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LIMITING BEHAVIOR OF THE STRONGLY DAMPED EXTENSIBLE BEAM EQUATION

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(Submitted by: Glenn Webb)

Abstract. Energy estimates are coupled with semigroup theory and the theory of cosine and sine operators to establish the convergence of solutions to the strongly damped extensible beam equation to solutions of the extensible beam equation.

1. Introduction. This paper addresses the question of the convergence of regularizing solutions to the solution of the extensible beam equation. The extensible beam equation was proposed by Wionovsky-Greiger [19] to describe the transverse deflection of a beam whose ends are held a fixed distance apart. It has the form

$$\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial^4 u}{\partial x^4} - \left(\beta + \kappa \int_0^1 (\partial u(\xi,t)/\partial\xi)^2\, d\xi\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{BE}$$

for $x \in [0,1]$, $t \geq 0$ with boundary conditions

$$u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0 \tag{1.1}$$

and initial data

$$u(x,0) = \phi(x), \quad x \in [0,1], \tag{1.2a}$$

$$u_t(x,0) = \psi(x), \quad x \in [0,1]. \tag{1.2b}$$

The boundary conditions correspond to the physical situation of hinged ends.

The strongly damped beam equation as described by Ball [2–4] has the form

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial^4 u}{\partial x^4} - \left(\beta + \kappa \int_0^1 (\partial u(\xi,t)/\partial\xi)^2\, d\xi\right)\frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial^5 u}{\partial x^4\partial t}\\ - \sigma\left(\int_0^1 (\partial u/\partial\xi\,\partial^2 u/\partial\xi\partial t)d\xi\right)\frac{\partial^2 u}{\partial x^2} + \delta\frac{\partial u}{\partial t} = 0,\end{aligned} \tag{DBE}$$

for $x \in [0,1]$, $t \geq 0$, and has the same boundary and initial data as BE.

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In both cases we require that α and κ be positive while the sign of β is unrestricted. The terms having nonnegative coefficients γ, σ and δ represent the mechanical effects of damping. We lose no generality and simplify matters by assuming $\alpha = 1$. Finally, we assume the damping terms σ and δ are continuously differentiable functions of γ with bounded derivatives and that $\sigma(0) = \delta(0) = 0$. Consequently, there exits $\mu > 0$ so that

$$0 \leq \sigma(\gamma) \leq \mu\gamma \tag{1.3a}$$

and

$$0 \leq \delta(\gamma) \leq \mu\gamma. \tag{1.3b}$$

The last assertion is needed only when the question of convergence is considered.

We shall show the solutions of DBE converge to solutions of BE as $\gamma \to 0$. Avrin [1] considered a similar question concerning the convergence of solutions to the strongly damped nonlinear Klein-Gordon equation. The principal tools of our analysis shall be semigroup theory, energy type estimates and the theory of cosine and sine operators. Well posedness for both BE and DBE has been given by the author [7], [8]; a more abstract approach is provided by Lightbourne and Rankin [11]. Of related interest is the work of Stahel [13] which uses sine and cosine operators to represent solutions to the dynamic von Karman equation.

We point out that the integral convergence argument occurring herein was used by Avrin [1] and a very similar one appears in Strauss [21]. Additionally Biler [20] handles the case of damping terms going to zero in a similar situation. The reason why Avrin's paper [22] does not include the case considered here is that the term

$$\left(\int_0^1 (\partial u(\xi,t)/\partial\xi)^2 \, d\xi\right) \frac{\partial^2 u}{\partial x^2}$$

must also be estimated. These estimates are provided in §3. An alternative approach to ours would be to establish this estimate and then apply the abstract theory of [22] in conjunction with the results surveyed in §2.

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2. Abstract setting for BE and DBE. In this section, we cast BE and DBE in a Hilbert space. Here we basically survey results appearing in [7], [8] and the interested reader is referred thereto for verification of the assertions of this section.

If $X = L^2(0,1)$, we define $A : X \to X$ in the following manner:

$$Au = u'''' \tag{2.1a}$$

with

$$\begin{aligned} D(A) = \{u \in X \mid u', u'', u''' \text{ absolutely continuous}, u'''' \in X, \\ u(a) = u''(a) = 0 \text{ for } a = 0 \text{ or } a = 1\}. \end{aligned} \tag{2.1b}$$

It is well known that A so defined is a positive self adjoint operator on X with eigenvalues $\{\lambda_n = (n\pi)^4 \mid n \in \mathbb{Z}^+\}$ and corresponding eigenfunctions $\{z_n(s) = \sqrt{2}\sin n\pi s\}$. Furthermore, we have the explicit representation

$$Au = \sum_{n=1}^{\infty} \lambda_n \langle u, z_n\rangle z_n, \tag{2.2}$$

where $\langle\, ,\, \rangle$ denotes the standard L_2 inner product.

The spectral calculus permits the straightforward computation of fractional powers of A. we have

$$A^v u = \sum_{n=1}^{\infty} (n\pi^4)^v \langle u, z_n \rangle z_n \tag{2.3}$$

for any real v. Moreover,

$$\int_0^1 |u'|^2 dx = \sum_{n=1}^{\infty} (n\pi)^4 \langle u, z_n \rangle^2 = \|A^{\frac{1}{4}} u\|^2. \tag{2.4}$$

The boundary conditions of BE and DBE are incorporated into the specification of the domain of A. It is not difficult to see that BE may be realized as the abstract Cauchy initial value problem

$$\ddot{x}(t) + Ax(t) + \left(\beta + \kappa \|A^{\frac{1}{4}} x(t)\|^2\right) A^{\frac{1}{2}} x(t) = 0 \tag{2.5a}$$

$$x(0) = \phi \tag{2.5b}$$

$$\dot{x}(0) = \psi \tag{2.5c}$$

and that DBE has abstract interpretation

$$\ddot{x}_\gamma(t) + \gamma A \dot{x}_\gamma(t) + A x_\gamma(t) + \left(\beta + \kappa \|A^{\frac{1}{4}} x_\gamma(t)\|^2\right) A^{\frac{1}{2}} x_\gamma(t)$$
$$+ \frac{\sigma}{2} \frac{d}{dt} \left(\|A^{\frac{1}{4}} x_\gamma(t)\|^2\right) A^{\frac{1}{2}} x_\gamma(t) + \delta \dot{x}_\gamma(t) = 0 \tag{2.6a}$$

$$x_\gamma(0) = \phi \tag{2.6b}$$

$$\dot{x}_\gamma(0) = \psi. \tag{2.6c}$$

We proceed to write (2.6 a–c) as a first order system. We make $D(A)$ into a Banach space X_A by imposing the Euclidean graph norm

$$\|\phi\|_A = \left(\|A\phi\|^2 + \|\phi\|^2\right)^{\frac{1}{2}}, \tag{2.7}$$

and introduce a new Hilbert space $\hat{X}$ by letting

$$\hat{X} = X_A \times X \tag{2.8a}$$

with

$$\|[\phi, \psi]\| = \left(\|\phi\|_A^2 + \|\psi\|^2\right)^{\frac{1}{2}}. \tag{2.8b}$$

We define $\hat{A}_\gamma : \hat{X} \to \hat{X}$ by

$$\hat{A}_\gamma[\phi, \psi] = [-\psi, A\phi + \gamma A \psi] \tag{2.9a}$$

for

$$[\phi, \psi] \in D(\hat{A}_\gamma) = D(A) \times D(A). \tag{2.9b}$$

In [8] we show that $-\hat{A}_\gamma$ is the infinitesimal generator of an analytic semigroup $\{\hat{T}_\gamma(t) \mid t \geq 0\}$ on $\hat{X}$. We define a nonlinear function $\hat{F}(\cdot)$ on $D(A^{\frac{1}{2}}) \times D(A^{\frac{1}{2}})$ by

$$\hat{F}_\gamma([\phi,\psi]) = \left[0, \left\{ \left(\beta + \kappa \|A^{\frac{1}{4}}\phi\|^2\right) A^{\frac{1}{2}}\phi + \langle A^{\frac{1}{4}}\phi, A^{\frac{1}{4}}\psi\rangle A^{\frac{1}{2}}\phi + \delta\psi \right\}\right]. \tag{2.10}$$

Formally, it is not difficult to see that the solution to (2.6 a–c) may be read off from the second component of the abstract Cauchy problem

$$\frac{d\hat{u}_\gamma}{dt} + \hat{A}_\gamma \hat{u}_\gamma + \hat{F}_\gamma(u_\gamma) = 0 \tag{2.11a}$$

$$\hat{u}_\gamma(0) = [\phi,\psi] \in D(A_\gamma). \tag{2.11b}$$

The following theorem is established in [8].

Theorem 2.12. *If $T > 0$ and $[\phi,\psi] \in D(\hat{A}_\gamma)$, there exists a unique $\hat{u}_\gamma : [0,T] \to D(\hat{A}_\gamma)$ which satisfies*

$$\hat{u}_\gamma(t) = \hat{T}_\gamma(t)\hat{u}_\gamma(0) - \int_0^T \hat{T}_\gamma(t-s)\hat{F}_\gamma(\hat{u}_\gamma(s))\,ds.$$

Moreover, $\hat{u}(\cdot)$ is continuously differentiable on $[0,1]$ *and satisfies* (2.11a–b). *Finally, if π_1 projects $\hat{X}$ onto its first component then $x_\gamma(t) = \pi_1(\hat{u}_\gamma(t))$ is the unique solution of* (2.6a–c).

Our convergence arguments will be facilitated by another representation of solutions to (2.6a–c) which involves cosine and sine operators. For

$$A^{\frac{1}{2}}u = \sum_{n=1}^{\infty} (n\pi)^2 \langle u, z_n\rangle z_n \tag{2.13}$$

we employ the spectral calculus to compute

$$\cos(tA^{\frac{1}{2}})u = \sum_{n=1}^{\infty} \cos((n\pi)^2 t)\langle u, z_n\rangle z_n \tag{2.14a}$$

$$\sin(tA^{\frac{1}{2}})u = \sum_{n=1}^{\infty} \sin((n\pi)^2 t)\langle u, z_n\rangle z_n \tag{2.14b}$$

for $u \in L_2(0,1)$ and $t \in \mathbb{R}$. It should be clear that $\{\cos(tA^{\frac{1}{2}}) \mid t \in \mathbb{R}\}$ and $\{\sin(tA^{\frac{1}{2}}) \mid t \in \mathbb{R}\}$ are families of bounded linear operators on $L_2(0,1)$. Cosine and sine operators satisfy the functional equations of the sine and cosine functions and are well known in the literature. For a thorough discussion of these subjects the reader is referred to Travis and Webb [14–17], Webb [18] and Goldstein [10]. It is not difficult to see that sine and cosine operators commute with powers of A. In particular, we have

$$A^{-\frac{1}{2}}\sin(tA^{\frac{1}{2}})u = \sin(tA^{\frac{1}{2}})A^{-\frac{1}{2}}u. \tag{2.15}$$

The following result which may be adapted from [15, Prop. 3.1] provides another variation of parameter formula for solutions to DBE.

Proposition 2.16. *Let $T > 0$ and $[\phi, \psi] \in D(A_\gamma)$. If $x_\gamma(\cdot) : [0,T] \to D(A_\gamma)$ is the unique solution to* (2.6a–c) *guaranteed by Theorem* 2.12, *then*

$$\begin{aligned} x_\gamma(t) &= \cos(tA^{\frac{1}{2}})\phi + A^{-\frac{1}{2}} \sin(tA^{\frac{1}{2}})\psi \\ &\quad - \int_0^t A^{-\frac{1}{2}} \sin\Big((t-s)A^{\frac{1}{2}}\{\gamma A\dot{x}(s) + \pi_2(\hat{F}(x(s), \dot{x}_\gamma(s))\}\Big)\, ds, \end{aligned} \tag{2.17}$$

where π_2 projects $\hat{X}$ onto its second component.

We now turn our attention to BE. We define an operator $\hat{A} : \hat{X} \to \hat{X}$ via

$$\hat{A}[\phi, \psi] = [-\psi, A\phi] \tag{2.18a}$$

with

$$D(\hat{A}) = D(A) \times D(A^{\frac{1}{2}}). \tag{2.18b}$$

It is known ([7]) that $-\hat{A}$ so defined is the infinitesimal generator of a strongly continuous group $\{\hat{T}(t) \mid t \in \mathbb{R}\}$ of linear transformations on $\hat{X}$. We define a nonlinear operator on $D(A^{\frac{1}{2}}) \times X$ by

$$F(\phi, \psi) = \left[0,\ \left(\beta + \kappa\|A^{\frac{1}{4}}\phi\|^2 A^{\frac{1}{2}}\phi\right)\right]. \tag{2.19}$$

Solutions to BE may be obtained from the second component of the abstract Cauchy initial value problem

$$\frac{d\hat{u}}{dt} + \hat{A}\hat{u} + \hat{F}(\hat{u}) = 0 \tag{2.20a}$$

$$\hat{u}(0) = [\phi, \psi] \in D(\hat{A}). \tag{2.20b}$$

The following appears in [7].

Theorem 2.21. *If $T > 0$ and $[\phi, \psi] \in D(\hat{A})$, there exists a unique $\hat{u}(\cdot) : [0,T] \to D(\hat{A})$ which satisfies*

$$\hat{u}(t) = \hat{T}(t)\hat{u}(0) - \int_0^t \hat{T}(t-s)\hat{F}(\hat{u}(s))\, ds.$$

Moreover, $\hat{u}(\cdot)$ is continuously differentiable on $[0,T]$ and satisfies (2.20a–b). *If $x(t) = \pi_1(\hat{u}(t))$, then $x(t)$ satisfies* (2.5a–c). *Finally, $x(t) = \pi_1(\hat{u}(t))$ has the variation of parameters representation*

$$\begin{aligned} x(t) &= \cos(tA^{\frac{1}{2}})\phi + A^{-\frac{1}{2}} \sin(tA^{\frac{1}{2}})\psi \\ &\quad - \int_0^t A^{-\frac{1}{2}} \sin((t-s)A^{\frac{1}{2}})\{\pi_2(\hat{F}(x(s), \dot{x}(s))\}ds, \end{aligned} \tag{2.22}$$

where π_2 projects $\hat{X}$ onto its second component.

We also have the following uniqueness assertion which follows from a straightforward Gronwall argument.

Proposition 2.23. *Let $[\phi, \psi] \in D(\hat{A})$ and $T > 0$. If $y(\cdot) : [0,T] \to X_{A^{\frac{1}{2}}} \times X$ is continuous and satisfies*

$$y(t) = \cos(tA^{\frac{1}{2}})\phi + A^{-\frac{1}{2}} \sin(tA^{\frac{1}{2}})\psi - \int_0^t A^{-\frac{1}{2}} \sin((t-s)A^{\frac{1}{2}})\{\pi_2(\hat{F}(x(s), \dot{x}(s))\}ds,$$

then $y(t) = x(t)$ where $x(t) = \pi_1(\hat{u}(t))$ and $\hat{u}(t)$ is the solution of Proposition 2.12.

3. Convergence result. We begin with a lemma that provides requisite a priori bounds for our convergence theory.

Proposition 3.1. *Let $[\phi, \psi] \in D(\hat{A}_\gamma)$ and $T > 0$. If $x_\gamma(\cdot)$ is the solution to* (2.6 a–c), *then there exists $M = M(\phi, \psi, T)$ which is independent of γ so that*

$$\|A^r x_\gamma(t)\| \leq M \quad \text{for } t \in [0,T], \quad r = \tfrac{1}{4},\ \tfrac{1}{2},\ \tfrac{3}{4},\ 1 \tag{3.2a}$$

$$\|A^s \dot{x}_\gamma(t)\| \leq M \quad \text{for } t \in [0,T], \quad s = \tfrac{1}{4},\ \tfrac{1}{2},\ 0. \tag{3.2b}$$

Proof: Our arguments will be abstract realizations of standard energy arguments. We compute the L_2 inner product of (2.6a) and $\dot{x}(t)$ to obtain

$$\begin{aligned}
&\langle \ddot{x}_\gamma(t), \dot{x}_\gamma(t)\rangle + \langle Ax_\gamma(t), \dot{x}_\gamma(t)\rangle + \gamma\langle A\dot{x}_\gamma(t), \dot{x}_\gamma(t)\rangle \\
&\quad + \left(\beta + \kappa\|A^{\frac{1}{4}}x_\gamma(t)\|^2\right)\langle A^{\frac{1}{2}}x_\gamma(t), \dot{x}_\gamma(t)\rangle \\
&\quad + \sigma\langle A^{\frac{1}{4}}x_\gamma(t), A^{\frac{1}{4}}\dot{x}_\gamma(t)\rangle\langle A^{\frac{1}{2}}x_\gamma(t), \dot{x}_\gamma(t)\rangle + \delta\|\dot{x}_\gamma(t)\|^2 \\
&= \frac{1}{2}\frac{d}{dt}(\|\dot{x}_\gamma(t)\|^2) + \frac{1}{2}\frac{d}{dt}(\|A^{\frac{1}{2}}x_\gamma(t)\|^2) + \gamma\langle A^{\frac{1}{2}}\dot{x}_\gamma(t), A^{\frac{1}{2}}\dot{x}_\gamma(t)\rangle \\
&\quad + \frac{\beta}{2}\frac{d}{dt}(\|A^{\frac{1}{4}}x_\gamma(t)\|^2) + \frac{\kappa}{4}\frac{d}{dt}(\|A^{\frac{1}{4}}x_\gamma(t)\|^4) + \frac{\sigma}{4}(\frac{d}{dt}\|A^{\frac{1}{4}}x_\gamma(t)\|^2)^2 + \delta\|\dot{x}_\gamma(t)\|^2 \\
&= 0.
\end{aligned} \tag{3.3}$$

We integrate (3.3) to get

$$\begin{aligned}
&\frac{1}{2}\|\dot{x}_\gamma(t)\|^2 + \frac{1}{2}\|A^{\frac{1}{2}}x_\gamma(t)\|^2 + \gamma\int_0^t \|A^{\frac{1}{2}}\dot{x}_\gamma(s)\|^2 ds + \frac{\beta}{2}\|A^{\frac{1}{4}}x_\gamma(t)\|^2 + \frac{\kappa}{4}\|A^{\frac{1}{4}}x_\gamma(t)\|^4 \\
&\quad \leq \frac{1}{2}\|\psi\|^2 + \|A^{\frac{1}{2}}\phi\|^2 + \frac{\beta}{2}\|A^{\frac{1}{4}}\phi\| + \frac{\kappa}{4}\|A^{\frac{1}{4}}\phi\|^4.
\end{aligned} \tag{3.4}$$

If $\beta < 0$, the quantities $\frac{\beta}{2}\|A^{\frac{1}{4}}x(t)\|^2 + \frac{\kappa}{4}\|A^{\frac{1}{4}}x(t)\|^4$ and $\frac{\beta}{2}\|A^{\frac{1}{4}}\phi\|^2 + \frac{\kappa}{4}\|A^{\frac{1}{4}}\phi\|^4$ may be negative. However, we observe that the function $y = \frac{b}{2}x^2 + \frac{\kappa}{4}x^4$ is bounded below and the bound depends only on β and κ. Thus (3.4) yields a priori bounds for $\|\dot{x}_\gamma(t)\|^2$, $\|A^{\frac{1}{2}}x_\gamma(t)\|^2$ and $\gamma\int_0^t \|A^{\frac{1}{2}}\dot{x}_\gamma(s)\|ds$. Because $A^{-\frac{1}{4}}$ is a bounded linear operator, we have bounds for $\|A^{\frac{1}{4}}x_\gamma(t)\|^2 = \|A^{-\frac{1}{4}}A^{\frac{1}{2}}x_\gamma(t)\|^2$. Elementary Hilbert space calculations allow us to bound $|\langle A^{\frac{1}{4}}x_\gamma(t), A^{\frac{1}{4}}\dot{x}_\gamma(t)\rangle| = |\frac{1}{2}\frac{d}{dt}(\|A^{\frac{1}{4}}x_\gamma(t)\|^2)|$ in terms of $\|\dot{x}_\gamma(t)\|^2$ and $\|A^{\frac{1}{2}}x_\gamma(t)\|$.

To continue with our analysis we multiply (2.6a) by $A^{\frac{1}{2}}\dot{x}_\gamma(t)$ and compute the L_2 inner product to observe that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|A^{\frac{1}{4}}\dot{x}_\gamma(t)\|^2)+\frac{1}{2}\frac{d}{dt}(\|A^{\frac{3}{4}}x_\gamma(t)\|^2)+\gamma\|A^{\frac{3}{4}}\dot{x}_\gamma(t)\|^2 \\ &+\left(\beta+\kappa\|A^{\frac{1}{4}}x_\gamma(t)\|^2\right)\langle A^{\frac{1}{2}}x_\gamma(t),A^{\frac{1}{2}}\dot{x}_\gamma(t)\rangle+\sigma\langle A^{\frac{1}{4}}x_\gamma(t),A^{\frac{1}{4}}\dot{x}_\gamma(t)\rangle \\ &\quad+\langle A^{\frac{1}{2}}x_\gamma(t),A^{\frac{1}{2}}\dot{x}_\gamma(t)\rangle+\delta\|A^{\frac{1}{4}}\dot{x}_\gamma(t)\|^2=0.\end{aligned} \tag{3.5}$$

We integrate to obtain

$$\begin{aligned}&\frac{1}{2}\|A^{\frac{1}{4}}\dot{x}_\gamma(t)\|^2+\frac{1}{2}\|A^{\frac{3}{4}}x_\gamma(t)\|^2+\gamma\int_0^t\|A^{\frac{3}{4}}\dot{x}_\gamma(s)\|^2ds \\ &\le\frac{1}{2}\|A^{\frac{1}{4}}\psi\|^2+\frac{1}{2}\|A^{\frac{3}{4}}\phi\|-\int_0^t\left(\beta+\kappa\|A^{\frac{1}{4}}x_\gamma(s)\|^2\right)\langle A^{\frac{1}{2}}x_\gamma(s),A^{\frac{1}{2}}\dot{x}_\gamma(s)\rangle\,ds \\ &\quad-\int_0^t\sigma\langle A^{\frac{1}{4}}x_\gamma(s),A^{\frac{1}{4}}\dot{x}_\gamma(s)\rangle\langle A^{\frac{1}{2}}x_\gamma(s),A^{\frac{1}{2}}\dot{x}_\gamma(s)\rangle\,ds.\end{aligned} \tag{3.5}$$

We shall estimate the two integrals on the right-hand side of (3.5). Integration by parts yields

$$\begin{aligned}&\left|\int_0^t\left(\beta+\kappa\|A^{\frac{1}{4}}x_\gamma(s)\|^2\right)\left\{\frac{1}{2}\frac{d}{ds}\|A^{\frac{1}{2}}x_\gamma(s)\|^2\right\}ds\right| \\ &\le\frac{1}{2}\left|\left(\beta+\kappa\|A^{\frac{1}{4}}x_\gamma(t)\|^2\right)\right|\|A^{\frac{1}{2}}x_\gamma(t)\|^2+\left|\beta+\kappa\|A^{\frac{1}{4}}\phi\|^2\right|\|A^{\frac{1}{2}}\phi\|^2 \\ &\quad+\int_0^t\kappa\|A^{\frac{1}{2}}x_\gamma(s)\|^2\left|\langle A^{\frac{1}{4}}x_\gamma(s),A^{\frac{1}{4}}\dot{x}_\gamma(s)\rangle\right|ds.\end{aligned} \tag{3.6}$$

The terms on the right-hand side of (3.6) are dominated by estimates of the previous paragraph. The second integral is estimated as follows

$$\begin{aligned}&\left|\int_0^t\sigma\langle A^{\frac{1}{4}}x_\gamma(s),A^{\frac{1}{4}}\dot{x}_\gamma(s)\rangle\langle A^{\frac{1}{2}}x_\gamma(s),A^{\frac{1}{2}}\dot{x}_\gamma(s)\rangle ds\right| \\ &\le\sup_{s\in[0,t]}\sigma\left\{\frac{1}{2}\frac{d}{ds}\left(A^{\frac{1}{4}}x_\gamma(s)\|^2\right)\right\}\frac{1}{2}\left\{\int_0^t\|Ax_\gamma(s)\|^2ds+\int_0^t\|A^{\frac{1}{2}}\dot{x}_\gamma(s)\|^2ds\right\}.\end{aligned} \tag{3.7}$$

Because $\sigma(\gamma)\le\mu_\gamma$, the term $\sigma\int_0^t\|A^{\frac{1}{2}}\dot{x}_\gamma(s)\|^2ds$ may be estimated independently of a particular choice of γ. Clearly the remaining terms may be estimated in terms which are independent of a particular choice of $\gamma\in[0,\gamma_{\max}]$ using the estimates from the previous calculation.

We now compute the L_2 inner product of (2.6a) with $A\dot{x}_\gamma(t)$ and integrate on $(0,t)$ to observe

$$\begin{aligned}&\frac{1}{2}\|A^{\frac{1}{2}}\dot{x}_\gamma(t)\|^2+\frac{1}{2}\|Ax_\gamma(t)\|^2+\gamma\int_0^t\|A\dot{x}_\gamma(s)\|^2ds \\ &\le\frac{1}{2}\|A^{\frac{1}{2}}\psi\|^2+\frac{1}{2}\|A\phi\|^2-\int_0^t\left(\beta+\kappa\|A^{\frac{1}{4}}x_\gamma(s)\|^2\right)\left(\langle A^{\frac{3}{4}}x_\gamma(s),A^{\frac{3}{4}}\dot{x}_\gamma(s)\rangle\right)ds \\ &\quad-\int_0^t\sigma\left(\langle A^{\frac{1}{4}}x_\gamma(s),A^{\frac{1}{4}}\dot{x}_\gamma(s)\rangle\right)\left(\langle A^{\frac{3}{4}}x_\gamma(s),A^{\frac{3}{4}}\dot{x}_\gamma(s)\rangle\right)ds.\end{aligned} \tag{3.8}$$

It is evident that the question of a priori bounds for the terms on the left-hand side of (3.8) will be a consequence of the existence of bounds for the two integral terms on the right. We see that

$$\begin{aligned}\left|\int_0^t \left(\beta+\kappa\|A^{\frac14}x_\gamma(s)\|^2\right)\left(\frac{d}{ds}\left(\frac12\|A^{\frac34}x_\gamma(s)\|^2\right)\right)ds\right| \qquad (3.9)\\ \le \frac12\left|\beta+\kappa\|A^{\frac14}x_\gamma(t)\|^2\right|\|A^{\frac34}x_\gamma(t)\|^2+\frac12\left|\beta+\kappa\|A^{\frac14}\phi\|\right|\|A^{\frac14}\phi\|^2\\ +\sup_{s\in[0,t]}\|A^{\frac34}x_\gamma(s)\|^2\int_0^t\kappa\left|\langle A^{\frac14}x_\gamma(s),A^{\frac14}\dot x_\gamma(s)\rangle\right|ds.\end{aligned}$$

Previous estimates provide bounds for the terms on the right-hand of (3.9). Finally, we consider the last integral term and observe that

$$\begin{aligned}\sigma\left|\int_0^t\langle A^{\frac14}x_\gamma(s),A^{\frac14}\dot x_\gamma(s)\rangle\langle A^{\frac34}x_\gamma(s),A^{\frac34}\dot x_\gamma(s)\rangle\,ds\right| \qquad (3.10)\\ \le\sigma\left(\sup_{s\in[0,t]}\left(\frac12\left(\frac{d}{ds}(\|A^{\frac14}x_\gamma(s)\|)\right)^2\right)\int_0^t\|A^{\frac34}x_\gamma(s)\|^2ds+\frac12\int_0^t\|A^{\frac34}\dot x_\gamma(s)\|^2ds\right).\end{aligned}$$

We now apply the a priori bounds obtained for (3.5) together with previous estimates to complete our proof.

We are now able to obtain our convergence result.

Theorem 3.11. *Let $[\phi,\psi]\in D(A)\times D(A^{\frac12})$, $T>0$ and $\{\gamma_n\}$ be a sequence of positive numbers converging to zero. Further assume that $\{[\phi_n,\psi_n]\}$ is contained in $D(A)\times D(A)$ and that $\lim_{n\to\infty}A\phi_n=A\phi$ and $\lim_{n\to\infty}A\psi_n=A\psi$. If $\{x_{\gamma_n}(\cdot)\}$ is the sequence of solutions to (2.6a–c) with initial conditions $x_{\gamma_n}(0)=\phi_n$ and $\dot x_{\gamma_n}(0)=\psi_n$, then*

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|A^{\frac12}x_\gamma(t)-A^{\frac12}x(t)\|=0. \qquad (3.13)$$

Proof: We point out that the a priori bounds of Proposition 3.1 depend only upon T and the initial data. Because $\gamma_n\to0$ and $A\phi_n\to A\phi$ and $A^{\frac12}\psi_n\to A^{\frac12}\psi$, we may assume that these bounds are independent of a particular choice of γ_n, ϕ_n, ψ_n and depend only upon $\|A^{\frac14}\phi\|$, $\|A^{\frac12}\phi\|$, $\|A^{\frac34}\phi\|$, $\|A\phi\|$, $\|\psi\|$, $\|A^{\frac14}\psi\|$, $\|A^{\frac12}\psi\|$ and $T>0$. It is not difficult to see that the operator

$$A^{-\frac12}u=\sum_{n=1}^{\infty}(n\pi)^{-2}\langle u,z_n\rangle z_n \qquad (3.13)$$

is compact on X. Therefore, the uniform boundedness of $\|Ax_{\gamma_n}(t)\|$ and the observation that $A^{\frac12}x_{\gamma_n}(t)=A^{-\frac12}Ax_{\gamma_n}(t)$ imply that the sequence $\{A^{\frac12}x_{\gamma_n}(t)\}$ is precompact for each $t\in[0,T]$. Because $\|\frac{d}{dt}(A^{\frac12}x_{\gamma_n}(t)\|=\|A^{\frac12}\dot x_{\gamma_n}(t)\|^2$ is uniformly bounded it is readily apparent that the sequence of functions $\{A^{\frac12}x_{\gamma_n}(\cdot)\}$ is equicontinuous. Therefore, we may apply the Arzela-Ascoli theorem to extract a subsequence $\{A^{\frac12}x_{\gamma_{n(k)}}(\cdot)\}$ and a function $z(\cdot)\in C([0,T],X)$ so that

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|A^{\frac12}x_{\gamma_{n(k)}}(t)-z(t)\|=0. \qquad (3.14)$$

Because $X_{A^{\frac{1}{2}}}$ is itself a Banach space we are guaranteed a $w(\cdot) \in C([0,T],X)$ so that $z(t) = A^{\frac{1}{2}}w(t)$.

For convenience we supress the subsequence notation and relabel the convergent subsequence as $\{A^{\frac{1}{2}}x_{\gamma_n}(\cdot)\}$. Because $A^{-\frac{1}{4}}$ is a bounded linear operator

$$\lim_{n\to\infty} \|A^{\frac{1}{4}}x_{\gamma_n}(t) - A^{\frac{1}{4}}w(t)\| = 0 \tag{3.15}$$

uniformly for $t \in [0,T]$. Therefore one may conclude

$$\lim_{n\to\infty} \left(\beta + \kappa\|A^{\frac{1}{4}}x_{\gamma_n}(t)\|^2\right) A^{\frac{1}{2}}x_{\gamma_n}(t) = \left(\beta + \kappa\|A^{\frac{1}{4}}w(t)\|^2\right) A^{\frac{1}{2}}w(t),$$

uniformly for $t \in [0,T]$. Additionally, our a priori bounds and (1.3a–b) dictate the uniform convergence for the expression

$$\sigma\langle A^{\frac{1}{4}}x_{\gamma_n}(t),\, A^{\frac{1}{4}}\dot{x}_{\gamma_n}(t)\rangle A^{\frac{1}{2}}x_{\gamma_n}(t) + \delta\dot{x}_{\gamma}(t)$$

to zero as $\gamma_n \to 0$. Finally, we see that for $0 \le s \le t \le T$

$$A^{-\frac{1}{2}}\sin\left((t-s)A^{\frac{1}{2}}\right)\left(\gamma_n A\dot{x}_{\gamma_n}(t)\right) = \sin\left((t-s)A^{\frac{1}{2}}\right)\left(\gamma_n A^{\frac{1}{2}}\dot{x}_{\gamma_n}(t)\right) \tag{3.17}$$

and that the a priori bound for $\|A^{\frac{1}{2}}\dot{x}_{\gamma}(t)\|$ guarantees the uniform convergence of (3.17) to zero. Thus we may compute the uniform limit in X to observe that $w(\cdot)$ satisfies

$$\begin{aligned} w(t) &= \cos(tA^{\frac{1}{2}})\phi + A^{-\frac{1}{2}}\sin(tA^{\frac{1}{2}})\psi \\ &\quad - \int_0^t A^{-\frac{1}{2}}\sin((t-s)A^{-\frac{1}{2}})\left\{\beta + \kappa\|A^{\frac{1}{4}}w(s)\|A^{\frac{1}{2}}w(s)\right\} ds. \end{aligned} \tag{3.16}$$

By virtue of the uniqueness of the variation of parameters formula we see that $w(t) = x(t)$ where $x(\cdot)$ is the solution guaranteed by Theorem 2.21. We have shown that every sequence contains a convergent subsequence and that the limit of each sequence is the same. Therefore we are guaranteed full sequential convergence and we have established our desired results.

Actually we have established a stronger type of convergence. If $v(t) \in D(A^{\frac{1}{2}})$ and $\sup_{t\in[0,T]} \|A^{\frac{1}{2}}v(t)\| < \infty$ then we observe that $\langle A^{\frac{1}{2}}v(t), v(t)\rangle = \|A^{\frac{1}{4}}v(t)\|^2$ is bounded and we may conclude the evaluation $v(t)(x) = v(t,x)$ is continuous in space via the one-dimensional Poincare lemma. We have the following corollary.

Corollary 3.17. *Assume the conditions of Theorem* 3.11 *hold. If* $u_{\gamma_n}(\,,\,)$ *and* $u(\,,\,)$ *are the realizations of* $x_\gamma(\cdot)$ *and* $x(\cdot)$ *respectively, then*

$$\lim_{n\to\infty} \sup_{y\in[0,1]} |u_{\gamma_n}(y,t) - u(y,t)| = 0$$

uniformly for $t \in [0,T]$.

Proof: The convergence of $A^{\frac{1}{2}}x_{\gamma_n}(\cdot)$ and $x_\gamma(\cdot)$ guarantees the convergence of $\langle A^{\frac{1}{4}}x_\gamma(\cdot), A^{\frac{1}{4}}x_\gamma(\cdot)\rangle$ to $\langle A^{\frac{1}{4}}x(\cdot), A^{\frac{1}{4}}x(\cdot)\rangle$. We observe that

$$\|\partial_y(u_{\gamma_n}(\cdot,t) - u(\cdot,t)\|^2 = \|A^{\frac{1}{4}}x_{\gamma_n}(t) - A^{\frac{1}{4}}x_\gamma(t)\|^2$$

and apply the Poincare inequality.

In conclusion, we point out that the semigroup associated with the linear portion of DBE is an analytic semigroup while a strongly continuous group is associated with the linear portion of BE. In this sense, we view our results as providing a parabolic approximation and regularization for a hyperbolic problem. As previously mentioned, Avrin [1] has obtained similar results for strongly damped Klein-Gordon equation and it is our opinion that the procedure will work for a wide variety of semilinear hyperbolic problems. It is hoped that this procedure will be useful for approximating the dynamics of hyperbolic problems.

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