourier Coefficients of Automorphic Forms and Application to Kloosterman Sums

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We present a numerical test for determining whether a given set of numbers is the set of Fourier coefficients of a Maass form, without knowing its eigenvalue. Our method extends directly to consideration of holomorphic newforms. The test is applied to show that the Kloosterman sums $\pm S(1,1;p)/\sqrt{p}$ are not the coefficients of a Maass form with small level and eigenvalue. Source code and the calculated Kloosterman sums are available electronically.

1. INTRODUCTION

Consider the *Kloosterman sums*, defined for primes p and α, β relatively prime to p by

$$S(\alpha,\beta;p) = \sum_{x\bar{x}\equiv 1\,(p)} e\left(\frac{\alpha x + \beta \bar{x}}{p}\right).$$

By a theorem of Weil, $|S(\alpha,\beta;p)| \le 2\sqrt{p}$. Moreover, Katz [1980] proved that $S(\alpha,1;p)/\sqrt{p}$ asymptotically follows the Sato–Tate measure

$$d\mu = \frac{1}{\pi}\sqrt{1-\tfrac{1}{4}x^2}\,dx,$$

as α varies over relatively prime residue classes (mod p), and $p \to \infty$. He conjectured that the same holds as the prime p varies, with α held fixed. (This conjecture is substantiated by our calculated data; see Section 3.) On the other hand, the Sato–Tate conjecture, first formulated independently by Sato and Tate in the context of elliptic curves and reformulated by Serre [1968], predicts this behavior of the Fourier coefficients of "typical" GL_2-eigenforms. (See [Hejhal and Arno 1993] for some numerical evidence for the Sato–Tate conjecture.)

Now, the $S(\alpha,\beta;p)$ can not be the Fourier coefficients of a holomorphic form, since they do not lie

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in a fixed number field. To see this, suppose that for some α and β, $S(\alpha,\beta;p)$ were contained in a fixed (necessarily abelian) finite extension $K/\mathbb{Q}$, for all p. Then K is contained in some cyclotomic extension $\mathbb{Q}(\zeta_m)$. Choose a prime $p>3$ not dividing m. Then $S(\alpha,\beta;p)\in\mathbb{Q}(\zeta_p)\cap\mathbb{Q}(\zeta_m)=\mathbb{Q}$, where $\zeta_p=e(1/p)$. Rearranging the series for $S(\alpha,\beta;p)$ we find

$$S(\alpha,\beta;p)=\sum_{t=0}^{p-1}\biggl(1+\Bigl(\frac{t^2-4\alpha\beta}{p}\Bigr)\biggr)\zeta_p^t.$$

However, this series has at least $\frac{p-1}{2}$ zero terms, which is impossible since the minimal polynomial for ζ_p, $1+X+\cdots+X^{p-1}$, has only non-zero coefficients.

In view of these facts, Katz asked [1980, Question 1.2.5.3] whether the numbers $\varepsilon S(\alpha,\beta;p)/\sqrt{p}$, with $\varepsilon=\pm1$, could be Fourier coefficients of a Maass form [Maass 1949; Bump 1997]. More precisely, is there a Maass cusp form and newform of some level N whose L-function has local factor

$$\bigl(1-\varepsilon S(\alpha,\beta;p)p^{-s-1/2}+p^{-2s}\bigr)^{-1}$$

for all primes p not dividing N? Actually, Katz only posed this question for $\varepsilon=1$, but it is not clear that the sign should not be -1. Indeed, Li [1999] has shown that an analogue of Katz's question, with $\varepsilon=-1$, holds over function fields.

In Section 3 we present a test for determining numerically when a set of numbers are the Fourier coefficients of a Maass form, and apply it to $a_\pm(p)=\pm S(1,1;p)/\sqrt{p}$, yielding the following partial negative answer to the question above:

Theorem 1.1. *If a Katz eigenform with coefficients $a_\pm(p)$ of level $N=2^\nu$ and eigenvalue λ exists, then $N(\lambda+3)>18.3\times10^6$.*

First, we recall some properties and state a general theorem about Maass forms.

2. A THEOREM ON MAASS FORMS

If f is a Maass cusp form of eigenvalue $\lambda=r^2+\frac14$ and a newform of level N then it has a Fourier expansion of the following form [Bump 1997, § 1.9]:

$$f(x+iy)=\sum_{n\in\mathbb{Z}\setminus\{0\}}\lambda_f(n)\sqrt{y}K_{ir}(2\pi|n|y)\,e^{2\pi inx},$$

where K_{ir} is the K-Bessel function. We have either $\lambda_f(-n)=\lambda_f(n)$, in which case f is called *even*, or $\lambda_f(-n)=-\lambda_f(n)$, in which case f is *odd*. The L-function of f, defined by

$$L(s,f)=\sum_{n=1}^{\infty}\lambda_f(n)n^{-s},$$

extends to be entire. Further, the function

$$\Lambda(s,f)=N^{s/2}\pi^{-s}\Gamma\Bigl(\frac{s+\varepsilon+ir}{2}\Bigr)\Gamma\Bigl(\frac{s+\varepsilon-ir}{2}\Bigr)L(s,f),$$

where $\varepsilon=0$ is f is even and 1 if f is odd, is entire and satisfies a functional equation of the form

$$\Lambda(s,f)=\pm\Lambda(1-s,f).$$

(The sign of the functional equation does not concern us.)

Now, let $F(x)=x^2e^{-x}$. For a large parameter Y we form the sum

$$S_{Y;N}=\sum_{(n,N)=1}\lambda_f(n)F\Bigl(\frac{n}{Y}\Bigr).$$

Assuming the Ramanujan conjecture, the $\lambda_f(n)$'s in this sum are $O_\varepsilon(n^\varepsilon)$ for all $\varepsilon>0$. Anyway, one may show using the Rankin–Selberg method [Bump 1997] that this holds on average, so the terms in the sum are significant for n up to a small constant times $Y\log Y$. If the $\lambda_f(n)$'s were "random" ±1 then $S_{Y;N}$ would typically be of size $\sqrt{Y}$. On the other hand, $S_{Y;N}$ may be written as an integral involving $L(s,f)$ and the Mellin transform of F. Using the functional equation, we will see that this integral affords much cancellation, making $S_{Y;N}$ very small. This is made precise by the following proposition, whose proof is postponed until Section 5.

Proposition 2.1. *Set $q=\prod_{p|N,\,p^2\nmid N}(1+p)$. Then*

$$\left|\frac{S_{Y;N}}{\sqrt{Y}}\right|<q\cdot\Bigl(\frac{N(\lambda+3)}{42.88Y}\Bigr)^2.$$

Remark. The exponent 2 in the right-hand side is the same as the order of vanishing of $F(x)$ at 0. By choosing a different F, it could be replaced by any higher power with different constants. However, in practice this is not useful; it is more important that $F(x)$ be easy to compute and that the constant in the denominator be large and easy to estimate.

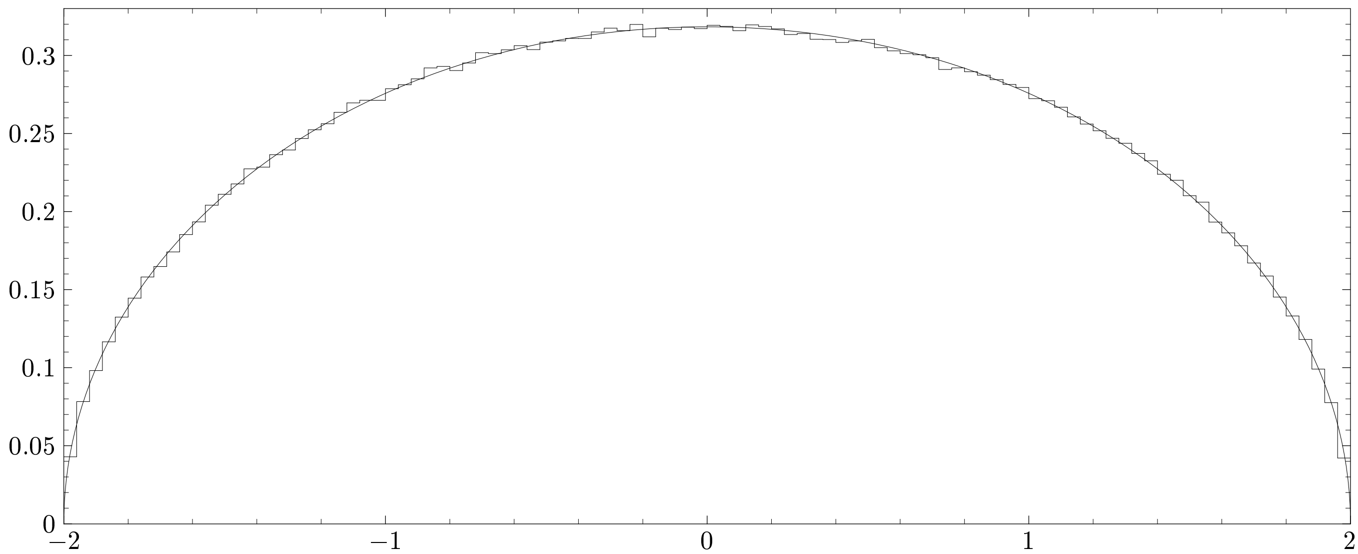

FIGURE 1. Histogram of $a_+(p)$ for $p \le 41161751$, compared with the Sato–Tate measure.

3. APPLICATION TO KLOOSTERMAN SUMS

We return to the numbers $a_\pm(p) = \pm S(1,1;p)/\sqrt{p}$. Suppose $a_\pm(p) = \lambda_f(p)$, the Fourier coefficients of a Maass cusp form f, for all primes p not dividing some N. If we extend $a_\pm(p)$ to $n \in \mathbb{Z}^+$ relatively prime to N by means of the Hecke relations

$$a_\pm(mn) = a_\pm(m)a_\pm(n), \quad \text{for } (m,n)=1,$$
$$a_\pm(p^k) = a_\pm(p)a_\pm(p^{k-1}) - a_\pm(p^{k-2}),$$

and compute the sum $S_{Y;N}$ assuming a particular value of N, then Proposition 2.1 gives a lower bound for the eigenvalue λ of f. If the $a_\pm(p)$ are not the coefficients of a Maass form, we expect $S_{Y;N}/\sqrt{Y}$ to be on the order of 1, thus giving a lower bound for λ on the order of Y. Clearly the larger we choose Y the better, for a large lower bound makes it unlikely that any such form could exist.

To perform the test, the numbers $a_\pm(p)$ were calculated for $p \le 41161751$ (the first $2.5 \times 10^6 + 1$ primes), using a network of 46 Linux PCs. The code for these calculations was written in assembly language and utilized the pentium processor's internal 80-bit (64-bit mantissa) floating point type to avoid excessive precision loss. This allows accurate calculation of $S_{Y;N}$ for $Y \le 2 \times 10^6$. Figure 1 shows a histogram of the calculated values of $a_+(p)$; they evidently follow the Sato–Tate measure $d\mu = \frac{1}{\pi}\sqrt{1-x^2/4}\,dx$. As further evidence, we list the first few moments in Table 1.

Figure 2, top part, is a plot of $S_{Y;1}/\sqrt{Y}$ for a_+ and a_- as Y varies between 1 and 2×10^6. Similarly, the bottom part shows $S_{Y;2}/\sqrt{Y}$. The plots show clear oscillatory behavior, with the width of oscillation growing roughly in proportion to Y. More importantly, there is no observed decay as Y grows. Correspondingly, we obtain good lower bounds (roughly proportional to Y) for most values of Y, as shown in Figure 3. The highest peaks are listed in Table 2.

Note that values of $S_{Y;2}$ yield information for N any power of 2. For example, assuming $\lambda = \frac{1}{4}$ (which is expected to be the only value for which the coefficients are algebraic; see [Casselman 1977]), we find $N > 2^{23}$ for a_+ and $N > 2^{24}$ for a_-. In general we have Theorem 1.1.

n	n-th moment of $a_\pm$	n-th Sato–Tate moment
1	0.00027391	0
2	1.00064651	1
3	0.00227780	0
4	2.00145955	2

TABLE 1. Kloosterman sum moments.

Coefficients	N	Y	λ
a_+	1	1.33×10^6	30.9×10^6
	2	1.04×10^6	9.35×10^6
a_-	1	1.12×10^6	27.5×10^6
	2	2.00×10^6	23.9×10^6

TABLE 2. Eigenvalue lower bounds.

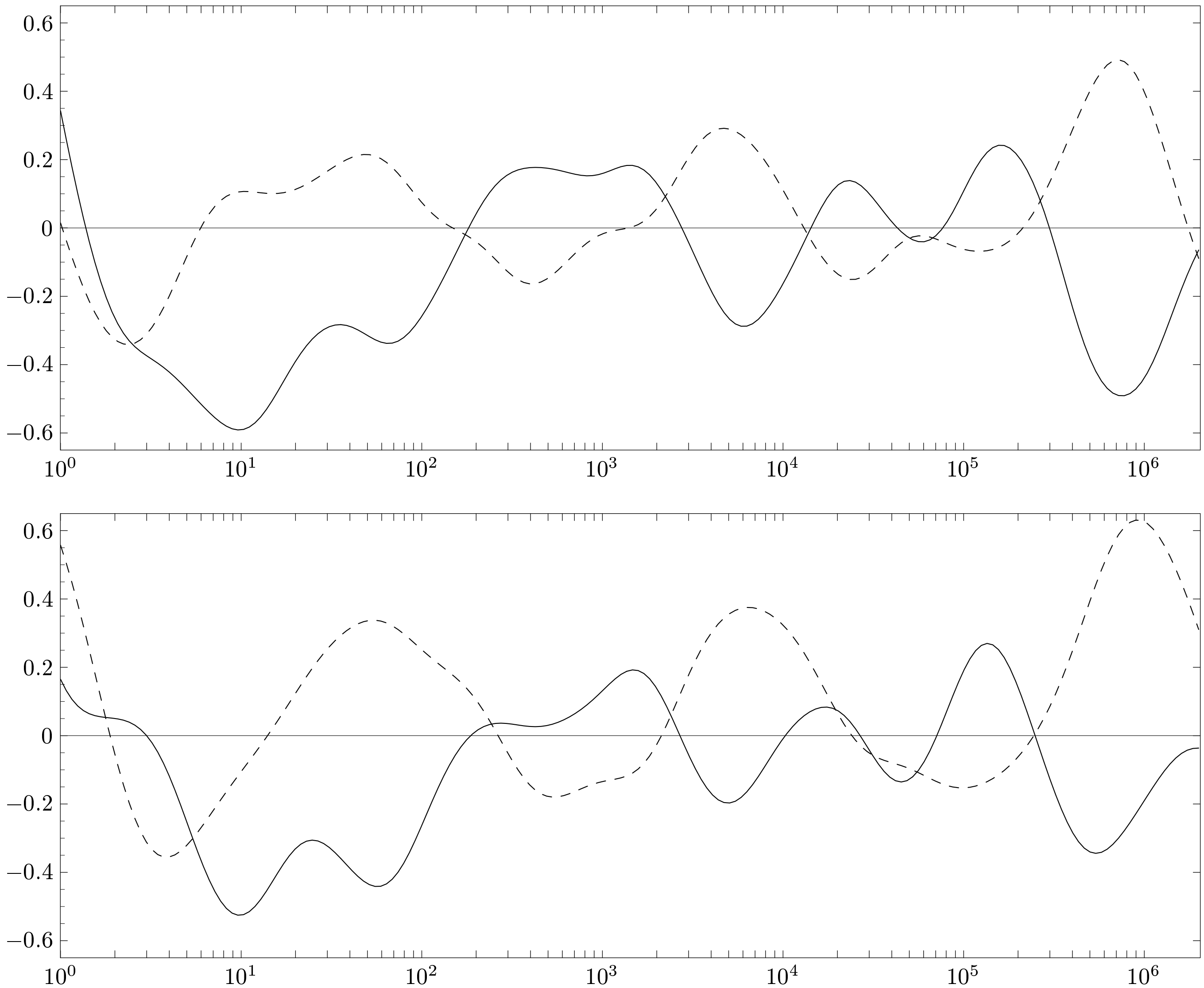

FIGURE 2. $S_{Y;1}/\sqrt{Y}$ (top) and $S_{Y;2}/\sqrt{Y}$ (bottom) for Y between 1 and 2×10^6. The full curves represent a_+ and the dashed curves a_-.

4. EXTENSIONS

There was nothing special about the use of Maass forms in Proposition 2.1. A similar analysis shows:

Proposition 4.1. *Suppose we start with a newform $f \in S_k(\Gamma_0(N))$. Then the resulting sum $S_{Y;N}$ must satisfy*

$$\left|\frac{S_{Y;N}}{\sqrt{Y}}\right| < q \cdot \left(\frac{N\bigl(k^2+2k+9\bigr)}{171.5Y}\right)^2.$$

This statement is essentially the same as Proposition 2.1, with λ replaced by $\frac{1}{4}k(k+2)$. We omit the proof, which is almost identical to that of Proposition 2.1.

For example, we test this inequality for the normalized newform in $S_2(\Gamma_0(11))$, whose Fourier coefficients are calculated by counting points on the elliptic curve $E : y^2+y = x^3-x^2$ over $\mathbb{F}_p$ [Knapp 1992, § XI.1]. (Specifically, $a(p) = \bigl(p+1-\#E(\mathbb{F}_p)\bigr)/\sqrt{p}$.) Choosing $Y = 10^4$, the right-hand side is approximately 1.31×10^{-7}, while we compute the left-hand side to be approximately 1.69×10^{-10}. (In this case we can not choose Y as large as before without running into precision problems; such factors must be taken into account when calculating the sum $S_{Y;N}$.) Thus, the test correctly predicts that the numbers $a(p)$ could be coefficients of a (holomorphic) cusp form of level 11.

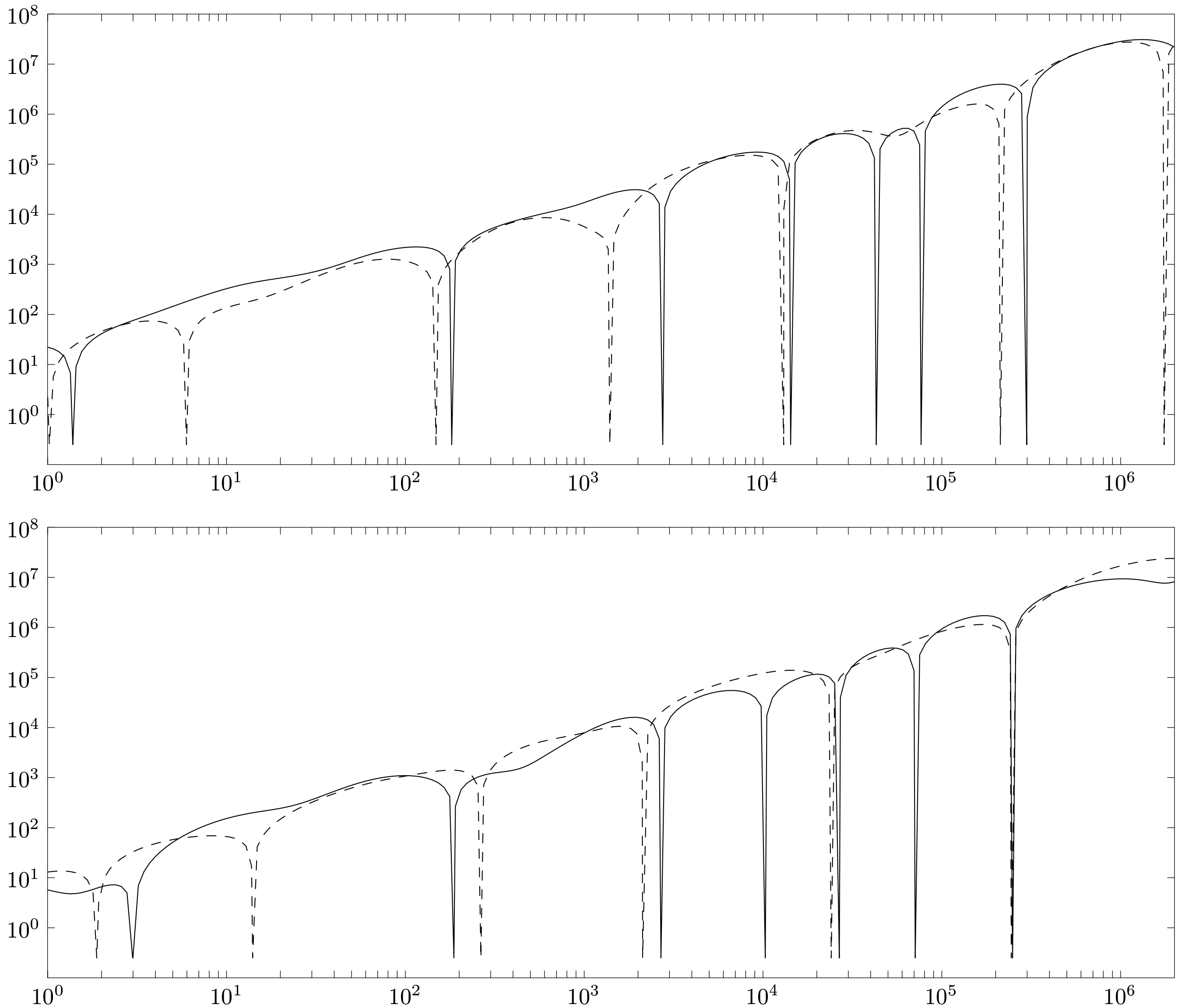

FIGURE 3. Eigenvalue lower bounds, $N = 1$ (top) and $N = 2$ (bottom). The full curves represent a_+ and the dashed curves a_-.

More generally, in examining the proof of Proposition 2.1, we see that the Euler product of our form f was not used in an essential way. In other words, we may apply the same techniques to a form given in the additive form $f(q) = \sum_{n=1}^{\infty} a(n)q^n$, as long as we have a priori bounds on the coefficients $a(n)$.

5. PROOF OF PROPOSITION 2.1

By Mellin inversion we have

$$F(x) = \frac{1}{2\pi i} \int_{\operatorname{Re} s=2} \Gamma(s+2)x^{-s}\, ds.$$

Thus,

$$\begin{aligned} S_{Y;N} &= \sum_{(n,N)=1} \lambda_f(n) \frac{1}{2\pi i} \int_{\operatorname{Re} s=2} \Gamma(s+2) \left(\frac{n}{Y}\right)^{-s} ds \\ &= \frac{1}{2\pi i} \int_{\operatorname{Re} s=2} \tilde{L}(s,f)\Gamma(s+2)Y^s\, ds, \end{aligned}$$

where we have defined

$$\tilde{L}(s,f) = L(s,f) \prod_{p|N,\, p^2\nmid N} \left(1 \pm p^{-s-1/2}\right),$$

that is, $L(s,f)$ with the local factors for primes $p|N$ removed. (Again, the sign of the local factors does not matter.)

Now shift the line of integration to $\operatorname{Re} s = -\frac{3}{2}$. The integrand has no poles to the right of $s = -2$, so

$$S_{Y;N} = \frac{1}{2\pi i}\int_{\operatorname{Re} s=-3/2} \tilde{L}(s,f)\Gamma(s+2)Y^s\,ds.$$

Replacing $\tilde{L}(s,f)$ by its definition and using the functional equation for $L(s,f)$, we have

$$S_{Y;N} = \pm\frac{1}{2\pi i}\int_{\operatorname{Re} s=-3/2}\Gamma(s+2)Y^s\big(\pi^{-1}\sqrt{N}\big)^{1-2s} \times\frac{\Gamma\Big(\dfrac{1-s+\varepsilon+ir}{2}\Big)}{\Gamma\Big(\dfrac{s+\varepsilon+ir}{2}\Big)}\frac{\Gamma\Big(\dfrac{1-s+\varepsilon-ir}{2}\Big)}{\Gamma\Big(\dfrac{s+\varepsilon-ir}{2}\Big)} \times L(1-s,f)\prod_{p|N,\,p^2\nmid N}\big(1\pm p^{-s-1/2}\big)\,ds.$$

Hence,

$$\left|\frac{S_{Y;N}}{\sqrt{Y}}\right| \le \frac{N^2}{2\pi^5Y^2}\int_{-\infty}^{\infty}\left|\Gamma\Big(\frac{1}{2}+it\Big)\right| \times\left|\frac{\Gamma\Big(\dfrac{5}{4}+\dfrac{\varepsilon}{2}+\dfrac{i(r-t)}{2}\Big)}{\Gamma\Big(-\dfrac{3}{4}+\dfrac{\varepsilon}{2}-\dfrac{i(r-t)}{2}\Big)}\frac{\Gamma\Big(\dfrac{5}{4}+\dfrac{\varepsilon}{2}-\dfrac{i(r+t)}{2}\Big)}{\Gamma\Big(-\dfrac{3}{4}+\dfrac{\varepsilon}{2}+\dfrac{i(r+t)}{2}\Big)}\right| \times\big|L(\tfrac{5}{2}-it)\big|\prod_{p|N,\,p^2\nmid N}\big|1\pm p^{1-it}\big|\,dt.$$

We treat the terms in absolute values separately.

First, if p is a prime not dividing N then we may write $\lambda_f(p) = 2\cos\theta$. (This again assumes the Ramanujan conjecture.) The Hecke relations yield

$$\big|\lambda_f(p^k)\big| = \left|\frac{\sin(k+1)\theta}{\sin\theta}\right| \le k+1.$$

For primes $p|N$, $\big|\lambda_f(p^k)\big| = p^{-k/2}$, so the preceding bound is trivially true. Thus, $|\lambda_f(n)| \le d(n)$ for all n, and the factor $|L(5/2-it)|$ may be bounded by

$$\big|L\big(\tfrac{5}{2}-it\big)\big| \le \sum_{n=1}^{\infty}\frac{d(n)}{n^{5/2}} = \zeta(\tfrac{5}{2})^2.$$

Second,

$$\left|\frac{\Gamma\Big(\dfrac{5}{4}+\dfrac{\varepsilon}{2}+\dfrac{it}{2}\Big)}{\Gamma\Big(-\dfrac{3}{4}+\dfrac{\varepsilon}{2}-\dfrac{it}{2}\Big)}\right| = \left|\frac{1}{4}+\frac{\varepsilon}{2}+\frac{it}{2}\right|\;\left|-\frac{3}{4}+\frac{\varepsilon}{2}+\frac{it}{2}\right| \le \tfrac{1}{4}\big(t^2+\tfrac{5}{4}\big).$$

Thus,

$$\begin{aligned}\left|\frac{\Gamma\Big(\dfrac{5}{4}+\dfrac{\varepsilon}{2}+\dfrac{i(r-t)}{2}\Big)}{\Gamma\Big(-\dfrac{3}{4}+\dfrac{\varepsilon}{2}-\dfrac{i(r-t)}{2}\Big)}\frac{\Gamma\Big(\dfrac{5}{4}+\dfrac{\varepsilon}{2}-\dfrac{i(r+t)}{2}\Big)}{\Gamma\Big(-\dfrac{3}{4}+\dfrac{\varepsilon}{2}+\dfrac{i(r+t)}{2}\Big)}\right| &\le \tfrac{1}{16}\big((t-r)^2+\tfrac{5}{4}\big)\big((t+r)^2+\tfrac{5}{4}\big)\\ &= \tfrac{1}{16}\big((r^2+\tfrac{1}{4}+t^2+1)^2-4t^2(r^2+\tfrac{1}{4})+t^2\big)\\ &= \tfrac{1}{16}\big(\lambda^2+2(1-t^2)\lambda+(t^4+3t^2+1)\big).\end{aligned}$$

Finally, $\prod_{p|N,\,p^2\nmid N}|1\pm p^{1-it}| \le q$ and

$$\big|\Gamma\big(\tfrac{1}{2}+it\big)\big| = \sqrt{\pi\operatorname{sech}\pi t}.$$

Altogether, we have the estimate

$$\left|\frac{S_{Y;N}}{\sqrt{Y}}\right| \le \frac{\zeta(5/2)^2qN^2}{16\pi^{9/2}Y^2} \times\int_0^{\infty}\big(\lambda^2+2\left(1-t^2\right)\lambda+\left(t^4+3t^2+1\right)\big)\sqrt{\operatorname{sech}\pi t}\,dt.$$

Now, put

$$A = \int_0^{\infty}\sqrt{\operatorname{sech}\pi t}\,dt = \frac{1}{M(\sqrt{2},1)} \approx 0.834627,$$

$$B = \int_0^{\infty}2\left(1-t^2\right)\sqrt{\operatorname{sech}\pi t}\,dt \approx 0.214955,$$

$$C = \int_0^{\infty}\left(t^4+3t^2+1\right)\sqrt{\operatorname{sech}\pi t}\,dt \approx 6.564697.$$

We then have

$$\begin{aligned}\left|\frac{S_{Y;N}}{\sqrt{Y}}\right| &\le \frac{\zeta(5/2)^2qN^2}{16\pi^{9/2}Y^2}\left(A\lambda^2+B\lambda+C\right)\\ &\le \frac{\zeta(5/2)^2qA}{16\pi^{9/2}}\left(\frac{N(\lambda+3)}{Y}\right)^2.\end{aligned}$$

Proposition 2.1 follows on noting that

$$\frac{\zeta(5/2)^2}{16\pi^{9/2}M(\sqrt{2},1)} < (42.88)^{-2}.$$

APPENDIX

It was established in [Sarnak 1987] that the Fourier coefficients of GL_2-cusp forms are dense in $\Omega = [-2,2]$. (Implicit in this is the assumption that the coefficients are normalized appropriately so as to lie in Ω.) The goal of this appendix will be to prove the following theorem, a kind of converse to Propositions 2.1 and 4, for the case of holomorphic forms.

Theorem A.1. *Suppose angles $\alpha_p \in \Omega' = [0,\pi]$ are given for primes $p \le P$. There is an absolute constant $c > 0$ such that, for any $\varepsilon > 0$, there exists a Maass cusp form of level* 1 *and eigenvalue at most*

$$\exp\left(\frac{cP^2}{\log P}\frac{\log(1+1/\varepsilon)}{\varepsilon}\right)$$

and whose Fourier coefficients $\lambda_p = 2\cos\theta_p$ satisfy $|\theta_p - \alpha_p| < \varepsilon$ for all $p \le P$. The same is true with "Maass cusp form" replaced by "holomorphic cusp form" and "eigenvalue" replaced by "weight".

Remarks. 1. The theorem shows that it impossible to completely disprove Katz's question numerically; for any given finite set of numbers in Ω, there is a Maass form whose Fourier coefficients are equal to those numbers to within the precision of our computer. Moreover, we get an idea of how sharp Proposition 2.1 is. Given a number P, the proposition says that there is some Maass form with eigenvalue on the order of about e^{P^2} with all coefficients below P greater than $\frac{1}{2}$, say. On the other hand, using the same coefficients, Proposition 2.1 gives a lower bound for the eigenvalue essentially on the order of P.

2. The proof for Maass forms is similar to the one for holomorphic forms, but is more complicated since the splitting of the trace formula is not as clean in the Maass form case; one must consider weighted averages of the trace over all Maass forms. We present the simpler proof only.

3. The result is effective. That is, one could determine an explicit value for the constant c.

More generally, let $S = \{p_1, \dots, p_r\}$ be a set of r prime numbers, and suppose $\alpha_p \in \Omega'$ is given for each $p \in S$. We will prove that there is a holomorphic form of the required type with weight k satisfying

$$\log k = O\left(r\frac{\log(1+1/\varepsilon)}{\varepsilon}\log\prod_{p\in S} p\right).$$

(Unless otherwise noted, the constants implied by all O-notations are absolute.) The stated result then follows from the prime number theorem.

Our main tool will be the Selberg trace formula, which allows one to calculate the trace of the Hecke operator $T_n(k)$ acting on $S_k(\mathrm{SL}_2(\mathbb{Z}))$. We follow closely the exposition of Serre [1997], with some simplifications thanks to the restriction to level 1 (which does not significantly affect the upper bound). First, we set some notation and recall a few facts; see [Serre 1997] for proofs.

Let μ_∞ denote the Sato–Tate measure on Ω,

$$d\mu_\infty = \frac{1}{\pi}\sqrt{1-\tfrac{1}{4}x^2}\,dx = \frac{2}{\pi}\sin^2\theta\,d\theta,$$

where $x = 2\cos\theta$ for $\theta \in \Omega'$.

Let $\langle f, \mu\rangle$ denote $\int f\,d\mu$. The space over which the integral is taken will be clear from context.

The polynomials X_n defined on Ω by

$$X_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}$$

are the orthonormal polynomials with respect to μ_∞. In other words, $\langle X_n X_m, \mu_\infty\rangle = \delta_{nm}$. Let

$$T_n'(k) = T_n(k)/n^{(k-1)/2}$$

denote the n-th Hecke operator, normalized so that its eigenvalues lie in Ω. In this context, the Hecke relations may be written $T_{p^m}'(k) = X_m(T_p'(k))$.

For $q \ge 1$, we have the family of unit measures

$$d\mu_q = \frac{q+1}{(q^{1/2}+q^{-1/2})^2 - x^2}\,d\mu_\infty.$$

For a prime p, we will see from Lemma A.3 that μ_p gives the the limiting distribution (as $k \to \infty$) of eigenvalues of $T_p'(k)$. As $q \to \infty$, $\mu_q \to \mu_\infty$, which is the limiting distribution of the eigenvalues of all $T_p'(k)$ taken together. For $q = 1$,

$$d\mu_1 = \frac{dx}{2\pi\sqrt{1-x^2/4}} = \frac{d\theta}{\pi}.$$

For $q > 1$, μ_q is comparable to μ_∞ in the sense that

$$\frac{q+1}{(q^{1/2}+q^{-1/2})^2}\mu_\infty(E) \le \mu_q(E) \le \frac{q+1}{(q^{1/2}-q^{-1/2})^2}\mu_\infty(E)$$

for any measurable set $E \subset \Omega$. Also, $\langle X_m, \mu_q\rangle = q^{-m/2}$ if m is even, and 0 if m is odd.

Define $s(k) = \dim S_k(\mathrm{SL}_2(\mathbb{Z}))$. Let $\delta_{S;k}$ be the measure on Ω^r that assigns weight $1/s(k)$ at each r-tuple of simultaneous eigenvalues $(\lambda_1, \dots, \lambda_r)$ of $T_{p_1}'(k), \dots, T_{p_r}'(k)$ and 0 everywhere else. Let μ_S be the measure $\mu_S = \mu_{p_1} \times \cdots \times \mu_{p_r}$. For an integer $n = p_1^{m_1}\cdots p_r^{m_r}$, define

$$Y_n(x_1, \dots, x_r) = X_{m_1}(x_1)\cdots X_{m_r}(x_r);$$

the Y_n are the orthonormal polynomials on Ω^r with respect to $\mu_\infty^{\times r}$ (the product of r copies of μ_∞). Also, $T'_n(k) = Y_n(T'_{p_1}(k), \dots, T'_{p_r}(k))$.

Now, for each $p \in S$, let I_p be the interval $\{x = 2\cos\theta \in \Omega : |\theta - \alpha_p| < \varepsilon\}$. Define $I_S = I_{p_1} \times \cdots \times I_{p_r} \subset \Omega^r$. For a function f on Ω^r we have

$$\int_{I_S} f\, d\delta_{S;k} = \frac{1}{s(k)} \sum_{(\lambda_1,\dots,\lambda_r)\in I_S} f(\lambda_1, \dots, \lambda_r),$$

where the sum is taken over all r-tuples of simultaneous eigenvalues of $T_{p_1}, \dots, T_{p_r}$ lying in I_S. On the other hand, the trace formula allows us to calculate $\int_{\Omega^r} f\, d\delta_{S;k}$ when f is a product of polynomials. Thus, it suffices to find such a function f which is small outside of I_S, yet with $\langle f, \delta_{S;k}\rangle$ large to show that $\int_{I_S} f\, d\delta_{S;k} > 0$. Ultimately, the upper bound for k will come from a bound on the degree of the constructed function.

To accomplish this, we need a few lemmas.

Lemma A.2. *Define $\delta_{n=\square}$ as 1 if n is a square and 0 otherwise. Then*

$$\operatorname{Tr} T'_n(k) = \delta_{n=\square} \frac{k-1}{12 n^{1/2}} + O\big(n^{3/2}\big).$$

Proof. Using the notation of [Cohen 1977; Schoof and van der Vlugt 1991], the Selberg trace formula is written

$$\operatorname{Tr} T_n(k) = A_1 + A_2 + A_3 + A_4,$$

where the $A_i = A_i(n,k)$ are expressions depending on n and k, as follows:

$$A_1 = \delta_{n=\square} \frac{k-1}{12} n^{k/2-1}$$

is the principal term;

$$A_2 = -\frac{1}{2} \sum_{t^2<4n} \frac{\rho^{k-1} - \overline{\rho}^{k-1}}{\rho - \overline{\rho}} \sum_f h_w\left(\frac{t^2-4n}{f^2}\right),$$

where

- t runs through all integers such that $t^2 < 4n$;
- ρ and $\overline{\rho}$ are the roots of the polynomial
$$X^2 - tX + n;$$
- f runs through the integers ≥ 1 such that f^2 divides $t^2 - 4n$ with
$$(t^2-4n)/f^2 \equiv 0, 1 \pmod 4;$$
- $h_w\big((t^2-4n)/f^2\big)$ is the class number of the imaginary quadratic field of discriminant $(t^2-4n)/f^2$, divided by 2 if this discriminant is -4 and by 3 if it is -3.

To bound this term, first note that $|\rho| = n^{1/2}$ and $|\rho - \overline{\rho}| = (4n - t^2)^{1/2} \geq 1$, so that

$$\left|\frac{\rho^{k-1} - \overline{\rho}^{k-1}}{\rho - \overline{\rho}}\right| \leq 2n^{(k-1)/2}.$$

Next, by Dirichlet's class number formula,

$$|h_w(D)| = \left|\frac{1}{D} \sum_{j=1}^{|D|} \Big(\frac{D}{j}\Big) j\right| = O(|D|).$$

Thus,

$$\sum_{t^2<4n} \sum_f h_w\left(\frac{t^2-4n}{f^2}\right) = O\big(n^{3/2}\big),$$

and $|A_2| = O\big(n^{k/2+1}\big)$. (One can get the sharper estimate $O_\varepsilon\big(n^{(k+1)/2+\varepsilon}\big)$ by using a better bound for the class number, as was done in [Brumer 1995, Lemma 4.1]; this won't be necessary here.) Further,

$$A_3 = -\frac{1}{2} \sum_{d|n} \inf(d, n/d)^{k-1} = O\big(n^{(k-1)/2} d(n)\big),$$

$$A_4 = \begin{cases} 0 & \text{if } k = 2, \\ d(n) & \text{otherwise,} \end{cases} = O\big(d(n)\big).$$

In all, we have $O\big(n^{k/2+1}\big)$. The lemma follows upon dividing by $n^{(k-1)/2}$. $\square$

Lemma A.3. $\langle Y_n, \delta_{S;k} - \mu_S\rangle = O\Big(\dfrac{n^{3/2}}{k}\Big)$.

Proof. We have

$$\langle Y_n, \delta_{S;k}\rangle = \frac{\operatorname{Tr} Y_n(T'_{p_1}(k), \dots, T'_{p_r}(k))}{s(k)} = \frac{\operatorname{Tr} T'_n(k)}{s(k)}.$$

$T'_1(k)$ is the identity map, so $s(k) = \operatorname{Tr} T'_1(k)$. Thus, applying Lemma A.2 twice,

$$\begin{aligned}\langle Y_n, \delta_{S;k}\rangle &= \frac{\delta_{n=\square}(k-1)/(12n^{1/2}) + O\big(n^{3/2}\big)}{(k-1)/12 + O(1)} \\ &= \frac{\delta_{n=\square} n^{-1/2} + O\big(n^{3/2}/k\big)}{1 + O(1/k)} \\ &= \delta_{n=\square} n^{-1/2} + O\Big(\frac{n^{3/2}}{k}\Big).\end{aligned}$$

But $\delta_{n=\square} n^{-1/2} = \langle Y_n, \mu_S\rangle$. $\square$

Lemma A.4. *There is a polynomial $g(x_1,\dots,x_r)$ on Ω^r with the following properties*:

1. $\langle g, \mu_S\rangle > (\varepsilon/5)^{2r}$;
2. $|g(x_1,\dots,x_r)| < (\varepsilon/10)^{2r}$ *for* $(x_1,\dots,x_r)$ *outside of* I_S;
3. $\langle g^2, \mu_\infty^{\times r}\rangle < (\varepsilon/10)^{-r}$;
4. g *admits the factorization*
$$g(x_1,\dots,x_r) = g_{p_1}(x_1)\cdots g_{p_r}(x_r)$$
with $\deg g_p = O\big(r\log(1+1/\varepsilon)/\varepsilon\big)$ *for all* $p\in S$.

Proof. It will be convenient to work with the variable $\theta\in\Omega'$. For each $p\in S$, let I'_p be the image of I_p in this variable, that is
$$I'_p = \{\theta\in\Omega' : |\theta-\alpha_p| < \varepsilon\}.$$
Note that I'_p has length at least ε. Let c_p be the center of I'_p (which is just α_p unless α_p is close to the boundary of Ω').

Let h_0 be the "triangle" function on $[-\pi,\pi]$ given by the convolution
$$h_0 = \left(\frac{8\pi}{\varepsilon}\right)^2 \chi_{[-\varepsilon/8,\varepsilon/8]} * \chi_{[-\varepsilon/8,\varepsilon/8]}.$$
In this context, convolution is defined by
$$(f*g)(\theta) = \int_{-\pi}^{\pi} f(\theta')g(\theta-\theta')\,\frac{d\theta'}{2\pi}.$$

For an even integer parameter m to be specified later, let φ_m be the smoothing kernel
$$\varphi_m = \left(\frac{4\pi m}{\varepsilon}\right)^m \underbrace{\chi_{[-\varepsilon/4m,\varepsilon/4m]} * \cdots * \chi_{[-\varepsilon/4m,\varepsilon/4m]}}_{m\text{ times}}.$$
Then the function $h_m = h_0*\varphi_m$ lies in $C^m([-\pi,\pi])$, is supported on $[-\varepsilon/2,\varepsilon/2]$, and has the absolutely convergent Fourier expansion
$$\begin{aligned} h_m(\theta) &= \sum_{n\in\mathbb{Z}} \hat h_m(n)e^{in\theta} \\ &= \sum_{n\in\mathbb{Z}} \operatorname{sinc}^2\left(\frac{n\varepsilon}{8}\right)\operatorname{sinc}^m\left(\frac{n\varepsilon}{4m}\right)e^{in\theta}. \end{aligned}$$

Now, for each prime $p\in S$, define
$$f_p(\theta) = \frac{h_m(\theta-c_p)+h_m(\theta+c_p)}{2}.$$
Then f_p has Fourier coefficients $\hat f_p(n)=\hat h_m(n)\cos c_p n$. Next, let g_p be the partial sum
$$g_p(\theta) = \sum_{|n|\le N}\hat f_p(n)e^{in\theta},$$
where N is to be chosen later. Since g_p is even, it may be written as a cosine series, and hence is a polynomial of degree N in $x=2\cos\theta$. Put
$$\begin{aligned} f(\theta_1,\dots,\theta_r) &= f_{p_1}(\theta_1)\cdots f_{p_r}(\theta_r), \\ g(\theta_1,\dots,\theta_r) &= g_{p_1}(\theta_1)\cdots g_{p_r}(\theta_r). \end{aligned}$$

Having completed this construction, all that remains is to choose the parameters m and N so that g satisfies the required properties. First, the error in approximating f by g is bounded by
$$\begin{aligned} &\big|g(\theta_1,\dots,\theta_r)-f(\theta_1,\dots,\theta_r)\big| \\ &\quad\le \Bigg|\sum_{\substack{(n_1,\dots,n_r)\in\mathbb{Z}^r\\ n_1,\dots,n_r\notin[-N,N]}}\prod_{i=1}^r \hat f_{p_i}(n_i)\Bigg| \\ &\quad\le \sum_{\substack{(n_1,\dots,n_r)\in\mathbb{Z}^r\\ n_1,\dots,n_r\notin[-N,N]}}\prod_{i=1}^r \hat h_m(n_i) \\ &\quad= h_m(0)^r - \Bigg(h_m(0)-\sum_{|n|>N}\hat h_m(n)\Bigg)^r \\ &\quad\le r h_m(0)^{r-1}\sum_{|n|>N}\hat h_m(n) \\ &\quad\le r h_0(0)^{r-1}\cdot 2\int_N^\infty \left(\frac{x\varepsilon}{8}\right)^{-2}\left(\frac{x\varepsilon}{4m}\right)^{-m}dx \\ &\quad= r\Big(\frac{8\pi}{\varepsilon}\Big)^{r-1}\frac{128(4m)^m}{\varepsilon^{m+2}(m+1)N^{m+1}}. \end{aligned}$$

Now choose m to be the smallest even integer $\ge r\log(1+1/\varepsilon)$, and choose N to be the smallest integer which makes the last line $< (\varepsilon/10)^{2r}$. We see that $N = O\big(r\log(1+1/\varepsilon)/\varepsilon\big)$. This establishes properties 2 and 4.

Property 3 follows easily since
$$\begin{aligned} \langle g^2,\mu_\infty^{\times r}\rangle &= \prod_{p\in S}\langle g_p^2,\mu_\infty\rangle \le \prod_{p\in S}\frac{2}{\pi}\int_0^\pi g_p^2\,d\theta \\ &= \prod_{p\in S}2\int_{-\pi}^{\pi} g_p^2\,\frac{d\theta}{2\pi} \le \prod_{p\in S}2\int_{-\pi}^{\pi} f_p^2\,\frac{d\theta}{2\pi} \\ &= \left(\int_{-\pi}^{\pi} h_m^2\,\frac{d\theta}{2\pi}\right)^r = (h_m*h_m)(0)^r \\ &\le (h_0*h_0)(0)^r = \left(\frac{8\pi}{3\varepsilon}\right)^r \\ &< \big(\tfrac{1}{10}\varepsilon\big)^{-r}. \end{aligned}$$

To prove property 1, we first obtain a lower bound for $\langle f_p, \mu_\infty \rangle$. We have

$$\begin{aligned}\langle f_p, \mu_\infty \rangle &= \frac{2}{\pi} \int_0^\pi f_p(\theta) \sin^2\theta \, d\theta \\ &= \int_{-\pi}^{\pi} f_p(\theta)(1-\cos 2\theta) \frac{d\theta}{2\pi} \\ &= \hat{f}_p(0) - \hat{f}_p(2) \ge 1 - \cos 2c_p = 2\sin^2 c_p.\end{aligned}$$

Now, c_p is at least $\frac{1}{2}\varepsilon$ away from the boundary of Ω'. Thus,

$$\langle f_p, \mu_\infty \rangle \ge 2\sin^2\left(\tfrac{1}{2}\varepsilon\right) \ge \frac{2}{\pi^2}\varepsilon^2.$$

Next,

$$\langle f_p, \mu_p \rangle \ge \frac{p+1}{(p^{1/2}+p^{-1/2})^2} \langle f_p, \mu_\infty \rangle \ge \frac{2}{3} \cdot \frac{2}{\pi^2}\varepsilon^2 > \left(\tfrac{3}{10}\varepsilon\right)^2.$$

Finally, $\langle g, \mu_S \rangle = \langle f, \mu_S \rangle + \langle g - f, \mu_S \rangle > \left(\frac{3}{10}\varepsilon\right)^{2r} - \left(\frac{1}{10}\varepsilon\right)^{2r} > \left(\frac{1}{5}\varepsilon\right)^{2r}$. This completes the proof. □

We now have what we need to finish the proof of the theorem. Let g be the function given by Lemma A.4 and put

$$M = \prod_{p \in S} p^{\deg g_p}.$$

Since g is a polynomial, we have the expansion

$$g(x_1, \dots, x_r) = \sum_{n \mid M} a_n Y_n(x_1, \dots, x_r),$$

where $a_n = \langle g Y_n, \mu_\infty^{\times r} \rangle$. Also, $g < \left(\frac{1}{10}\varepsilon\right)^{2r}$ outside of I_S, so that

$$\begin{aligned}\int_{I_S} g \, d\delta_{S;k} &\ge \langle g, \delta_{S;k} \rangle - \left(\tfrac{1}{10}\varepsilon\right)^{2r} \\ &= \langle g, \mu_S \rangle - \left(\tfrac{1}{10}\varepsilon\right)^{2r} + \langle g, \delta_{S;k} - \mu_S \rangle \\ &> \left(\tfrac{1}{5}\varepsilon\right)^{2r} - \left(\tfrac{1}{10}\varepsilon\right)^{2r} + \langle g, \delta_{S;k} - \mu_S \rangle.\end{aligned}$$

We use Lemma A.3 to bound the last term. We have, for an appropriate constant c_1,

$$\begin{aligned}\left|\langle g, \delta_{S;k} - \mu_S \rangle\right| &= \left| \sum_{n \mid M} a_n \langle Y_n, \delta_{S;k} - \mu_S \rangle \right| \\ &\le \left(\sum_{n \mid M} a_n^2 \right)^{1/2} \left(\sum_{n \mid M} \frac{c_1^2 n^3}{k^2} \right)^{1/2} \\ &= \langle g^2, \mu_\infty^{\times r} \rangle^{1/2} \frac{c_1 \sigma_3(M)^{1/2}}{k} \\ &\le \left(\tfrac{1}{10}\varepsilon\right)^{-r/2} \frac{c_1 M^2}{k}.\end{aligned}$$

Altogether,

$$\int_{I_S} g \, d\delta_{S;k} > \left(\tfrac{1}{10}\varepsilon\right)^{2r} - \left(\tfrac{1}{10}\varepsilon\right)^{-r/2} \frac{c_1 M^2}{k}.$$

Now, the right-hand side is positive as long as $k > c_1 M^2 (\frac{1}{10}\varepsilon)^{-5r/2}$. Thus, we may take

$$\begin{aligned}\log k &= O\big(\log M + r \log(1 + 1/\varepsilon)\big) \\ &= O\left(r \frac{\log(1+1/\varepsilon)}{\varepsilon} \log \prod_{p \in S} p \right),\end{aligned}$$

which was to be proved.

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ELECTRONIC AVAILABILITY

Source code implementing the algorithms described here and the calculated Kloosterman sums are available at http://www.math.princeton.edu/~arbooker/klsum/.

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