Computing Hecke Eigenvalues Below the Cohomological Dimension

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CONTENTS

The author was partially supported by a Columbia University Faculty Research grant and NSF Grant DMS 96–27870.

AMS Subject Classification: 11F67, 11F75, 11H55, 11Y16

Keywords: Hecke operators, cohomology of arithmetic groups, modular symbols, sharbly complex, automorphic forms, LLL-reduction, Voronoi-reduction

Let Γ be a torsion-free finite-index subgroup of $\mathrm{SL}_n(\mathbb{Z})$ or $\mathrm{GL}_n(\mathbb{Z})$, and let ν be the cohomological dimension of Γ. We present an algorithm to compute the eigenvalues of the Hecke operators on $H^{\nu-1}(\Gamma;\mathbb{Z})$, for n = 2, 3, and 4. In addition, we describe a modification of the modular symbol algorithm of Ash and Rudolph for computing Hecke eigenvalues on $H^{\nu}(\Gamma;\mathbb{Z})$.

1. INTRODUCTION

1.1. Let Γ be a finite-index subgroup of $\mathrm{SL}_n(\mathbb{Z})$ or $\mathrm{GL}_n(\mathbb{Z})$, and let $\mathcal{M}$ be a $\mathbb{Z}\Gamma$-module. The group cohomology $H^*(\Gamma;\mathcal{M})$ plays an important role in number theory, through its connection with automorphic forms and representations of the absolute Galois group. For an introduction to this conjectural framework, see [Ash 1992a].

For $n = 2$ and Γ a congruence subgroup, the arithmetic nature of $H^*(\Gamma;\mathcal{M})$ has been decisively confirmed [Shimura 1971]. For higher dimensions the picture is mysterious, although several compelling examples for $n = 3$ have appeared recently in the literature. In [Ash et al. 1991], rational cohomology classes of certain $\Gamma \subset \mathrm{GL}_3(\mathbb{Z})$ are related to modular Galois representations. Many more examples of this phenomenon appear in [Ash and McConnell 1992]. In [van Geemen and Top 1994; van Geemen et al. 1997], rational cohomology classes of certain congruence groups are related to the Hasse–Weil zeta functions of certain surfaces. Finally, in [Allison et al. 1998] torsion classes in the cohomology of $\Gamma = \mathrm{GL}_3(\mathbb{Z})$ with twisted coefficients are linked to modular Galois representations, and in [Ash and Tiep 1997] the arithmetic nature of many of these classes is proven.

1058-6458/2000 $0.50 per page

1.2. In all cases, the arithmetic significance of

$$H^*(\Gamma; \mathcal{M})$$

is revealed through the *Hecke operators*. These are endomorphisms of the cohomology associated to certain finite-index subgroups of Γ. The eigenvalues of these linear maps provide a "signature" for the cohomology, which one hopes can be matched to number-theoretic data. Thus to test these conjectures, or to search for counterexamples, it is crucially important to be able to compute Hecke eigenvalues.

1.3. In general, computing these eigenvalues is a difficult problem. Essentially the only technique available in the literature is the *modular symbol algorithm* (Section 3.1), due to Manin [1972] (in the case $n = 2$) and Ash and Rudolph [1979] (for $n \geq 3$). Using this algorithm one can compute the Hecke action on $H^\nu(\Gamma; \mathcal{M})$, where ν is the *cohomological dimension* of Γ. That is, ν is the smallest number such that $H^i(\Gamma; \mathcal{M}) = 0$ for $i > \nu$ and any $\mathcal{M}$.

In particular, if $\Gamma \subset \mathrm{SL}_3(\mathbb{Z})$ or $\mathrm{GL}_3(\mathbb{Z})$, then $\nu = 3$. This is the focus of [Ash et al. 1991; Ash and McConnell 1992; van Geemen and Top 1994; van Geemen et al. 1997; Allison et al. 1998; Ash and Tiep 1997]. For certain congruence groups Γ, the groups $H^3(\Gamma; \mathbb{Q})$ and $H^2(\Gamma; \mathbb{Q})$ contain *cuspidal* cohomology classes [Ash et al. 1984]. In a certain sense these classes are the most interesting constituents of the cohomology. A Lefschetz duality argument [Ash and Tiep 1997, Theorem 3.1] shows that the cuspidal eigenclasses in $H^2(\Gamma; \mathbb{Q})$ have the same eigenvalues as those in $H^3(\Gamma; \mathbb{Q})$, and therefore the modular symbol algorithm suffices to compute Hecke eigenvalues in this dimension.

Now suppose $\Gamma \subset \mathrm{SL}_4(\mathbb{Z})$, so that $\nu = 6$. In this case $H^6(\Gamma; \mathbb{Q})$ does not contain cuspidal classes, and one is interested in $H^5(\Gamma; \mathbb{Q})$. Again one wants to compute Hecke eigenvalues, and since Lefschetz duality relates H^6 to H^3 and not H^5, the modular symbol algorithm doesn't apply. Thus one has the natural problem of devising an algorithm for this context.

1.4. The purpose of this article is to describe an algorithm that — in practice — allows computation of the Hecke action on $H^{\nu-1}(\Gamma; \mathbb{Z})$, where Γ is a torsion-free subgroup of $\mathrm{SL}_n(\mathbb{Z})$ or $\mathrm{GL}_n(\mathbb{Z})$, and $n \leq 4$. We emphasize that the phrase "in practice" is to be taken literally.

Let us be more precise. To represent elements of $H^{\nu-1}(\Gamma; \mathbb{Z})$, we use chains in the *sharbly complex* S_* (Section 2.5). The sharbly complex (whose name was introduced by Lee Rudolph, in honor of the authors of [Lee and Szczarba 1976]) is a complex of infinite Γ-modules such that the homology of the complex of coinvariants $(S_*)_\Gamma$ is naturally isomorphic to $H^*(\Gamma; \mathbb{Z})$. Furthermore, S_* has a natural Hecke action (Section 2.10) that passes to $(S_*)_\Gamma$. Hence the sharbly complex provides a convenient setting to study the cohomology as a Hecke module.

Both S_* and $(S_*)_\Gamma$ are infinitely generated, while $H^*(\Gamma; \mathbb{Z})$ is finitely generated. Hence, for practical computations, we must identify a finite subset of $(S_*)_\Gamma$ that spans the cohomology. For $n \leq 4$ and $H^{\nu-1}$, a spanning set is provided by the *reduced sharblies* (Section 4), which form a subcomplex of S_*. Unfortunately, the Hecke operators do not preserve this subcomplex. Thus to compute eigenvalues, we must describe an algorithm that writes a general sharbly cycle as a sum of reduced sharbly cycles.

1.5. So suppose ξ is a sharbly cycle mod Γ representing a class in $H^{\nu-1}(\Gamma; \mathbb{Z})$. There is a function $\| \ \| : S_* \to \mathbb{Z}$ such that ξ is reduced if and only if $\|\xi\| = 1$ (Definition 3.2). Algorithm 4.13 describes a process that takes ξ as input and produces a cycle ξ' homologous to ξ in $(S_*)_\Gamma$. Geometrically, the algorithm acts by applying the modular symbol algorithm simultaneously over all of ξ. Of course, to be useful for eigenvalue computations, we want that if $\|\xi\| > 1$, then $\|\xi'\| < \|\xi\|$.

We cannot prove that the output ξ' will satisfy this inequality. However, for $n \leq 4$ — the cases of practical interest — this inequality has always held. More precisely, in computer experiments (Section 5) with both random data and 1-sharbly cycles for $n \leq 4$, Algorithm 4.13 has always successfully written a general 1-sharbly cycle as a sum of reduced 1-sharbly cycles. Currently we are applying Algorithm 4.13 in joint work with Avner Ash and Mark McConnell to decompose $H^5(\Gamma; \mathbb{Q})$ as a Hecke module for certain congruence groups $\Gamma \subset \mathrm{SL}_4(\mathbb{Z})$ [Ash et al. 2000]. Details of these computations will appear in a later publication.

1.6. Here is a guide to this paper. In Section 2 we recall the topological and combinatorial background necessary for computing $H^*(\Gamma;\mathbb{Z})$. We discuss the reduction theory due to Voronoi [1908] and the sharbly complex, as well as the Hecke operators and how they interact with the sharbly complex. In Section 3 we recall the modular symbol algorithm, and describe two new conjectural techniques to implement it (Conjectures 3.5 and 3.9). These techniques link the modular symbol algorithm to Voronoi reduction and LLL-reduction, and are conjectured to be true in all dimensions. We also include proofs of the conjectures in special cases. Then in Section 4 we present Algorithm 4.13 and prove that, given a sharbly cycle ξ mod Γ as input, the output ξ' is a homologous cycle mod Γ (Theorem 4.15). We also discuss conditions under which we expect $\|\xi'\| < \|\xi\|$ (Conjecture 4.18). Finally, in Section 5, we describe experiments we performed to generate evidence for Conjectures 3.5, 3.9, and 4.18.

2. BACKGROUND

In this section we describe the topological tools we use to study $H^*(\Gamma;\mathbb{Z})$: the Voronoi polyhedron and the sharbly complex. We present these objects in the context of $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$. However, all of what we say applies with minor modification to $\Gamma \subset \mathrm{GL}_n(\mathbb{Z})$.

2.1. Let V be the $\mathbb{R}$-vector space of symmetric $n \times n$ matrices, and let $C \subset V$ be the cone of positive-definite matrices. The linear group $G = \mathrm{SL}_n(\mathbb{R})$ acts on C by $(g, c) \mapsto g \cdot c \cdot g^t$, and the stabilizer of any given point is isomorphic to SO_n.

Let X be C mod homotheties. The G-action on C commutes with the homotheties and induces a transitive G-action on X. The stabilizer of any fixed point of X is again SO_n. After choosing a basepoint, we may identify X with the global Riemannian symmetric space $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n$, which is a contractible, noncompact, smooth manifold of real dimension $N = \frac{1}{2}n(n+1) - 1$.

The group $\mathrm{SL}_n(\mathbb{Z})$ acts on X via the G-action, and does so properly discontinuously. Hence if $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$ is any torsion-free subgroup, the quotient $\Gamma\backslash X$ is a real noncompact manifold, and is an Eilenberg–Mac Lane space for Γ. We may then identify the group cohomology $H^*(\Gamma;\mathbb{Z})$ with $H^*(\Gamma\backslash X;\mathbb{Z})$. Although the dimension of $\Gamma\backslash X$ is N, it can be shown that $H^i(\Gamma\backslash X;\mathbb{Z}) = 0$ if $i > N - n + 1$ [Borel and Serre 1973, Theorem 11.4.4]. The number $\nu = N - n + 1$ is called the *cohomological dimension* of Γ.

2.2. Recall that a point in $\mathbb{Z}^n$ is said to be *primitive* if the greatest common divisor of its coordinates is 1. In particular, a primitive point is nonzero. Let $\mathcal{P} \subset \mathbb{Z}^n$ be the set of primitive points. Any $v \in \mathcal{P}$, written as a column vector, determines a rank-one quadratic form $q(v) \in \bar{C}$ by $q(v) = v \cdot v^t$.

Definition 2.3. The *Voronoi polyhedron* Π is the closed convex hull of the points $q(v)$, as v ranges over $\mathcal{P}$.

Note that, by construction, $\mathrm{SL}_n(\mathbb{Z})$ acts on Π. The cones over the faces of Π form a fan $\mathcal{V}$ that induces a Γ-admissible decomposition of C [Ash 1977, p. 117]. Essentially, this means that Γ acts on $\mathcal{V}$; that each cone is spanned by a *finite* collection of points $q(v)$ where $v \in \mathcal{P}$; and that mod Γ there are only finitely many orbits in $\mathcal{V}$. The fan $\mathcal{V}$ provides a reduction theory for C in the following sense: any point $x \in C$ is contained in a unique $\sigma \in \mathcal{V}$.

Given $\sigma \in \mathcal{V}$, let $\operatorname{vert}\sigma$ be the set of all $v \in \mathcal{P}$ such that $q(v)$ is a vertex of the face of Π generating σ. For later use, we record the following theorem of Voronoi:

Theorem 2.4 [Voronoi 1908]. *Let E be the standard basis of $\mathbb{Z}^n$, and let Σ be the cone spanned by the $n(n+1)/2$ points $q(e_i)$ and $q(e_i - e_j)$, where $e_i \in E$ and $1 \le i < j \le n$. Then Σ occurs as a top-dimensional cone in $\mathcal{V}$ for all n.*

2.5. We now discuss an algebraic tool to compute $H^*(\Gamma;\mathbb{Z})$. The material in this section closely follows [Ash 1994].

Recall that the *Steinberg module* $\mathrm{St}(n)$ is the $\mathbb{Z}\Gamma$-module $H^\nu(\Gamma;\mathbb{Z}\Gamma)$.

Theorem 2.6 [Ash 1994]. *The Steinberg module is isomorphic to the module of formal $\mathbb{Z}$-linear combinations of the elements $[v_1,\dots,v_n]^*$, where each $v_i \in \mathbb{Q}^n$ is nonzero, mod the relations*:

1. *If τ is a permutation on n letters, then*
$$[v_1,\dots,v_n]^* = \mathrm{sgn}(\tau)[\tau(v_1),\dots,\tau(v_n)]^*,$$
where $\mathrm{sgn}(\tau)$ is the sign of τ.
2. *If $q \in \mathbb{Q}^\times$, then $[qv_1, v_2,\dots,v_n]^* = [v_1,\dots,v_n]^*$.*

3. *If the v_i are linearly dependent, then*
$$[v_1, \dots, v_n]^* = 0.$$
4. *If $v_0, \dots, v_n$ are nonzero points in $\mathbb{Q}^n$, then*
$$\sum_i (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n]^* = 0.$$

By Borel–Serre duality [1973, § 11.4], if Γ is torsion-free, then for any $\mathbb{Z}\Gamma$-module $\mathcal{M}$ we have a natural isomorphism
$$\Phi : H^k(\Gamma; \mathcal{M}) \longrightarrow H_{\nu-k}(\Gamma; \mathrm{St}(n) \otimes \mathcal{M}). \tag{2–1}$$
Hence one may compute $H^*(\Gamma; \mathbb{Z})$ by computing the homology of a $\mathbb{Z}\Gamma$-free resolution of $\mathrm{St}(n) \otimes \mathbb{Z}$. Such a resolution is provided by the sharbly complex.

Definition 2.7 [Ash 1994]. The *sharbly complex* is the chain complex $\{S_*, \partial\}$ given by the following data:

1. For $k \geq 0$, S_k is the module of formal $\mathbb{Z}$-linear combinations of elements
$$\boldsymbol{u} = [v_1, \dots, v_{n+k}],$$
where each $v_i \in \mathcal{P}$, mod the relations:
   a. If τ is a permutation on $(n+k)$ letters, then
   $$[v_1, \dots, v_{n+k}] = \mathrm{sgn}(\tau)[\tau(v_1), \dots, \tau(v_{n+k})],$$
   where $\mathrm{sgn}(\tau)$ is the sign of τ.
   b. If $q = \pm 1$, then
   $$[qv_1, v_2 \dots, v_{n+k}] = [v_1, \dots, v_{n+k}].$$
   c. If the rank of the matrix $(v_1, \dots, v_{n+k})$ is less than n, then $\boldsymbol{u} = 0$.
2. The boundary map $\partial : S_k \to S_{k-1}$ is
$$[v_1, \dots, v_{n+k}] \longmapsto \sum_{i=1}^{n+k} (-1)^i [v_1, \dots, \hat{v}_i, \dots, v_{n+k}].$$

The elements $\boldsymbol{u} = [v_1, \dots, v_{n+k}]$ are called *k-sharblies*. A 0-sharbly is also called a *modular symbol*. By abuse of notation, we will often use the same symbol $\boldsymbol{u}$ to denote a k-sharbly and the k-sharbly chain $1 \cdot \boldsymbol{u}$. The obvious left action of Γ on S_* commutes with ∂.

Proposition 2.8 [Ash 1994]. *The complex $\{S_*, \partial\}$ is a $\mathbb{Z}\Gamma$-free resolution of* $\mathrm{St}(n)$, *with the map $S_0 \to \mathrm{St}(n)$ given by $\boldsymbol{u} \mapsto \boldsymbol{u}^*$.*

For any $k \geq 0$, let $(S_k)_\Gamma$ be the module of Γ-coinvariants. This is the quotient of S_k by the relations of the form $\gamma \cdot \boldsymbol{u} - \boldsymbol{u}$, where $\gamma \in \Gamma$, $\boldsymbol{u} \in S_k$. This is also a complex with the induced boundary, which we denote by ∂_Γ. Proposition 2.8 and (2–1) imply that $H^k(\Gamma; \mathbb{Z})$ is naturally isomorphic to $H_{\nu-k}((S_*)_\Gamma)$.

2.9. Now we recall the definition of the Hecke operators. More details can be found in [Shimura 1971, Chapter 3].

Fix an arithmetic group $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$. Given $g \in GL_n(\mathbb{Q})$, let $\Gamma^g = g^{-1}\Gamma g$ and $\Gamma' = \Gamma \cap \Gamma^g$. Then $[\Gamma : \Gamma']$ and $[\Gamma^g : \Gamma']$ are finite. The inclusions $\Gamma' \to \Gamma$ and $\Gamma' \to \Gamma^g$ determine a diagram

$$\begin{tikzcd} & \Gamma' \backslash X \arrow[dl, "s"'] \arrow[dr, "t"] & \\ \Gamma \backslash X & & \Gamma \backslash X \end{tikzcd}$$

Here $s(\Gamma' x) = \Gamma x$ and t is the composition of $\Gamma' x \mapsto \Gamma^g x$ with left multiplication by g. This diagram is the *Hecke correspondence* associated to g. It can be shown that, up to isomorphism, the Hecke correspondence depends only on the double coset $\Gamma g \Gamma$.

Because the maps s and t are proper, they induce a map on cohomology:
$$T_g := t_* s^* : H^*(\Gamma \backslash X; \mathbb{Z}) \to H^*(\Gamma \backslash X; \mathbb{Z}).$$
This is the *Hecke operator* associated to g. We let $\mathcal{H}_\Gamma$ be the $\mathbb{Z}$-algebra generated by the Hecke operators, with product given by composition.

For an example, let $\Gamma = \mathrm{SL}_n(\mathbb{Z})$. Then $\mathcal{H}_\Gamma$ decomposes as a tensor product
$$\mathcal{H}_\Gamma = \bigotimes_{p \text{ prime}} \mathcal{H}_p.$$
Each $\mathcal{H}_p$ is a polynomial ring generated by the double cosets
$$T_p(k, n) = \Gamma \, \mathrm{diag}(1, \dots, 1, \underbrace{p, \dots, p}_{k}) \, \Gamma. \tag{2–2}$$

2.10. Now let $u \in H^k(\Gamma; \mathbb{Z})$ be a cohomology class. Choose $g \in GL_n(\mathbb{Q})$, and let $T_g \in \mathcal{H}$ be the Hecke operator associated to g. We want to explicitly describe the action of T_g on u in terms of the sharbly complex.

Choose $\xi \in S_k$ such that ξ is a cycle mod Γ and $\Phi^{-1}(\xi) = u$. Write $\xi = \sum n(\boldsymbol{u})\boldsymbol{u}$, where $n(\boldsymbol{u}) \in \mathbb{Z}$, and almost all $n(\boldsymbol{u}) = 0$. The double coset $\Gamma g \Gamma$ decomposes as

$$\Gamma g \Gamma = \coprod_{h \in I} \Gamma h$$

for some set $I \subset \mathrm{GL}_n(\mathbb{Q})$. Note that I is finite. We have a map $S_k \to S_k$ given by

$$T_g : \xi \longmapsto \sum_{h \in I} n(\boldsymbol{u}) h \cdot \boldsymbol{u}. \tag{2–3}$$

One can show that the right-hand side of (2–3) is a well-defined cycle mod Γ, and that under Φ this cycle passes to $T_g(u)$.

In general, $I \not\subset \mathrm{SL}_n(\mathbb{Z})$. Thus the Hecke operators do not preserve the subcomplex of S_* generated by $\mathcal{V}$.

3. MODULAR SYMBOLS

In this section we recall the Ash–Rudolph modular symbol algorithm and present our conjectural implementations of it.

3.1. Let ξ be a k-sharbly chain, and write

$$\xi = \sum n(\boldsymbol{u}) \boldsymbol{u},$$

where $n(\boldsymbol{u}) \in \mathbb{Z}$ and almost all $n(\boldsymbol{u}) = 0$. Let $\operatorname{supp} \xi$ be the set of k-sharblies $\{\boldsymbol{u} \mid n(\boldsymbol{u}) \neq 0\}$. Let $Z(\xi)$ be the set of all modular symbols that appear as a submodular symbol of some $\boldsymbol{u} \in \operatorname{supp} \xi$. In other words, $\boldsymbol{v} \in Z(\xi)$ if and only if there is a $\boldsymbol{u} = [v_1, \dots, v_{n+k}] \in \operatorname{supp} \xi$ such that $\boldsymbol{v} = [v_{i_1}, \dots, v_{i_n}]$ for $\{i_1, \dots, i_n\} \subset \{1, \dots, n+k\}$.

Definition 3.2. Given any modular symbol

$$\boldsymbol{v} = [v_1, \dots, v_n],$$

let

$$\|\boldsymbol{v}\| = |\det(v_1, \dots, v_n)|.$$

We extend this to $\|\ \| : S_k \to \mathbb{Z}$ by setting

$$\|\xi\| = \mathrm{Max}_{\boldsymbol{v} \in Z(\xi)} \{\|\boldsymbol{v}\|\}.$$

We say ξ is *reduced* if $\|\xi\| = 1$. In the special case that $\xi = \boldsymbol{v}$ is a modular symbol, we say that ξ is a *unimodular symbol.*

Note that $\|\ \|$ is well-defined modulo the relations in Definition 2.7.

The reduced k-sharbly chains form a finitely generated subgroup of $(S_k)_\Gamma$. In general, the image of this subgroup under the map $S_k \to H^{\nu-k}(\Gamma; \mathbb{Z})$ does not generate. However, we have the following result:

Theorem 3.3 [Ash and Rudolph 1979]. *The restriction of $S_0 \to H^\nu(\Gamma; \mathbb{Z})$ to the subgroup generated by the unimodular symbols is surjective.*

Proof. We present the proof of [Ash and Rudolph 1979]. It suffices to show that any modular symbol is equivalent mod ∂S_1 to a sum of unimodular symbols.

Let $\boldsymbol{v} = [v_1, \dots, v_n]$, and suppose that $\|\boldsymbol{v}\| > 1$. Let $w \in \mathbb{Z}^n$ be any point not in the lattice generated by the v_i. (Such a point exists since $\|\boldsymbol{v}\| > 1$.) Let $\boldsymbol{v}_i$ be the modular symbol obtained by replacing v_i with w in $\boldsymbol{v}$. Applying relation (4) from Theorem 2.6, we have

$$\boldsymbol{v} = \sum (-1)^{i+1} \boldsymbol{v}_i \tag{3–1}$$

in $S_0/\partial S_1$. We claim w can be modified so that $0 \leq \|\boldsymbol{v}_i\| < \|\boldsymbol{v}\|$, and at least one $\boldsymbol{v}_i$ satisfies $\|\boldsymbol{v}_i\| \neq 0$. This proves the theorem, because after repeating the argument finitely many times, we can write $\boldsymbol{v}$ as a sum of unimodular symbols.

To prove the claim, write $w = \sum q_i v_i$, where $q_i \in \mathbb{Q}$. We have $\|\boldsymbol{v}_i\| = |q_i| \|\boldsymbol{v}\|$. If we modify w by subtracting integral multiples of the v_i, we can ensure $0 \leq |q_i| < 1$. Furthermore, at least one $q_i \neq 0$ since w was originally chosen not to lie in the lattice generated by the v_i. □

3.4. Given a modular symbol $\boldsymbol{v}$, the set of *candidates* of $\boldsymbol{v}$ is the set

$$\operatorname{cand} \boldsymbol{v} = \bigl\{ w \in \mathbb{Z}^n \bigm| w \neq 0 \text{ and } w = \textstyle\sum q_i v_i, \quad \text{where } 0 \leq |q_i| < 1 \bigr\}.$$

The set $\operatorname{cand} \boldsymbol{v}$ contains exactly the points that may be used to construct the homology (3–1) so that the resulting modular symbols are closer to unimodularity.

For application of Theorem 3.3 to Hecke eigenvalue computations, we need to construct a candidate for any $\boldsymbol{v}$ with $\|\boldsymbol{v}\| > 1$. We now discuss two conjectural ways to do this. These are useful for three reasons:

1. The conjectures will play an important role in our algorithm to compute the Hecke action on $H^{\nu-1}(\Gamma; \mathbb{Z})$.

2. The candidates produced by these methods are efficient in practice, in the sense that $\|\boldsymbol{v}_i\|$ from (3–1) will be much smaller than $\|\boldsymbol{v}\|$.
3. Conjecture 3.9 provides an explicit polynomial-time implementation of the modular symbol algorithm.

Write $\boldsymbol{v} = [v_1, \dots, v_n]$, and let $b(\boldsymbol{v})$ be the point $\sum v_i v_i^t$. One can show $b(\boldsymbol{v}) \in C$ since $\|\boldsymbol{v}\| \neq 0$. Recall that if $\sigma \in \mathcal{V}$, then $\operatorname{vert}\sigma \subset \mathcal{P}$ is the set of primitive points corresponding to the face of Π that generates σ (Section 2.2).

Conjecture 3.5. *Let $\boldsymbol{v}$ be a modular symbol with $\|\boldsymbol{v}\| > 1$. Let $\sigma \in \mathcal{V}$ be a top-dimensional cone containing $b(\boldsymbol{v})$. Then*

$$\operatorname{cand}\boldsymbol{v} \cap \operatorname{vert}\sigma \neq \varnothing.$$

Remark 3.6. The cone σ can be computed using the *Voronoi reduction algorithm* [Voronoi 1908, § 27ff].

3.7. Although geometrically attractive, the use of Conjecture 3.5 in practice suffers from two disadvantages. First, to the best of our knowledge, the complexity of the Voronoi reduction algorithm is unknown. Second, the structure of Π is difficult to determine.[1] An alternative uses LLL-reduction, which we now recall.

Definition 3.8 [Cohen 1993, Section 2.6]. Let $B = \{b_1, \dots, b_n\}$ be an ordered basis of $\mathbb{R}^n$, and let $B^* = \{b_1^*, \dots, b_n^*\}$ be the orthogonal (not orthonormal) basis obtained from B using the Gram–Schmidt process. Let

$$\mu_{i,j} = (b_i \cdot b_j^*)/(b_j^* \cdot b_j^*), \quad \text{where } 1 \leq j < i \leq n.$$

Then B is *LLL-reduced* if the following inequalities hold:

1. $|\mu_{i,j}| \leq 1/2$, for $1 \leq j < i \leq n$.
2. $|b_i^* + \mu_{i,i-1} b_{i-1}^*|^2 \geq (3/4)|b_{i-1}^*|^2$.

Furthermore, a quadratic form is said to be LLL-reduced if it is the Gram matrix of an LLL-reduced basis.

[1] However, for $n \leq 4$, the structure of Π is well understood. An elegant technique to index the faces using configurations in projective space (in the sense of [Hilbert and Cohn-Vossen 1952]) can be found in [McConnell 1991]. To the best of our knowledge, the complete structure of Π is unknown for any other n, although much is known for $5 \leq n \leq 8$ (see [Conway and Sloane 1988] and the references there).

We emphasize that the basis B in Definition 3.8 is ordered. Changing the order of B changes B^*, which affects the conditions of the definition.

Conjecture 3.9. *Let $\boldsymbol{v}$ be a modular symbol with $\|\boldsymbol{v}\| > 1$, and suppose that $b(\boldsymbol{v})$ is an LLL-reduced quadratic form. Let E be the standard basis for $\mathbb{Z}^n$. Then*

$$\operatorname{cand}\boldsymbol{v} \cap E \neq \varnothing.$$

Remark 3.10. To apply Conjecture 3.9 in practice, one finds a matrix $\gamma \in \mathrm{GL}_n(\mathbb{Z})$ such that $b(\gamma \cdot \boldsymbol{v})$ is LLL-reduced, and then a candidate for $\boldsymbol{v}$ will be in $\gamma^{-1}E$.

3.11. We can prove the conjectures in some cases. We begin by describing a geometric interpretation of what it means for $w \in E$ to be a candidate for $\boldsymbol{v}$.

Let $\boldsymbol{v} = [v_1, \dots, v_n]$ be a modular symbol, and fix an ordering of the v_i. Let A be the matrix with columns v_i, and let $B = \{b_1, \dots, b_n\}$ be the basis made up of the rows of A. Then one easily checks that the quadratic form $b(\boldsymbol{v})$ is the Gram matrix of B.

Lemma 3.12. *Let $w = e_k \in E$. For $1 \leq i \leq n$, let $\boldsymbol{v}_i$ be the modular symbol constructed from $\boldsymbol{v}$ and w as in* (3–1). *Also for $1 \leq i \leq n$, let $B_i \subset \mathbb{R}^{n-1}$ be the set of $(n-1)$ vectors obtained by projecting $B \smallsetminus \{b_k\}$ into P_i, where P_i is the span of $E \smallsetminus \{e_i\}$. Then the following statements are equivalent:*

1. $\|\boldsymbol{v}_i\| < \|\boldsymbol{v}\|$ *for* $1 \leq i \leq n$.
2. $\operatorname{vol} B_i < \operatorname{vol} B$ *for* $1 \leq i \leq n$.

Here the volume in P_i is normalized so that the fundamental domains of $\mathbb{Z}^n \subset \mathbb{R}^n$ and $\mathbb{Z}^n \cap P_i$ each have volume 1.

Proof. We have $\|\boldsymbol{v}\| = \operatorname{vol} B = |\det A|$. Furthermore, after choosing e_k, we observe that $\|\boldsymbol{v}_i\|$ and $\operatorname{vol} B_i$ are the absolute value of the determinant of the same $(n-1) \times (n-1)$ minor of A. □

Lemma 3.13. *Let $\boldsymbol{v}$ and B be as above, and assume $\|\boldsymbol{v}\| > 1$. If $|b_n^*| > 1$, then $e_n \in \operatorname{cand}\boldsymbol{v}$.*

Proof. First note that $\operatorname{vol} B = \prod |b_i^*|$, since B^* is orthogonal. Since $|b_n^*| > 1$ and $\operatorname{vol} B = \|\boldsymbol{v}\| > 1$, we have

$$\prod_{i<n} |b_i^*| < \operatorname{vol} B.$$

Now let B_i be the projection of $B \setminus \{b_n\}$ into the coordinate hyperplane P_i, as in Lemma 3.12. Clearly $\operatorname{vol} B_i \leq \prod_{i<n} |b_i^*|$. Hence by Lemma 3.12, $\|\boldsymbol{v}_i\| < \|\boldsymbol{v}\|$, and $e_n \in \operatorname{cand} \boldsymbol{v}$. $\square$

Proposition 3.14. *Suppose* $\|\boldsymbol{v}\| > 1$ *and* $b(\boldsymbol{v})$ *is a diagonal quadratic form. Then Conjectures* 3.5 *and* 3.9 *are true.*

Proof. First we show that Conjecture 3.9 is true. Since $b(\boldsymbol{v})$ is a diagonal quadratic form, we have $B = B^*$, and the μ_{ij} from Definition 3.8 vanish. Thus $\operatorname{vol} B = \prod |b_i| > 1$, and $|b_i| \geq 1$ for all i since B is integral.

Assume first that B satisfies $|b_i| \leq |b_j|$ for $i \leq j$. This implies $|b_n| > 1$, and by Lemma 3.13 we have $e_n \in \operatorname{cand} \boldsymbol{v}$, and Conjecture 3.9 is true.

Now drop the assumption that B is ordered by increasing lengths. We can multiply $\boldsymbol{v}$ by a permutation matrix γ so that B satisfies $|b_i| \leq |b_j|$ for $i \leq j$. This means that $\gamma^{-1} e_n \in \operatorname{cand} \boldsymbol{v}$. Since $\gamma^{-1} e_n \in E$, Conjecture 3.9 follows.

Finally, in this case Conjecture 3.9 implies Conjecture 3.5. Since $b(\boldsymbol{v})$ is diagonal, it lies in the cone σ spanned by $\{q(e) \mid e \in E\}$. This cone is a proper face of the cone Σ from Theorem 2.4, and hence $b(\boldsymbol{v}) \in \Sigma$. Since $E \subset \operatorname{vert} \Sigma$, the result follows. $\square$

Using standard estimates on B and B^*, we can find a lower bound on $\|\boldsymbol{v}\|$ so that Conjecture 3.9 is true.

Proposition 3.15. *Suppose that* $\|\boldsymbol{v}\| > 2^{n(n-1)/2}$. *Then Conjecture* 3.9 *is true.*

Proof. We show that $\|\boldsymbol{v}\| > 2^{n(n-1)/2}$ guarantees $|b_n^*| > 1$, which by Lemma 3.13 implies $e_n \in \operatorname{cand} \boldsymbol{v}$. According to [Cohen 1993, Theorem 2.6.2], B satisfies

$$\prod_j |b_j| \geq \|\boldsymbol{v}\|$$

and

$$|b_j| \leq 2^{(n-1)/2} |b_n^*|, \quad \text{for } j = 1, \ldots, n.$$

Hence

$$2^{n(n-1)/2} |b_n^*| \geq \prod_j |b_j| \geq \|\boldsymbol{v}\|.$$

Solving for $|b_n^*|$, we see $\|\boldsymbol{v}\| > 2^{n(n-1)/2}$ ensures that $|b_n^*| > 1$, which proves the claim. $\square$

Theorem 3.16. *Conjecture* 3.5 *is true for* $n = 2$ *and* $n = 3$.

Proof. We use Lemma 3.13 and direct investigation of the reduction domains. First we recall some facts about reduction theory in these dimensions. For convenience we use $\mathrm{GL}_n(\mathbb{Z})$ instead of $\mathrm{SL}_n(\mathbb{Z})$.

For $n \leq 3$ the cone Σ from Theorem 2.4 is the only top-dimensional Voronoi cone modulo $\mathrm{GL}_n(\mathbb{Z})$. According to [Conway and Sloane 1992], $b(\boldsymbol{v}) \in \sigma$ if and only if B is an *obtuse superbase*. By definition, this means the following. Let $b_0 = -\sum b_i$, and let $\bar{B} = B \cup \{b_0\}$. Then $\bar{B}$ satisfies

$$b_i \cdot b_j \leq 0 \quad \text{for } 0 \leq i < j \leq n.$$

The set $\Sigma \cap C$ is not a fundamental domain for $\mathrm{GL}_n(\mathbb{Z})$ acting on C. In fact, the stabilizer $\Gamma(\Sigma) \subset \mathrm{GL}_n(\mathbb{Z})$ is a finite group, which for $n = 2$ has order 6 and for $n = 3$ has order 24. By placing additional conditions on the basis B, we can describe a fundamental domain T for $\Gamma(\Sigma)$ acting on Σ.

First we consider the case $n = 2$. The cone Σ is a 3-dimensional cone inside the cone $\bar{C}$, and is spanned by $q(e_1)$, $q(e_2)$, and $q(e_1 - e_2)$. Figure 1 shows a 2-dimensional affine slice of $\bar{C}$, with Σ divided into fundamental domains for $\Gamma(\Sigma)$. The shaded region T is half of the classical fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ acting on C.

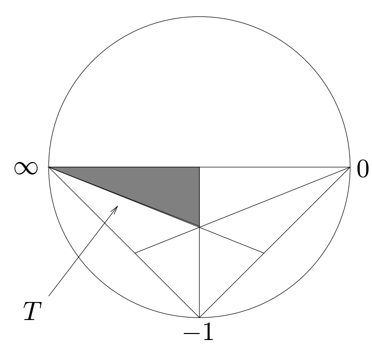


FIGURE 1. Two-dimensional affine slice of $\bar{C}$.

Now we claim that if $b(\boldsymbol{v}) \in T$ and $\|\boldsymbol{v}\| \geq 2$, then $e_2 \in \operatorname{cand}(\boldsymbol{v})$. This implies the theorem for $n = 2$, because multiplying by elements of $\Gamma(\Sigma)$ stabilizes $\operatorname{vert} \Sigma$.

To prove the claim, we present another way to picture bases in the region T. If $B = (b_1, b_2)$, then $b(\boldsymbol{v}) \in T$ if and only if B appears as in Figure 2. In this figure, we have fixed b_1, and b_2 must be in the infinite shaded region S that lies above the semicircle of radius $|b_1|$. Points in S correspond to ways to complete b_1 to an obtuse superbase satisfying the additional inequalities $|b_0| \geq |b_2| \geq |b_1|$.

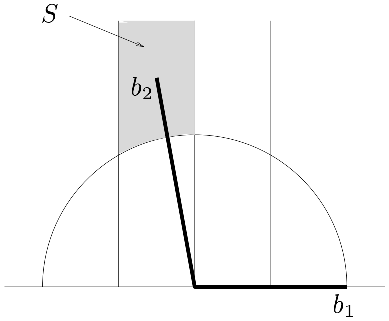


FIGURE 2. Relative position of the elements of a basis $B = (b_1, b_2)$ such that $b(\boldsymbol{v}) \in T$.

Now consider the orthogonal basis B^* constructed from B. We have $b_1 = b_1^*$. It is easy to compute that $|b_2^*| \geq \sqrt{3}|b_1|/2$ for all $b_2 \in S$, and that the minimum occurs when b_2 is at the lower left corner of S. Hence if $|b_1| \geq \sqrt{2}$, we have $|b_2^*| > 1$, and by Lemma 3.13 we have $e_2 \in \operatorname{cand} \boldsymbol{v}$.

Since B is integral, the remaining possibility is $|b_1| = 1$. However, this implies that b_2 lies along the right edge of S, and hence $b_2^* = b_2$. If $|b_2| = 1$, then $\|\boldsymbol{v}\| = 1$. Thus $|b_2| > 1$, and again $e_2 \in \operatorname{cand} \boldsymbol{v}$. This proves the theorem for $n = 2$.

The argument for $n = 3$ is similar, although the reduction domain is more complicated. Now Σ is 5-dimensional, and the fundamental domain T can be described as follows. As before, fix b_1, and take b_2 to lie in the 2-dimensional region S from the $n = 2$ case. Together b_1 and b_2 determine the Dirichlet-Voronoi domain $pqrstu$ (see Figure 3). Let Z be the intersection of $pqrstu$ with

$$\{x = \lambda_1 b_1 + \lambda_2 b_2 \mid x \cdot b_1 \leq 0, x \cdot b_2^* \leq 0\}.$$

Then if $b(\boldsymbol{v}) \in T$, the point b_3 must lie in the 3-dimensional region consisting of the points on or outside the hemisphere of radius $|b_2|$ that project to Z. Figure 3 shows the basis B, and Figure 4 shows Z for different choices of b_2. Altogether T is a 5-dimensional family of obtuse superbases that can be described by additional inequalities similar to those for $n = 2$.

We want to find conditions that imply $|b_3^*| > 1$, which will imply $e_3 \in \operatorname{cand} \boldsymbol{v}$. Clearly the minimum value of $|b_3^*|$ occurs when $|b_1| = |b_2| = |b_3|$. Then for any fixed b_2, the value of $|b_3^*|$ will be smallest

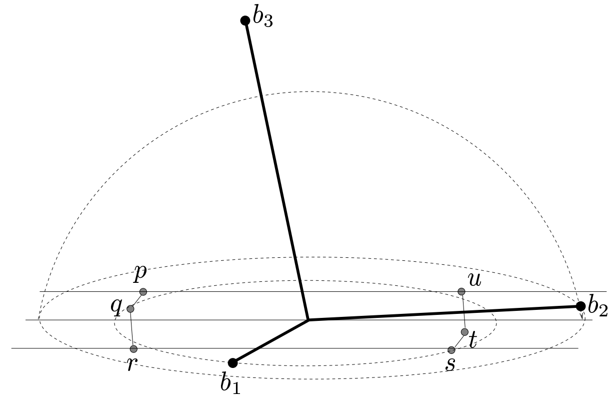


FIGURE 3. The basis B in the case $n = 3$.

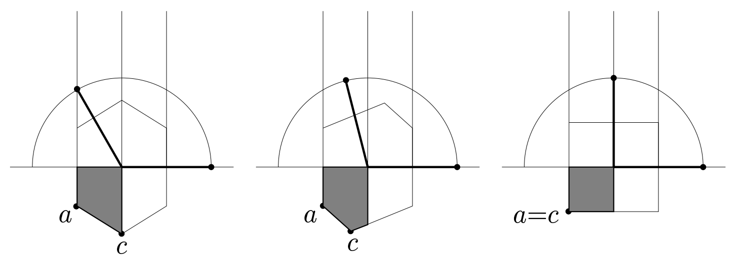


FIGURE 4. Schematics of Z for different choices of b_2.

when b_3 projects to the vertices a or c of Z shown in Figure 4.

So consider the set of bases satisfying

1. $|b_1| = |b_2| = |b_3|$,
2. $0 \geq b_1 \cdot b_2 \geq -|b_1|^2/2$, and
3. b_3 projects to either a or c in Figure 4.

It is not difficult to show that the minimal value of $|b_3^*|$ in this family occurs when $a = c$, or when $b_1 \cdot b_2 = 0$. For this basis, $|b_3^*| = |b_1|/\sqrt{2}$. Hence if $|b_1| > \sqrt{2}$, we have $e_3 \in \operatorname{cand} \boldsymbol{v}$.

The remaining cases are $|b_1| = 1$ or $\sqrt{2}$. As for $n = 2$, it is straightforward, although tedious, to check that for any basis in T satisfying these conditions, we have either $\operatorname{cand} \boldsymbol{v} \cap \operatorname{vert} \Sigma \neq \varnothing$ or $\|\boldsymbol{v}\| = 1$. □

Remark 3.17. For $n = 4$, there is only one other type of top-dimensional Voronoi cone mod $\mathrm{GL}_4(\mathbb{Z})$, which corresponds to Voronoi's *second perfect form* [Voronoi 1908, § 34]. This cone corresponds to the lattice D_4. We are not aware of a useful characterization of the bases appearing in this cone.

4. ONE-SHARBLIES

In this section we describe our technique to compute the Hecke action on $H^{\nu-1}(\Gamma;\mathbb{Z})$.

4.1. Let $\xi = \sum n(\boldsymbol{u})\boldsymbol{u}$ be a k-sharbly chain, where $n(\boldsymbol{u}) \in \mathbb{Z}$, and almost all $n(\boldsymbol{u}) = 0$. Recall that a k-sharbly is said to be reduced if and only if all its submodular symbols are unimodular (Definition 3.2).

In general the reduced k-sharblies do not span $H^{\nu-k}(\Gamma;\mathbb{Z})$ (Section 5.9). However, according to [McConnell 1991], $H^{\nu-1}(\Gamma;\mathbb{Z})$ is spanned by reduced 1-sharblies if $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$ and $n \leq 4$. Hence to compute the Hecke action on $H^{\nu-1}(\Gamma;\mathbb{Z})$ it suffices to describe an algorithm that takes as input a 1-sharbly cycle ξ and produces as output a cycle ξ' satisfying:

(a) The classes of ξ and ξ' in $H^{\nu-1}(\Gamma;\mathbb{Z})$ are the same.

(b) $\|\xi'\| < \|\xi\|$ if $\|\xi\| > 1$.

We will present an algorithm satisfying (a) in Algorithm 4.13; in Conjecture 4.18 we claim that the algorithm satisfies (b) for $n \leq 4$. To simplify the exposition, in Sections 4.2–4.9 we describe the algorithm for $n = 2$. This case is arithmetically uninteresting — we are describing how to compute the Hecke action on $H^0(\Gamma;\mathbb{Z})$ — but the geometry faithfully reflects the situation for all n. We defer presentation for general n to Section 4.10.

4.2. Fix $n = 2$, let $\xi \in S_1$ be a 1-sharbly cycle mod Γ for some $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, and suppose that ξ is not reduced. We want to construct a cycle ξ' homologous to ξ, such that $\|\xi'\| < \|\xi\|$. Since ξ is not reduced, there exist $\boldsymbol{v} \in Z(\xi)$ with $\|\boldsymbol{v}\| > 1$. Hence we want to perform the modular symbol algorithm simultaneously over *all* of $\operatorname{supp}\xi$ while constructing ξ'. This leads to two problems:

1. How should one choose candidates for the submodular symbols of ξ? Is the usual modular symbol algorithm sufficient for this?
2. Given ξ and a collection of candidates for its submodular symbols, how does one assemble the data into ξ'?

Although these questions appear to be independent, they are in fact coupled. To answer the first, we claim that candidates should be chosen using either Conjecture 3.5 or 3.9; we indicate why in Section 4.7. We discuss the second in Sections 4.3–4.5.

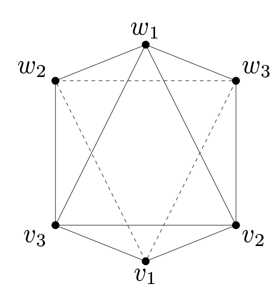
4.3. Suppose first that all $\boldsymbol{v} \in Z(\xi)$ are nonunimodular. We begin by selecting candidates for each $\boldsymbol{v} \in Z(\xi)$ using either Conjecture 3.5 or 3.9, and we make these choices Γ-equivariantly. This means the following. Suppose $\boldsymbol{u}, \boldsymbol{u}' \in \operatorname{supp}\xi$ and $\boldsymbol{v} \in \operatorname{supp}(\partial\boldsymbol{u})$ and $\boldsymbol{v}' \in \operatorname{supp}(\partial\boldsymbol{u}')$ are modular symbols such that $\boldsymbol{v} = \gamma\cdot\boldsymbol{v}'$ for some $\gamma \in \Gamma$. Then we select $w \in \operatorname{cand}\boldsymbol{v}$ and $w' \in \operatorname{cand}\boldsymbol{v}'$ such that $w = \gamma \cdot w'$.

We can do this because if $\boldsymbol{v}$ is a modular symbol and $w \in \operatorname{cand}\boldsymbol{v}$, then $\gamma\cdot w \in \operatorname{cand}(\gamma\cdot\boldsymbol{v})$ for any $\gamma \in \Gamma$. Since there are only finitely many Γ-orbits in $Z(\xi)$, we can choose candidates Γ-equivariantly by selecting them for some set of orbit representatives.

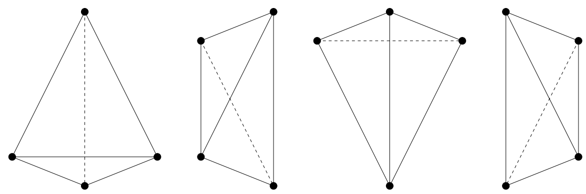
It is important to note that Γ-equivariance is the only "non-local" criterion we use when selecting candidates. In particular, there is a priori no relationship among the 3 candidates chosen for any $\boldsymbol{u} \in \operatorname{supp}\xi$.

4.4. Now we want to use the candidates and the 1-sharblies in ξ to build ξ'. Choose $\boldsymbol{u} = [v_1, v_2, v_3] \in \operatorname{supp}\xi$, and denote the candidate for $[v_i, v_j]$ by w_k, where $\{i, j, k\} = \{1, 2, 3\}$. We use the v_i and the w_i to build a 2-sharbly chain $\eta(\boldsymbol{u})$ as follows.

Let P be an octahedron in $\mathbb{R}^3$. Label the vertices of P with the v_i and w_i such that the vertex labelled v_i shares no edge with the vertex labelled w_i. Now subdivide P into four tetrahedra without adding new vertices. This can be done by connecting two opposite vertices, say those with labels v_1 and w_1, by a new edge:



Now use the four tetrahedra to construct $\eta(\boldsymbol{u})$ as follows. For each tetrahedron T, take the labels of four vertices and arrange them into a quadruple. If we orient P, then we can use the induced orientation on T to order the four primitive points. In this way, each T determines a 2-sharbly, and $\eta(\boldsymbol{u})$ is defined to be the sum. For example, if we use the decomposition



we have

$$\eta(\boldsymbol{u}) = [v_1, v_2, w_3, w_1] + [v_1, w_3, w_2, w_1] + [v_1, w_2, v_3, w_1] + [v_1, v_3, v_2, w_1]. \quad (4\text{–}1)$$

Now repeat this construction for all $\boldsymbol{u} \in \operatorname{supp}\xi$, and let $\eta = \sum n(\boldsymbol{u})\eta(\boldsymbol{u})$. Finally, let $\xi' = \xi + \partial\eta$.

4.5. By construction, ξ' is a cycle mod Γ in the same class as ξ. We claim in addition that no submodular symbols from ξ appear in ξ'. To see this, consider $\partial\eta(\boldsymbol{u})$. From (4–1), we have

$$\begin{aligned}\partial\eta(\boldsymbol{u}) = &-[v_1, v_2, v_3] + [v_1, v_2, w_3] \\ &+ [v_1, w_2, v_3] + [w_1, v_2, v_3] - [v_1, w_2, w_3] \\ &- [w_1, v_2, w_3] - [w_1, w_2, v_3] + [w_1, w_2, w_3]. \quad (4\text{–}2)\end{aligned}$$

Note that this is the boundary in S_*, not $(S_*)_\Gamma$. Furthermore, it's easy to see that $\partial\eta(\boldsymbol{u})$ is independent of which pair of opposite vertices of P we connected to define $\eta(\boldsymbol{u})$.

From (4–2), we see that in $\xi + \partial\eta$, the 1-sharbly $-[v_1, v_2, v_3]$ is canceled by $\boldsymbol{u} \in \operatorname{supp}\xi$. Consider the 1-sharblies in (4–2) of the form $[v_i, v_j, w_k]$. We claim these 1-sharblies vanish in $\partial_\Gamma\eta$.

To see this, suppose that $\boldsymbol{u},\boldsymbol{u}' \in \operatorname{supp}\xi$, and suppose $\boldsymbol{v} = [v_1, v_2] \in \operatorname{supp}\partial\boldsymbol{u}$ equals $\gamma \cdot \boldsymbol{v}'$ for some $\boldsymbol{v}' = [v_1', v_2'] \in \operatorname{supp}\partial\boldsymbol{u}'$. Since the candidates were chosen Γ-equivariantly, we have $w = \gamma \cdot w'$. This means that the 1-sharbly $[v_1, v_2, w] \in \partial\eta(\boldsymbol{u})$ will be canceled mod Γ by $[v_1', v_2', w'] \in \partial\eta(\boldsymbol{u}')$. Hence, in passing from ξ to ξ', the effect in $(S_*)_\Gamma$ is to replace $\boldsymbol{u}$ with *four* 1-sharblies in $\operatorname{supp}\xi'$:

$$\begin{aligned}[v_1, v_2, v_3] \longmapsto &-[v_1, w_2, w_3] \\ &-[w_1, v_2, w_3] - [w_1, w_2, v_3] + [w_1, w_2, w_3]. \quad (4\text{–}3)\end{aligned}$$

Note that in (4–3), there are no 1-sharblies of the form $[v_i, v_j, w_k]$.

Remark 4.6. For implementation purposes, it is not necessary to explicitly construct η. Rather, one may work directly with (4–3).

4.7. Why do we expect ξ' to satisfy $\|\xi'\| < \|\xi\|$? First of all, in the right hand side of (4–3) there are no submodular symbols of the form $[v_i, v_j]$. In fact, any submodular symbol involving a point v_i also includes a candidate used to reduce the $[v_i, v_j]$.

However, consider the submodular symbols of the form $[w_i, w_j]$ in (4–3). Since there is no relationship among the w_i, one has no reason to believe that these modular symbols are closer to unimodularity than those in $\boldsymbol{u}$. Indeed, one might expect that these modular symbols satisfy $\|[w_i, w_j]\| \geq \|\boldsymbol{u}\|$. This is the content of problem 2 from Section 4.2.

We claim that — in practice — if one uses Conjecture 3.5 or 3.9 to select candidates, then these new modular symbols will be very close to unimodularity. In fact, usually they are trivial or satisfy $\|[w_i, w_j]\| = 1$. To us, it seems that Conjectures 3.5 and 3.9 select candidates "uniformly" over $\operatorname{supp}\xi$, although we will not attempt to make this notion precise.

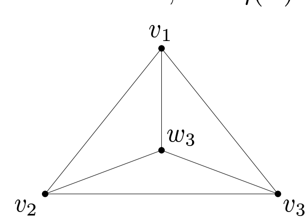
Remark 4.8. To ensure $\|\xi'\| < \|\xi\|$, one must also choose the *best* candidate offered by the conjectures in a suitable sense (Section 4.16).

4.9. In the previous discussion we assumed that no submodular symbols of any $\boldsymbol{u} \in \operatorname{supp}\xi$ were unimodular. Now we discuss what to do if some are. As before, pick candidates for the nonunimodular symbols. There are three cases to consider.

First, all submodular symbols of $\boldsymbol{u}$ may be unimodular. In this case there are no candidates, and (4–3) becomes

$$[v_1, v_2, v_3] \longmapsto [v_1, v_2, v_3]. \quad (4\text{–}4)$$

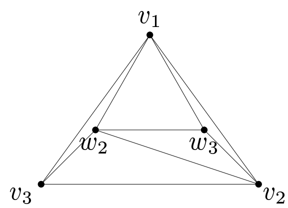
Second, one submodular symbol of $\boldsymbol{u}$ may be nonunimodular, say the symbol $[v_1, v_2]$. In this case we take P to be a tetrahedron, and $\eta(\boldsymbol{u}) = [v_1, v_2, v_3, w_3]$:



As before, $[v_1, v_2, w_3]$ vanishes in the boundary of η mod Γ, and (4–3) becomes

$$[v_1, v_2, v_3] \mapsto -[v_1, v_3, w_3] + [v_2, v_3, w_3].$$

Finally, two submodular symbols of $\boldsymbol{u}$ may be nonunimodular, say $[v_1, v_2]$ and $[v_1, v_3]$. In this case we take P to be the cone on a square:



To construct $\eta(\boldsymbol{u})$ we must choose a decomposition of P into tetrahedra. Since P has a non-simplicial face we must make a choice that affects ξ'. If we choose to subdivide P by connecting the vertex labelled v_2 with the vertex labelled w_2, we obtain

$$[v_1, v_2, v_3] \longmapsto [v_2, w_2, w_3] + [v_2, v_3, w_2] + [v_1, v_3, w_2].$$

4.10. We now describe the procedure for general n. First we recall some facts about convex polytopes. Proofs can be found in [Ziegler 1995].

Let P be a d-dimensional convex polytope embedded in $\mathbb{R}^d$. The *facets* of a d-polytope P are the faces of dimension $(d-1)$. The *cone on* P is the polytope cP constructed as follows. Choose a linear embedding $i : \mathbb{R}^d \to \mathbb{R}^{d+1}$ and let $x \in \mathbb{R}^{d+1} \smallsetminus \mathbb{R}^d$. Then cP is the convex hull of x and $i(P)$. One can show that the combinatorial type of cP is independent of the choice of x or i. We also write $c^0P := P$ and $c^k(P) := c(c^{k-1}P)$.

Let E be the standard basis of $\mathbb{R}^n$. Then the $(n-1)$-*simplex* Δ_{n-1} is the convex hull of E, and the n-*crosspolytope* β_n is the convex hull of $-E \cup E$. Write $E = \{e_i\}$, and let $P(n, j)$ be the convex hull of E and the j points $\{-e_k \mid 1 \le k \le j \le n\}$.

Lemma 4.11. *The polytope $P(n, j)$ is isomorphic to the iterated cone $c^{n-j}\beta_j$.*

Proof. By definition, the convex hull of $A := \{\pm e_k \mid 1 \le k \le j\}$ is β_j. The remaining vertices of $P(n, j)$ are the points $B := \{-e_k \mid j+1 \le k \le n\}$. Since B is linearly independent, and is also linearly independent of the linear span of A, the lemma follows easily by induction. □

Lemma 4.12. *There exist j distinct subdivisions of $P(n, j)$ into simplices without adding new vertices.*

Proof. This follows immediately from Lemma 4.11. Any such subdivision of β_j is formed by connecting one of the j pairs of vertices not already connected by an edge of β_j, and any such subdivision of $c^{n-j}\beta_j$ is formed by subdividing β_j first. □

Algorithm 4.13. Let Γ be a torsion-free subgroup, and let $\xi = \sum n(\boldsymbol{u})\eta(\boldsymbol{u})$ be a 1-sharbly cycle mod Γ representing a class in $H^{\nu-1}(\Gamma; \mathbb{Z})$. The output of this algorithm is a class $\xi' \in H^{\nu-1}(\Gamma; \mathbb{Z})$.

A. *Choose candidates.* For each $\boldsymbol{u} \in \operatorname{supp}\xi$, and for each $\boldsymbol{v} \in \operatorname{supp}\partial\boldsymbol{u}$ with $\|\boldsymbol{v}\| > 1$, choose a candidate $w(\boldsymbol{v})$. Make these choices Γ-equivariantly over all of $\operatorname{supp}\xi$ as in Section 4.3. For each $\boldsymbol{u} \in \operatorname{supp}\xi$, we let $C(\boldsymbol{u})$ be the set $\{w(\boldsymbol{v}) \mid \boldsymbol{v} \in \operatorname{supp}\partial\boldsymbol{u}\}$.

B. *Shift candidates.* Choose

$$\boldsymbol{u} = [v_1, \dots, v_{n+1}] \in \operatorname{supp}\xi,$$

and set $j = \#C(\boldsymbol{u})$. Apply relation (1) from Definition 2.7 so that the j submodular symbols

$$\left\{ \boldsymbol{v}^i = [v_1, \dots, \hat{v}_i, \dots, v_{n+1}] \;\middle|\; 1 \le i \le j \right\}$$

satisfy $\|\boldsymbol{v}^i\| > 1$. Write w_i for $w(\boldsymbol{v}^i)$, and let $n'(\boldsymbol{u})$ be the new coefficient of $\boldsymbol{u}$ in ξ.

C. *Construct 2-sharblies.* Let $P = P(n+1, j)$ be the polytope from Lemma 4.11, and choose a subdivision of P into simplices without adding new vertices as in Lemma 4.12. Orient P so that the induced orientation on the face spanned by $e_1, \dots, e_{n+1}$ is the opposite of the orientation given by the standard ordering of $e_1, \dots, e_{n+1}$. Via the correspondence

$$e_k \longleftrightarrow v_k \quad \text{for } 1 \le k \le n+1,$$
$$-e_k \longleftrightarrow w_k \quad \text{for } 1 \le k \le j,$$

and the orientation of P, use the subdivision of P to construct a 2-sharbly chain $\eta(\boldsymbol{u})$.

D. *Continue.* Complete steps **B** and **C** for all $\boldsymbol{u} \in \operatorname{supp}\xi$.

E. *Terminate.* Set

$$\eta = \sum_{\boldsymbol{u} \in \operatorname{supp}\xi} n'(\boldsymbol{u})\eta(\boldsymbol{u}),$$

and define $\xi' := \partial\eta + \xi$.

4.14. Now we want to describe how ξ' is related to ξ, and in particular in what sense ξ' is closer to unimodularity than ξ. Let $\boldsymbol{u} \in \operatorname{supp} \xi$, and let $\eta(\boldsymbol{u})$ be the 2-sharbly chain constructed above. Define

$$\partial\eta_{\mathrm{old}}(\boldsymbol{u}) = -[v_1, \dots, v_{n+1}],$$

$$\partial\eta_{\mathrm{side}}(\boldsymbol{u}) = \sum_{k=1}^{j} [v_1, \dots, \hat{v_k}, \dots, v_{n+1}, w_k],$$

$$\partial\eta_{\mathrm{new}}(\boldsymbol{u}) = \partial\eta(\boldsymbol{u}) - \partial\eta_{\mathrm{old}}(\boldsymbol{u}) - \partial\eta_{\mathrm{side}}(\boldsymbol{u}).$$

Note that $\partial\eta_{\mathrm{old}}(\boldsymbol{u})$ and $\partial\eta_{\mathrm{side}}(\boldsymbol{u})$ contain all the submodular symbols of $\boldsymbol{u}$ that are nonunimodular.

Theorem 4.15. *The cycle ξ' constructed in Algorithm 4.13 is homologous to ξ. If $\boldsymbol{u} \in \operatorname{supp} \xi$ and $\boldsymbol{v} \in \operatorname{supp} \partial\boldsymbol{u}$ with $\|\boldsymbol{v}\| > 1$, then $\boldsymbol{v}$ does not appear as submodular symbol of ξ' in the following sense*:

$$\xi' = \sum_{\boldsymbol{u} \in \operatorname{supp} \xi} n'(\boldsymbol{u}) \partial\eta_{\mathrm{new}}(\boldsymbol{u}).$$

Proof. It is clear that ξ' is homologous to ξ mod Γ. To see the rest of the statement, first note that we have chosen orientations so that

$$\xi + \partial\eta = \sum_{\boldsymbol{u} \in \operatorname{supp} \xi} n'(\boldsymbol{u})(\partial\eta_{\mathrm{side}}(\boldsymbol{u}) + \partial\eta_{\mathrm{new}}(\boldsymbol{u})).$$

Hence we must show

$$\partial\eta_{\mathrm{side}}(\boldsymbol{u}) = 0 \bmod \Gamma.$$

We claim this follows since the candidates we chosen Γ-equivariantly over all of $\operatorname{supp}(\xi)$. Indeed, any 1-sharbly in $\operatorname{supp}(\partial\eta_{\mathrm{side}}(\boldsymbol{u}))$ is built from a certain candidate and a 0-sharbly in $\operatorname{supp}(\partial\xi)$. An investigation of the orientations we chose in the construction of $\partial\eta$ and the fact that $\partial_\Gamma(\xi) = 0$ show that $\partial\eta_{\mathrm{side}}(\boldsymbol{u}) = 0 \bmod \Gamma$. □

4.16. To conclude this section we discuss conditions under which we expect $\|\xi'\| < \|\xi\|$. First we clarify Remark 4.8.

Let $\boldsymbol{v}$ be a modular symbol, and let $w \in \operatorname{cand} \boldsymbol{v}$. Let $\{\boldsymbol{v}_i\}$ be the modular symbols from (3–1) constructed using $\boldsymbol{v}$ and w. Define an integer $\mu(w)$ by

$$\mu(w) = \mathrm{Max}_{i=1,\dots,n} \{\|\boldsymbol{v}_i\|\}.$$

Definition 4.17. Let $S \subset \operatorname{cand} \boldsymbol{v}$. Then $w \in S$ is a *good candidate from S* if

$$\mu(w) = \mathrm{Min}_{w' \in S} \{\mu(w')\}.$$

Furthermore, we say that w is a good candidate chosen using Conjecture 3.5 (respectively Conjecture 3.9) if w is a good candidate for the (conjecturally nonempty) intersection indicated in Conjecture 3.5 (respectively Conjecture 3.9).

Good candidates are not necessarily unique.

Conjecture 4.18. *Suppose $n \leq 4$, and let ξ and ξ' be as in Algorithm 4.13. Assume that $\|\xi\| > 1$. Then if each $w(\boldsymbol{v})$ from step A of Algorithm 4.13 is a good candidate chosen using Conjecture 3.5 or 3.9, then $\|\xi'\| < \|\xi\|$.*

5. EXPERIMENTS

We conclude by describing experiments that we performed to test Conjectures 3.5, 3.9, and 4.18. These experiments were performed at MIT and Columbia at various times from 1995 to 1998, on Sun (SunOS) and Intel (Linux) workstations. We are grateful to these departments for making this equipment and support available.

Before we describe the experiments, we remark that all trials completed successfully, and no counterexamples to the conjectures were found.

5.1. The first experiments we performed addressed Conjectures 3.5 and 3.9. Because of implementation difficulties mentioned immediately after Remark 3.6, we were only able to test Conjecture 3.5 in dimensions ≤ 4. However, we were able to test Conjecture 3.9 in dimensions ≤ 40, thanks to *LLL*-reduction code available in GP-Pari and LiDIA.

- We began by testing finding candidates for random modular symbols for $\mathrm{SL}_n(\mathbb{Z})$. A random square integral matrix m was constructed with entries chosen some fixed range. If $\det m \neq 0$, then we attempted to find a candidate for the modular symbol formed from the columns of m. We tried to test matrices with small determinant, since for these modular symbols the set of candidates is small.

1. For $n = 4$ we verified Conjecture 3.5 on approximately 20000 matrices.
2. For $2 \leq n \leq 20$, we verified Conjecture 3.9 on approximately 20000 matrices from each dimension, and for $21 \leq n \leq 40$ we tested Conjecture 3.9 on approximately 1000 matrices from each dimension. In these tests we rejected those matrices

whose determinants were outside the range specified by Proposition 3.15.

• Instead of random modular symbols, we tested coset representatives of the double cosets in (2–2) for different dimensions and values of p and k. We used the standard coset representatives found in [Krieg 1990].

1. For $T_p(1,3)$, $T_p(2,3)$, $T_p(1,4)$, and $T_p(3,4)$, we tested all primes $p \leq 97$ using both conjectures (again discarding those outside the range of Proposition 3.15).
2. For $T_p(2,4)$, we tested all primes $p \leq 67$ using both conjectures.
3. For dimensions $5 \leq n \leq 10$, we verified Conjecture 3.9 on representatives of $T_p(1,n)$ for $p = 2, 3$.

• Finally, we performed complete reduction of random modular symbols. In the previous experiments, we only verified that a candidate for a given modular symbol could be found using our conjectures. In this case, we stored the resulting modular symbols on a stack and iterated the process until all modular symbols were unimodular. Due to the large number of modular symbols produced, we limited our tests of Conjecture 3.9 to dimensions ≤ 10, and tested only medium-sized determinants, typically with absolute value less than 20. We verified Conjecture 3.5 on approximately 2000 modular symbols and Conjecture 3.9 on approximately 1000 modular symbols from each dimension.

5.2. To test Conjecture 4.18, we wanted to mimic the experiments in Section 5.1. This cannot be done naively for the following reason. A single modular symbol is automatically a cycle mod Γ, but for a 1-sharbly chain ξ to be a cycle mod Γ, nontrivial conditions must be met. Furthermore, Algorithm 4.13 uses these conditions in an essential way to decrease $\|\xi\|$.

This dilemma has two resolutions. Either we must test Conjecture 4.18 on cycles for specific groups $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$, or we must design an implementation of Algorithm 4.13 that is "local," i.e. operates on a single 1-sharbly at a time. The first solution is not feasible if one wishes to test many 1-sharbly cycles, because such cycles are very difficult to construct. Hence we must take the second approach.

Definition 5.3. Let $\boldsymbol{u}$ be a basis element of S_k. Then a *lift* for $\boldsymbol{u}$ is an $n \times (n+k)$ integral matrix M with primitive columns such that $[M_1, \ldots, M_{n+k}] = \boldsymbol{u}$, where M_i is the ith column of M.

Let ξ be a k-sharbly cycle mod Γ. We claim that ξ may be encoded as a finite collection of 4-tuples $(\boldsymbol{u}, n(\boldsymbol{u}), \{\boldsymbol{v}\}, \{M(\boldsymbol{v})\})$, where:

1. $\boldsymbol{u} \in \operatorname{supp} \xi$.
2. $n(\boldsymbol{u}) \in \mathbb{Z}$.
3. $\{\boldsymbol{v}\} = \operatorname{supp} \partial \boldsymbol{u}$.
4. $\{M(\boldsymbol{v})\}$ is a set of lifts for $\{\boldsymbol{v}\}$. These lifts are chosen so they satisfy the following Γ-equivariance condition. Suppose that for $\boldsymbol{u}, \boldsymbol{u}' \in \operatorname{supp} \xi$ we have $\boldsymbol{v} \in \operatorname{supp}(\partial \boldsymbol{u})$ and $\boldsymbol{v}' \in \operatorname{supp}(\partial \boldsymbol{u}')$ satisfying $\boldsymbol{v} = \gamma \cdot \boldsymbol{v}'$ for some $\gamma \in \Gamma$. Then we require $M(\boldsymbol{v}) = \gamma M(\boldsymbol{v}')$.

Clearly any cycle can be represented by such data, although the representation is far from unique.

5.4. Let $\psi = (\boldsymbol{u}, n(\boldsymbol{u}), \{\boldsymbol{v}\}, \{M(\boldsymbol{v})\})$ be a 4-tuple that is part of a cycle ξ. We claim that we can choose candidates for $\{\boldsymbol{v}\}$ that will the equivariance condition in Section 4.3 without knowing the rest of ξ.

To see this, recall that a square matrix $M = (M_{ij})$ with $\det M \neq 0$ is in *Hermite normal form* if $M_{ij} = 0$ for $i < j$, and $0 \leq M_{ij} < M_{ii}$ for $i > j$. Furthermore, if $\det M > 0$, then $M_{ii} > 0$. It is standard that for any M, the orbit $\mathrm{GL}_n(\mathbb{Z}) \cdot M$ contains only one element in Hermite normal form [Cohen 1993, 2.4.2].

Now to choose a candidate w for $\boldsymbol{v} \in \operatorname{supp}(\partial \boldsymbol{u})$, we compute the Hermite normal form $M_0(\boldsymbol{v})$ of $M(\boldsymbol{v})$ first, and input $M_0(\boldsymbol{v})$ into Conjecture 3.5 or 3.9 to compute w. If $M(\boldsymbol{v}) = \gamma M(\boldsymbol{v}')$, then $M_0(\boldsymbol{v}) = M_0(\boldsymbol{v}')$. Hence by using lifts we guarantee that candidate selection is Γ-equivariant, even though the choices are made locally.

5.5. This means that we can think of a random 4-tuple ψ as being a piece of some unknown cycle ξ mod Γ, and can test Algorithm 4.13 by trying to write ψ as a collection of reduced 4-tuples. To complete the discussion, we must say how lifts are chosen for the submodular symbols of $\partial \eta(\boldsymbol{u})$ that survive to ξ'.

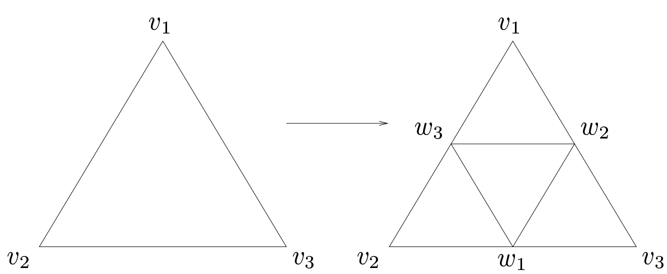
Definition 5.6. Let $\boldsymbol{u} = [v_1, \ldots, v_{n+1}]$ be a 1-sharbly, and let

$$\boldsymbol{v}^i = [v_1, \ldots, \hat{v}_i, \ldots, v_{n+1}], \quad \text{for } 1 \le i \le n+1$$

be the submodular symbols in $\operatorname{supp} \partial \boldsymbol{u}$. Suppose that $\|\boldsymbol{v}^i\| > 1$ for $1 \le i \le j \le n+1$, and let $W = \{w_i \mid 1 \le i \le j\}$ be the set of candidates. Let U be the set $\{v_1, \ldots, v_{n+1}\} \cup W$. Let $\boldsymbol{v} = [u_1, \ldots, u_n]$ be a modular symbol with $u_i \in U$.

1. The modular symbol $\boldsymbol{v}$ is called an *outer submodular symbol of* $\boldsymbol{u}$ if exactly one $u_i \in W$.
2. The modular symbol $\boldsymbol{v}$ is called an *inner submodular symbol of* $\boldsymbol{u}$ if two or more $u_i \in W$.

Here is the meaning behind Definition 5.6. For convenience suppose $n = 2$ and $j = 3$, and consider what happens when we apply the algorithm to $\boldsymbol{u}$. We can think of $\boldsymbol{u}$ as being a triangle with vertices labelled by v_1, v_2, and v_3. With this picture, to apply (4–3), we can think of subdividing the triangle into four new triangles, with the new vertices labelled by the candidates W:



Now we discuss the relevance of inner and outer to our implementation. For an inner submodular symbol $\boldsymbol{v}$, we can choose any lift we like, as long as we choose the *same* lift for any other 1-sharbly in (4–3) containing $\boldsymbol{v}$. If $\boldsymbol{v}$ is an outer submodular symbol, however, we must be more careful. In particular, consider the preceding figure. The lift $M([v_1, v_2])$ was chosen using the Γ-action, and we must choose $M([v_1, w_3])$ and $M([v_2, w_3])$ to reflect this.

In practice, we can do the following. If $\boldsymbol{v} \in Z(\boldsymbol{u})$, then each outer submodular symbol $\boldsymbol{v}_i$ arising from $\boldsymbol{v}$ is obtained by replacing the ith primitive point of $\boldsymbol{v}$ with w. We construct $M(\boldsymbol{v}_i)$ by replacing the corresponding column of $M(\boldsymbol{v})$ with w, and say that the lifts $\{M(\boldsymbol{v}_i)\}$ are *inherited*.

Remark 5.7. One might think that we could avoid computing Hermite normal forms, by just applying Conjecture 3.5 or 3.9 directly. But this will not necessarily determine a unique representative of the orbit $\mathrm{GL}_n(\mathbb{Z}) \cdot M(\boldsymbol{v})$, since this orbit may not uniquely meet the Voronoi and LLL reduction domains.

5.8. Now we describe the tests we performed to investigate Conjecture 4.18.

- We generated random 1-sharblies ξ with randomly chosen lifts. Using both modular symbol conjectures we constructed candidates for ξ and verified that $\|\xi'\| < \|\xi\|$. Because we only investigated dimensions 2, 3, and 4, we were able to test many ξ, approximately 10000 per trial for 50 trials.
- We also tested all Hecke images within certain ranges associated to certain "standard" reduced 1-sharblies. It is easy to see that mod $\mathrm{SL}_n(\mathbb{Z})$, any reduced 1-sharbly has the form

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & \varepsilon_1 \\ 0 & 1 & \ldots & 0 & \varepsilon_2 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & \varepsilon_n \end{pmatrix}, \tag{5–1}$$

where the number of columns is $(n+1)$, and the last column is

$$(\varepsilon_1, \ldots, \varepsilon_n) = (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0)$$

where $k = 2, \ldots, n$.

Using these 1-sharblies and randomly chosen lifts, we tested all Hecke images that lay within the following ranges:

1. For $T_p(1,3)$, $T_p(2,3)$, $T_p(1,4)$, and $T_p(3,4)$, we tested all primes $p \le 97$ using Conjectures 3.5 and 3.9.
2. For $T_p(2,4)$, we tested all primes $p \le 67$ using Conjectures 3.5 and 3.9.

We repeated this experiment 10 times to vary the lifts used.

- We tested complete reduction of randomly chosen 1-sharblies with lifts. At each step, the new 1-sharblies inherited lifts as described in Section 5.4. This introduces the possibility that for some initial choice of lifts, iteration of the algorithm could fail to terminate. However, this situation never arose. In 50 trials with approximately 10000 randomly chosen 1-sharblies, the complete reduction always terminated successfully.

• After testing with random data, we computed the Hecke action on cuspidal cycles occurring in $H^2(\Gamma_0(53);\mathbb{Q})$, where $\Gamma_0(53) \subset \mathrm{GL}_3(\mathbb{Z})$ is the subgroup of matrices with bottom row equivalent to $(0,0,*)$ mod 53. These cycles, or rather their Lefschetz duals, were first discovered and investigated in [Ash et al. 1984].

We computed the characteristic polynomials of the Hecke operators $T_p(1,3)$ for $p \leq 13$. We found that these polynomials matched those in [Ash et al. 1984], which is consistent with the duality argument of [Ash and Tiep 1997, Theorem 3.1].

• Finally, in current work we are using the algorithm to compute the Hecke action on $H^5(\Gamma_0(N);\mathbb{Q})$, where $\Gamma_0(N) \subset \mathrm{SL}_4(\mathbb{Z})$ is the subgroup of matrices with bottom row congruent to $(0,0,0,*)$ mod N [Ash et al. 2000]. At the time of this writing, we have completed computations for prime levels $N \leq 31$. We have computed the characteristic polynomials for the Hecke operators $T_p(k,4)$ for $1 \leq k \leq 3$ and a range of p. In all cases the program wrote the Hecke image of a 1-sharbly cycle as sum of reduced 1-sharbly cycles.

For these GL_3 and SL_4 tests, I was helped and encouraged enormously by Mark McConnell, who provided data for the cycles generated by his program SHEAFHOM [McConnell 1998], and computed the characteristic polynomials.

5.9. We conclude with some remarks and open problems.

• In general, if one wishes to implement the modular symbol algorithm, Conjecture 3.9 is much more efficient to work with than Conjecture 3.5. Voronoi reduction is somewhat difficult to program and requires a substantial amount of preliminary computation. On the other hand, high-quality computer code for *LLL*-reduction is available from a variety of sources.

• Algorithm 4.13 can be adapted to work on sharbly cycles $\xi \in S_{n+k}$ with $k > 1$. In particular, we can describe the analogues of the polytopes $P(n,j)$ used in the construction of ξ': their facets involve iterated cones on *hypersimplices* [Ziegler 1995, Example 0.11]. In practice this is not useful for computing Hecke eigenvalues, since we cannot expect in general that $\|\xi'\| < \|\xi\|$.

• Throughout the description of Algorithm 4.13, we used the determinant as a measure of "nonunimodularity" of a 1-sharbly. Ultimately this approach suffers from several shortcomings:

- For $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$ with $n \geq 4$, we must use a nonreduced sharbly cycle to write a nontrivial element of $H^0(\Gamma;\mathbb{Z})$.
- One would like to compute Hecke eigenvalues in $H^*(\Gamma;\mathbb{Z})$ for more exotic Γ. For example, especially interesting is $\Gamma \subset \mathrm{SL}_n(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers in a number field $K/\mathbb{Q}$. If $\mathcal{O}_K$ is not a euclidean domain, then there is no obvious notion of a *primitive* vector. One can still define the analogue of the sharbly complex, and can use the determinant to define a Γ-finite subset of sharblies [Gunnells 2000a], but a practical modular symbol algorithm is unknown in general.

A different approach is to use the relative position of a sharbly with respect to Π instead of the determinant. This is carried out in [Gunnells 1999] and [Gunnells and McConnell 1999] for all arithmetic groups for which Π is available. It would be nice to fuse the approach of these articles and the approach described here.

• If Γ is not torsion-free, then our results hold if we use cohomology with rational coefficients. However, one can also consider the equivariant cohomology $H^*_\Gamma(\Gamma;\mathbb{Z})$, and can formulate conjectures about the arithmetic significance of equivariant torsion classes [Ash 1992b]. Can Algorithm 4.13 be modified to compute the Hecke action on $H^{\nu-1}_\Gamma(\Gamma;\mathbb{Z})$?

• The modular symbol algorithm can be generalized to Sp_{2n} [Gunnells 2000b], and there is a cell complex that can be used to compute $H_*(\Gamma)$, where $\Gamma \subset \mathrm{Sp}_4(\mathbb{Z})$ [MacPherson and McConnell 1993]. Is there a "symplectic" sharbly complex, and can an algorithm be devised to compute Hecke eigenvalues on $H^{\nu-1}(\Gamma)$?

6. ACKNOWLEDGEMENTS

We thank Romuald Dabrowski and Lee Rudolph for helpful discussions. We are especially grateful to Avner Ash and Mark McConnell for much enthusiastic help and advice during this project. We thank the referee, whose many comments vastly improved the exposition. Finally, we thank Bob MacPherson for originally suggesting the problem and for much encouragement and support.

The experiments performed in this paper to develop Algorithm 4.13 and to test Conjectures 3.5, 3.9, and 4.18 were implemented using several software packages: GP-Pari [Batut et al. 1998], LiDIA [LiDIA n.d.], Mathematica, and SHEAFHOM [McConnell 1998].

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Received December 1, 1998; accepted in revised form October 10, 1999