

## A WEIGHTED SEMILINEAR ELLIPTIC EQUATION INVOLVING CRITICAL SOBOLEV EXPONENTS

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**Abstract.** In this paper we prove the existence of a positive radial solution of the problem

$$-\Delta u = r^\sigma |u|^{p-1} u + \lambda r^\alpha u, \quad \text{in } B_R \subset \mathbf{R}^N \quad (r = |x|)$$

for  $\lambda$  in a suitable (and almost optimal) range. Here  $N \geq 3$ ,  $\alpha, \sigma \geq -2$  and  $p = (N + 2 + 2\sigma)/(N - 2)$  corresponds to the critical Sobolev exponent  $p + 1 = (2N + 2\sigma)/(N - 2)$ . Our result extends the previous one due to Brézis and Nirenberg when  $\sigma = \alpha = 0$ .

**0. Introduction.** In a previous paper [8] we considered the problem

$$\begin{cases} -\frac{1}{r^\gamma} (r^\gamma u')' = r^\sigma |u|^{q-1} u & \text{in } (0, 1) \\ u(1) = 0, \quad \int_0^1 r^\gamma |u'|^2 dr < \infty \\ u > 0. \end{cases} \tag{0.1}$$

We recall some of the results we obtained there.

“If  $\gamma > 1$  then the problem has exactly one weak solution for  $1 < q < \frac{\gamma+3+2\sigma}{\gamma-1}$  and no weak solution for  $q > (\gamma + 3 + 2\sigma)/(\gamma - 1)$ .”

In this paper we shall deal exactly with the critical case, namely,  $q = p = (\gamma + 3 + 2\sigma)/(\gamma - 1)$ . Instead of (0.1) we consider the more general problem

$$\begin{cases} -\frac{1}{r^\gamma} (r^\gamma u')' = r^\sigma |u|^{p-1} u + \lambda r^\alpha u & \text{in } (0, 1) \\ u(1) = 0, \quad \int_0^1 r^\gamma |u'|^2 dr < \infty \\ u > 0 \end{cases} \tag{0.2}$$

where  $\gamma > 1$ ,  $\sigma, \alpha > -2$ .

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In order to describe our results we need to explain some notation. Let us consider the linear eigenvalue problems

$$-\frac{1}{r^\gamma}(r^\gamma u')' = \lambda r^\alpha u, \quad u(1) = 0, \quad \int_0^1 r^\gamma |u'|^2 dr < \infty \tag{0.3}$$

and

$$-\frac{1}{r^{2-\gamma}}(r^{2-\gamma} v')' = \mu r^\alpha v, \quad v(1) = 0, \quad \int_0^1 r^{2-\gamma} |v'|^2 dr < \infty \tag{0.4}$$

which should be understood in a suitable weak sense.

We denote by  $\lambda_1(\alpha, \gamma)$  the least eigenvalue of (0.3) which exists for  $\gamma > 1, \alpha > -2$  and by  $\mu_1(\alpha, \gamma)$  the first eigenvalue of (0.4) which exists for  $1 < \gamma < \alpha + 3$ . We can now state our main results.

- A. There exists no weak solution of (0.2) for  $\lambda \leq 0$  or  $\lambda \geq \lambda_1(\alpha)$ .
- B. If  $\gamma > \alpha + 3$  then (0.2) has at least a weak solution if and only if  $\lambda \in (0, \lambda_1(\alpha))$ .
- C. If  $1 < \gamma < \alpha + 3$  then  $\mu_1(\alpha, \gamma) < \lambda_1(\alpha, \gamma)$  and problem (0.2) has a weak solution for each  $\lambda \in (\mu_1(\alpha, \gamma), \lambda_1(\alpha, \gamma))$ .

The proof of these results uses a variational method together with the techniques of Brézis-Nirenberg [3] in order to overcome the difficulties raised by the lack of compactness due to the critical Sobolev exponent  $p + 1 = (2\gamma + 2 + 2\sigma)/(\gamma - 1)$ .

Our results can be directly applied to elliptic PDEs yielding a twofold generalization of some of the results of the Brézis-Nirenberg paper [3]. The paper is divided into three sections. Section 1 deals with weighted Sobolev spaces. In particular, here is proved the existence of a best Sobolev constant in a critical Sobolev imbedding and we compute it explicitly. Section 2 is the core of the paper. Here are stated and proved the existence results for (0.2). Section 3 is devoted to the proof of the inequality  $\mu_1(\alpha, \gamma) < \lambda_1(\alpha, \gamma)$  for  $1 < \gamma < \alpha + 3$ .

**1. Imbedding theorems for weighted Sobolev spaces. Existence of a best constant in the critical case.** We first recall some known facts about weighted Sobolev spaces which we have stated and proved in [8] in a form suitable for our purposes. Let  $R \in (0, +\infty]$ ,  $E_\gamma^R$  is the closure of the set

$$S = \{u \in C^1[0, R] : u \equiv 0 \text{ in a neighborhood of } R\}$$

in the norm  $\|\cdot\|_{\gamma,R}$  defined by

$$\|u\|_{\gamma,R} = \left( \int_0^R r^\gamma |u'|^2 dr \right)^{1/2}.$$

When  $R = 1$  we write simply  $\|u\|_\gamma$ ,  $L_\theta^q(0, R)$  is the weighted  $L^q(0, R; r^\theta dr)$ .

We recall the following results (see [8] for a proof).

**Radial Lemma.** *There exists  $C = C(\gamma) > 0$  such that for  $u \in E_\gamma^R$*

$$|u(r)| \leq \frac{C}{r^{(\gamma-1)/2}} \|u\|_{\gamma,R}, \quad \forall r \in (0, R).$$

**Imbedding Lemma.** *Let  $\gamma > 1$ ,  $R < \infty$  and  $\theta > \max(-1, \gamma - 2)$ . Then  $E_\gamma^R \hookrightarrow L_\theta^q(0, R)$  continuously if and only if*

$$\frac{\theta + 1}{q} \geq \frac{\gamma - 1}{2}. \tag{1.1}$$

**Compactness Lemma.** *Let  $\gamma > 1$ ,  $R < \infty$  and  $\theta > \max(-1, \gamma - 2)$ . The imbedding  $E_\gamma^R \hookrightarrow L_\theta^q(0, R)$  is compact if*

$$\frac{\theta + 1}{q} > \frac{\gamma - 1}{2}. \tag{1.1'}$$

We are interested mainly in the critical case; i.e., the situation  $(\theta+1)/q = (\gamma-1)/2$  hence

$$q = \gamma^* = \frac{2(\theta + 1)}{\gamma - 1}. \tag{1.2}$$

The continuity of the critical imbedding is equivalent to the existence of a constant  $K > 0$  such that

$$\|u\|_{L_{\theta}^{\gamma^*}(0,R)} \leq K \|u\|_{\gamma,R}. \tag{1.3}$$

In fact (1.3) holds also with  $R = \infty$  (see Maz'ya [7], Sect. 1.3.1). Set

$$S(\gamma, \theta, R) = \inf\{\|u\|_{\gamma,R}^2 \mid u \in E_\gamma^R, \|u\|_{L_{\theta}^{\gamma^*}(0,R)} = 1\}. \tag{1.4}$$

Following the ideas in Aubin [1] we can prove the following result.

**Proposition 1.1.** *Let  $\theta > \gamma - 2 > -1$ . Then in (1.4) with  $R = \infty$  the infimum  $S(\gamma, \theta, \infty)$  is achieved by the function*

$$\widetilde{U}(r) = \frac{C}{(1 + r^{2+\sigma})^{(\gamma-1)/(2+\sigma)}} = C \cdot U(r), \quad \sigma = \theta - \gamma \tag{1.5}$$

where  $C$  is a normalization constant; i.e., such that  $\|\widetilde{U}\|_{L_{\theta}^{\gamma^*}(0,\infty)} = 1$ .

**Proof:** The proof will be carried out in two steps.

**Step 1.** If the infimum is achieved then it can also be achieved by a positive decreasing function. Let us assume the infimum is achieved. Obviously  $\|u\|_{\gamma,\infty} = \| |u| \|_{\gamma,\infty}$  and  $\|u\|_{L_{\theta}^{\gamma^*}(0,\infty)} = \| |u| \|_{L_{\theta}^{\gamma^*}(0,\infty)}$ . (The former equality is a consequence of a variant of Stampacchia's lemma; see [6] for a proof.) Hence the infimum can also be reached by positive functions. The functions that realize the infimum satisfy the Euler-Lagrange equation

$$-\frac{1}{r^\gamma} (r^\gamma u')' = \lambda r^\sigma |u|^{\gamma^*-2} u, \tag{1.6}$$

where  $\lambda \in \mathbf{R}^*$  is a Lagrange multiplier. We may assume  $u > 0$ . By the Radial Lemma we infer

$$\lim_{r \rightarrow \infty} u(r) = 0. \tag{1.7}$$

We make the change of variables  $s = r^{-(\gamma-1)}$  and we denote  $v(s) = u(r)$ . Then  $v$  satisfies the following equation

$$v_{ss} + \frac{1}{(\gamma - 1)^2} \frac{\lambda v^{\gamma^* - 1}}{s^{2+(2+\sigma)/(\gamma-1)}} = 0 \quad \text{in } (0, \infty) \tag{1.8}$$

with  $v(0) = 0$  and  $v_s = dv/ds$ . Hence the function  $\lambda v$  is concave. One of the following two situations may occur.

- A.  $\lambda < 0$ . Then  $v$  is convex and hence  $v'(0)$  exists and  $v'(0) \geq 0$ . Therefore  $v$  is increasing and obviously  $u$  is decreasing.
- B.  $\lambda > 0$ . Then  $v$  is concave and since  $v > 0$  near  $\infty$  we get  $v_s(\infty) = 0$  and consequently  $v$  is increasing. Again we get that  $u$  is decreasing.

**Step 2.** The infimum is achieved by the function (1.5). It is easily seen that the function (1.5) satisfies (1.6) with a suitable  $\lambda$ . The statement of Step 2 follows from a sharp result due to G.A. Bliss [2]. We state a special case of it.

**Lemma 1.2.** *Let  $q > 2$  and let  $h(x) \geq 0$  be a measurable real-valued function such that  $\int_0^\infty h^2(x) dx$  is finite. Set  $g(x) = \int_0^x g(t) dt$ . Then*

$$\left( \int_0^\infty g^q(x) x^{\alpha-q} dx \right) \leq K \left( \int_0^\infty h^2(x) dx \right)^{q/2} \tag{1.9}$$

where  $q = 2\alpha - 2$  and  $K = 1/(q - \alpha - 1) [(\alpha\Gamma(q/\alpha))/(\Gamma(1/\alpha)\Gamma((q - 1)/\alpha))]^\alpha$ .

Here  $\Gamma$  is Euler's gamma function. The relation (1.9) holds with equality for every function  $h(x)$  of the form

$$H_\alpha(x) = \frac{C}{(dx^\alpha + 1)^{(\alpha+1)/\alpha}}. \tag{1.10}$$

We see that (1.9) can be restated as

$$\left( \int_0^\infty g^q(x) x^{\alpha-q} dx \right)^{1/q} \leq K^{1/q} \int_0^\infty |g'(x)|^2 dx, \tag{1.9'}$$

for every increasing function  $g$  such that  $g(0) = 0$  and  $g' \in L^2(0, \infty)$ . If in (1.9') we make the change in variables  $x = 1/r^\nu$ ,  $u(r) = g(x)$  then we get

$$\nu^{1/q} \left( \int_0^\infty u^q(r) r^{((\nu q)/2) - 1} dr \right)^{1/q} \leq K^{1/q} \nu^{-1} \left( \int_0^\infty |u'(r)|^2 r^{\nu+1} dr \right)^{1/2} \tag{1.11}$$

for every positive decreasing function  $u$  such that  $u(r) \rightarrow 0$  as  $r \rightarrow \infty$ . If in (1.11) we further specialize  $\nu$  and  $q$  such that  $\nu + 1 = \gamma$ ,  $\frac{\nu q}{2} - 1 = \theta$  we get  $\nu = \gamma - 1$  and  $q = (2(\theta + 1))/(\gamma - 1)$  and we obtain the critical Sobolev imbedding. Moreover (1.11) becomes equality when  $d/dx u(r) = H_\alpha(x)$ , where  $x = r^{-(\gamma-1)}$ . This happens when  $u(r) = U(r)$  where  $U$  is given by (1.5).

**Remark 1.3.** The result above allows one to compute the exact value of  $S(\gamma, \theta, \infty)$ . In fact

$$S(\gamma, \theta, \infty) = \frac{\|U\|_{\gamma, \infty}^2}{\|U\|_{L^q_{\gamma^*}(0, \infty)}^2} \tag{1.12}$$

$$\begin{aligned} \|U\|_{\gamma,\infty}^2 &= \int_0^\infty r^\gamma |U'|^2 dr, \quad U'(r) = (\gamma - 1) \frac{r^{1+\sigma}}{(1 + r^{2+\sigma})^{(\gamma+1+\sigma)/(2+\sigma)}} \\ \|U\|_{\gamma,\infty}^2 &= (\gamma - 1)^2 \int_0^\infty \frac{r^{\gamma+2+2\sigma}}{(1 + r^{2+\sigma})^{2(\gamma+1+\sigma)/(2+\sigma)}} dr \\ &= \frac{(\gamma - 1)^2}{2 + \sigma} \int_0^\infty \frac{s^{(\gamma+3+2\sigma)/(2+\sigma)-1}}{(1 + s)^{2(\gamma+1+\sigma)/(2+\sigma)}} ds \end{aligned}$$

so that

$$\|U\|_{\gamma,\infty}^2 = \frac{(\gamma - 1)^2}{2 + \sigma} \frac{\Gamma(\frac{\gamma+3+2\sigma}{2+\sigma}) \Gamma(\frac{\gamma-1}{2+\sigma})}{\Gamma(\frac{2(\gamma+1+\sigma)}{2+\sigma})}, \tag{1.13}$$

where we have used the formula

$$\int_0^\infty \frac{x^{m-1} dx}{(1 + x)^{m+n}} = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m + n)}, \quad \forall m, n > 0 \tag{1.14}$$

(cf. Dwight [4]). We now compute in a similar way

$$\begin{aligned} \|U\|_{L_\theta^{\gamma^*}(0,\infty)}^{\gamma^*} &= \int_0^\infty \frac{r^\theta}{(1 + r^{2+\sigma})^{2(\theta+1)/(2+\sigma)}} dr \quad (\theta = \gamma + \sigma) \\ &= \frac{1}{2 + \sigma} \int_0^\infty \frac{s^{(\gamma+\sigma+1)/(2+\sigma)-1}}{(1 + s)^{2(\gamma+\sigma+1)/(2+\sigma)}} ds = \frac{1}{2 + \sigma} \frac{[\Gamma(\frac{\gamma+1+\sigma}{2+\sigma})]^2}{\Gamma(\frac{2(\gamma+1+\sigma)}{2+\sigma})} \end{aligned} \tag{1.15}$$

where again we have used (1.14). It is a straightforward observation that  $S(\gamma, \theta, R)$  is invariant under rescaling so that it does not really depend on  $R$ . We set

$$S(\gamma, \theta) := S(\gamma, \theta, R), \quad R \in (0, \infty).$$

**2. Existence of a positive solution.** In this section we consider the existence question for the following boundary value problem

$$\begin{cases} -\frac{1}{r^\gamma} (r^\gamma u')' = r^\sigma |u|^{p-1} u + \lambda r^\alpha u & \text{in } (0, 1), \lambda \in R, \\ \alpha > -2, \sigma > -2, u \in E_\gamma, u > 0 \end{cases} \tag{2.1}$$

where  $p = (\gamma + 3 + 2\sigma)/(\gamma - 1)$ ,  $\gamma > 1$ ,  $\sigma > -2$  so that the imbedding  $E_\gamma \hookrightarrow L_\theta^{p+1}$ ,  $\theta = \gamma + \sigma$  is noncompact.

We shall look for a weak solution of (2.1); i.e., a function  $u \in E_\gamma$ ,  $u > 0$  such that

$$\int_0^1 r^\gamma u' \varphi' dr = \int_0^1 [r^\theta u^p \varphi + \lambda r^{\gamma+\alpha} u \varphi] dr, \quad \forall \varphi \in E_\gamma. \tag{2.2}$$

**Remark 2.1.** (2.1) has no weak solution for  $\lambda \leq 0$ .

This will follow from a Pohozaev-type argument similar to that used in [8]. Assume the contrary; i.e., there exists a weak solution  $u$  of (2.1) with  $\lambda \leq 0$ . In (2.2) we set  $\varphi = u$  and we get

$$\int_0^1 r|u'|^2 dr = \int_0^1 [r^\theta u^{p+1} + \lambda r^{\gamma+\alpha} u^2] dr. \tag{2.3}$$

By standard elliptic regularity,  $u \in C^2(0, 1)$  so that  $u$  is a classical solution of

$$-(r^\gamma u')' = r^\theta u^p + \lambda r^{\gamma+\alpha} u, \quad \text{in } (0, 1), \quad u(1) = 0. \tag{2.4}$$

We multiply (2.4) with  $ru'$  and we get

$$\begin{aligned} & r^{\gamma+1} u' u'' + \gamma r^\gamma |u'|^2 + r^{\theta+1} u^p u' + \lambda r^{\gamma+\alpha} u u' = 0 \\ & = \frac{1}{2} r^{\gamma+1} \frac{d}{dr} |u'|^2 + r^\gamma |u'|^2 + \frac{1}{p+1} r^{\theta+1} \frac{d}{dr} (u^{p+1}) + \frac{\lambda}{2} r^{\gamma+\alpha} \frac{d}{dr} (u^2) = 0. \end{aligned}$$

We integrate the last inequality by parts on  $(\varepsilon, 1)$ . We obtain

$$\begin{aligned} & \frac{1}{2} |u'(1)|^2 - \frac{1}{2} \varepsilon^{\gamma+1} |u'(\varepsilon)|^2 - \frac{1}{p+1} \varepsilon^{\theta+1} u^{p+1}(\varepsilon) - \frac{\lambda}{2} \varepsilon^{\gamma+\alpha+1} u^2(\varepsilon) \\ & + \frac{\gamma-1}{2} \int_\varepsilon^1 r^\gamma |u'(r)|^2 dr - \frac{\theta+1}{p+1} \int_\varepsilon^1 r^\theta u^{p+1}(r) dr \\ & - \frac{\lambda(\gamma+\alpha+1)}{2} \int_\varepsilon^1 r^{\gamma+\alpha} u^2(r) dr = 0. \end{aligned} \tag{2.5}$$

Since  $u \in E_\gamma \cap L_\theta^{p+1}$  we get that on a subsequence  $\varepsilon_k \rightarrow 0$

$$\varepsilon_k^{\gamma+1} |u'(\varepsilon_k)|^2 + \varepsilon_k^{\theta+1} u^{p+1}(\varepsilon_k) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \tag{2.6}$$

By the Radial Lemma  $u^2(\varepsilon) \leq \text{const.} \varepsilon^{-(\gamma-1)}$  so that

$$\varepsilon^{(\gamma+\alpha+1)} u^2(\varepsilon) \leq \text{const.} \varepsilon^{\alpha+2} = O(1) \quad \text{as } \varepsilon \rightarrow 0, \text{ since } \alpha > -2. \tag{2.7}$$

If in (2.5) we let  $\varepsilon = \varepsilon_k \rightarrow 0$  we infer by (2.6), (2.7)

$$\begin{aligned} & \frac{1}{2} |u'(1)|^2 + \frac{\gamma-1}{2} \int_0^1 r^\gamma |u'(r)|^2 dr - \frac{\theta+1}{p+1} \int_0^1 r^\theta u^{p+1}(r) dr \\ & - \frac{\lambda(\gamma+\alpha+1)}{2} \int_0^1 r^{\gamma+\alpha} u^2(r) dr = 0. \end{aligned} \tag{2.8}$$

From (2.3) and (2.8) we infer

$$\frac{1}{2} |u'(r)|^2 + \left( \frac{\gamma-1}{2} - \frac{\theta+1}{p+1} \right) \int_0^1 r^\gamma |u'(r)|^2 dr - \frac{\lambda(\alpha+2)}{2} \int_0^1 r^{\gamma+\alpha} u^2(r) dr = 0$$

and finally since  $(\gamma - 1)/2 = (\theta + 1)/(p + 1)$

$$\frac{1}{2}|u'(1)|^2 = \frac{\lambda(\alpha + 2)}{2} \int_0^1 r^{\gamma+\alpha} u^2(r) dr \leq 0. \tag{2.9}$$

We get  $u'(1) = 0$ . We know that also  $u(1) = 0$ . Therefore it follows—according to the uniqueness in a Lipschitzian Cauchy problem—that  $u \equiv 0$ .

A special part in our considerations will be played by the following generalized engenvalue problem

$$-\frac{1}{r^\gamma}(r^\gamma u')' = \lambda r^\alpha u, \quad \lambda \in \mathbf{R}, \alpha > -2 \quad u \in E_\gamma, \tag{2.10}$$

which is meant in the following generalized sense

$$\int_0^1 r^\gamma u' \varphi' dr = \lambda \int_0^1 r^{\gamma+\alpha} u \varphi dr, \quad \forall \varphi \in E_\gamma. \tag{2.11}$$

Since for  $\alpha > -2$  the imbedding  $E_\gamma \hookrightarrow L^2_{\gamma+\alpha}$  is compact one can prove in a standard manner the following facts:

- (F1) The spectrum of (2.10) consists of an unbounded sequence of positive eigenvalues  $0 < \lambda_1(\alpha, \gamma) < \lambda_2(\alpha, \gamma) \leq \lambda_3(\alpha, \gamma) < \dots \rightarrow \infty$ , each of them having finite multiplicity.
- (F2) The eigenvalue  $\lambda_1(\alpha, \gamma)$  is simple and the corresponding eigenspace is generated by a positive eigenfunction.
- (F3) For every  $\gamma > 1$  the mapping  $\alpha \rightarrow \lambda_1(\alpha, \gamma)$  is decreasing and continuous.

(For a proof of these by now classical statements we refer the reader to the work of D.J. de Figueiredo [5].)

When there is no possibility of confusion we shall write  $\lambda_1(\alpha)$  instead of  $\lambda_1(\alpha, \gamma)$ .

**Remark 2.2.** Problem (2.1) has no weak solution for  $\lambda \geq \lambda_1(\alpha, \gamma)$ ,  $\gamma > 1$ ,  $\alpha > -2$ .

Indeed, we set in (2.2)  $\varphi = \varphi_1$ . We get

$$\int_0^1 r^\gamma u' \varphi_1' dr = \int_0^1 [r^\theta u^p \varphi_1 + \lambda r^{\gamma+\alpha} u \varphi_1] dr.$$

If in (2.11) we set  $\varphi = u$  and  $\lambda = \lambda_1(\alpha)$  we get

$$\int_0^1 r^\gamma u' \varphi_1' dr = \lambda_1 \int_0^1 r^{\gamma+\alpha} u \varphi_1 dr.$$

We infer

$$0 \geq (\lambda - \lambda_1) \int_0^1 r^{\gamma+\alpha} u \varphi_1 dr = \int_0^1 r^\theta u^p \varphi_1 dr.$$

Due to the fact that  $\varphi_1 > 0$  it follows  $u \equiv 0$ .

From the two remarks above we see that a necessary condition for the existence of a solution of (2.1) is that  $\lambda \in (0, \lambda_1(\alpha))$ .

Following the ideas of Brézis-Nirenberg [3] we shall consider the minimization problem

$$S_\lambda = S_\lambda(\gamma, \theta) = \inf\{\|u\|_\gamma^2 - \lambda\|u\|_{L^2_{\gamma+\alpha}}^2; \|u\|_{L^{p+1}_\theta} = 1\}. \tag{2.12}$$

If  $u$  is a solution of (2.12) we may assume  $u \geq 0$  for otherwise we replace  $u$  by  $|u|$ . Then  $u$  satisfies

$$-\frac{1}{r^\gamma}(r^\gamma u')' - \lambda r^\alpha u = S_\lambda r^\sigma u^p \quad \text{in } (0, 1), \quad u \in E_\gamma$$

in the weak sense (2.2). It follows that if  $S_\lambda > 0$  then  $ku$  satisfies (2.2) for some appropriate constant  $k > 0$  (namely  $k = S_\lambda^{1/(p-1)}$ ). Thus in order to obtain a solution of (2.1) it is sufficient to check the following conditions.

$$\text{Problem (2.12) has a solution.} \tag{2.13}$$

$$S_\lambda > 0. \tag{2.14}$$

Condition (2.14) holds if and only if  $\lambda < \lambda_1(\alpha)$  which is (according to Remark 2.2) a necessary condition for the existence of a solution of (2.1).

A sufficient condition so that (2.13) holds is supplied by the following result.

**Proposition 2.3.** *If  $S_\lambda < S$  then the minimization problem (2.12) has a solution.*

The proof of this result follows the same lines as the proof of Lemma 1.2 in the paper of Brézis-Nirenberg [3] so we omit it.

Our task is to check when  $S < S_\lambda$ . We follow the arguments used in [3] Lemma 1.1 and Lemma 1.3. We treat separately two cases.

**2.1. The case  $\gamma \geq \alpha + 3$ .** The main result of this subsection is the following.

**Proposition 2.4.** *Let  $\gamma \geq \alpha + 3$ . Then  $S_\lambda(\gamma, \theta) < S(\gamma, \theta)$  for every  $\lambda > 0$ .*

**Proof:** We shall estimate the ratio

$$Q_\lambda(u_\varepsilon) = \frac{\|u\|_\gamma^2 - \lambda\|u\|_{L^2_{\gamma+\alpha}}^2}{\|u_\varepsilon\|_{L^{p+1}_\theta}^2}, \quad \theta = \gamma + \sigma$$

with

$$u_\varepsilon(r) = \frac{\varphi(r)}{(\varepsilon + r^{2+\sigma})^{(\gamma-1)/(2+\sigma)}}, \quad \varepsilon > 0 \tag{2.15}$$

where  $\varphi \in C^\infty[0, 1)$  is a fixed function such that  $\varphi(r) \equiv 1$  for  $r$  in some neighborhood of 0 and  $\varphi(r) \equiv 0$  in some neighborhood of 1.

We claim that as  $\varepsilon \rightarrow 0$  we have

$$\|u_\varepsilon\|_\gamma^2 = \frac{K_1}{\varepsilon^{(\gamma-1)/(2+\sigma)}} + O(1) \tag{2.16}$$

$$\|u_\varepsilon\|_{L^{p+1}_\theta}^2 = \frac{K_2}{\varepsilon^{(\gamma-1)/(2+\sigma)}} + O(1) \tag{2.17}$$



$$\|u_\varepsilon\|_{L^2_{\gamma+\alpha}}^2 = \begin{cases} \frac{K_3}{\varepsilon^{(\gamma-3-\sigma)/(2+\sigma)}} + O(1), & \gamma > \alpha + 3 \\ K_3 |\log \varepsilon|, & \gamma = \alpha + 3 \end{cases} \quad (2.18)$$

where  $K_1, K_2, K_3$  are positive constants such that  $K_1/K_2 = S$ .

**Verification of (2.16).**

$$u'_\varepsilon(r) = \frac{\varphi'(r)}{(\varepsilon + r^{2+\sigma})^{(\gamma-1)/(2+\sigma)}} - (\gamma-1) \frac{r^{1+\sigma} \varphi(r)}{(\varepsilon + r^{2+\sigma})^{(\gamma+1+\sigma)/(2+\sigma)}}$$

Since  $\varphi \equiv 1$  near 0 it follows that

$$\begin{aligned} \|u_\varepsilon\|_\gamma^2 &= \int_0^1 r^\gamma |u'_\varepsilon|^2 dr = (\gamma-1)^2 \int_0^1 \frac{r^{\gamma+2+2\sigma}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}} dr + O(1) \\ &= \frac{1}{\varepsilon^{(\gamma-1)/(2+\sigma)}} (\gamma-1)^2 \int_0^{\varepsilon^{-1/(2+\sigma)}} \frac{s^{\gamma+2+2\sigma}}{(1+s^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}} ds \\ &= \frac{K_1}{\varepsilon^{(\gamma-1)/(2+\sigma)}} + O(1) \end{aligned}$$

where

$$K_1 = (\gamma-1)^2 \int_0^\infty \frac{s^{\gamma+2+2\sigma}}{(1+s^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}} ds = \|U\|_{\gamma,\infty}^2$$

and  $U$  is given by (1.5).

**Verification of (2.17).**

$$\begin{aligned} \int_0^1 u_\varepsilon^{p+1} r^\theta dr &= \int_0^1 \frac{r^{\gamma+\sigma} \varphi^{p+1}(r)}{(\varepsilon + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}} dr \\ &= \int_0^1 \frac{\varphi^{p+1}(r) - 1}{(\varepsilon + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}} dr + \int_0^1 \frac{r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}} dr \\ &= \int_0^1 \frac{r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}} dr + O(1) = \frac{K'_2}{\varepsilon^{(\gamma+1+\sigma)/(2+\sigma)}} + O(1) \end{aligned}$$

where

$$K'_2 = \int_0^\infty \frac{r^{\gamma+\sigma}}{(1+r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}} dr = \|U\|_{L^{p+1}_{\theta}(0,\infty)}^{p+1}.$$

Thus (2.17) follows with  $K_2 = \|U\|_{L^{p+1}_{\theta}(0,\infty)}^2$  and  $K_1/K_2 = S(\gamma, \theta)$ .

**Verification of (2.18).** We have

$$\begin{aligned} \|u_\varepsilon\|_{L^2_{\gamma+\alpha}}^2 &= \int_0^1 \frac{[\varphi^2(r) - 1] r^{\gamma+\alpha}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}} dr + \int_0^1 \frac{r^{\gamma+\alpha}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}} dr \\ &= \int_0^1 \frac{r^{\gamma+\alpha}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}} dr + O(1). \end{aligned}$$

When  $\gamma > \alpha + 3$  we have

$$\begin{aligned} \int_0^1 \frac{r^{\gamma+\alpha}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}} dr &= \frac{1}{\varepsilon^{(\gamma-3-\alpha)/(2+\sigma)}} \int_0^{\varepsilon^{-1/(2+\sigma)}} \frac{s^{\gamma+\alpha}}{(1 + s^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}} ds \\ &= \frac{1}{\varepsilon^{(\gamma-3-\alpha)/(2+\sigma)}} \int_0^\infty \frac{s^{\gamma+\alpha}}{(1 + s^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}} ds + O(1) \end{aligned}$$

and thus (2.18) follows with

$$K_3 = \int_0^\infty \frac{s^{\gamma+\alpha}}{(1 + s^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}} ds.$$

When  $\gamma = \alpha + 3$  we have

$$\begin{aligned} \int_0^1 \frac{r^{2\alpha+3}}{(\varepsilon + r^{2+\sigma})^{(2(\alpha+2))/(2+\sigma)}} dr &= \frac{1}{2 + \sigma} \int_0^1 \left( \frac{s}{\varepsilon + s} \right)^{(2(\alpha+2))/(2+\sigma)} \frac{ds}{s} \\ &= \frac{1}{2 + \sigma} |\log \varepsilon| + O(1) \end{aligned}$$

and thus (2.18) follows with  $K_3 = 1/(2 + \sigma)$ .

Combining (2.16), (2.17) and (2.18) we get

$$Q_\lambda(u_\varepsilon) = \begin{cases} S - \lambda \frac{K_3}{K_2} \varepsilon^{(2+\alpha)/(2+\sigma)} + O(\varepsilon^{(\gamma-1)/(2+\sigma)}), & \gamma > \alpha + 3 \\ S - \lambda \frac{K_3}{K_2} \varepsilon^{(2\alpha+2)/(2+\sigma)} |\log \varepsilon| + O(\varepsilon^{(2\alpha+2)/(2+\sigma)}), & \gamma = \alpha + 3. \end{cases}$$

In all cases we deduce that  $Q_\lambda(u_\varepsilon) < S$  provided  $\varepsilon > 0$  is small enough. ■

We can state the following result

**Theorem 2.5.** *Let  $\gamma \geq \alpha + 3$ . There exists a solution of (2.1) if and only if  $0 < \lambda < \lambda_1(\alpha)$ .*

**2.2. The case  $\gamma < \alpha + 3$ .** Of special interest to us will be the following eigenvalue problem

$$-\frac{1}{r^{2-\gamma}}(r^{2-\gamma}u')' = \mu r^\alpha u, \quad u \in E_{2-\gamma} \tag{2.19}$$

which should be understood in the weak sense (2.11).

We shall consider the following more general eigenvalue problem

$$-\frac{1}{r^\beta}(r^\beta u')' = \mu r^\alpha u, \quad u \in E_\beta, \quad \beta < 1 \tag{2.20}$$

which is understood in the  $E_\beta$ -weak sense

$$\int_0^1 r^\beta u' \varphi' dr = \mu \int_0^1 r^{\alpha+\beta} u \varphi dr, \quad \forall \varphi \in E_\beta. \tag{2.20'}$$

If one makes the change in variables  $r = s^{1/(1-\beta)}$ ,  $u(r) = v(s)$ , then (2.20) reduces to

$$v'' = \frac{\mu}{(1-\beta)^2} s^{(\alpha+2)/(1-\beta)-2} v, \quad v \in E_0 \tag{2.21}$$

understood in  $E_0$ -weak sense.

The decisive remark is the following

**Lemma 2.6.** *The space  $E_0$  is compactly imbedded in  $L^2_\delta$  for every  $\delta > -1$ .*

**Proof:** Indeed  $E_0$  is compactly imbedded in  $L^p_0$  for every  $p > 1$  according to Kondrachov's theorem. Now let  $\delta > -1$ .

$$\|u\|_{L^2_\delta}^2 = \int_0^1 r^\delta u^2 ds \leq \left( \int_0^1 r^{p\delta} dr \right)^{1/p} \left( \int_0^1 u^{2q} dr \right)^{1/q}$$

for some  $p, q > 1$  such that  $p\delta > -1$  and  $(1/p) + (1/q) = 1$ . Hence  $\|u\|_{L^2_\delta} \leq \text{const.}\|u\|_{L^{2q}}$  and therefore  $E_0$  is compactly imbedded in  $L^2_\delta$ . ■

From the compactness result stated above it follows that if in (2.21)  $(\alpha + 2)/(\beta - 1) - 2 > -1$  (or simply  $\alpha + \beta + 1 > 0$ ) then the spectrum is discrete, positive, the first eigenvalue  $\mu_1(\alpha, \beta)$  is simple and the corresponding eigenspace is generated by a positive function  $\psi_1$ .

**Lemma 2.7.** *Let  $\psi_1$  be defined as above. Then  $\psi_1(r) - \psi_1(0) = O(r^{\alpha+2})$  as  $r \rightarrow 0$  and  $\psi_1(0) > 0$ .*

**Proof:** Since  $\psi_1 \in E_\beta$  and  $\beta < 1$  it follows that  $\psi_1 \in L^\infty(0, 1)$  and a simple argument shows that  $\psi_1 \in C[0, 1]$ . Using the same arguments as in Proposition 2.4 of [8] it follows that

$$\psi_1(r) - \psi_1(0) = O(r^{\alpha+2}).$$

Suppose that  $\psi_1(0) = 0$ . If we make the change of variables  $r = s^{1/(1-\beta)}$ ,  $\omega(s) = \psi_1(r)$  then  $\omega(s)$  satisfies

$$-\omega''(s) = \frac{\mu_1}{(1-\beta)^2} s^\delta \omega \quad \text{in } (0, 1), \quad \delta = \frac{\alpha + 2}{1 - \beta} - 2 \quad \omega \in E_0$$

in  $E_0$ -weak sense. By the strong maximum principle we infer  $\omega'(0) > 0$  since  $\omega(0) = 0$ . Invoking once again the arguments of Proposition 2.4 in [8] we infer  $\omega(s) = O(s^{\delta+2})$  as  $s \rightarrow 0$ . Since  $\delta > -1$  it follows that  $\omega'(0) = 0$ . Contradiction! The lemma is proved. ■

Thus the eigenvalue problem (2.19) leads to a positive discrete spectrum when  $2 - \gamma + \alpha + 1 > 0$ ; i.e.,  $\gamma < \alpha + 3$ .

We denote the least eigenvalue with  $\mu_1(\alpha, \gamma)$  and the corresponding eigenfunction with  $\psi_1$  such that  $\psi_1 > 0$

$$\psi_1(0) = 1, \quad \psi_1(r) - 1 = O(r^{\alpha+2}) \text{ as } r \rightarrow 0. \tag{2.22}$$

Our next task is to estimate

$$Q_\lambda(u_\epsilon) = \frac{\|u_\epsilon\|_\gamma^2 - \lambda \|u_\epsilon\|_{L^{2_{\gamma+\alpha}}}^2}{\|u_\epsilon\|_{L^{p_{\gamma+\alpha}}}^2}$$

where

$$u_\epsilon(r) = \frac{\psi(r)}{(\epsilon + r^{2+\sigma})^{(\gamma-1)/(2+\sigma)}}, \quad \psi(r) = \psi_1(r).$$

First we estimate the terms that enter  $Q_\lambda(u_\epsilon)$ .

**Estimation of  $\|u_\varepsilon\|_\gamma^2$ .**

$$u'(r) = \frac{\psi'(r)}{(\varepsilon + r^{2+\sigma})^{(\gamma-1)/(2+\sigma)}} - (\gamma - 1) \frac{r^{1+\sigma}\psi(r)}{(\varepsilon + r^{2+\sigma})^{(\gamma+1+\sigma)/(2+\sigma)}}$$

so that

$$\begin{aligned} \|u_\varepsilon\|_\gamma^2 = & \int_0^1 \left[ \frac{|\psi'(r)|^2 r^\gamma}{(\varepsilon + r^{2+\sigma})^{2(\gamma-1)/(2+\sigma)}} - 2(\gamma - 1) \frac{r^{\gamma+1+\sigma}\psi(r)\psi'(r)}{(\varepsilon + r^{2+\sigma})^{2(\gamma+\sigma)/(2+\sigma)}} \right. \\ & \left. + (\gamma - 1)^2 \frac{r^{\gamma+2+2\sigma}\psi^2(r)}{(\varepsilon + r^{2+\sigma})^{2(\gamma+2+2\sigma)/(2+\sigma)}} \right] dr. \end{aligned}$$

We integrate by parts the second term above and we get

$$\begin{aligned} \|u_\varepsilon\|_\gamma^2 = & \int_0^1 \frac{|\psi'(r)|^2 r^\gamma}{(\varepsilon + r^{2+\sigma})^{2(\gamma-1)/(2+\sigma)}} dr \\ & + \varepsilon(\gamma - 1)(\gamma + 1 + \sigma) \int_0^1 \psi^2(r) \frac{r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})^{2(\gamma+1+\sigma)/(2+\sigma)}} dr \\ & \int_0^1 \psi^2(r) \frac{r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})^{2(\gamma+1+\sigma)/(2+\sigma)}} dr \\ = & \int_0^1 \frac{r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})^{2(\gamma+1+\sigma)/(2+\sigma)}} dr + \int_0^1 [\psi^2(r) - 1] \frac{r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})^{2(\gamma+1+\sigma)/(2+\sigma)}} dr \\ = & I_1 + I_2 \end{aligned}$$

$$I_1 = \frac{K'_1}{\varepsilon^{(\gamma+1+\sigma)/(2+\sigma)}} + O(1)$$

where

$$K'_1 = \int_0^\infty \frac{r^{\gamma+\sigma}}{(1 + r^{2+\sigma})^{2(\gamma+1+\sigma)/(2+\sigma)}} dr. \tag{2.23}$$

By (2.22) we get

$$|I_2| \leq \text{const.} \int_0^1 \frac{r^{\gamma+\sigma+\alpha+2}}{(\varepsilon + r^{2+\sigma})^{2(\gamma+1+\sigma)/(2+\sigma)}} = O(f(\varepsilon))$$

where

$$f(\varepsilon) = \begin{cases} \varepsilon^{-(\gamma+\sigma-\alpha-1)/(2+\sigma)}, & \gamma + \sigma > \alpha + 1 \\ |\log \varepsilon|, & \gamma + \sigma = \alpha + 1 \\ 1, & \gamma + \sigma < \alpha + 1. \end{cases}$$

We infer that

$$\|u_\varepsilon\|^2 = \int_0^1 \frac{|\psi^1(r)|^2 r^\gamma}{(\varepsilon + r^{2+\sigma})^{2(\gamma-1)/(2+\sigma)}} dr + \frac{1}{\varepsilon^{(\gamma-1)/(2+\sigma)}} + O(\varepsilon) + O(\varepsilon f(\varepsilon)),$$

$\varepsilon \rightarrow 0$ .

A simple computation shows that  $O(\varepsilon) + O(\varepsilon(f(\varepsilon))) = O(\varepsilon f(\varepsilon))$ ,  $\varepsilon \searrow 0$  so that

$$\|u_\varepsilon\|_\gamma^2 = \int_0^1 \frac{|\psi'(r)|^2 r^\gamma}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}} dr + \frac{K_1}{\varepsilon^{(\gamma-1)/(2+\sigma)}} + O(\varepsilon f(\varepsilon)) \tag{2.24}$$

as  $\varepsilon \rightarrow 0$ , where

$$K_1 = (\gamma - 1)(\gamma + 1 + \sigma)K'_1 \tag{2.25}$$

and

$$\varepsilon f(\varepsilon) = \begin{cases} \varepsilon^{(\alpha+3-\gamma)/(2+\sigma)}, & \gamma + \sigma > \alpha + 1 \\ \varepsilon |\log \varepsilon| & \gamma + \sigma = \alpha + 1 \\ \varepsilon, & \gamma + \sigma < \alpha + 1. \end{cases}$$

**Estimation of  $\|u_\varepsilon\|_{L^p_\theta}$ .**

$$\begin{aligned} \|u_\varepsilon\|_{L^{p+1}_\theta}^{p+1} &= \int_0^1 \frac{\psi^{p+1}(r)r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}} = \int_0^1 \frac{r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}} dr \\ &= \int_0^1 \frac{[\psi^{p+1}(r) - 1]r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma+1+\sigma))/(2+\sigma)}} dr = I_1 + I_2. \end{aligned}$$

As above we infer

$$\|u_\varepsilon\|_{L^{p+1}_\theta}^{p+1} = \frac{K'_1}{\varepsilon^{(\gamma+1+\sigma)/(2+\sigma)}} + O(g(\varepsilon))$$

where

$$g(\varepsilon) = \begin{cases} \varepsilon^{-(\gamma+\sigma-\alpha-1)/(2+\sigma)}, & \gamma + \sigma > \alpha + 1 \\ |\log \varepsilon|, & \gamma + \sigma = \alpha + 1 \\ 1, & \gamma + \sigma < \alpha + 1. \end{cases}$$

Hence

$$\|u_\varepsilon\|_{L^{p+1}_\theta}^2 = \frac{K_2}{\varepsilon^{(\gamma-1)/(2+\sigma)}} + O(g(\varepsilon)) \tag{2.26}$$

with

$$K_2 = [K'_1]^{2/(p+1)}. \tag{2.27}$$

**Estimation of  $\|u_\varepsilon\|_\gamma^2 - \lambda \|u_\varepsilon\|_{L^2_{\gamma+\alpha}}$ .** By (2.24) we get

$$\begin{aligned} \|u_\varepsilon\|_\gamma^2 - \lambda \|u_\varepsilon\|_{L^2_{\gamma+\alpha}}^2 &= \frac{K_1}{\varepsilon^{(\gamma-1)/(2+\sigma)}} + \int_0^1 \frac{|\psi'(r)|^2 r^\gamma}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}} dr \\ &\quad - \lambda \int_0^1 \frac{\psi^2(r)r^{\gamma+\alpha}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}} + O(\varepsilon f(\varepsilon)) \end{aligned}$$

$$\begin{aligned} & \int_0^1 \frac{|\psi'(r)|^2 r}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}} dr - \lambda \int_0^1 \frac{\psi^2(r) r^{\gamma+\alpha}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}} dr \\ &= \int_0^1 \left[ |\psi'(r)|^2 r^{2-\sigma} - \lambda r^{\alpha+2-\gamma} \psi^2(r) \right] \frac{r^{2\gamma-2}}{(\varepsilon + r^{2+\sigma})^{(2(\gamma-1))/(2+\sigma)}} dr \\ &= \int_0^1 R(\psi) N(\varepsilon, r) dr = \int_0^1 R(\psi) dr + \int_0^1 R(\psi) [N(\varepsilon, r) - 1] dr \\ &= \int_0^1 R(\psi) dr + O(\varepsilon). \end{aligned}$$

We infer

$$\|u_\varepsilon\|_\gamma^2 - \lambda \|u_\varepsilon\|_{L^2_{\gamma+\alpha}}^2 = \frac{K_1}{\varepsilon^{(\gamma-1)/(2+\sigma)}} + \int_0^1 R(\psi) dr + O(\varepsilon f(\varepsilon)) \quad \text{as } \varepsilon \rightarrow 0 \quad (2.28)$$

where again we have used the fact that  $O(\varepsilon) + O(\varepsilon f(\varepsilon)) = O(\varepsilon f(\varepsilon))$  as  $\varepsilon \searrow 0$ . By (2.26) and (2.28) we infer

$$Q_\lambda(u_\varepsilon) = \frac{K_1}{K_2} + \varepsilon^{(\gamma-1)/2} \int_0^1 R(\psi) dr + O(\varepsilon^{(\gamma-1)/2} \varepsilon f(\varepsilon)).$$

From  $\gamma < \alpha + 3$  we get  $\varepsilon f(\varepsilon) = o(1)$  and therefore

$$= \frac{K_1}{K_2} + \varepsilon^{(\gamma-1)/(2+\sigma)} \int_0^1 R(\psi) dr + o(\varepsilon^{(\gamma-1)/(2+\sigma)}) \quad \text{as } \varepsilon \searrow 0. \quad (2.29)$$

If we recall our definition of  $\psi$  we get

$$\begin{aligned} \int_0^1 R(\psi) dr &= \int_0^1 [r^{2-\gamma} |\psi'(r)|^2 - \lambda r^{2+\alpha-\gamma} \psi^2(r)] dr \\ &= (\mu_1(\alpha) - \lambda) \int_0^1 r^{\alpha+2-\gamma} \psi^2(r) dr = (\mu_1(\alpha) - \lambda) K_3, \end{aligned}$$

where  $K_3$  is a positive constant. This yields

$$Q_\lambda(u_\varepsilon) = \frac{K_1}{K_2} + K_3 (\mu_1(\alpha) - \lambda) \varepsilon^{(\gamma-1)/(2+\sigma)} + o(\varepsilon^{(\gamma-1)/(2+\sigma)}) \quad \text{as } \varepsilon \searrow 0. \quad (2.30)$$

Now we claim that  $K_1/K_2 = S$ . Indeed  $S = \frac{\|U\|_{\gamma, \infty}^2}{\|U\|_{L^{p+1}}^2}$ . By (2.23), (2.27) we see that  $K_2 = \|U\|_{L^{p+1}}^2$ . We have to check that  $K_1 = \|U\|_{\gamma, \infty}^2$ . From (1.13) we get

$$\|U\|_{\gamma, \infty}^2 = \frac{(\gamma - 1)^2}{2 + \sigma} \frac{\Gamma\left(\frac{\gamma+3+2\sigma}{2+\sigma}\right) \Gamma\left(\frac{\gamma-1}{2+\sigma}\right)}{\Gamma\left(\frac{2(\gamma+1+\sigma)}{2+\sigma}\right)}.$$

$K_1$  can be easily computed using (1.14) and we get

$$K_1 = \frac{(\gamma - 1)(\gamma + 1 - \sigma)}{(2 + \sigma)^2} \frac{[\Gamma\left(\frac{\gamma+\sigma+1}{2+\sigma}\right)]^2}{\Gamma\left(\frac{2(\gamma+1+\sigma)}{2+\sigma}\right)}.$$

So that we actually have to check

$$(\gamma + 1 + \sigma) \left[ \Gamma \left( \frac{\gamma + 1 + \sigma}{2 + \sigma} \right) \right]^2 = (\gamma - 1) \Gamma \left( \frac{\gamma + 3 + 2\sigma}{2 + \sigma} \right) \Gamma \left( \frac{\gamma - 1}{2 + \sigma} \right). \tag{2.31}$$

From the well-known relation  $\Gamma(x + 1) = x\Gamma(x)$  we infer

$$\Gamma \left( \frac{\gamma + 1 + \sigma}{2 + \sigma} \right) = \frac{\gamma - 1}{2 + \sigma} \Gamma \left( \frac{\gamma - 1}{2 + \sigma} \right)$$

and

$$\Gamma \left( \frac{\gamma + 3 + 2\sigma}{2 + \sigma} \right) = \frac{\gamma + 1 + \sigma}{2 + \sigma} \Gamma \left( \frac{\gamma + 1 + \sigma}{2 + \sigma} \right).$$

(2.31) follows from the equalities above. Hence

$$\frac{K_1}{K_2} = S. \tag{2.32}$$

By (2.30) and (2.32) we get

$$\text{If } \lambda > \mu_1(\alpha) \text{ then } S_\lambda < S. \tag{2.33}$$

Now we can state

**Proposition 2.8.** *If  $\mu_1(\alpha) < \lambda_1(\alpha)$  ( $\alpha > -2, 1 < \gamma < \alpha + 3$ ) then the problem possesses at least a solution for each  $\lambda \in (\mu_1(\alpha), \lambda_1(\alpha))$ .*

There is still a question we must answer, namely when does the following spectral equality hold?

$$\mu_1(\alpha) < \lambda_1(\alpha). \tag{2.34}$$

We recall that  $\mu_1(\alpha) = \mu_1(\alpha, \gamma)$  and  $\lambda_1(\alpha) = \lambda_1(\alpha, \gamma)$ , are the least eigenvalues of the following eigenvalue problems

$$-\frac{1}{r^{2-\gamma}}(r^{2-\gamma}\psi')' = \mu r^\alpha \psi, \quad \psi \in E_{2-\gamma}, \tag{2.35}$$

and respectively

$$-\frac{1}{r^\gamma}(r^\gamma\psi')' = \lambda r^\alpha \varphi, \quad \varphi \in E_\gamma \tag{2.36}$$

where  $\alpha > -2, 1 < \gamma < \alpha + 3$  and these eigenvalue problems are understood in the weak sense (2.20') and (2.11).

The aim of our next section is to answer this question.

**3. Proof of the spectral inequality.** The main result of this section is the following.

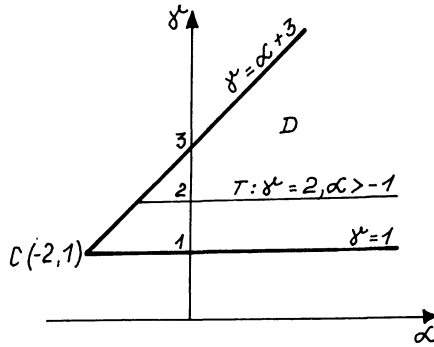


Figure 1.

**Proposition 3.1.**  $\mu_1(\alpha, \gamma) < \lambda_1(\alpha, \gamma)$  for all  $\alpha > -2, 1 < \gamma < \alpha + 3$ .

The proof will be carried out in several steps. Let us first denote by  $D$  the set

$$D = \{(\alpha, \gamma) \mid 1 < \gamma < \alpha + 3\}.$$

In the  $(\alpha, \gamma)$ -plane  $D$  looks like Figure 1.

Let us denote by  $S$  the set

$$S = \{(\alpha, \gamma) \in D \mid \mu_1(\alpha, \gamma) < \lambda_1(\alpha, \gamma)\}.$$

We also consider the following semigroup of transformations of the  $(\alpha, \gamma)$ -plane  $(H_\beta)_{\beta > 0}$  where the  $H_\beta$  are given by the law

$$(\alpha, \gamma) \xrightarrow{H_\beta} (\alpha_\beta, \gamma_\beta) = \beta(\alpha + 2, \gamma - 1) + (-2, 1). \tag{3.1}$$

$(H_\beta)_{\beta > 0}$  is a semigroup of homoteties of pole  $C(-2, 1)$ . Clearly,  $H_\beta(D) \subset D \forall \beta > 0$ .

**Proof of Proposition 3.1.**

**Step 1.**  $H_\beta(S) \subset S, \forall \beta > 0$ . We have to prove that if  $\mu_1(\alpha, \gamma) < \lambda_1(\alpha, \gamma)$  then also

$$\mu_1(\alpha_\beta, \gamma_\beta) < \lambda_1(\alpha_\beta, \gamma_\beta).$$

Indeed, let us make the change of variables  $r = s^\beta$  in the equations (2.35) and (2.36). We get

$$-\frac{1}{s^{2-\gamma_\beta}}(s^{2-\gamma_\beta}\psi')' = \beta^2\mu s^{\alpha_\beta}\psi, \quad \psi \in E_{2-\gamma_\beta}, \tag{3.2}$$

and respectively

$$-\frac{1}{s^{\gamma_\beta}}(s^{\gamma_\beta}\varphi')' = \beta^2\lambda s^{\alpha_\beta}\varphi, \quad \varphi \in E_{\gamma_\beta} \tag{3.3}$$

where  $\alpha_\beta$  and  $\gamma_\beta$  are given by (3.1). This yields  $\mu_1(\alpha_\beta, \gamma_\beta) = \beta^2\mu_1(\alpha, \gamma)$  and  $\lambda_1(\alpha_\beta, \gamma_\beta) = \beta^2\lambda_1(\alpha, \gamma)$  and thus  $\mu_1(\alpha, \gamma) < \lambda_1(\alpha, \gamma)$  if and only if  $\mu_1(\alpha_\beta, \gamma_\beta) < \lambda_1(\alpha_\beta, \gamma_\beta)$ .

**Step 2.**  $\mu_1(\alpha, 2) < \lambda_1(\alpha, 2)$ , for all  $\alpha > -1$ . We denote by  $\psi$  and respectively  $\varphi$  eigenfunctions corresponding to  $\mu_1$  and  $\lambda_1$  such that  $\psi, \varphi > 0$  in  $(0, 1)$ . Then  $\psi$  and  $\varphi$  satisfy

$$-\psi'' = \mu_1 r^\alpha \psi, \quad \varphi \in E_0 \tag{3.4}$$



and respectively

$$-\frac{1}{r^2}(r^2\varphi')' = \lambda_1 r^\alpha \varphi, \quad \psi \in E_2. \tag{3.5}$$

As in Lemma 2.7 we can prove that

$$\varphi(0) > 0 \tag{3.6}$$

$$\varphi(r) - \varphi(0) = O(r^{\alpha+2}) \quad \text{as } r \searrow 0. \tag{3.7}$$

Hence  $\varphi'(0) = 0$  and  $\varphi \in E_0$ . From (3.5) we infer

$$-\varphi'' = \lambda_1 r^\alpha + \frac{2}{r}\varphi'. \tag{3.8}$$

Multiplying (3.8) with  $\varphi$  and then integrating on  $(0, 1)$  we get

$$\int_0^1 |\varphi'|^2 dr = \lambda_1 \int_0^1 r^\alpha \varphi^2 dr + \int_0^1 \frac{2\varphi\varphi'}{r} dr.$$

Since

$$\int_0^1 |\varphi'|^2 dr \geq \mu_1 \int_0^1 r^\alpha \varphi^2 dr$$

it follows

$$\mu_1 \int_0^1 r^\alpha \varphi^2 dr \leq \lambda_1 \int_0^1 r^\alpha \varphi^2 dr + \int_0^1 \frac{2\varphi\varphi'}{r} dr.$$

It is a routine exercise using (3.5) to prove that  $\varphi' < 0$ . Thus

$$\mu_1 \int_0^1 r^\alpha \varphi^2 dr < \lambda_1 \int_0^1 r^\alpha \varphi^2 dr$$

hence  $\mu_1 < \lambda_1$ .

**Step 3.**  $S = D$ . We have proved so far that the half line  $(T) : \alpha > -1, \gamma = 2$  lies in  $S$  (see Figure 1). Hence by Step 1 we get  $\bigcup_{\beta>0} H_\beta(T) \subset S$ . It is an easy geometric remark that

$$\bigcup_{\beta>0} H_\beta(T) = D \quad (\text{see Figure 1}).$$

Proposition 3.1 is proved. ■

Now we can state the following result.

**Theorem 3.2.** a) For  $\gamma \geq \alpha + 3, \alpha > -2$  problem (2.1) has a solution if and only if  $\lambda \in (0, \lambda_1(\alpha, \gamma))$ .

b) For  $1 < \gamma < \alpha + 3, \alpha > -2$  we have  $\mu_1(\alpha, \gamma) < \lambda_1(\alpha, \gamma)$  and problem (2.1) has a solution if  $\lambda \in (\mu_1(\alpha, \beta), \lambda_1(\alpha, \gamma))$  and no solution if  $\lambda \leq 0$  or  $\lambda \geq \lambda_1$ .

**Remark 3.3.** In the paper [3] of Brézis-Nirenberg it was proved that if  $\gamma = 2, \alpha = 0$  then problem (2.1) has a solution if and only if  $\lambda \in (\pi^2/4, \pi^2)$ . An easy

computation shows that  $\mu_1(0, 2) = \pi^2/4$  and  $\lambda_1(0, 2) = \pi^2$  and thus our result seems optimal. It is therefore natural to ask the following question:

Is it true that in the case  $1 < \gamma < \alpha + 3$  problem (2.1) has no solution for  $\lambda < \mu_1(\alpha, \gamma)$ ?

**Remark 3.4.** We consider the problem

$$\begin{aligned} -\Delta u &= r^\sigma |u|^{p-1} u + \lambda r^\alpha u \quad \text{in } B_R \subset \mathbf{R}^N \quad (r = |x|) \\ u &= 0 \quad \text{on } \partial B_R \\ u &> 0 \end{aligned} \tag{3.9}$$

$\alpha, \sigma \geq -2, p + 1 = (2N + 2\sigma)/(N - 2)$ .

By a weak solution of (3.8) we mean a function  $u \in H_0^1(B_R)$   $u > 0$  satisfying

$$\int_{B_R} \nabla u \nabla \varphi = \int_{B_R} r^\sigma |u|^{p-1} u \varphi \, dx + \lambda \int_{B_R} r^\alpha u \varphi \, dx, \quad \forall \varphi \in H_0^1(B_R). \tag{3.10}$$

It is easily seen that at least at a formal level the radial solutions of (3.9) satisfy

$$-\frac{1}{r^{N-1}}(r^{N-1}u') = r^\sigma |u|^{p-1} u + \lambda r^\alpha u \quad \text{in } (0, R), \quad u' = du/dr, \quad u > 0, \quad u \in E_{N-1}. \tag{3.11}$$

We claim that weak solutions of (3.11) in the sense of (2.2) satisfy (3.10). Indeed, if  $u$  satisfies (3.11), then  $u$  satisfies

$$-\Delta u = r^\sigma |u|^{p-1} u + \lambda r^\alpha u \quad \text{in } B_R \setminus \{0\} \tag{3.12}$$

by standard elliptic regularity. Let  $\varphi \in H_0^1(\Omega)$ . We multiply (3.12) with  $\varphi$  and then we integrated on  $D_\varepsilon = \{x \in \mathbf{R}^N : \varepsilon \leq |x| \leq R\}$  using Green’s formula

$$\int_{D_\varepsilon} \nabla u \nabla \varphi \, dx - \int_{D_\varepsilon} r^\sigma |u|^{p-1} u \varphi \, dx - \lambda \int_{D_\varepsilon} r^\alpha u \varphi \, dx + \int_{|x|=\varepsilon} u'(\varepsilon) \varphi(x) \, dS = 0.$$

Here  $ds$  is the surface element on  $|x| = \varepsilon$ ,  $dS = \sum_{N-1} \varepsilon^{N-1} d\theta$ ;  $d\theta$  is the surface element on  $[|x| = 1]$  and  $\sum_{N-1}$  is the  $(N - 1)$ -dimensional measure of the sphere  $|x| = 1$ . We get

$$\begin{aligned} \int_{D_\varepsilon} \nabla u \nabla \varphi \, dx - \int_{D_\varepsilon} r^\sigma |u|^{p-1} u \varphi \, dx - \lambda \int_{D_\varepsilon} r^\alpha u \varphi \, dx \\ + \sum_{N-1} \int_{|x|=\varepsilon} \varepsilon^{N-1} u'(\varepsilon) \varphi(\varepsilon, \theta) \, d\theta = 0. \end{aligned} \tag{3.13}$$

Since  $u \in E_{N-1}$  we deduce that  $\varepsilon^N u'(\varepsilon)^2 \rightarrow 0$  on a subsequence  $\varepsilon = \varepsilon_k \rightarrow 0$ . Hence  $\varepsilon_k^{N-1} u'(\varepsilon_k) \rightarrow 0$  since  $N > 1$ . In (3.13) we let  $\varepsilon = \varepsilon_k \rightarrow 0$ . This yields

$$\int_{B_R} \nabla u \cdot \nabla \varphi \, dx - \int_{B_R} r^\sigma |u|^{p-1} u \varphi \, dx - \lambda \int_{B_R} r^\alpha u \varphi \, dx = 0;$$

i.e., exactly (3.10). Now it is easy to formulate existence results for the problem (3.9). Let  $\lambda_1(\alpha)$  be the first eigenvalue of the following eigenvalue problem

$$-\Delta u = \lambda r^\alpha u \quad \text{in } B_R, \quad u = 0 \quad \text{on } \partial B_R. \tag{3.14}$$

As in Figueiredo [5] we infer that  $\lambda_1(\alpha)$  is simple and the corresponding eigenfunction may be chosen positive. We deduce that in fact  $\lambda_1(\alpha) = \lambda_1(\alpha, N - 1)$  where  $\lambda_1(\alpha, N - 1)$  was defined at (2.11). Let  $\mu_1(\alpha)$  be the first eigenvalue of the eigenvalue problem (2.35) with  $\gamma = N - 1$ .

We can now state

**Theorem 3.5.** a) For  $N \geq \alpha + 4$ ,  $\alpha \geq -2$  problem (3.9) has a radial weak solution if and only if  $\lambda \in (0, \lambda_1(\alpha))$ .

b) For  $2 < N < \alpha + 4$ ,  $\alpha > -2$ , we have  $\mu_1(\alpha) < \lambda_1(\alpha)$  and if  $\lambda \in (\mu_1(\alpha), \lambda_1(\alpha))$  problem (3.9) has a radial weak solution. It has no radial weak solution if  $\lambda \leq 0$ ,  $\lambda \geq \lambda_1$ .

Concerning the regularity of radial weak solutions it can be easily proved that these are classical solutions of (3.9). Indeed  $u \in E_{N-1}$ . Let  $\theta = (N-1) + (\sigma+1)/2$ ,  $q = (p-1)N/2$ . Then, a simple computation shows that  $\frac{(N-1)-1}{2} = \frac{\theta+1}{q}$  so that by the imbedding lemma we get that  $u \in L_\theta^q(0, R)$ ; i.e.,  $r^{(N-1)r^{(\sigma+1)/2}}|u|^{(p-1)N/2} \in L^1(0, R)$  which can be restated as  $r^\sigma|u|^{p-1} \in L^{N/2}(B_R)$ . Hence if we set  $a(r) = r^\sigma|u|^{p-1}$  then  $u$  is a weak solution of

$$-\Delta u = a(r)u + \lambda r^\alpha u \quad \text{in } H_0^1(B_R)$$

with  $a \in L^{N/2}(B_R)$ .

We can now proceed as in Brézis-Nirenberg [3] to infer that  $u \in L^\infty(B_R)$  and hence by standard elliptic regularity  $u \in C^2(B_R)$ .

#### REFERENCES

- [1] T. Aubin, "Nonlinear Analysis on Manifolds; Monge-Ampère Equations," Springer-Verlag, 1982.
- [2] G.A. Bliss, *An integral inequality*, J. London Math. Soc. 5 (1930), 40-46.
- [3] H. Brézis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. 36 (1983), 437-477.
- [4] H.D. Dwight, "Tables of Integrals and Other Mathematical Data," MacMillan, New York, 1964.
- [5] D.G. de Figueiredo, *Positive solutions of semilinear elliptic problems*, in "Differential Equations," Lect. Notes in Math. 957, Springer-Verlag, 1982.
- [6] D. Kinderlehrer and G. Stampacchia, "An Introduction to Variational Inequalities and their Applications," Academic Press, 1980.
- [7] V.G. Maz'ya, "Sobolev Spaces," Springer-Verlag, 1985.
- [8] L.I. Nicolaescu, *Existence and regularity for a singular semilinear Sturm-Liouville problem*, Differential Integral Equations, to appear.