

A TRANSFORMATION OF RIGHT-DEFINITE S-HERMITIAN SYSTEMS TO CANONICAL SYSTEMS

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Abstract. It is shown that right-definite S -hermitian boundary value problems (im Normalfall), which were defined and thoroughly studied by Schäfke and Schneider in [5, 6], can be reduced to canonical systems with selfadjoint boundary conditions in such a way that the transformed boundary conditions become a special case of those considered by Dijkstra, Langer and de Snoo in [1-3].

1. In this note, the first order system of differential equations

$$F_{11}y' + F_{12}y = \lambda(G_{11}y' + G_{12}y) \quad (1)$$

on the compact interval $I = [a, b]$ is considered. As in [5, 6] we suppose that the $n \times n$ -matrix functions F_{1j} , G_{1j} as well as S_{1j} below, $j = 1, 2$, are continuous on I and that $F_{11}(x) - \lambda G_{11}(x)$ is invertible for all $x \in I$ and $\lambda \in \mathbb{R}$. The problem (1) is called S -hermitian (im Normalfall) with respect to the differential operator

$$S_1y := S_{11}y' + S_{12}y$$

if there exists a continuously differentiable $n \times n$ -matrix function H on I such that for all real λ the relationship

$$\begin{aligned} & \begin{pmatrix} F_{11} - \lambda G_{11} & F_{12} - \lambda G_{12} \\ S_{11} & S_{12} \end{pmatrix}^* \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} F_{11} - \lambda G_{11} & F_{12} - \lambda G_{12} \\ S_{11} & S_{12} \end{pmatrix} \\ &= \begin{pmatrix} 0 & H \\ H & H' \end{pmatrix} \end{aligned} \quad (2)$$

holds on I and $H(x)$ is invertible and skew-hermitian for all $x \in I$. If Y denotes the $n \times n$ -matrix function which is the solution of the initial value problem

$$F_{11}Y' + F_{12}Y = 0, \quad Y(a) = I_n$$

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on I , then the matrix function H satisfies the equation

$$H(x) = Y(x)^{\ast -1} H(a) Y(x)^{-1}$$

and the relationship (2) can be written as

$$\begin{pmatrix} F_{11} - \lambda G_{11} & F_{12} - \lambda G_{12} \\ S_{11} & S_{12} \end{pmatrix} = \begin{pmatrix} I_n & -\lambda W_1 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ L & I_n \end{pmatrix} \begin{pmatrix} F_{11} & F_{12} \\ 0 & \overset{\circ}{S}_{12} \end{pmatrix} \quad (3)$$

where

$$\overset{\circ}{S}_{12} = -F_{11}^{\ast -1} H \quad (4)$$

and W_1, L are continuous $n \times n$ -matrix functions such that $W_1(x), L(x)$ are hermitian and $W_1(x)L(x)$ is nilpotent for all $x \in [a, b]$, (cf. [6], p.,72). The problem (1), (2) is called right-definite if $W_1 \geq 0$ on I .

It is the aim of this note to show that a right-definite S -hermitian system can be reduced to a canonical system, that is to the situation where $F_{12} = G_{11} = 0, F_{11} = J$ (J is an $n \times n$ -matrix, independent of x , with the properties $J^{\ast} = J^{-1} = -J$) and $G_{12} = \Delta$ (a hermitian nonnegative matrix function on I). This reduction is done in two steps: The first one is to reduce the system (1) to a situation where $G_{11} = 0$; after a suitable symmetrization, a result of [4] can be applied to this new system which transforms it into a canonical system.

In [5, 6] the system (1),(2) is considered together with S -hermitian boundary conditions which depend linearly on the eigenvalue parameter λ . We show in section 3 that, by the transformation of the system (1), (2) to a canonical system, these S -hermitian boundary conditions become a special case of those conditions considered in [1] for canonical systems.

Similar results hold in the singular case when $I = [a, b)$ and the right endpoint b is in the limit point case for (1). Then the problem (1), (2) with a boundary condition at the regular point b can be included in the situation studied in [3], even in the case of arbitrary defect numbers. This will be considered elsewhere.

2. Proposition 1. *Suppose that the S -hermitian system (1) is right definite. If the function y satisfies (1), then it satisfies also the equation*

$$F_{11}y' + F_{12}y = \lambda W_1 \overset{\circ}{S}_{12}y \quad (5)$$

with $\overset{\circ}{S}_{12}$ and W_1 from (4), (3), and

$$\begin{pmatrix} F_{11} & F_{12} - \lambda W_1 \overset{\circ}{S}_{12} \\ 0 & \overset{\circ}{S}_{12} \end{pmatrix}^{\ast} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} F_{11} & F_{12} - \lambda W_1 \overset{\circ}{S}_{12} \\ 0 & \overset{\circ}{S}_{12} \end{pmatrix} = \begin{pmatrix} 0 & H \\ H & H' \end{pmatrix}. \quad (6)$$

That is, the problem (5) is again S -hermitian with $S_{11} = 0, S_{12} = \overset{\circ}{S}_{12}$ and with the same function H as the original problem.

Proof: Consider $x_0 \in I$. As $W_1(x_0) \geq 0$, there exists an orthogonal decomposition of \mathbb{C}^n :

$$\mathbb{C}^n = \mathcal{L}_1 \oplus \mathcal{L}_2,$$

such that with respect to this decomposition we have

$$W_1(x_0) = \begin{pmatrix} \widehat{W}_1(x_0) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with } \widehat{W}_1(x_0) > 0 \text{ if } \mathcal{L}_1 \neq \{0\}.$$

With a corresponding matrix representation of $L(x_0)$ it follows

$$W_1(x_0)L(x_0) = \begin{pmatrix} \widehat{W}_1(x_0)L_{11}(x_0) & \widehat{W}_1(x_0)L_{12}(x_0) \\ 0 & 0 \end{pmatrix}.$$

As this product is nilpotent we have $\sigma(\widehat{W}_1(x_0)L_{11}(x_0)) = \{0\}$. For arbitrary square matrices A, B the eigenvalues of AB and BA coincide. It follows that $\sigma(\widehat{W}_1(x_0)^{1/2}L_{11}(x_0)\widehat{W}_1(x_0)^{1/2}) = \{0\}$, hence $L_{11}(x_0) = 0$,

$$W_1(x_0)L(x_0) = \begin{pmatrix} 0 & \widehat{W}_1(x_0)L_{12}(x_0) \\ 0 & 0 \end{pmatrix}$$

and $W_1(x_0)L(x_0)W_1(x_0) = 0$. This relationship is independent of the choice of x_0 , therefore

$$W_1(x)L(x)W_1(x) = 0 \quad \text{for all } x \in I. \tag{7}$$

The relationship (3) is equivalent to

$$G_{11} = W_1LF_{11}, \quad G_{12} = W_1LF_{12} + W_1\overset{\circ}{S}_{12}, \tag{8}$$

$$S_{11} = LF_{11}, \quad S_{12} = LF_{12} + \overset{\circ}{S}_{12}, \tag{9}$$

and it follows that

$$G_{11} = W_1S_{11}, \quad G_{12} = W_1S_{12}. \tag{10}$$

Now the system (1) can be written as

$$(I_n - \lambda G_{11}F_{11}^{-1})F_{11}y' = \lambda G_{12}y - F_{12}y,$$

and it follows from (8) and (7) that

$$\begin{aligned} F_{11}y' &= (I_n - \lambda G_{11}F_{11}^{-1})^{-1}(\lambda G_{12}y - F_{12}y) = (I_n - \lambda W_1L)^{-1}(\lambda G_{12}y - F_{12}y) \\ &= (I_n + \lambda W_1L)(\lambda G_{12}y - F_{12}y). \end{aligned}$$

Further, (7) and (8) imply

$$W_1LG_{12} = W_1L(W_1LF_{12} + W_1\overset{\circ}{S}_{12}) = 0,$$

and we find

$$F_{11}y' = \lambda G_{12}y - \lambda W_1LF_{12}y - F_{12}y = \lambda W_1\overset{\circ}{S}_{12}y - F_{12}y,$$

that is (5). It remains to show that also (6) holds, which is equivalent to the relations

$$H = -F_{11}^* \overset{\circ}{S}_{12}, \tag{11}$$

$$H' = \overset{\circ}{S}_{12}^* F_{12} - F_{12}^* \overset{\circ}{S}_{12}. \tag{12}$$

The relationship (11) coincides with (4), and (2) yields

$$-F_{12}^* S_{12} + S_{12}^* F_{12} = H'.$$

With the second relationship in (9) and $L = L^*$ this implies (12). The proposition is proved.

Remark. Recall that the (complex) symplectic group $\Gamma(2n)$ is the set of all $2n \times 2n$ -matrices \mathfrak{D} such that

$$\mathfrak{D}^* \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \mathfrak{D} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

or, equivalently,

$$\mathfrak{D} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \mathfrak{D}^* = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

The relationships we have shown in the proof of Proposition 1 imply that

$$\mathfrak{A}(x) := \begin{pmatrix} I_n + \lambda W_1(x)L(x) & 0 \\ -L(x) & I_n - \lambda L(x)W_1(x) \end{pmatrix} \in \Gamma(2n)$$

(this is an immediate consequence of (7)), and that the relationship

$$\mathfrak{A} \begin{pmatrix} F_{11} - \lambda G_{11} & F_{12} - \lambda G_{12} \\ S_{11} & S_{12} \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} - \lambda W_1 \overset{\circ}{S}_{12} \\ 0 & \overset{\circ}{S}_{12} \end{pmatrix} \tag{13}$$

holds (this is a consequence of (7), (8), (9) and (10)). Thus the equivalence of (2) and (6) also follows if in (2) we replace $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ by $\mathfrak{A}^* \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \mathfrak{A}$ and observe (13).

Now we symmetrize the equation (5) by multiplying it from the left with $\overset{\circ}{S}_{12}^*$:

$$\overset{\circ}{S}_{12}^* F_{11} y' + \overset{\circ}{S}_{12}^* F_{12} y = \lambda \overset{\circ}{S}_{12}^* W_1 \overset{\circ}{S}_{12} y. \tag{14}$$

In the relation (6), this corresponds to a transformation with the matrix

$$\mathfrak{D}(x) = \begin{pmatrix} \overset{\circ}{S}_{12}^*(x) & 0 \\ 0 & \overset{\circ}{S}_{12}^{-1}(x) \end{pmatrix} \in \Gamma(2n),$$

that is, the matrix

$$\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

on the left hand side of (6) is replaced by

$$\mathfrak{D}^* \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \mathfrak{D},$$

and we obtain

$$\begin{aligned} & \begin{pmatrix} \mathring{S}_{12}^* F_{11} & \mathring{S}_{12}^* F_{12} - \lambda \mathring{S}_{12}^* W_1 \mathring{S}_{12} \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} \mathring{S}_{12}^* F_{11} & \mathring{S}_{12}^* F_{12} - \lambda \mathring{S}_{12}^* W_1 \mathring{S}_{12} \\ 0 & I_n \end{pmatrix} \\ &= \begin{pmatrix} 0 & H \\ H & H' \end{pmatrix}. \end{aligned} \tag{15}$$

We introduce the matrix functions

$$Q := -iH, \quad A := \mathring{S}_{12}^* W_1 \mathring{S}_{12}, \quad B := \operatorname{Re}(\mathring{S}_{12}^* F_{12}). \tag{16}$$

Then Q has a continuous derivative, A and B are continuous on I . The values of Q and B are hermitian matrices, the values of Q are invertible and A is nonnegative as the problem (1) was supposed to be right definite: $W_1 \geq 0$. With these functions, the equation (14) becomes

$$\frac{i}{2} \{ (Qy)' + Qy' \} = \lambda Ay + By. \tag{17}$$

Now, as in [4] we represent $Q(x)$ as

$$iQ(x) = q(x)^* J q(x) \tag{18}$$

where J is an $n \times n$ -matrix with the properties $J^* = -J = J^{-1}$. From the construction of the $n \times n$ -matrix function q in [4] it follows that this function is again continuously differentiable. With the matrix solution Z of the initial value problem

$$JZ' = q^{*-1} \{ B + \frac{1}{2}(q^* J q' - q'^* J q) \} q^{-1} Z, \quad Z(a) = I_n \tag{19}$$

and

$$S(x) := q(x)^{-1} Z(x), \quad \tilde{y}(x) = S(x)^{-1} y(x) \quad (x \in I) \tag{20}$$

the equation (17) becomes

$$J\tilde{y}' = \lambda \Delta \tilde{y} \tag{21}$$

where J is given by (18), and

$$\Delta(x) := Z(x)^* q(x)^{*-1} A(x) q(x)^{-1} Z(x) \geq 0 \quad (x \in I).$$

Thus we have proved:

Proposition 2. *Suppose that the S -hermitian system (1) is right-definite. If the function y is a solution of (1), then the function \tilde{y} , defined by (20), is a solution of the canonical differential equation (21). The canonical differential equation (21) is again S -hermitian, and the relationship (2) becomes now*

$$\begin{pmatrix} J & -\lambda\Delta \\ 0 & I_n \end{pmatrix}^* \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} J & -\lambda\Delta \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}. \quad (22)$$

Below we shall use the following relation:

$$S(x)^* H(x) S(x) = J \quad (x \in I). \quad (23)$$

Indeed, from (16), (18), (20), we obtain

$$\begin{aligned} S(x)^* H(x) S(x) &= iS(x)^* Q(x) S(x) = S(x)^* q(x)^* J q(x) S(x) \\ &= Z(x)^* J Z(x) = J, \end{aligned} \quad (24)$$

where the last equality is a well-known property of the solution Z of the initial value problem (19). The relation (24) is equivalent to

$$S(x) J S(x)^* = -H(x)^{-1} \quad (x \in I). \quad (25)$$

3. Now we consider boundary conditions of the form

$$(F_{21} - \lambda G_{21})y(a) + (F_{22} - \lambda G_{22})y(b) = 0, \quad (26)$$

with $n \times n$ -matrices F_{2j}, G_{2j} , $j = 1, 2$. Recall that the boundary condition (26) for the system (1),(2) is called S -hermitian if there exist $n \times n$ -matrices S_{21}, S_{22} such that the relationship

$$\begin{aligned} &\begin{pmatrix} F_{21} - \lambda G_{21} & F_{22} - \lambda G_{22} \\ S_{21} & S_{22} \end{pmatrix}^* \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} F_{21} - \lambda G_{21} & F_{22} - \lambda G_{22} \\ S_{21} & S_{22} \end{pmatrix} \\ &= \begin{pmatrix} H(a) & 0 \\ 0 & -H(b) \end{pmatrix} \end{aligned} \quad (27)$$

holds for all real λ . Then there exists a hermitian $n \times n$ -matrix W_2 such that

$$G_{21} = W_2 S_{21}, \quad G_{22} = W_2 S_{22}, \quad (28)$$

(see [6], p. 72). If we introduce the function $\tilde{y} = S^{-1}y$ according to (20), the boundary condition for the canonical system (21) becomes

$$(\tilde{F}_{21} - \lambda \tilde{G}_{21})\tilde{y}(a) + (\tilde{F}_{22} - \lambda \tilde{G}_{22})\tilde{y}(b) = 0, \quad (29)$$

with

$$\tilde{F}_{21} = F_{21}S(a), \quad \tilde{F}_{22} = F_{22}S(b), \quad \tilde{G}_{21} = G_{21}S(a), \quad \tilde{G}_{22} = G_{22}S(b). \quad (30)$$

Multiplying the relationship (27) by

$$\begin{pmatrix} S(a)^* & 0 \\ 0 & S(b)^* \end{pmatrix}$$

from the left and by

$$\begin{pmatrix} S(a) & 0 \\ 0 & S(b) \end{pmatrix}$$

from the right and observing (30) it follows

$$\begin{aligned} & \begin{pmatrix} \tilde{F}_{21} - \lambda\tilde{G}_{21} & \tilde{F}_{22} - \lambda\tilde{G}_{22} \\ \tilde{S}_{21} & \tilde{S}_{22} \end{pmatrix}^* \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} \tilde{F}_{21} - \lambda\tilde{G}_{21} & \tilde{F}_{22} - \lambda\tilde{G}_{22} \\ \tilde{S}_{21} & \tilde{S}_{22} \end{pmatrix} \\ &= \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} \end{aligned} \quad (31)$$

with

$$\tilde{S}_{21} = S_{21}S(a), \quad \tilde{S}_{22} = S_{22}S(b). \quad (32)$$

Therefore, the canonical system (21) with the boundary condition (29) is S -hermitian with the companion matrix $\tilde{H}(x) \equiv J$ ($x \in I$) (observe (22)). The relationships (28), (30), and (32) imply that

$$\tilde{G}_{21} = W_2\tilde{S}_{21}, \quad \tilde{G}_{22} = W_2\tilde{S}_{22}, \quad (33)$$

that is, the hermitian matrix W_2 does not change under the transformation of (1) into a canonical system. Hence, if the boundary condition (26) is right-definite (that is $W_2 \geq 0$), also the boundary condition (29) for the canonical system is right-definite.

The next proposition implies that the S -hermitian boundary condition (29) is a special case of the boundary condition considered in [1].

Proposition 3. *If the boundary condition (29) is S -hermitian for the canonical system (21) (that is (31) is satisfied), then, with $\tilde{A}(\lambda) := \tilde{F}_{21} - \lambda\tilde{G}_{21}$, $\tilde{B}(\lambda) := \tilde{F}_{22} - \lambda\tilde{G}_{22}$, it holds:*

- (i) $\text{rank}(\tilde{A}(\lambda), \tilde{B}(\lambda)) = n \quad (\lambda \in \mathbf{R});$
- (ii) $\tilde{A}(\lambda)J\tilde{A}(\lambda)^* - \tilde{B}(\lambda)J\tilde{B}(\lambda)^* = 0 \quad (\lambda \in \mathbf{R});$
- (iii) $K_{\tilde{A}, \tilde{B}}(\lambda, l) := \frac{\tilde{A}(\bar{l})J\tilde{A}(\bar{l})^* - \tilde{B}(\bar{l})J\tilde{B}(\bar{l})^*}{\lambda - \bar{l}} = -\tilde{F}_{21}J\tilde{G}_{21}^* + \tilde{F}_{22}J\tilde{G}_{22}^*$

and this expression coincides with W_2 in (33):

$$W_2 = -\tilde{F}_{21}J\tilde{G}_{21}^* + \tilde{F}_{22}J\tilde{G}_{22}^*. \quad (34)$$

Proof: With $\tilde{\mathfrak{A}}(\lambda) := \begin{pmatrix} \tilde{F}_{21} - \lambda\tilde{G}_{21} & \tilde{F}_{22} - \lambda\tilde{G}_{22} \\ \tilde{S}_{21} & \tilde{S}_{22} \end{pmatrix}$ the equation (31) is equivalent to

$$\tilde{\mathfrak{A}}(\lambda) \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} \tilde{\mathfrak{A}}(\lambda)^* = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \quad (\lambda \in \mathbf{R}),$$

that is

$$\tilde{A}(\lambda)J\tilde{A}(\lambda)^* - \tilde{B}(\lambda)J\tilde{B}(\lambda)^* = 0, \tag{35}$$

$$\tilde{A}(\lambda)J\tilde{S}_{21}^* - \tilde{B}(\lambda)J\tilde{S}_{22}^* = -I_n, \tag{36}$$

$$\tilde{S}_{21}J\tilde{S}_{21}^* - \tilde{S}_{22}J\tilde{S}_{22}^* = 0.$$

The relation (36) implies (i), and (ii) is equivalent to (35). The first equality in (iii) is easy to check. Further, (31) implies

$$\tilde{G}_{21}^*\tilde{S}_{21} - \tilde{S}_{21}^*\tilde{G}_{21} = \tilde{G}_{22}^*\tilde{S}_{21} - \tilde{S}_{22}^*\tilde{G}_{21} = \tilde{G}_{22}^*\tilde{S}_{22} - \tilde{S}_{22}^*\tilde{G}_{22} = 0,$$

and (36) yields

$$-\tilde{F}_{21}J\tilde{S}_{21}^* + \tilde{F}_{22}J\tilde{S}_{22}^* = I_n.$$

Therefore, we find for $j = 1, 2$

$$(-\tilde{F}_{21}J\tilde{G}_{21}^* + \tilde{F}_{22}J\tilde{G}_{22}^*)\tilde{S}_{2j} = (-\tilde{F}_{21}J\tilde{S}_{21}^* + \tilde{F}_{22}J\tilde{S}_{22}^*)\tilde{G}_{2j} = \tilde{G}_{2j},$$

that is, the matrix defined in the first relationship of (iii) has the properties of W_2 in (33). As this matrix is uniquely determined, the relationship (34) follows.

Remark 1. The statement (iii) of Proposition 3 implies that the boundary condition (26) or (29) is right definite if and only if the kernel $K_{\tilde{A},\tilde{B}}(\lambda, l)$ is nonnegative definite, that is, the extending space in the construction of [1] is equipped with a positive definite inner product.

Remark 2. The relations (25) and (30) imply that the matrix W_2 in (34) can be expressed in terms of the matrices in the boundary condition (26):

$$W_2 = F_{21}H(a)^{-1}G_{21}^* - F_{22}H(b)^{-1}G_{22}^*.$$

Remark 3. If $\tilde{A}(\lambda), \tilde{B}(\lambda)$ are λ -linear $n \times n$ -matrix functions as in Proposition 3 with the properties (i) and (ii), there do not necessarily exist matrices $\tilde{S}_{21}, \tilde{S}_{22}$ such that (31) holds. According to ([5], p.245), for this it is necessary and sufficient that we have additionally

$$\text{rank} \begin{pmatrix} \tilde{F}_{21} & \tilde{F}_{22} \\ \tilde{G}_{21} & \tilde{G}_{22} \end{pmatrix} = n + \text{rank}(\tilde{G}_{21}, \tilde{G}_{22}).$$

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